Robust Cooperative Control and Optimization of Multi-Agent Systems

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Statement of Originality

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

Date

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Abstract

Multi-agent systems have been extensively studied in modern control theory due to their wide range of applications in robotics, transportation networks, and smart grid, etc. In this thesis, we investigate control and optimization problems in networked multi-agent systems from the perspectives of distributed coordinated control, distributed optimization and game algorithm design, and network controllability.

In the first part of this thesis, we study robust cooperative control of multi-agent systems. Firstly, we consider the robust finite-time connectivity-preserving consensus tracking and formation control problems. An integral sliding mode based framework is proposed to simultaneously achieve disturbance rejection, finite-time convergence, and connectivity preservation for double-integrators with bounded disturbances and a virtual leader. The result is further extended to formation tracking. Secondly, we propose a continuous filter-based consensus protocol for a class of high-order multi-agent systems to deal with model uncertainties and disturbances in the agent dynamics. Sufficient conditions are given to guarantee asymptotic consensus tracking results. An output feedback control algorithm is further proposed by using only position information.

In the second part, we study distributed optimization and game algorithm design problems. Firstly, we consider distributed quadratic optimization problem where the optimal solution and the objective functions are both assumed to be time-varying. When there exists a local compact convex constraint set for each agent, by using projected gradient methods, we prove that the tracking errors are uniformly ultimately bounded (UUB) with arbitrarily small bounds. Next, we consider a distributed Nash equilibrium seeking problem for a nonsmooth noncooperative game. Each player has a convex cost function and is subject to multiple shared constraints. The objective is to design a Nash equilibrium seeking law such that each player minimizes its own cost function in a distributed way. Both class-C_2 objective functions and locally Lipschitz objective functions are studied.

In the third part, the network controllability problem is investigated. Specifically, we study the controllability of a class of antagonistic networks with not only positive weights but also negative ones. This kind of network model has a wide range of applications in social network science. Nodes connected with a positive edge can be viewed as friends and linked with a negative edge can be viewed as enemies. We present a necessary condition to characterize the controllability and analyze the relationship between an antagonistic network and an all-positive network in terms of controllability.

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Symbols and Acronyms

$\mathbb{R}$	set of reals
$\mathbb{R}^{\geq 0}$	set of nonnegative reals
$\mathbb{R}^n$	set of n-dimensional real column vectors
$\mathbb{R}^{n\times n}$	set of $n \times n$ real matrices
$\mathbf{1}$	column vector with all elements being 1
$\mathbf{0}$	column vector with all elements being 0
I_n	$n \times n$ identity matrix
$0_{m\times n}$	$m \times n$ zero matrix
$[a_i]\in \mathbb{R}^N$	column vector defined as $[a_i] = [a_1, \cdots, a_N]^T$
$[a_{ij}]\in \mathbb{R}^{N\times N}$	matrix with the element in i-th row and j-th column being a_{ij}
A^T	transpose of matrix A
A^{-1}	inverse of matrix A
$\|\cdot\|$	2-norm
$\|\cdot\|_1$	1-norm
$A \otimes B$	Kronecker product of matrices A, B

$\mathbf{sig}^{\alpha}(e)$	it equals to $\|e\|^{\alpha}\left(e/\|e\|\right)$, if $e \neq \mathbf{0}$, and $\mathrm{sig}^{\alpha}(e) = \mathbf{0}$ if $e = \mathbf{0}$
(A, b)	for a set $A \subset \mathbb{R}^n$ and a vector $b \in \mathbb{R}^n$, it denotes a set satisfying $(A, b) = \{(a^T, b^T)^T, a \in A\}$
(A, B)	for two sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$, it denotes a set satisfying $(A, B) = \{(a^T, b^T)^T, a \in A, b \in B\}$
$A + b$	for a set $A \subset \mathbb{R}^n$ and a vector $b \in \mathbb{R}^n$, it denotes a set satisfying $A + b = \{a + b, a \in A\}$
$b^T A$	for a set $A \subset \mathbb{R}^n$ and a vector $b \in \mathbb{R}^n$, it denotes a set satisfying $b^T A = \{b^T a, a \in A\}$
$\mathbf{im}(P)$	vector space generated by the columns of the matrix P
$R \times \mathbf{im}(P)$	for a square matrix R, it denotes a vector space $\{Rx : x \in \mathrm{im}(P)\}$
$B \backslash A$	for two sets A and B, it denotes the relative complement of A in B
RISE	Robust Integral of the Sign of the Error
ISM	Integral Sliding Mode
UUB	Uniformly Ultimately Bounded
KKT	Karush-Kuhn-Tucker
EP	Equitable Partition
AEP	Almost Equitable Partition
GAEP	Generalized Almost Equitable Partition

Chapter 1

Introduction

1.1 Background, Motivation and Objectives

1.1.1 Distributed Multi-Agent Coordination

Multi-agent systems have attracted much attention in the recent decade, as it has a wide range of potential applications in both civilian and military areas. Typical multi-agent systems include automated guided vehicles (AGVs) [3], unmanned air vehicles (UAVs) [4], and sensor networks [5], to name a few.

Distributed multi-agent coordination studies the distributed control problems in multi-agent systems. The objective is to make a team of agents collaborate with each other to achieve desired behaviors in a cooperative fashion by distributed control methods. The research on this topic can be simply classified into several directions, such as consensus, formation, distributed optimization, and distributed estimation, etc [6]. In a consensus problem, the states of a group of agents reach an agreement through neighboring information exchange. The "agreement" can be a position (velocity, acceleration, etc) [7], a trajectory [8], a voltage (frequency) [9], or a common opinion [10], etc. Formation control, in general, refers to the control of a group of autonomous vehicles or robots to achieve a desired shape. The shape might be fixed [11] or change with time [12]. Optimization is to find an optimal solution

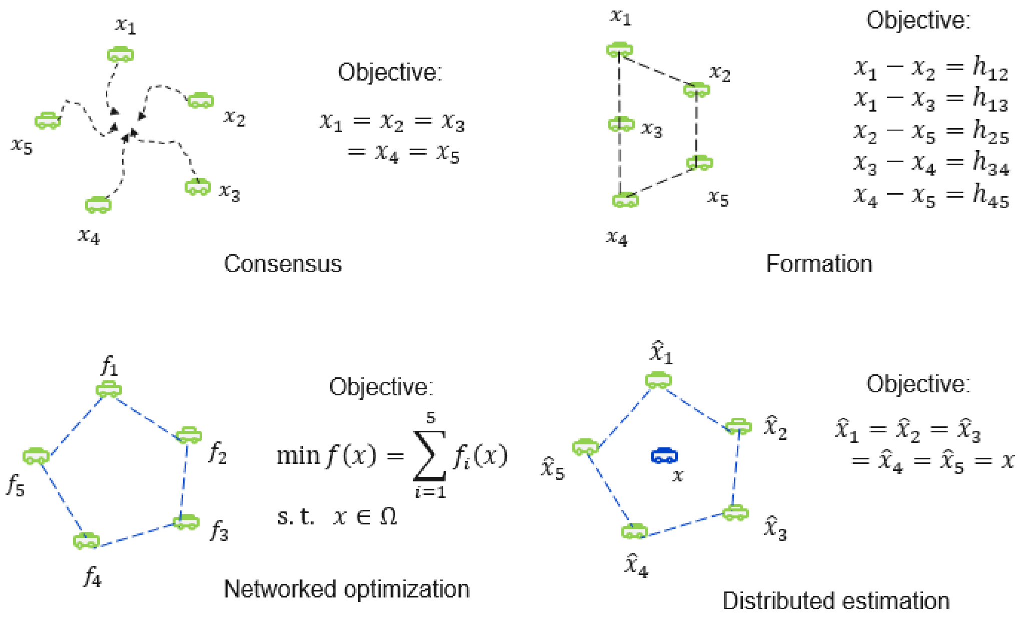

Figure 1.1: Multi-agent coordination problems.

of a cost function subject to some given constraints [6]. For a distributed multi-agent optimization problem, the cost function is a sum of a series of local objective functions in the network. Distributed estimation techniques are designed for the case where the global information is not available to all the agents. Some classical estimation methods, e.g., Kalman filter and sliding mode, were studied in the state-of-the-art distributed control architecture [13, 14]. The objectives of these problems in a mathematical form can be described in Fig. 1.1.

Distributed Robust Cooperative Control

In practical application of multi-agent systems, uncertainties and disturbances are always surrounded, especially for systems working in complex environments. For example, for autonomous underwater vehicles (AUVs) that work in the seabed environment, there are a variety of uncertain factors, such as hydrodynamic coefficients. Also, when navigating in the ocean, they are often influenced by external disturbances such as ocean currents.

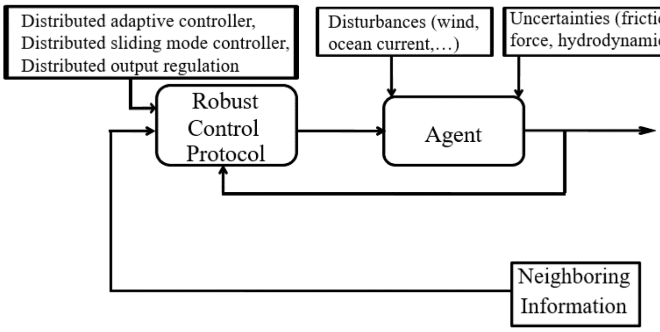

Figure 1.2: Framework of distributed robust cooperative control problems.

According to different mathematical expressions of the uncertainties and disturbances, different approaches have been developed including adaptive control [15], sliding mode control [16], and output regulation [17], etc. Adaptive control usually requires the uncertainties to be linearly parameterizable, sliding mode can be used to deal with bounded disturbances, and output regulation works for external disturbance with a specific form. For multi-agent systems, the disturbances usually come from external sources, e.g., wind, wave, current, and earth magnetic disturbance [18]. Uncertainties mainly refer to the state-related uncertainties in the modelling process, and the uncertainty function may be related to both the agent itself and other agents. Uncertainties studied in the existing literature include hydrodynamic coefficients [19], parameter uncertainties in the inertial structure [20], and friction effects [21], etc. In order to improve system stability and performance, control algorithms designed for multi-agent systems against uncertainties and disturbances should be considered.

Distributed Optimization and Nash Equilibrium Seeking

In a distributed optimization problem, all the agents collaborate with each other to optimize a common objective function, which is composed of the local objective of each agent. The communication of the network is distributed and the agents can only communicate with their neighbors. Typical applications of distributed optimization techniques include economic dispatch in smart grid [22, 23], formation flight [24], and demand response [25], etc. Most of the distributed optimization methods are designed based on discrete-time algorithm updating. For example, in [26], a projected subgradient method was developed for a constrained optimization problem. In [27], a discrete-time distributed optimization algorithm was proposed for a multi-agent system which consists of multiple clusters of agents. In [28], a randomized incremental subgradient method was developed. Recently, continuous-time distributed optimization algorithms have attracted many interests. Most of these works are developed using consensus and saddle-point-like dynamics [22, 29–32].

Noncooperative games, as one of the most important branch of game theory, are widely applied in modelling and decision-making of demand response [33], resource allocation [34], and road pricing [35], to list only a few. Compared with cooperative games, noncooperative games focus on competitive decision-making involving several players and any cooperation must be self-enforcing. For demand response problems in smart grids, it is important to understand the demand of energy for both the energy provider and end-users in a city. Obviously, the consumption of the electricity is different at different time of the day. One may expect that the demand of electricity is much higher in the evening (i.e., peak time) than the morning. In this case, the energy provider may frequently meet the challenge of generating the power at full capacity. Therefore, it is desirable to come up with some pricing strategy to the users, such that the peak time can be shifted and the demand of energy can be relatively equal over the day. Identifying the Nash equilibrium of this energy consumption game will enable more efficient power generation and management.

Nash equilibrium is a critical concept in noncooperative game theory, which refers to a state where no player can improve its utility by changing its strategy, if the other players maintain their current strategies [36]. Although noncooperative game theory has been studied for many years, how to design efficient and effective algorithms to find the Nash equilibria is still an interesting problem.

1.1.2 Network Controllability

For distributed multi-agent coordination problems considered in Section 1.1.1, each node in the network is assumed to be equipped with a controller; and the objective is just to design an effective algorithm for each node such that the network acts as one expects. For a small-scale network, adding a controller to each node might be possible. However, for large-scale networks, controlling each node becomes impractical or even impossible. For example, Pokec, an online social network in Slovakia, contains 1632803 nodes and 30622564 edges [1]. Some questions arise concerning the control of such networks, e.g., whether and how the whole network can be driven to any desired state by only a subset of the nodes. The concept of controllability in the control theory field provides a theoretical approach to study this problem. For a linear system

$$\dot{x} = Ax + Bu, \tag{1.1}$$

it is said to be (state) controllable if and only if it can be driven from any initial state to any desired state [37]. Imagine that in (1.1), the variable x represents the state of the network, Ax represents a certain network intrinsic property, matrix B represents the selection matrix which indicates how we select the controlled nodes, and u represents the control. In this case, the controllability of system (1.1) is equivalent to the controllability of the network.

In social networks, the interactions involve both positive and negative relation-

[1] http://snap.stanford.edu/data/soc-pokec.html

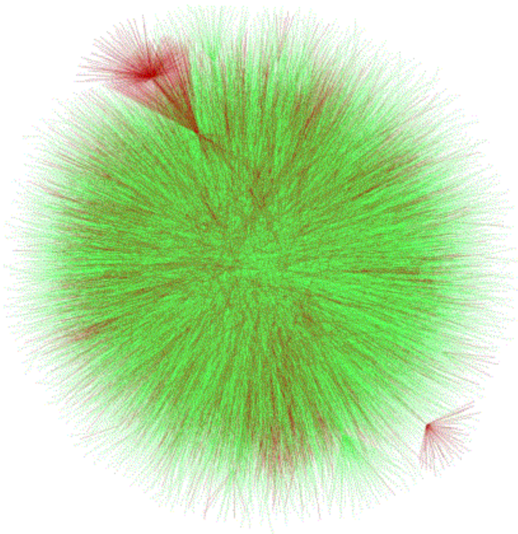

Figure 1.3: The Slashdot Zoo social network [1].

ships. For example, “agreement” and “disagreement”, “trust” and “distrust”, and “friendship” and “hostility”, etc [38]. An example of such networks is the Slashdot Zoo [1] (shown in Fig. 1.3), which is a social network of users of Slashdot (slashdot.org). The relationships among users are represented by a directed graph with nodes indicating users and edges standing for “friend” and “foe” relationships. The network contains 77,985 nodes and 510,157 relationships, with “friend” relationships shown in green and “foe” relationships in red. Controllability, in such a network, can be used to study the possibility of selecting some agents to influence others and finally form a desired social behavior, e.g., a common opinion [39].

A network with positive and negative links can be modelled by a signed graph [1,40]. However, in the network controllability research, especially for those utilizing multi-agent system models, the tools are mainly developed based on unsigned graph theory. The objective of our network controllability research is to model and analyze the controllability of networks with both positive and negative interactions using signed graphs. Specifically, relationships between the controllability of a signed graph network and an unsigned graph network will be analyzed, and some concepts in the unsigned graphs will be generalized to signed graphs.

1.2 Literature Review

1.2.1 Robust Finite-Time Connectivity Preserving Consensus Tracking and Formation Control

As mentioned earlier, various methods have been developed to deal with uncertainties and disturbances in multi-agent systems. In [41], a distributed finite-time consensus algorithm for second-order nonlinear multi-agent systems was proposed based on terminal sliding mode control. In [42], the authors proposed a continuous and distributed consensus protocol to achieve asymptotic consensus tracking of a second-order multi-agent system with disturbances and unmodelled dynamics. A pinning control protocol was proposed in [43] to achieve synchronization of interconnected systems with identical nonlinear dynamics. Adaptive control methods were also utilized to address this issue in [15, 44–48].

Network connectivity plays an important role in the distributed control of multi-agent systems, e.g. multi-robot rendezvous. The network topology may be changed during motion evolutions since the agent usually has only limited sensing and communication capabilities. If there are no mechanisms to ensure connectivity preservation, these algorithms may fail when the agents are out of the sensing region. Potential function based methods are widely applied in the connectivity preservation. In [49], a distributed gradient method was developed to maintain the initial network topology for a first-order multi-robot system. In [50], the robot's dynamics was considered to have non-holonomic constraints. In [51], a general class of bounded potential functions were designed to achieve connectivity preservation for a second-order multi-agent system.

There have been some algorithms that consider both disturbance rejection and connectivity preservation, such as [52, 53]. However, most of these works didn't specify the convergence rate of the algorithms. When considering the convergence rate and robustness to external disturbances, finite-time control laws usually have better performance [54]. Finite-time consensus problems were investigated in [2, 41, 55–66]. Nonsmooth tools were presented in [55] to analyze finite-time stability of continuous-time systems where the differential equations have a discontinuous right-hand side, and then extended the results to networked finite-time consensus. For second-order systems, the authors in [56] proposed a binary consensus protocol which only requires sign information between neighbors. Considering intrinsic nonlinear dynamics, some nonlinear consensus protocols using odd functions were developed in [60] and [66], and homogeneity theory is used to prove the stability. The finite-time rendezvous problem was proposed in [62]. Integrator-type dynamics with Lipschitz nonlinearities was investigated in [2]. Disturbance rejection for first-order systems was considered in [67].

1.2.2 Robust Consensus Tracking of High-Order Multi-Agent Systems

The consensus problem for high-order multi-agent systems has recently drawn much attention. For example, the authors in [45] considered the synchronization problem of high-order systems with unknown nonlinear dynamics and disturbances. A robust adaptive control law was proposed to ensure all nodes achieve synchronization to a leader node in the sense of cooperatively UUB. In [68], the high-order consensus problem was turned into a normal H_∞ control problem, and conditions for H_∞ consensus were obtained under undirected and directed communication topologies, respectively. In [69], a multi-surface sliding control scheme was designed to guarantee finite-time consensus for high-order uncertain nonlinear systems. In [70], the authors studied a consensus tracking problem for high-order linear multi-agent systems under switching directed topology and missing control inputs.

Most of the existing algorithms were developed by using full state information (e.g., position, velocity, and acceleration, etc.). However, in many cases, it is unrealistic and costly to equip all agents with sensors to collect information of velocity, acceleration and even higher-order states. Therefore, an output feedback control algorithm using only position information is an interesting and important problem. It is challenging to consider the distributed output feedback control problem for multiple coordinated agents with high-order dynamics in the presence of uncertainties. In [71], the authors studied the distributed containment control problem for second-order multi-agent systems without velocity measurements in the presence of multiple leaders. The authors in [72] investigated the distributed coordination problem for second-order multi-agent systems with intrinsic nonlinear dynamics using only relative position measurements. In [65], the authors investigated the distributed finite-time consensus tracking problem for coupled harmonic oscillators using both state and output feedback control. In [73], the authors proposed a cyclic-small-gain approach to solve the distributed output-feedback control problem of a class of nonlinear multi-agent systems.

1.2.3 Distributed Time-Varying Optimization

The time-varying optimization problem, as an extension of static optimization, considers that the objective function varies with time. The developed algorithm should be able to track the trajectory of the optimal solution with acceptable variations. A general distributed time-varying optimization problem can be formulated as follows:

$$\begin{aligned} &\min\ f(x,t) = \sum_{i=1}^{N} f_i(x,t), \text{ for } x \in X^n \\ &\text{such that } g_i(x,t) > 0, h_i(x,t) = 0, \end{aligned} \tag{1.2}$$

where $f(x,t) \in \mathbb{R}$ is the global objective function of the network, $f_i(x,t) \in \mathbb{R}$ is the local objective function of agent i, x is the state variable of all agents, and g_i and h_i are the inequality constraint and equality constraint of agent i, respectively. In [74], the authors proposed a sliding mode based method to solve the time-varying quadratic optimization problem. In [75], a finite-time observer based method was proposed to solve the time-varying optimization problem for systems with first-order and second-order dynamics. However, both algorithms assumed that the optimal solution can be selected from the whole real domain.

A similar problem that has been studied for many years is the dynamic multi-objective optimization problem [76], [77], which is usually formulated as follows:

$$\begin{aligned} &\min f(x,t) = [f_1(x,t), f_2(x,t), \cdots, f_m(x,t)]^T, \text{ for } x \in X^n \\ &\text{such that } g(x,t) > 0, h(x,t) = 0, \end{aligned} \tag{1.3}$$

where $f(x,t) \in \mathbb{R}^m$ is the objective function vector, and $g(x,t)$ and $h(x,t)$ are two time-varying constraints. Compared with distributed time-varying optimization problems, most of the algorithms in this area were developed based on evolutionary algorithms [78], and used centralized communication.

1.2.4 Distributed Generalized Nash Equilibrium Seeking for Noncooperative Games

How to find a Nash equilibrium is an interesting and important problem for non-cooperative games. In [79], an iterative steepest descent algorithm was proposed for numerical approximation of local Nash equilibria. In [80], an adaptive learning technique was used to iteratively compute the Nash equilibrium of a multi-player game. Extremum seeking based methods were proposed to search Nash equilibrium [81–83].

Compared with the conventional Nash games, the generalized Nash games assume that each player's feasible set can be constrained by the rival players' strategies [84–86]. A convex generalized Nash game usually has continuous strategy spaces and the actions are coupled through both objective functions and constraints. It was first formally proposed by [87], followed by [88] studying economic equilibria. Since then, there have been many studies working on the existence of a generalized Nash equilibrium, and algorithms to compute it. For instance, a method involving variational inequalities was presented in [89].

Most of the aforementioned literature require that each player can obtain the s-trategies of its opponents. In practice, especially in some multi-agent networks, the agent might have limited interaction (communication) with other agents. The recently proposed algorithm in [74] solved a distributed Nash equilibrium seeking problem for unconstrained noncooperative games based on average consensus and singular perturbation theory. In [90], discrete-time adaptive algorithms were presented to solve Nash equilibrium seeking problems, where the objective functions were required to be smooth.

1.2.5 Controllability of Multi-Agent Systems with Antagonistic Interactions

In order to reduce the complexity of the control of complex networks, people are interested in finding how to drive the whole network to desired states only by a subset of nodes, which induces the network controllability problem. The multi-agent network with antagonistic interactions is the system model in our network controllability research. In this section, relevant works on the two areas are presented.

Network Controllability Research Classification

Basically, the studies on network controllability can be divided into two lines, i.e., state controllability and structural controllability. A linear system is (state) controllable if and only if the controllability matrix $[B, AB, \cdots, A^{n-1}B]$ is of full rank. A structured matrix is a matrix where the elements are fixed zeros or independent parameters. A structured system (A, B) is structurally controllable if it is controllable for a certain selection of parameters in matrices A and B.

There are two key issues for the network controllability research. The first one is to find a method to characterize if the network is controllable. The second one is to choose the minimum number of leaders to control the whole network. Obviously, if the first problem can be solved, the second problem will become easier. However, till now, for a general network structure, only necessary or sufficient conditions (based on graph theory) can be found [2, 16, 91–96] to verify the state controllability of the network. Specifically, some works focused on the graph-theoretical characterization of lower and upper bounds on the rank of controllability matrix (e.g., [96]), and some works focused on necessary and sufficient conditions on the controllability of some special graphs (e.g., path graph, cycle graph, cartesian product graph [97–99]). State controllability methods were used in these literature since they required an accurate model of the network system (e.g., a multi-agent dynamical system model).

Structural controllability provides another way to solve the two problems. The studies in this area can provide an important reference for complex network researchers to study the network property, especially for the vulnerability and network centrality studies. For the controllability characterization problem, necessary and sufficient conditions for structural controllability were proposed in [100]. While for the minimum control set problem, [101] presented a maximum matching algorithm to get the minimum leader set. [102] considered some nodes that cannot be controlled (i.e., a special form of matrix B). [103] studied output structural controllability (i.e., for $y = Cx$, they considered a special C). In addition, some works investigated the strongly structural controllability problem [104, 105], where a structured system (A, B) is called strongly structurally controllable if it is controllable for every selection of parameters of A and B.

Networks with Antagonistic Interactions

Research so far on state controllability of networks focuses on all-positive networks, i.e., networks with edges all being non-negative. Recently, networks with antagonistic interactions have attracted many interests. As mentioned before, for social networks, it is common to have both positive and negative edges where a positive edge can be viewed as a connection to a friend while a negative edge can be viewed as a connection to an enemy [106], [107]. In studies on consensus problems of multi-agent systems, recently, there are also some models having both positive and negative edges. For instance, in [108], the authors modeled an antagonistic multi-agent system based on the signed graph. It was shown that in structurally balanced networks, the states of the agents asymptotically converge to a common nonzero value but with opposite signs, while in structurally unbalanced graphs, all the states converge to zero. In [109], the authors extended the results of [108] by considering signed graphs with a directed spanning tree. Similar problems have been studied in [110–113]. Although networks with possibly negative weights can be studied using the structural controllability scheme, the network model depends on the structure of the graph rather than a particular physical realization [105]. Moreover, even if a system is structurally controllable, it is not necessarily controllable from the control system perspective.

1.3 Contributions

The contributions of this dissertation are listed as follows:

- A distributed connectivity preserving coordination scheme is established, so that all the agents not only maintain the connectivity of the network, but also achieve robust finite-time consensus tracking of the leader by using a gradient-based potential method. The result is further extended to a formation control problem. Integral sliding mode is used to deal with the disturbances in the dynamics, compared with traditional sliding mode, the reaching phase is eliminated, which avoids the effect of disturbances on network connectivity before the system reaches the sliding surface. The robust distributed consensus tracking problem for high-order systems with unmodelled dynamics and unknown disturbances is studied. A distributed control law based on an adaptive identifier is developed for each agent to achieve robust coordinated control of a high-order multi-agent system. Adaptive gains are used to eliminate the dependence on prior knowledge of the upper bound of the uncertainties. Based on Lyapunov stability analysis method and graph theory, sufficient conditions are established, under which the proposed distributed algorithm can lead to asymptotic robust consensus tracking of the leader.

- A distributed continuous-time updating law is proposed to solve the formulated time-varying optimization problem, where the constraints are handled through a projected gradient based design. A continuous-time distributed algorithm is developed to seek the normalized Nash equilibrium of a generalized convex game with nonsmooth objective functions, including class-C^2 and locally Lipschitz objective functions. Gradient and subgradient methods are used to generate the distributed control law for games with class-C^2 and locally Lipschitz objective functions, respectively. A consensus based method is adopted to estimate the non-neighboring information while nonsmooth analysis and singular perturbation are combined to analyze the system performance.

- The network controllability problem for networks with positive and negative links is formulated mathematically, based on the multi-agent system model. The concept of the almost equatable partition (AEP) in graph theory is extended to the generalized equatable partition (GAEP). Based on this, a graph-theoretical characterization of an upper bound on the rank of the controllability matrix is developed, together with a necessary condition for the network to be controllable. Relationships between the controllability of a structural balanced network and that of the corresponding unsigned graph network are analyzed.

Chapter 2

Preliminaries

In this chapter, preliminaries on graph theory and nonsmooth analysis are introduced. In addition, two lemmas on inequalities and an invariance-like theorem are presented, respectively.

2.1 Graph Theory

2.1.1 Unsigned Graphs

Denote $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ as an undirected graph, where $\mathcal{V} = \{1, ..., N\}$ and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ indicate the set of vertices and edges, respectively. An edge is an ordered pair $(i, j) \in \mathcal{E}$ if agent j can be directly supplied with information from agent i. $\mathcal{N}_i = \{j \in \mathcal{V} \,|(j, i) \in \mathcal{E}\}$ denotes the neighborhood set of vertex i. Graph $\mathcal{G}$ is connected if there is an undirected path between every pair of distinct agents. A matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ denotes the adjacency matrix of $\mathcal{G}$, where $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$ else $a_{ij} = 0$. In this thesis, we suppose $a_{ii} = 0$. A matrix $L \triangleq D - A \in \mathbb{R}^{N\times N}$ is called the Laplacian matrix of $\mathcal{G}$, where $D = [d_{ii}] \in \mathbb{R}^{N\times N}$ is a diagonal matrix with $d_{ii} = \sum_{j=1}^{N} a_{ij}$.

The following lemma provides some spectral properties of the Laplacian matrix L.

Lemma 2.1. *[114] Zero is an eigenvalue of L. Given a Laplacian matrix L for an undirected graph $\mathcal{G}$, it has a simple eigenvalue zero with an associated eigenvector $\mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^N$ denotes a unitary column vector, if and only if $\mathcal{G}$ is connected. Meanwhile, the nonzero eigenvalues of L are positive.*

2.1.2 Signed Graphs

[40, 108] Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \sigma\}$ represent a signed graph, where $\mathcal{V} = \{1, ..., N\}$ and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ indicate the set of vertices and edges, respectively, and σ: $\mathcal{E} \to \{+, -\}$ is the mapping of the edges to the signs $\{+, -\}$. $\mathcal{N}_i = \{j \in \mathcal{V} \,|(j, i) \in \mathcal{E}\}$ represents the neighborhood set of vertex i. An $N \times N$ matrix $A = [a_{ij}]$ represents the adjacency matrix of $\mathcal{G}$, where $a_{ij} \in \{0,\ 1,-1\}$. If $a_{ij} = 1$, agent j is called a positive neighbor of agent i; if $a_{ij} = -1$, agent j is called a negative neighbor of agent i; else $a_{ij} = 0$, agent j is not a neighbor of agent i. We further assume that there is no self-loop in the graph, i.e., $a_{ii} = 0$. If $a_{ij} \geq 0$ for all i and j, the graph is called an all-positive graph. Let $L \triangleq D - A \in \mathbb{R}^{N\times N}$ be the signed Laplacian matrix of $\mathcal{G}$, where $D = [d_{ii}] \in \mathbb{R}^{N\times N}$ is a diagonal matrix with $d_i = \sum_{j=1}^{N} |a_{ij}|$ being the degree of i. Thus, the entries of the matrix L can be written as

$$[L]_{ij} = \begin{cases} \sum_{s=1}^{N} |a_{i,s}| & i = j, \\ -a_{i,j} & i \neq j. \end{cases} \tag{2.1}$$

2.1.3 Graph Partition

For an undirected graph $\mathcal{G}$ with the vertex set $\mathcal{V}$, a subset V of $\mathcal{V}$ is called a class (also called a cell [16, 96]). If a class contains only one node, it is called a singleton class. If a class contains more than one node, it is called a non-singleton class. A class V_1 is called a subclass of class V_2 if for any node $i \in V_1$, $i \in V_2$. A collection of classes $\pi = \{V_1, V_2, \cdots, V_k\}$ is called a partition if $V_i \cap V_j = \varnothing$ for all $i \neq j$ and

$\cup_i V_i = \mathcal{V}$. The $N \times k$ matrix $P(\pi) = [P_{ij}]$ is called the characteristic matrix of π, where

$$P_{ij} \triangleq \begin{cases} 1 & i \in V_j, \\ 0 & i \notin V_j. \end{cases}$$

Example 2.1. *For a partition* $\pi = \{\{1,\ 2\}, \{3,\ 4,\ 5\}\}$, *its characteristic matrix* $P(\pi) \in \mathbb{R}^{5\times 2}$ *can be written as*

$$P(\pi) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

2.2 Nonsmooth Analysis

[115, 116] A function $f : [0, a) \to [0, \infty)$ is said to belong to class-$\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. It is said to belong to class-$\mathcal{K}_\infty$ if it belongs to class-$\mathcal{K}$, $a = \infty$ and $\lim_{r\to\infty} f(r) = \infty$. A function $f(x)$ is said to belong to class-C^2 if the derivatives $f^{(1)}(x), f^{(2)}(x)$ exist and are continuous.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz at $x \in \mathbb{R}^n$ if there exists a neighborhood Ω of x and $C_x \in \mathbb{R}_{\geq 0}$ such that $|f(y) - f(z)| \leq C_x \|y - z\|$ for $y, z \in \Omega$. f is locally Lipschitz on $\mathbb{R}^n$ if it is locally Lipschitz for all $x \in \mathbb{R}^n$. f is globally Lipschitz on $\mathbb{R}^n$ if for $y, z \in \mathbb{R}^n$, there exists $C \in \mathbb{R}^{\geq 0}$ such that $|f(y) - f(z)| \leq C \|y - z\|$.

The generalized gradient of f is $\partial_x f(x) \triangleq co\{\lim_{k\to\infty} \nabla_x f(x_k) | x_k \to x,\ x_k \notin \Xi_f \cup S\}$, where Ξ_f is the set of points where f is not differentiable and S is any set of measure zero. The subdifferential $\partial_x f$ of f at $x_0 \in \mathbb{R}^n$ is the set of vectors s satisfying $f(y) \geq f(x_0) + \langle s, y - x_0 \rangle$, for all $y \in \mathbb{R}^n$, denoted as $\partial_x f(x_0)$. Any element $s \in \partial_x f(x_0)$ is called a subgradient of f at x_0. If $f(x)$ is convex, the generalized gradient coincides with the subdifferential $\partial_x f$ [115].

A set-valued map $M\colon X \to \mathbb{R}^n$ is locally bounded if for any compact $H \subseteq X$, there exists a compact set $H' \subseteq \mathbb{R}^n$ such that $M(H) := \{y|y \in M(x), x \in H\} \subset H'$. A set-valued map $M\colon X \to \mathbb{R}^n$ is upper semicontinuous if for each $x \in X$, and $\epsilon > 0$, there exists $\delta > 0$ such that for all x' with $||x' - x|| < \delta$, $M(x') \subset M(x) + \epsilon\mathcal{B}$, where $\mathcal{B}$ is the open unit ball with appropriate dimension.

Lemma 2.2. *[117] Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be convex, and locally Lipschitz in $\mathbb{R}^n$. Then the subdifferential $\partial_x f(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is upper semicontinuous and locally bounded at $x_0 \in \mathbb{R}^n$. Moreover, $\partial_x f(x_0)$ is nonempty, compact, and convex.*

Lemma 2.3. *(Theorem 4.5 in [118], Corollary 3 in [119]) Supposing that $F(x(t))$ is a nonempty compact convex subset for every $x(t) \in \mathbb{R}^n$ and is upper semicontinuous on $\mathbb{R}^n$, if the origin of the differential inclusion $\dot{x}(t) \in F(x(t))$ is globally asymptotically stable, then there exists a smooth converse Lyapunov function V_1 and class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that $\alpha_1(\|x\|) \leq V_1(x) \leq \alpha_2(\|x\|)$ and $\max_{w\in F(x)} \langle \nabla V_1(x), w\rangle \leq -V_1(x)$.*

2.3 Inequalities and An Invariance-like Theorem

Lemma 2.4. *(1) [59] Let $\delta_1, \cdots, \delta_N \geq 0$ and $0 < p \leq 1$, then $(\sum_{i=1}^{N}\delta_i)^p \leq \sum_{i=1}^{N}\delta_i^p \leq N^{1-p}(\sum_{i=1}^{N}\delta_i)^p$. (2) Let $\delta_1, \cdots, \delta_N \geq 0$ and $0 < p < q$, then[1] $(\sum_{i=1}^{N}\delta_i^q)^{\frac{1}{q}} \leq (\sum_{i=1}^{N}\delta_i^p)^{\frac{1}{p}}$. (3) [120] Let $\delta_1, \cdots, \delta_N \geq 0$ and $p > 1$, then $\sum_{i=1}^{N}\delta_i^p \geq N^{1-p}(\sum_{i=1}^{N}\delta_i)^p$. (4) (Young's Inequality) Let $\delta_1, \delta_2, c > 0$ and p, $q > 1$. If $\frac{1}{p} + \frac{1}{q} = 1$, then $\delta_1\delta_2 \leq c^p\frac{\delta_1^p}{p} + c^{-q}\frac{\delta_2^q}{q}$.*

Lemma 2.5. *[121] Let $\mathcal{D} \subset\mathbb{R}^n$ be a domain containing $x = 0$ and suppose $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x, uniformly in t, on $[0, \infty) \times \mathcal{D}$.*

[1] This conclusion can be easily obtained by noting that $(\sum_{i=1}^{N}\delta_i^q)^{\frac{p}{q}} \leq \sum_{i=1}^{N}(\delta_i^q)^{\frac{p}{q}}$.

Furthermore, suppose $f(t,0)$ *is uniformly bounded for all* $t \geq 0$*. Let* $V : [0,\infty) \times \mathcal{D} \to \mathbb{R}$ *be a continuously differentiable function such that*

$$W_1(x) \leq V(t,x) \leq W_2(x),$$
$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W(x),$$

$\forall t \geq 0$, $\forall x \in \mathcal{D}$*, where* $W_1(x)$ *and* $W_2(x)$ *are continuous positive definite functions and* $W(x)$ *is a continuous positive semidefinite function on* $\mathcal{D}$*. Choose* $r > 0$ *such that* $\mathcal{B}_r \subset \mathcal{D}$ *and let* $\rho < \min_{\|x\|=r} W_1(x)$*. Then all solutions of* $\dot{x} = f(t,x)$ *with* $x(0) \in \{x \in \mathcal{B}_r \mid W_2(x) \leq \rho\}$ *are bounded and satisfy*

$$W(x(t)) \to 0 \text{ as } t \to \infty.$$

Moreover, if all the assumptions hold globally and $W_1(x)$ *is radially unbounded, the statement is true for all* $x(0) \in \mathbb{R}^n$*.*

Part I

Distributed Cooperative Control

Chapter 3

Robust Connectivity Preserving Consensus Tracking and Formation Control

3.1 Introduction

The application of finite-time stability to consensus problems in multi-agent systems was first studied in [116], where the agents have first-order dynamics. In [122], the finite-time consensus problems for second-order systems were investigated by using homogeneity theory. In [41], the authors studied the finite-time consensus tracking control problems for second-order systems with bounded disturbances. Most of the existing algorithms designed for consensus/rendezvous/formation control problems assume that the network topology keeps invariant during motion evolutions. In practice, the agent usually has only limited sensing and communication capabilities. If there are no mechanisms to ensure connectivity preservation, these algorithms may fail when the agents are out of the sensing region. In [49], a distributed gradient method was developed to maintain the initial network topology for a first-order multi-robot system. In [50], the robot's dynamics was considered to have non-holonomic constraints. In [51], a general class of bounded potential functions were designed to achieve connectivity preservation for a second-order multi-agent system. By observing the fact that there are little literature that consider both connectivity preservation and finite-time convergence, in this chapter, we consider the robust finite-time connectivity preservation problem for a second-order leader-following multi-agent system with bounded nonlinearities and disturbances. We propose an integral sliding mode based framework to achieve robust finite-time consensus and formation tracking, and meanwhile maintain the connectivity of the communication network.

The main contributions of this chapter can be summarized as follows:

- The integral sliding mode control [123] is firstly used to deal with connectivity preserving consensus problems. By using this control method, the reaching phase of traditional sliding mode is eliminated and the disturbance is rejected from the beginning time, which guarantees that the disturbance will not affect the network connectivity. While using the traditional sliding mode, the disturbance may destroy the connectivity before the system reaches the sliding mode manifold.

- The model we studied is a second-order system, which is more complicated than first-order leaderless systems addressed in the literature [62] and [2]. The nonzero acceleration of the leader can be handled by the integral sliding mode controller.

- The method is developed according to odd function based consensus protocols for second-order multi-agent systems, where we design potential functions to achieve connectivity preservation. One of the difficulties is to estimate the settling time. In the existing literature such as [60] and [66], homogeneity theory is used to prove the stability, which leads to the difficulty in the settling time estimation. In this chapter, by properly selecting the Lyapunov function and potential functions, the upper bound of the settling time can be written as an expression of the initial conditions.

- Inspired by the derived consensus tracking controller, we propose a finite-time connectivity preserving formation control approach based on a new potential function. Compared with the existing studies on the finite-time formation tracking problem (e.g., [63], [64], [124]), the proposed method is robust to bounded disturbances and can preserve the network pattern.

The rest of this chapter is organized as follows. In Section 3.2, the robust finite-time connectivity preserving consensus tracking problem is formulated and solved. In Section 3.3, the formation tracking control problem is studied. In Section 3.4, simulation results show the effectiveness of the proposed algorithms and finally Section 3.5 concludes the chapter.

3.2 Robust Finite-Time Connectivity Preserving Consensus Tracking

3.2.1 Problem Formulation

Consider a second-order multi-agent system with N followers. The dynamics of the follower i ($i \in \{1, ..., N\}$) are described by

$$\begin{aligned}\dot{x}_i(t) =& v_i(t),\\ \dot{v}_i(t) =& u_i(t) + f_i(x_i, v_i, t) + d_i(t),\end{aligned} \tag{3.1}$$

where $x_i(t) \in \mathbb{R}^n$ represents the position state of agent i, $v_i(t) \in \mathbb{R}^n$ represents the velocity state of agent i, $u_i(t) \in \mathbb{R}^n$ represents the control input, $f_i(x_i, v_i, t) \in \mathbb{R}^n$ is an unknown nonlinear function and $d_i(t) \in \mathbb{R}^n$ is the disturbance. Let $x(t) = [x_1^T(t), \cdots, x_N^T(t)]^T \in \mathbb{R}^{Nn}$ be the position vector and $v(t) = [v_1^T(t), \cdots, v_N^T(t)]^T \in \mathbb{R}^{Nn}$ be the velocity vector.

The leader for system (3.1) has the following dynamics

$$\dot{x}_0(t) = v_0(t),\ \dot{v}_0(t) = f_0(x_0, v_0, t), \tag{3.2}$$

where $x_0(t) \in \mathbb{R}^n$ represents the position state, $v_0(t) \in \mathbb{R}^n$ represents the velocity state, and $f_0(x_0, v_0, t) \in \mathbb{R}^n$ is the acceleration state.

In reality, the agent usually has a limited communication capability and can only communicate with agents within its information range. If the relative distance of two neighboring agents are larger than this range, the communication link may be lost. Suppose that the initial connections are established according to the distances, i.e., $\mathcal{E}(0) = \{(i,\ j) : \|x_i(0) - x_j(0)\| < R,\ i \in \{0, 1,\ ...,\ N\},\ j \in \{1,\ ...,\ N\}\}$. The initial information exchange among the $N+1$ agents is represented by graph $\mathcal{G}(0) = \{\mathcal{V}, \mathcal{E}(0)\}$ with $\mathcal{V} = \{0, 1, ..., N\}$. Let $\mathcal{G}_F(0) = \{\mathcal{V}_F, \mathcal{E}_F(0)\}$ be the subgraph of the followers where $\mathcal{V}_F = \{1, ..., N\}$ and $\mathcal{E}_F(0) \subset \mathcal{E}(0)$.

The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ is defined as $a_{ij} > 0$ if $\|x_i(t) - x_j(t)\| < R$ and $\|x_i(0) - x_j(0)\| < R$, and $a_{ij} = 0$ elsewise. The access of agents to the leader's trajectory signal is represented by a diagonal matrix $B = [b_i] \in \mathbb{R}^{N\times N}$ where $b_i = 1$ if $\|x_i(t) - x_0(t)\| < R$ and $\|x_i(0) - x_0(0)\| < R$, and $b_i = 0$ elsewise. Let $H = L + B$ be the information exchange matrix with L being the Laplacian of $\mathcal{G}_F$.

The following assumptions will be used in the subsequent stability analysis.

Assumption 3.1. $\mathcal{G}_F(0)$ *is connected and at least one agent has access to the leader's information.*

Assumption 3.2. *[56]* $||f_i(x_i, v_i, t) + d_i - f_0(x_0, v_0, t)||_\infty \leq c$, $i \in \{1, ..., N\}$, *where* c *is a known constant.*

Remark 3.1. *This assumption implies that the difference between nonlinear dynamics of the follower and the leader cannot be very large. Also, if both* $f_i(x_i, v_i, t)$ *and* $f_0(x_0, v_0, t)$ *are bounded, this assumption is satisfied. A similar assumption is used in [56]. It was specified that such a model can describe more tracking tasks in practical applications (e.g., a low Earth orbit satellite formation task).*

Definition 3.1. ***(Robust Finite-time Connectivity Preserving Consensus Tracking)*** *Consider a multi-agent system composed of* N *followers with dynamics (3.1) and a leader with dynamics (3.2). Each agent can sense only up to a distance* R *from it. Suppose that Assumptions 3.1-3.2 hold. The robust finite-time connectivity preserving consensus tracking problem is to design a distributed state-feedback control law* $u_i(t)$ *using local and neighboring information, such that the system has the following properties: 1) the connectivity of the graph* $\mathcal{G}(t)$ *is preserved for all* $t \geq 0$*; 2) there exists a time* T *such that* $x_i(t) = x_0(t)$ *and* $v_i(t) = v_0(t)$ *for all* $t > T$, $i \in \{1, ..., N\}$.

3.2.2 Control Design

Consider a distributed integral sliding mode control law with the following form

$$u_i = u_{nomi} + u_{disconi}, \tag{3.3}$$

where the nominal control input $u_{nomi}(t)$ determines the performance of the system without disturbance and an additional discontinuous control input $u_{disconi}(t)$ deals with the disturbances [123]. The nominal control input $u_{nomi}(t)$ is designed as

$$u_{nomi} = -\sum_{j=1}^{N} k_1 a_{ij} \varphi(\|x_i - x_j\|) \mathrm{sig}^\alpha(x_i - x_j)$$

$$
\begin{aligned}
&- k_1 b_i \varphi(\|x_i - x_0\|)\mathrm{sig}^{\alpha}(x_i - x_0) \\
&- \sum_{j=1}^{N} k_2 a_{ij} \mathrm{sig}^{\beta}(v_i - v_j) \\
&- k_2 b_i \mathrm{sig}^{\beta}(v_i - v_0),
\end{aligned} \tag{3.4}
$$

where $\varphi(\cdot)$ is a nonnegative potential function to be designed later, k_1, k_2, α and β are positive constants.

Let $\vartheta_i(t) = v_i(t) - v_i(0)$ with $v_i(0)$ being the initial velocity of agent i, $i \in \{0,\ 1,\ ...,\ N\}$. The discontinuous control input $u_{disconi}(t)$ is designed as

$$
u_{disconi} = -k_3 \mathrm{sgn}(s_i), \tag{3.5}
$$

where k_3 is a positive constant and the sliding manifold $s_i(t) \in \mathbb{R}^n$, $i \in \{1,\ ...,\ N\}$ is designed as

$$
\begin{aligned}
s_i =& \sum_{j=1}^{N} a_{ij}(\vartheta_i - \vartheta_j) + b_i(\vartheta_i - \vartheta_0) \\
&- \int_0^t \{\sum_{j=1}^{N} a_{ij}(u_{nomi} - u_{nomj}) + b_i u_{nomi}\} d\tau.
\end{aligned} \tag{3.6}
$$

The nonnegative potential function $\varphi(\|x_i - x_j\|)$ is designed as a function of the distance between agent i and agent j. In order to guarantee finite-time convergence as well as connectivity preservation, we consider a class of potential functions $\varphi(\|x_i - x_j\|), \|x_i - x_j\| \in [0, R]$ satisfying the following conditions:

B1: $\varphi(\|x_i - x_j\|)$ is continuous for $\|x_i - x_j\| \in [0, R]$;

B2: $c_1 \le \varphi(\|x_i - x_j\|) \le c_2$ for $\|x_i - x_j\| \in [\varpi, R]$ and $\varphi(\|x_i - x_j\|) = c_1$ for $\|x_i - x_j\| \in [0, \varpi]$, where c_1 and c_2 are positive constant and $\varpi \in (0, R)$;

B3: $\int_0^R \varphi(s) s^{\alpha} ds > Q$, where Q is a positive constant.

Remark 3.2. *The condition **B1** guarantees $\varphi(\|x_i - x_j\|)$ is integrable, **B2** guaran-*

*tees that the consensus is achieved in a finite time, and **B3** is for the preservation of the initial network connection. A potential function satisfying **B1**, **B2**, and **B3** can be selected as* $\varphi(s) = \frac{(\frac{R}{2})^{1-\alpha}(\frac{3R}{2}+\frac{2R^2}{Q})}{(\frac{R}{2}+\frac{R^2}{Q})^2}$ *if* $s \in [0, \frac{R}{2}]$ *and* $\frac{s^{1-\alpha}(2R-s+\frac{2R^2}{Q})}{(R-s+\frac{R^2}{Q})^2}$ *if* $s \in (\frac{R}{2}, R]$.

3.2.3 Stability Analysis

The robust finite-time connectivity preservation problem can be split into three subproblems: 1) to guarantee that the states of agents are on the sliding manifold; 2) to achieve connectivity preservation on this sliding manifold; 3) to enable finite-time convergence for all the agents. The derived results on the three subproblems are given in the subsequent analysis.

Lemma 3.1. *Suppose that Assumptions 3.1-3.2 hold and* $\mathcal{G}(t) = \mathcal{G}(0)$ *for* $t \in [0, t_1)$. *Then, each agent will maintain its trajectory on the sliding manifold, i.e.,* $s_i(t) = 0$, $i \in \{1, ..., N\}$, $t \in [0, t_1)$, *provided that the control gain* k_3 *satisfies* $k_3 > c$.

Proof. Denote $s(t) = [s_1^T(t), \cdots, s_N^T(t)]^T$, $\vartheta(t) = [\vartheta_1^T(t), \cdots, \vartheta_N^T(t)]^T$, $u_{nom}(t) = [u_{nom1}^T(t), \cdots, u_{nomN}^T(t)]^T$, $f = [f_1^T, \cdots, f_N^T]^T$, and $d(t) = [d_1^T(t), \cdots, d_N^T(t)]^T \in \mathbb{R}^{Nn}$. Let $\tilde{H} = H \otimes I_n$, $t \in [0, t_1)$. Then, (3.6) can be rewritten in a compact form as $s = \tilde{H}(\vartheta - \mathbf{1}\otimes\vartheta_0) - \int_0^t \tilde{H}u_{nom}(\tau)d\tau$. According to [41], $\tilde{H}$ is symmetric and positive definite. Taking the derivative and pre-multiplying both sides by $\tilde{H}^{-1}$, one gets $\tilde{H}^{-1}\dot{s} = -k_3\mathrm{sgn}(s) + f + d - \mathbf{1}\otimes\dot{v}_0$. Let $V_1(s,t) = \frac{1}{2}s^T\tilde{H}^{-1}s$. Taking the time derivative of V_1 gives $\dot{V}_1 = s^T(-k_3\mathrm{sgn}(s) + f + d - \mathbf{1}\otimes f_0) \leq -(k_3 - c)\left\|s\right\|_1 \leq -\sqrt{\frac{2}{\lambda_{\max}(\tilde{H}^{-1})}}(k_3 - c)V_1^{\frac{1}{2}}$, where $\lambda_{\max}(\tilde{H}^{-1})$ denotes the maximum eigenvalue of $\tilde{H}^{-1}$. Furthermore, since $s(0) = \mathbf{0}$, it can be obtained that $s(t) = \mathbf{0}$, $t \in [0, t_1)$, which implies that the system trajectory resides on the sliding manifold in $[0, t_1)$. □

Lemma 3.2. $\mathcal{G}(t)$ *remains invariant for all* $t \geq 0$, *provided that the control parameters* k_1, k_2, k_3 *satisfy* $k_1 > 0$, $k_2 > 0$, $k_3 > c$, *and* Q *is selected to satisfy*

$W_0 < 2k_1 \min\{a_{ij}, b_i \mid a_{ij} \neq 0, b_i \neq 0\}Q$, *where* W_0 *is the initial value of an energy function.*

Proof. Assume that there exists a time t_2 such that $\mathcal{G}(t_2^-) = \mathcal{G}(0)$ and $\mathcal{G}(t_2) \neq \mathcal{G}(0)$. According to Lemma 3.1, $s_i(t) = s_i(0)$ for $t \in [0, t_2)$, $i \in \{1, ..., N\}$. Let $\zeta_i(t) = x_i(t) - x_0(t)$ and $\eta_i(t) = v_i(t) - v_0(t)$. Then, based on (3.6), on the sliding manifold, we have

$$\begin{aligned}\ddot{\zeta}_i =& -\sum_{j=1}^{N} k_1 a_{ij} \varphi(\|\zeta_i - \zeta_j\|) \mathrm{sig}^{\alpha}(\zeta_i - \zeta_j) \\ & - k_1 b_i \varphi(\|\zeta_i\|) \mathrm{sig}^{\alpha}(\zeta_i) \\ & - \sum_{j=1}^{N} k_2 a_{ij} \mathrm{sig}^{\beta}(\eta_i - \eta_j) - k_2 b_i \mathrm{sig}^{\beta}(\eta_i).\end{aligned} \tag{3.7}$$

Denote $\zeta(t) = [\zeta_1^T(t), \cdots, \zeta_N^T(t)]^T$ and $\eta(t) = [\eta_1^T(t), \cdots, \eta_N^T(t)]^T \in \mathbb{R}^{Nn}$. Define a non-negative energy function $W(\zeta, \eta)$ in $[0, t_2)$ as

$$\begin{aligned}W(\zeta, \eta) =& \sum_{i=1}^{N} \sum_{j=1}^{N} k_1 \int_0^{\|\zeta_i - \zeta_j\|} a_{ij} \varphi(s) s^{\alpha} ds \\ & + 2\sum_{i=1}^{N} k_1 \int_0^{\|\zeta_i\|} b_i \varphi(s) s^{\alpha} ds + \sum_{i=1}^{N} \eta_i^T \eta_i,\end{aligned}$$

with the initial value $W_0 = W(\zeta(0), \eta(0)) < 2k_1 \min\{a_{ij}, b_i \mid a_{ij} \neq 0, b_i \neq 0\}Q$. Note that W is continuous and differentiable in $[0, t_2)$. The time derivative of W satisfies

$$\begin{aligned}\dot{W} =& 2\sum_{i=1}^{N} \sum_{j=1}^{N} k_1 a_{ij} \varphi(\|\zeta_i - \zeta_j\|) \dot{\zeta}_i^T \mathrm{sig}^{\alpha}(\zeta_i - \zeta_j) \\ & + 2\sum_{i=1}^{N} \eta_i^T \dot{\eta}_i + 2\sum_{i=1}^{N} k_1 b_i \varphi(\|\zeta_i\|) \dot{\zeta}_i^T \mathrm{sig}^{\alpha}(\zeta_i).\end{aligned} \tag{3.8}$$

Substituting (3.7) into (3.8), we have

$$\dot{W} = -\sum_{i=1}^{N}\sum_{j=1}^{N} k_2 a_{ij}(\eta_i - \eta_j)^T \mathrm{sig}^{\beta}(\eta_i - \eta_j) \\ - 2\sum_{i=1}^{N} k_2 b_i \eta_i^T \mathrm{sig}^{\beta}(\eta_i) \le 0, \tag{3.9}$$

which implies that $W(t) \le W_0 < 2k_1 \min\{a_{ij}, b_i | a_{ij} \neq 0, b_i \neq 0\}Q$ for $t \in [0, t_2)$. If an edge is lost in t_2, then there must be at least a pair of neighbors i, j such that $\|\zeta_i - \zeta_j\| \to R$ as $t \to t_2^-$. Since $\int_0^R \varphi(s)s^{\alpha}ds > Q$, $W(t_2^-) > 2k_1 \min\{a_{ij}, b_i | a_{ij} \neq 0, b_i \neq 0\}Q > W_0$, which contradicts to $W(t) \le W_0$ for $t \in [0, t_2)$. Therefore, no edge will be lost at t_2. □

Theorem 3.1. *Consider a group of N mobile agents with dynamics (3.1), each being controlled by (3.3)-(3.6). Let (3.2) be the leader's trajectory. Suppose that Assumptions 3.1-3.2 hold, $k_3 > c$, and Q, k_1, k_2, α and β are selected satisfying*

$$W_0 < 2k_1 \min\{a_{ij}, b_i | a_{ij} \neq 0, b_i \neq 0\}Q, \\ 0 < \alpha < 1,\ (\frac{k_1 c_1}{1+\alpha})^{\frac{3+\alpha}{2(1+\alpha)}} \lambda_{\min}(\breve{H}_0)^{\frac{3+\alpha}{4}} > \frac{2\rho\theta_1^{\frac{3+\alpha}{2}}}{3+\alpha}, \\ 1 > \frac{\rho(1+\alpha)}{(3+\alpha)\theta_1^{\frac{3+\alpha}{1+\alpha}}}, \beta = \frac{2\alpha}{1+\alpha}, \gamma_1 \ge 0,\ \gamma_2 \ge 0, \tag{3.10}$$

with $\gamma_1 = \rho k_1 c_1 \lambda_{\min}(\breve{H}_2)^{\frac{(1+\alpha)}{2}} - \rho\frac{d_{\max}}{1+\alpha}N^{\frac{1-\alpha}{2}}\theta_2^{1+\alpha}k_2^{1+\alpha}$, $\gamma_2 = \frac{3+\alpha}{2(1+\alpha)}k_2\lambda_{\min}(\breve{H}_1)^{\frac{(1+\beta)}{2}} - \rho - \rho\frac{\alpha\lambda_{\max}(H)}{1+\alpha}\theta_2^{-\frac{(1+\alpha)}{\alpha}}$, where ρ, θ_1, θ_2 are positive constants that can be arbitrarily selected, $\breve{H}_0$, $\breve{H}_1$ and $\breve{H}_2$ are positive definite matrices, then there exists a time $T > 0$ such that the robust connectivity preserving consensus tracking is achieved. Furthermore, if $(\gamma_1)^{\frac{3+\alpha}{2(1+\alpha)}} \ge (\delta)^{\frac{3+\alpha}{2(1+\alpha)}}\omega_1, (\gamma_2)^{\frac{3+\alpha}{2(1+\alpha)}} \ge (\delta)^{\frac{3+\alpha}{2(1+\alpha)}}\omega_2$, where $\omega_1 = 2^{\frac{1-\alpha}{2(1+\alpha)}}(\frac{k_1 c_2}{1+\alpha})^{\frac{3+\alpha}{2(1+\alpha)}}N^{1-\alpha}\lambda_{\max}(\breve{H}_0)^{\frac{3+\alpha}{4}} + \frac{2\rho\theta_1^{\frac{3+\alpha}{2}}}{3+\alpha}$ and $\omega_2 = 2^{\frac{1-\alpha}{2(1+\alpha)}} + \frac{\rho(1+\alpha)}{(3+\alpha)}\theta_1^{-\frac{3+\alpha}{1+\alpha}}$, and δ is an arbitrary positive constant, then the robust finite-time connectivity preserving consensus tracking is achieved with the settling time satisfying $T \le$

$\delta^{-1}\frac{3+\alpha}{1-\alpha}(\omega_1^{\frac{1-\alpha}{3+\alpha}}\|x(0) - x_0(0)\otimes \mathbf{1}_N\|^{\frac{1-\alpha}{2}} + \omega_2^{\frac{1-\alpha}{3+\alpha}}\|v(0) - v_0(0)\otimes \mathbf{1}_N\|^{\frac{1-\alpha}{1+\alpha}})$.

Proof. Motivated by [56], we define a Lyapunov candidate $V = W^{\frac{3+\alpha}{2(1+\alpha)}} + \rho\sum_{i=1}^{N}\zeta_i^T\eta_i$. According to Lemma 2.4 (4), we have $\rho\sum_{i=1}^{N}\zeta_i^T\eta_i \geq -\rho\|\zeta\|\|\eta\| \geq -\frac{2\rho\theta_1^{\frac{3+\alpha}{2}}}{3+\alpha}\|\zeta\|^{\frac{3+\alpha}{2}} - \frac{\rho(1+\alpha)}{(3+\alpha)}\theta_1^{-\frac{3+\alpha}{1+\alpha}}\|\eta\|^{\frac{3+\alpha}{1+\alpha}}$. By Lemma 2.4 (2),

$$
\begin{aligned}
W^{\frac{3+\alpha}{2(1+\alpha)}} \geq {} & [(\frac{k_1c_1}{1+\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\|\zeta_i-\zeta_j\|^{1+\alpha} + \\
& \frac{2k_1c_1}{1+\alpha}\sum_{i=1}^{N}b_i\|\zeta_i\|^{1+\alpha})^{\frac{3+\alpha}{2(1+\alpha)}\frac{2(1+\alpha)}{3+\alpha}} \\
& + (\sum_{i=1}^{N}\|\eta_i\|^2)^{\frac{3+\alpha}{2(1+\alpha)}\frac{2(1+\alpha)}{3+\alpha}}]^{\frac{1}{\frac{2(1+\alpha)}{3+\alpha}}} \\
\geq {} & (\frac{k_1c_1}{1+\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\|\zeta_i-\zeta_j\|^{1+\alpha} + \\
& \frac{2k_1c_1}{1+\alpha}\sum_{i=1}^{N}b_i\|\zeta_i\|^{1+\alpha})^{\frac{3+\alpha}{2(1+\alpha)}} + (\sum_{i=1}^{N}\|\eta_i\|^2)^{\frac{3+\alpha}{2(1+\alpha)}}.
\end{aligned}
$$

By Lemma 2.4 (1),

$$
\begin{aligned}
W^{\frac{3+\alpha}{2(1+\alpha)}} \geq {} & (\frac{k_1c_1}{1+\alpha})^{\frac{3+\alpha}{2(1+\alpha)}}(\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}^{\frac{2}{1+\alpha}}\|\zeta_i-\zeta_j\|^2 \\
& + \sum_{i=1}^{N}2^{\frac{2}{1+\alpha}}b_i^{\frac{2}{1+\alpha}}\|\zeta_i\|^2)^{\frac{3+\alpha}{4}} + (\sum_{i=1}^{N}\|\eta_i\|^2)^{\frac{3+\alpha}{2(1+\alpha)}} \\
\geq {} & (\frac{k_1c_1}{1+\alpha})^{\frac{3+\alpha}{2(1+\alpha)}}\lambda_{\min}(\breve{H}_0)^{\frac{3+\alpha}{4}}\|\zeta\|^{\frac{3+\alpha}{2}} + \|\eta\|^{\frac{3+\alpha}{1+\alpha}},
\end{aligned}
$$

where $\breve{H}_0$ has the same structure with H where a_{ij} is replaced by $a_{ij}^{\frac{2}{1+\alpha}}$ and b_i is replaced by $2^{\frac{2}{1+\alpha}}b_i^{\frac{2}{1+\alpha}}$.

Thus,

$$
V \geq [(\frac{k_1c_1}{1+\alpha})^{\frac{3+\alpha}{2(1+\alpha)}}\lambda_{\min}(\breve{H}_0)^{\frac{3+\alpha}{4}} - \frac{2\rho\theta_1^{\frac{3+\alpha}{2}}}{3+\alpha}]\|\zeta\|^{\frac{3+\alpha}{2}}
$$

$$+ (1 - \frac{\rho(1+\alpha)}{(3+\alpha)\theta_1^{\frac{3+\alpha}{1+\alpha}}}) \|\eta\|^{\frac{3+\alpha}{1+\alpha}}.$$

Similarly, based on Lemma 2.4 (1) and (3), we have

$$\begin{aligned} W^{\frac{3+\alpha}{2(1+\alpha)}} \leq\ & 2^{\frac{1-\alpha}{2(1+\alpha)}} (\frac{k_1 c_2}{1+\alpha})^{\frac{3+\alpha}{2(1+\alpha)}} N^{1-\alpha} \lambda_{\max}(\breve{H})^{\frac{3+\alpha}{4}} \|\zeta\|^{\frac{3+\alpha}{2}} \\ & + 2^{\frac{1-\alpha}{2(1+\alpha)}} \|\eta\|^{\frac{3+\alpha}{1+\alpha}}, \end{aligned}$$

and

$$V \leq \omega_1 \|\zeta\|^{\frac{3+\alpha}{2}} + \omega_2 \|\eta\|^{\frac{3+\alpha}{1+\alpha}}.$$

Taking the derivative of V gives

$$\begin{aligned} \dot{V} \leq & -\frac{3+\alpha}{2(1+\alpha)} k_2 W^{\frac{1-\alpha}{2(1+\alpha)}} (\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \|\eta_i - \eta_j\|^{1+\beta} \\ & + 2\sum_{i=1}^{N} b_i \|\eta_i\|^{1+\beta}) + \rho \|\eta\|^2 - \rho k_1 (\sum_{i=1}^{N} \zeta_i^T \times \\ & \sum_{j=1}^{N} a_{ij} c_1 \mathrm{sig}^{\alpha}(\zeta_i - \zeta_j) + \sum_{i=1}^{N} \zeta_i^T b_i c_1 \mathrm{sig}^{\alpha}(\zeta_i)) \\ & - \rho k_2 (\sum_{i=1}^{N} \zeta_i^T \sum_{j=1}^{N} a_{ij} \mathrm{sig}^{\beta}(\eta_i - \eta_j) \\ & + \sum_{i=1}^{N} b_i \zeta_i^T \mathrm{sig}^{\beta}(\eta_i)). \end{aligned}$$

Noting that $W^{\frac{1-\alpha}{2(1+\alpha)}} \geq \|\eta\|^{\frac{1-\alpha}{1+\alpha}}$, based on Lemma 2.4 (1),

$$\begin{aligned} \dot{V} \leq & -\frac{3+\alpha}{2(1+\alpha)} k_2 W^{\frac{1-\alpha}{2(1+\alpha)}} \lambda_{\min}(\breve{H}_1)^{\frac{(1+\beta)}{2}} \|\eta\|^{1+\beta} \\ & + \rho \|\eta\|^2 - \rho k_1 c_1 \lambda_{\min}(\breve{H}_2)^{\frac{(1+\alpha)}{2}} \|\zeta\|^{1+\alpha} \\ & + \rho (\frac{d_{\max}}{1+\alpha} N^{\frac{1-\alpha}{2}} \theta_2^{1+\alpha} k_2^{1+\alpha} \|\zeta\|^{1+\alpha} \\ & + \frac{\alpha \lambda_{\max}(H)}{1+\alpha} \theta_2^{-\frac{(1+\alpha)}{\alpha}} \|\eta\|^2) \end{aligned}$$

$$
\begin{aligned}
&\leq -(\rho k_1 c_1 \lambda_{\min}(\breve{H}_2)^{\frac{(1+\alpha)}{2}} - \rho \frac{d_{\max}}{1+\alpha} N^{\frac{1-\alpha}{2}} \\
&\times k_2^{1+\alpha} \theta_2^{1+\alpha}) \left\|\zeta\right\|^{1+\alpha} - (\frac{3+\alpha}{2(1+\alpha)} k_2 \lambda_{\min}(\breve{H}_1)^{\frac{(1+\beta)}{2}} \\
&- \rho - \rho \frac{\alpha \lambda_{\max}(H)}{(1+\alpha)} \theta_2^{-\frac{(1+\alpha)}{\alpha}}) \left\|\eta\right\|^2 \\
&\leq -((\gamma_1)^{\frac{3+\alpha}{2(1+\alpha)}} \left\|\zeta\right\|^{\frac{3+\alpha}{2}} + (\gamma_2)^{\frac{3+\alpha}{2(1+\alpha)}} \left\|\eta\right\|^{\frac{3+\alpha}{1+\alpha}})^{\frac{2(1+\alpha)}{3+\alpha}} \\
&\leq -\delta V^{\frac{2(1+\alpha)}{3+\alpha}},
\end{aligned}
$$

where $\breve{H}_1$, $\breve{H}_2$ are two positive definite matrices.

Based on [54], suppose that the control gains are selected satisfying Theorem 3.1, then the system is finite-time stable. The settling time T can be estimated as

$$
\begin{aligned}
T &\leq \delta^{-1} \frac{3+\alpha}{1-\alpha} V(0)^{\frac{1-\alpha}{3+\alpha}} \\
&\leq \delta^{-1} \frac{3+\alpha}{1-\alpha} (\omega_1^{\frac{1-\alpha}{3+\alpha}} \left\|x(0) - x_0(0) \otimes \mathbf{1}_N\right\|^{\frac{1-\alpha}{2}} \\
&+ \omega_2^{\frac{1-\alpha}{3+\alpha}} \left\|v(0) - v_0(0) \otimes \mathbf{1}_N\right\|^{\frac{1-\alpha}{1+\alpha}}).
\end{aligned}
$$

□

Remark 3.3. *The discontinuous control (3.5) deals with both bounded disturbances (nonlinearities) and nonzero leader's acceleration. To guarantee the integrability of u_{nomi}, the nominal control (3.4) uses continuous nonlinear function based protocols. Some similar nonlinear protocols are proposed in [60] and [66]. Different from these works, the nonlinear function in this chapter is designed for connectivity preservation. Furthermore, the upper bound of the settling time can be estimated according to the initial conditions.*

3.3 Robust Finite-Time Connectivity Preserving Formation Tracking

In this section, we apply the result of the previous section to address a formation tracking problem. Consider a multi-agent system composed of N followers governed by (3.1) and a leader governed by (3.2). The objective of each follower is to track the leader while maintaining a certain desired geometric formation. Denote $h = [h_1^T, \cdots, h_N^T]^T$ where $h_i \in \mathbb{R}^n, i \in \{1, ..., N\}$ represents the configuration of agent i in the desired formation with respect to the leader. Let $h_{ij} = h_i - h_j$ be the desired position of agent i with respect to agent j. Then the problem can be described as follows

Definition 3.2. ***(Robust Finite-Time Connectivity Preserving Formation Tracking)*** *Consider a multi-agent system composed of N followers with dynamics (3.1) and a leader with dynamics (3.2). Each follower i is assigned with a desired formation vector h_i, $i \in \{1, ..., N\}$. Suppose that 3.1-3.2 hold, the robust finite-time formation tracking is achieved if the system has the following properties: 1) the connectivity of the sensing graph $\mathcal{G}(0)$ is preserved for all $t \geq 0$; 2) $x_i(t) - h_i \to x_0$ and $v_i(t) \to v_0(t)$ in a finite time T, $i \in \{1, ..., N\}$.*

Design a distributed control law in the similar way as (3.3) where the nominal control input $u_{nomi}(t)$ is designed as

$$\begin{aligned} u_{nomi} = & -\sum_{j=1}^{N} k_1 a_{ij} \varphi_{ij}(\|x_i - x_j - h_{ij}\|) \text{sig}^{\alpha}(x_i \\ & - x_j - h_{ij}) - k_1 b_i \varphi_{ij}(\|x_i - x_0 - h_i\|) \\ & \times \text{sig}^{\alpha}(x_i - x_0 - h_i) - \sum_{j=1}^{N} k_2 a_{ij} \\ & \times \text{sig}^{\beta}(v_i - v_j) - k_2 b_i \text{sig}^{\beta}(v_i - v_0), \end{aligned} \tag{3.11}$$

and the discontinuous control input $u_{disconi}(t)$ remains the same as that in (3.5). In order to achieve the formation tracking control objective, the potential function $\varphi_{ij}(s)$ is defined as follows

$$\varphi_{ij}(s) = \begin{cases} \frac{\varpi^{1-\alpha}(2R-2||h_{ij}||-\varpi+\frac{2(R-||h_{ij}||)^2}{Q})}{(R-||h_{ij}||-\varpi+\frac{(R-||h_{ij}||)^2}{Q})^2}, s \in [0, \varpi], \\ \frac{s^{1-\alpha}(2R-2||h_{ij}||-s+\frac{2(R-||h_{ij}||)^2}{Q})}{(R-||h_{ij}||-s+\frac{(R-||h_{ij}||)^2}{Q})^2}, s \in (\varpi, R-||h_{ij}||]. \end{cases}$$

where $\varpi \in (0, R-||h_{ij}||)$ is a positive constant, $h_{i0} = h_i$, $i \in \{1, ..., N\}$ and $j \in \{0, 1, ..., N\}$. In addition, the initial edges are generated by $\mathcal{E}(0) = \{(i,j) : \|x_i(0) - x_j(0)\| < R - 2\|h_{ij}\|,\ i,j \in \{1, ..., N\}\} \cup \{(0,j) : \|x_0(0) - x_j(0)\| < R - 2\|h_j\|,\ j \in \{1, ..., N\}\}$. It can be verified that the connectivity for the initial network topology can be preserved.

Lemma 3.3. *Under the control input designed in (3.3), (3.5), (3.6) and (3.11), if Assumptions 3.1-3.2 hold, $\mathcal{G}_F(0)$ is initially connected and at least one follower has access to the leader's information, then $\mathcal{G}(t)$ is invariant for all $t \geq 0$, provided that Q is sufficiently large, $k_3 > c$, the control parameters k_1, k_2, k_3 satisfy $k_1 > 0$, $k_2 > 0$, $k_3 > c$.*

Proof. It can be obtained that $\|x_i(0) - x_j(0) - h_{ij}\| < R-||h_{ij}||$. Let $\gamma = \int_0^{\|x_i-x_j-h_{ij}\|} \varphi_{ij}(s)s^{\alpha}ds$. Then $\gamma \to Q$ as $\|x_i - x_j - h_{ij}\| \to R - ||h_{ij}||$, where $Q = \int_0^{R-||h_{ij}||} \varphi_{ij}(s)s^{\alpha}ds$. Following similar analysis as in Lemma 3.2, one can obtain that the edge set $\mathcal{E}(t) = \{(i,j) \mid ||x_i - x_j - h_{ij}|| < R - ||h_{ij}||\}$ is invariant for the trajectory of the closed-loop system. Furthermore, the connectivity of the network is maintained since $\|x_i - x_j - h_{ij}\| < R-||h_{ij}||$ implies $|\|x_i - x_j\| - ||h_{ij}||| < R - ||h_{ij}||$ and $\|x_i - x_j\| < R$. □

Following a similar analysis as that in Section 3.2, we conclude that the agents converge to the desired formation in a finite time T. The following theorem summarizes the main result of this section.

Theorem 3.2. *Suppose that Assumptions 3.1-3.2 hold. Using control inputs determined by (3.3), (3.5), (3.6) and (3.11), where Q is sufficiently large, $k_3 > c$, the control parameters k_1, k_2, α, β are selected satisfying the conditions in Theorem 3.1, the problem described in Definition 3.1 can be solved with the settling time $T \leq \delta^{-1}\frac{3+\alpha}{1-\alpha}(\omega_1^{\frac{1-\alpha}{3+\alpha}}\|x(0) - h - x_0(0) \otimes \mathbf{1}_N\|^{\frac{1-\alpha}{2}} + \omega_2^{\frac{1-\alpha}{3+\alpha}}\|v(0) - v_0(0) \otimes \mathbf{1}_N\|^{\frac{1-\alpha}{1+\alpha}})$.*

3.4 Simulation

Consider a group of 6 mobile robots, moving in the plane with the following dynamics

$$\dot{x}_i = v_i,\ \dot{v}_i = m_i^{-1}(u_i - f_i(x_i, v_i) + d_i),$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t)]^T$, $m_i = 1.5$, $f_i(x_1, x_2)$ is the unknown friction [21] satisfying

$$f_i(x_i, v_i) = 0.25(\tanh(100v_i) - \tanh(10v_i)),$$

and d_i is the disturbance satisfying $d_i(t) = 0.2 * i\sin(t)$. The leader's trajectory is $\dot{x}_0 = v_0$, $\dot{v}_0(t) = [0.2\sin(t), 0.1\sin(t)]^T$ with $x(0) = [-2, -2]^T$ and $v(0) = [1, 2]^T$. The sensing radius is $R = 5$. Initial positions and velocities of the 6 robots are given by $x(0) = [-8, 4,\ -6, 4,\ -4, 2,\ -2, 0,\ 0, 0,\ 2, 2]^T$, $v(0) = [0, 2,\ 1, 3,\ 3, 4,\ 2, 1,\ 2, 5,\ 5, 0]^T$.

The first simulation is a consensus example, where each agent is controlled by (3.3)-(3.6). Fig. 3.1 shows the initial network topology of the robots. Let $Q = 2000$, $\alpha = 0.5$, $\beta = 2/3$, $k_1 = 15$, $k_2 = 10$, $k_3 = 10$. Fig. 3.2 shows the convergence of the positions and the distances among initially connected robots, and Fig. 3.3 shows the convergence of the velocities.

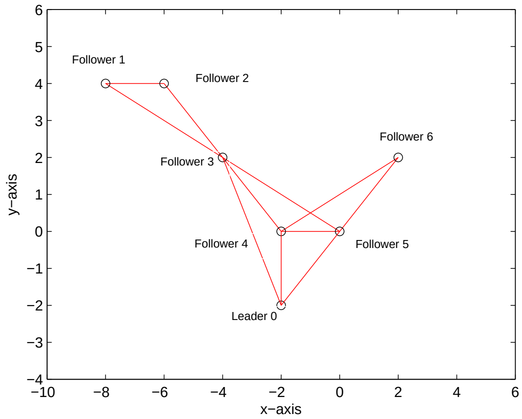

Figure 3.1: The initial graph in the consensus example where the hollow dots represent the robots and the lines indicate the available communication among agents.

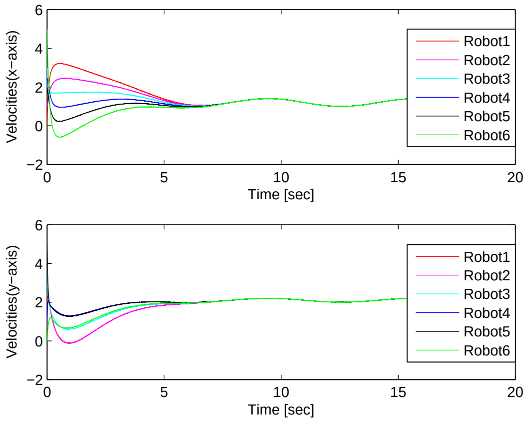

Figure 3.2: Top plot: positions of the followers under control input (3.3)-(3.6). Bottom plot: distances among initially connected robots.

In order to verify the effectiveness of the proposed sliding mode control, we add the same disturbance signal as in our paper to the agent dynamics in [2], and adopt similar control parameters for the two control methods. The simulation results of the distances among initially connected robots under the two algorithms are shown in Fig. 3.4 and Fig. 3.5. It can be shown that the introduced sliding mode control term works so that the system has better performance against the disturbances. The simulation results of the velocities of the followers (x-axis and y-axis) under the algorithms in [2] are shown in Fig. 3.6 and Fig. 3.7. Compared with Fig. 3.3,

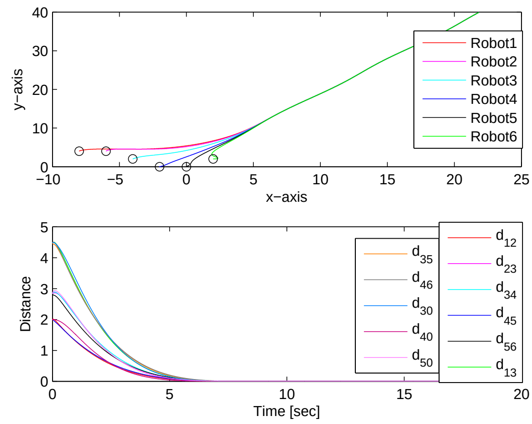

Figure 3.3: Velocities of the followers (x-axis and y-axis) under control input (3.3)-(3.6).

it can be shown that the proposed methods are more accurate in velocity control under disturbances.

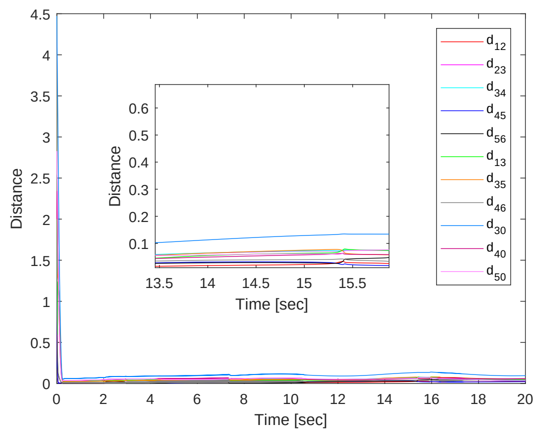

Figure 3.4: The distances among initially connected robots under the algorithms in [2].

The second simulation is a formation tracking example. The six robots are required to achieve a hexagonal formation described by $h = [-1, 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, 0, \frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}]^T$. Fig. 3.8 shows the initial network topology of the robots. Let $Q = 2000$, $\alpha = 0.5$, $\beta = 2/3$, $k_1 = 15$, $k_2 = 10$, $k_3 = 10$. The simulation results are shown in Figs. 3.9-3.10. It is observed that the formation tracking is achieved in a finite time.

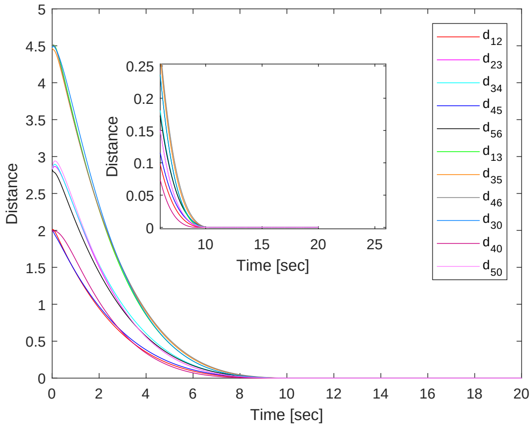

Figure 3.5: The distances among initially connected robots under the algorithms proposed in this chapter.

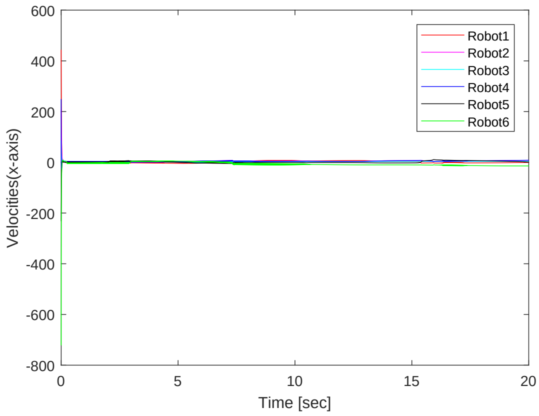

Figure 3.6: Velocities of the followers (x-axis) under the algorithms in [2].

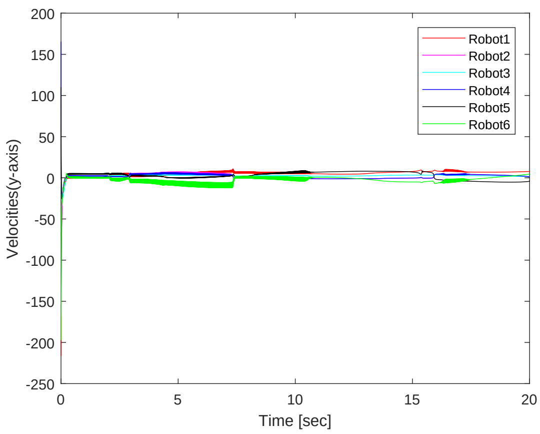

Figure 3.7: Velocities of the followers (y-axis) under the algorithms in [2].

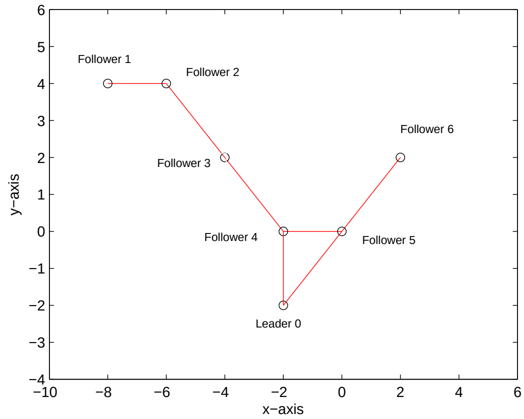

Figure 3.8: The initial graph in the formation example where the hollow dots represent the robots and the lines indicate the available communication among agents.

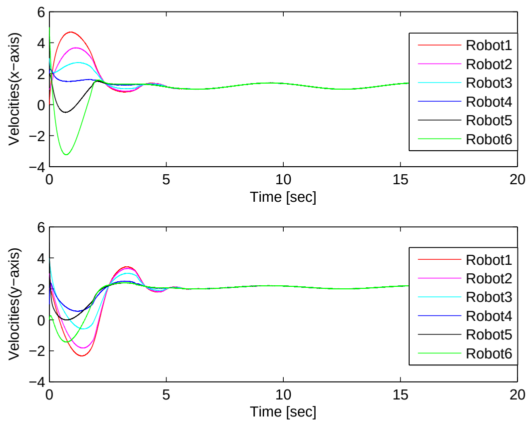

Figure 3.9: Top plot: positions of the followers under control input (3.3), (3.5), (3.6) and (3.11). Bottom plot: distances among initially connected robots.

3.5 Conclusions

In this chapter, we investigated the finite-time consensus and formation tracking problems for second-order multi-agent systems with connectivity preservation and disturbance rejection. Distributed control laws were designed to achieve finite-time consensus tracking or formation tracking and meanwhile maintain the connectivity of the communication network. Sliding mode control together with artificial potential field were utilized to achieve finite-time convergence in the presence of bounded nonlinearities disturbances. Note that in this chapter, the agent's dynamics is mod-

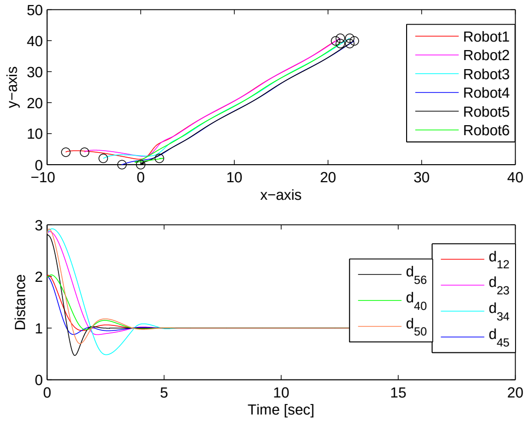

Figure 3.10: Velocities of the followers (x-axis and y-axis) under control input (3.3), (3.5), (3.6) and (3.11).

eled as a second-order system. In practice, there are many systems with higher-order dynamics. In next chapter, we will develop robust coordinated control law for a class of high-order multi-agent system.

Chapter 4

Robust Consensus Tracking for A Class of High-Order Multi-Agent Systems

4.1 Introduction

When considering disturbance rejection in consensus problems, sliding mode control [41], adaptive control [15], and output regulation [17] are widely used. One drawback of sliding mode based control is that the discontinuous control may cause chattering behavior in the plant. Although there exist some approaches that are used to eliminate this effects (e.g., boundary layer techniques [125]), these techniques may downgrade the tracking performance (e.g., UUB results). To simultaneously achieve chattering avoidance and asymptotic stability, the super-twisting control [61] and the robust integral of the sign of the errors (RISE) [8, 42] are developed. Most of the existing literature consider only first-order or second-order dynamics. As s-tated in Chapter 1, low-order models may not be able to accurately describe the agents' dynamics for many applications, such as a power generator model [126].

Output feedback control laws are desirable considering the control efficiency. This chapter focuses on the robust consensus tracking problem for a class of high-order multi-agent systems with unmodelled dynamics and unknown disturbances, by using continuous control. The main contributions of this chapter are summarized as below.

1. The proposed algorithm studies high-order systems with unmodelled dynamics and unknown disturbances, which are more challenging than first-order and second-order systems (e.g., [8], [42]) from the perspective of control design.

2. The proposed state feedback controller does not require prior knowledge of the upper bounds of the uncertain dynamics and disturbances. In contrast, the uncertainties in [8, 42, 68, 69] are upper bounded by some known constants or known functions.

3. Different from the approaches in [41] and [69], the proposed control law in this chapter is continuous and chattering-free.

4. The developed robust output feedback consensus tracking algorithm relies only on local information and does not require model information.

The rest of this chapter is organized as follows. In Section 4.2, the robust high-order consensus tracking problem is formulated. In Section 4.3, a distributed state feedback consensus tracking algorithm is designed, and Lyapunov methods are used to prove semi-global asymptotic consensus tracking. In Section 4.4, robust output feedback consensus tracking is considered. In Section 4.5, examples are presented to show the effectiveness of the proposed algorithms. Finally, Section 4.6 concludes this chapter.

4.2 Problem Formulation

Consider a high-order multi-agent system with N agents. The dynamics of the i-th agent is described by

$$\begin{aligned}
\dot{x}_i^1 &= x_i^2,\\
\dot{x}_i^2 &= x_i^3,\\
&\vdots\\
\dot{x}_i^{n-1} &= x_i^n,\\
\dot{x}_i^n &= u_i + f_i(x_i) + d_i,
\end{aligned} \tag{4.1}$$

where $x_i = [x_i^1(t), \cdots, x_i^n(t)]^T \in \mathbb{R}^n$, $i \in \{1, ..., N\}$, indicates the state vector of agent i, $x_i^j(t)$ represents the j-th state of the i-th agent, $u_i \in \mathbb{R}$ represents the control input, $f_i(x_i) \in \mathbb{R}$ represents the unmodelled dynamics, and $d_i(t) \in \mathbb{R}$ denotes the disturbance. The control objective is to design a distributed control law under which the states of the N agents reach consensus and track a desired signal. Suppose the desired tracking signal is generated by

$$\begin{aligned}
\dot{x}_d^1 &= x_d^2,\\
\dot{x}_d^2 &= x_d^3,\\
&\vdots\\
\dot{x}_d^{n-1} &= x_d^n,\\
\dot{x}_d^n &= f_d(t, x_d),
\end{aligned} \tag{4.2}$$

where $x_d = [x_d^1(t), \cdots, x_d^n(t)]^T \in \mathbb{R}^n$ represents the state vector of the tracking signal, and $f_d(t, x_d) \in \mathbb{R}$ is piecewise continuous in t and locally Lipschitz in x_d.

The following assumptions will be used in the subsequent stability analysis.

Assumption 4.1. *The communication graph $\mathcal{G}$ is a connected undirected graph and at least one agent has access to the desired trajectory.*

Assumption 4.2. *The disturbance term $d_i(t)$ and its first-order and second-order time derivatives are bounded (i.e., $d_i(t)$, $\dot{d}_i(t)$, $\ddot{d}_i(t) \in \mathcal{L}_\infty$, $i \in \{1, \ldots, N\}$).*

Remark 4.1. *Similar assumptions are used in existing literature [8, 21, 42], which implies that the disturbances are sufficiently smooth and bounded. In practice, there are some disturbances that can be modelled satisfying this condition (e.g., frictions [21]).*

Assumption 4.3. *For $\xi(t) \in \mathbb{R}^n$, if $\xi(t)$ is bounded (i.e., $\xi(t) \in \mathcal{L}_\infty$), then the function $f_i(\xi)$ and its first-order and second-order derivatives with respect to ξ are bounded (i.e., $f_i(\xi)$, $\dfrac{\partial f_i(\xi)}{\partial \xi}$, $\dfrac{\partial^2 f_i(\xi)}{\partial^2 \xi} \in \mathcal{L}_\infty$, $i \in \{1, \ldots, N\}$).*

Remark 4.2. *This assumption requires that the functions are bounded with respect to the states. For example, the function $f_1(\xi) = \|\xi - \mathbf{1}\|^2$ satisfies this condition.*

Assumption 4.4. *The desired signal $x_d(t)$ is third-order differentiable and bounded to its third-order derivative (i.e., $x_d(t) \in \mathcal{L}_\infty$, $\dot{x}_d(t) \in \mathcal{L}_\infty$, $\ddot{x}_d(t) \in \mathcal{L}_\infty$, $\dddot{x}_d(t) \in \mathcal{L}_\infty$).*

Remark 4.3. *This assumption implies that the trajectory of the leader should be sufficiently smooth. It is mild for many practical applications such as guidance and navigation.*

4.3 Robust State Feedback Consensus Tracking

4.3.1 Control Development

Since not all the agents have access to the desired trajectory information, we define the state tracking errors as

$$\begin{aligned} e_i^1 &= \sum_{j=1}^{N} a_{ij}(x_i^1 - x_j^1) + b_i(x_i^1 - x_d^1), \\ e_i^2 &= \dot{e}_i^1 + e_i^1, \\ e_i^3 &= \dot{e}_i^2 + e_i^2 + e_i^1, \\ &\vdots \\ e_i^{n-1} &= \dot{e}_i^{n-2} + e_i^{n-2} + e_i^{n-3}, \\ e_i^n &= \dot{e}_i^{n-1} + e_i^{n-1} + e_i^{n-2}, \end{aligned} \tag{4.3}$$

where $e_i^1(t), \cdots, e_i^n(t) \in \mathbb{R}$, $i \in \{1, ..., N\}$, are the tracking errors and b_i represents the access of agent i to the desired trajectory. If the i-th agent has access to the desired trajectory, $b_i = 1$; else, $b_i = 0$.

Then we define an auxiliary error signal $e_{fi}(t) \in \mathbb{R}$ by

$$e_{fi} = e_i^n + \alpha e_i^{n-1}, \tag{4.4}$$

where α is a constant that will be determined later. The following distributed state feedback control law is designed for each agent (4.1)

$$u_i = \hat{f}_i, \tag{4.5}$$

where $\hat{f}_i(t)$ is designed as

$$\dot{\hat{f}}_i(t) = -k_1(\dot{e}_{fi}(t) + \beta e_{fi}(t)) - (k_{2i}(t) + k_3)\,\mathrm{sgn}(e_{fi}(t)), \tag{4.6}$$

where β, k_1, k_3 are constant scalars that will be determined later, and $k_{2i}(t)$ is a time-varying gain which is designed as

$$\dot{k}_{2i}(t) = \mathrm{sgn}(e_{fi}(t))(\dot{e}_{fi}(t) + \beta e_{fi}(t)). \tag{4.7}$$

Since $\dot{e}_{fi}(t)$ is not measurable, the dynamic equations (4.6) and (4.7) cannot be used directly to generate $\hat{f}_i(t)$ and $k_{2i}(t)$. Instead, the following integral form will be used to generate $\hat{f}_i(t)$:

$$\hat{f}_i(t) = -k_1\left(e_{fi}(t) - e_{fi}(0)\right) - \int_0^t ((k_{2i}(\tau) + k_3)\mathrm{sgn}(e_{fi}(\tau)) + k_1\beta e_{fi}(\tau))d\tau. \tag{4.8}$$

Similarly, $k_{2i}(t)$ can be obtained by

$$k_{2i}(t) = |e_{fi}(t)| - |e_{fi}(0)| + \beta\int_0^t |e_{fi}(\tau)|\,d\tau. \tag{4.9}$$

Remark 4.4. *The differential equations given in (4.6) and (4.7) are continuous except when $e_{fi}(t) = 0$. The existence of solutions can be established by using Filippov's theory of differential inclusions [127]. In the proposed control law (4.5) and (4.8)-(4.9), the term $(k_{2i}(t) + k_3)\,sgn(e_{fi}(t))$ is used to dominate the uncertainties. The time-varying gain $k_{2i}(t)$ is used to remove the dependence of the control gains on the upper bounds of the uncertainties.*

To facilitate the subsequent analysis, we define the concatenated vectors $X_i(t)$, $E_i(t)$, $E_f(t)$, $U(t)$, $\hat{F}(t)$, $F(t)$, $D(t)$, $K_2(t) \in \mathbb{R}^N$ as

$$X_i = [x_1^i, x_2^i, \cdots, x_N^i]^T,$$

$$E_i = [e_1^i, e_2^i, \cdots, e_N^i]^T,$$

$$
\begin{aligned}
E_f &= [e_{f1}, e_{f2}, \cdots, e_{fN}]^T, \\
U &= [u_1, u_2, \cdots, u_N]^T, \\
\hat{F} &= [\hat{f}_1, \hat{f}_2, \cdots, \hat{f}_N]^T, \\
F &= [f_1, f_2, \cdots, f_N]^T, \\
D &= [d_1, d_2, \cdots, d_N]^T, \\
K_2 &= [k_{21}, k_{22}, \cdots, k_{2N}]^T.
\end{aligned}
$$

Define three matrices B, $\text{sgn}(E_f)$, and H as $B = diag[b_1,\ b_2,\ \cdots,\ b_N]$, $\text{sgn}(E_f) = diag[\text{sgn}(e_{f1}),\ \cdots,\ \text{sgn}(e_{fN})]$, $H = L + B$, respectively. Then, the following conclusion holds.

Lemma 4.1. *[42] If $\mathcal{G}$ is a connected undirected graph and at least one agent has access to the desired trajectory, then H is symmetric and positive definite.*

From (4.3), we get

$$
\begin{aligned}
E_1 &= HX_1 - Bx_d^1\mathbf{1}, \\
E_2 &= \dot{E}_1 + E_1, \\
E_3 &= \dot{E}_2 + E_2 + E_1, \\
&\vdots \\
E_n &= \dot{E}_{n-1} + E_{n-1} + E_{n-2}.
\end{aligned} \tag{4.10}
$$

Based on (4.10), $E_i(t)$ can be expressed in terms of $E_1(t)$ as

$$
E_i = E_1^{(i-1)} + \sum_{j=0}^{i-2} m_{ij} E_1^{(j)},\ i \in \{2, ..., n\}, \tag{4.11}
$$

where $E_1^{(j)}(t)$ represents the j-th order derivative of $E_1(t)$, and m_{ij} are some known constants regarding i.

Specially, for the n-th state,

$$\begin{aligned} E_n &= E_1^{(n-1)} + \sum_{j=0}^{n-2} m_{nj} E_1^{(j)} \\ &= HX_n - Bx_d^n \mathbf{1} + \sum_{j=0}^{n-2} m_{nj} E_1^{(j)}. \end{aligned} \tag{4.12}$$

Lemma 4.2. *Assume that $\mathcal{G}$ represents a connected undirected graph and at least one agent has access to the desired trajectory information $x_d(t)$, then robust consensus tracking of (4.1) can be enabled in the sense of $X_i(t) = x_d^i(t)\mathbf{1}$, $i \in \{1, ..., n\}$, provided that $E_1(t), \cdots, E_n(t) = 0$ as $t \to \infty$.*

Proof. From (4.10), we have $\dot{E}_1(t), \dot{E}_2(t), \cdots, \dot{E}_{n-1}(t) = 0$ if $E_1(t), \cdots, E_n(t) = 0$. Since $E_1(t) = 0$, $LX_1 + BX_1 = Bx_d^1\mathbf{1}$. According to Lemma 2.1, $L\mathbf{1} = 0$ for a connected undirected graph. Thus,

$$LX_1 + BX_1 = L\mathbf{1}x_d^1 + Bx_d^1\mathbf{1} = (L + B)\, x_d^1\mathbf{1}. \tag{4.13}$$

Since H has full rank based on Lemma 4.1, multiplying H^{-1} on both sides of (4.13) leads to $X_1 = x_d^1\mathbf{1}$. Similarly, based on $\dot{E}_1(t), \dot{E}_2(t), \cdots, \dot{E}_{n-1}(t) = 0$, we have $X_i = x_d^i\mathbf{1}$, $i \in \{2, ..., n\}$. □

Based on (4.4), (4.6) and (4.7), one gets

$$E_f = E_n + \alpha E_{n-1}, \tag{4.14}$$

$$\dot{\hat{F}} = -k_1(\dot{E}_f + \beta E_f) - \text{sgn}(E_f)(K_2 + k_3\mathbf{1}), \tag{4.15}$$

$$\dot{K}_2 = \text{sgn}(E_f)(\dot{E}_f + \beta E_f). \tag{4.16}$$

Based on (4.10), (4.12) and (4.14), the derivative of $E_f(t)$ satisfies

$$\begin{aligned}\dot{E}_f &= \dot{E}_n + \alpha\dot{E}_{n-1} = H\dot{X}_n - B\dot{x}_d^n\mathbf{1} + \sum_{j=0}^{n-2} m_{nj}E_1^{(j+1)} + \alpha(E_n - E_{n-1} - E_{n-2}) \\ &= H(F + D + U) - B\dot{x}_d^n\mathbf{1} + \sum_{j=0}^{n-2} m_{nj}E_1^{(j+1)} + \alpha(E_n - E_{n-1} - E_{n-2}). \end{aligned} \tag{4.17}$$

To facilitate the subsequent analysis, we define an auxiliary vector $r(t) \in \mathbb{R}^N$ as

$$r = H^{-1}(\dot{E}_f + \beta E_f). \tag{4.18}$$

From (4.10), (4.14) and (4.18), we have

$$\dot{E}_n = \dot{E}_f - \alpha\dot{E}_{n-1} = Hr - \beta E_f - \alpha(E_n - E_{n-1} - E_{n-2}). \tag{4.19}$$

Based on (4.19), taking the derivative of $r(t)$ gives

$$\begin{aligned}\dot{r} &= H^{-1}(\ddot{E}_f + \beta\dot{E}_f) \\ &= -k_1(\dot{E}_f + \beta E_f) - \mathrm{sgn}(E_f)(K_2 + k_3\mathbf{1}) + \dot{F} + \dot{D} - H^{-1}B\ddot{x}_d^n\mathbf{1} \\ &+ H^{-1}\sum_{j=0}^{n-2} m_{nj}E_1^{(j+2)} + \beta H^{-1}\dot{E}_f + \alpha H^{-1}(\dot{E}_n - \dot{E}_{n-1} - \dot{E}_{n-2}) \\ &= -k_1 Hr - \mathrm{sgn}(E_f)(K_2 + k_3\mathbf{1}) + \tilde{N} + N_d + H^{-1}\sum_{j=0}^{n-2} m_{nj}E_1^{(j+2)} \\ &+ \beta(r - \beta H^{-1}E_f) + \alpha H^{-1}(\dot{E}_n - \dot{E}_{n-1} - \dot{E}_{n-2}), \end{aligned} \tag{4.20}$$

where $\tilde{N}(t)$ and $N_d(t)$ are defined as

$$\begin{aligned}\tilde{N} &= \dot{F} - \dot{F}_d, \\ N_d &= \dot{F}_d + \dot{D} - H^{-1}B\ddot{x}_d^n\mathbf{1},\end{aligned}$$

and $\dot{F}_d = \dot{F}(x_d)$.

Based on (4.10) and (4.11), $E_1^{(i)}(t)$ can be rewritten as

$$E_1^{(i)} = E_{i+1} + \sum_{j=1}^{i} c_{ij} E_j,$$
$$E_1^{(n)} = \dot{E}_n + \sum_{j=1}^{n} c_{nj} E_j, \tag{4.21}$$

where $i \in \{2, ..., n-1\}$, and c_{ij} are some known constants regarding i. According to (4.21) we have

$$\begin{aligned}
&H^{-1} \sum_{j=0}^{n-2} m_{nj} E_1^{(j+2)} = H^{-1} \sum_{j=0}^{n-3} m_{nj} E_1^{(j+2)} + m_{n(n-2)} H^{-1} E_1^{(n)} \\
&= H^{-1} \sum_{j=0}^{n-3} m_{nj} (E_{j+3} + \sum_{i=1}^{j+2} c_{(j+2)i} E_i) + m_{n(n-2)} H^{-1} E_1^{(n)} \\
&= H^{-1} \sum_{j=0}^{n-3} m_{nj} (E_{j+3} + \sum_{i=1}^{j+2} c_{(j+2)i} E_i) + m_{n(n-2)} H^{-1} (\dot{E}_n + \sum_{j=1}^{n} c_{nj} E_j) \\
&= \sum_{s=1}^{n} g_s H^{-1} E_s + m_{n(n-2)} H^{-1} \dot{E}_n,
\end{aligned} \tag{4.22}$$

where g_s, $s \in \{1, ..., n\}$, is defined by

$$\sum_{s=1}^{n} g_s E_s = \sum_{j=0}^{n-3} m_{nj} (E_{j+3} + \sum_{i=1}^{j+2} c_{(j+2)i} E_i) + m_{n(n-2)} \sum_{j=1}^{n} c_{nj} E_j.$$

Remark 4.5. *g_s is a known constant and can be determined according to m_{ij} described in (4.11) and c_{ij} described in (4.21).*

Based on (4.10), (4.19), and (4.22), the closed-loop system (4.20) can be rewritten as

$$\begin{aligned}
\dot{r} = &-k_1 H r - \operatorname{sgn}(E_f)(K_2 + k_3 \mathbf{1}) + \tilde{N} + N_d + \beta(r - \beta H^{-1} E_f) + \sum_{s=1}^{n} g_s H^{-1} E_s \\
&+ m_{n(n-2)} H^{-1} \{Hr - \beta E_f - \alpha(E_n - E_{n-1} - E_{n-2})\}
\end{aligned}$$

$$
\begin{aligned}
&+\alpha H^{-1}\{Hr-\beta E_f-\alpha(E_n-E_{n-1}-E_{n-2})\}\\
&-\alpha H^{-1}(E_n-E_{n-1}-E_{n-2})-\alpha H^{-1}(E_{n-1}-E_{n-2}-E_{n-3})\\
&=-k_1Hr-\mathrm{sgn}(E_f)(K_2+k_3\mathbf{1})+\mu r-\mu\beta H^{-1}E_f+\sum_{j=1}^{n}h_jH^{-1}E_j+\tilde{N}+N_d,
\end{aligned}
\tag{4.23}
$$

where $\mu=\alpha+m_{n(n-2)}+\beta$ is a constant, and h_j, $j\in\{1,...,n\}$, is defined as

$$
h_j=\begin{cases}
g_j,\ j\in\{1,...,n-4\},\\
g_j+\alpha,\ j=n-3,\\
g_j+(2+m_{n(n-2)})\alpha+\alpha^2,\ j=n-2,\\
g_j+m_{n(n-2)}\alpha+\alpha^2,\ j=n-1,\\
g_j-(1+m_{n(n-2)})\alpha-\alpha^2,\ j=n.
\end{cases}
\tag{4.24}
$$

Based on Assumptions 4.2-4.4, $||N_d(t)||$ can be upper bounded as

$$||N_d||\le c_1, \tag{4.25}$$

$$||\dot{N}_d||\le c_2, \tag{4.26}$$

where $||\cdot||$ represents the norm of a vector and c_1 and c_2 are unknown constants. Based on Mean Value Theorem, the function $\tilde{N}$ can be upper bounded as ([128])

$$||\tilde{N}||\le\rho(||z||)||z||, \tag{4.27}$$

where $z(E_1,\cdots,E_{n-1},E_f,r)$ is defined as

$$z\triangleq[E_1^T,\cdots,E_{n-1}^T,E_f^T,r^T]^T, \tag{4.28}$$

and $\rho(\cdot)$ is a positive, globally invertible, nondecreasing function.

The following lemmas will be used in the stability analysis.

Lemma 4.3. *A function $P(t) \in \mathbb{R}$ defined as below is positive semidefinite*

$$P(t) = k_4 E_f^T(0) sgn(E_f(0))\mathbf{1} - E_f^T(0) N_d(0) - p(t), \tag{4.29}$$

where $p(t) \in \mathbb{R}$ is the solution of $\dot{p} = r^T H(N_d - k_4 sgn(E_f)\mathbf{1})$, provided that k_4 satisfies the following inequality

$$k_4 > c_1 + \frac{1}{\beta} c_2, \tag{4.30}$$

where c_1 and c_2 are described in (4.25) and (4.26).

Proof. The proof is similar to Lemma 5 in [42], thus we omit it here. □

Remark 4.6. *k_4 is only used for analysis and not utilized in the control law described in (4.3)-(4.7). One advantage of the proposed algorithm is that the bound information of the uncertainties is not needed.*

Lemma 4.4. *A function $Q(t) \in \mathbb{R}$ defined as below is positive semidefinite*

$$Q(t) = k_3 \left\| E_f(0) \right\|_1 - q(t), \tag{4.31}$$

where $q(t) \in \mathbb{R}$ is generated by $\dot{q} = -k_3 \dot{E}_f^T sgn(E_f)\mathbf{1}$, $||\cdot||_1$ represents the 1-norm of a vector, and k_3 is a positive constant.

Proof.

$$\begin{aligned}
q(t) &= -\int_0^t k_3 \dot{E}_f^T(\tau) \mathrm{sgn}(E_f(\tau))\mathbf{1} d\tau \\
&= -k_3 E_f^T \mathrm{sgn}(E_f)\mathbf{1} \mid_0^t \\
&= -k_3 (E_f^T(t) \mathrm{sgn}(E_f(t))\mathbf{1} - E_f^T(0) \mathrm{sgn}(E_f(0))\mathbf{1})
\end{aligned}$$

$$= -k_3(\|E_f(t)\|_1 - \|E_f(0)\|_1)$$
$$\leq k_3 \|E_f(0)\|_1 .$$

Thus, $Q(t) \geq 0$. □

In order to facilitate the stability analysis, let $k_1 = k_{1a} + k_{1b}$, where $k_{1a} > 0$ and $k_{1b} > 0$. Next, we propose sufficient conditions for the system in (4.1) to guarantee robust semi-global asymptotic consensus tracking.

4.3.2 Stability Analysis

Theorem 4.1. *Consider the high-order multi-agent system (4.1). If Assumptions 4.1-4.4 hold, then the control algorithms described in (4.3)-(4.5) and (4.8)-(4.9) can ensure semi-global asymptotic consensus tracking of the desired trajectory (4.2), provided that the control gains k_{1a}, k_3, α, β are selected to satisfy the following conditions:*

$$k_{1a} > \frac{1}{\lambda^2_{\min}(H)}(\mu\lambda_{\max}(H) + \frac{\mu^2\beta^2}{2} + \sum_{j=1}^{n}\frac{h_j^2}{2} + \frac{h_n^2\alpha^2}{2} + \frac{\lambda^2_{\max}(H)}{2}),$$
$$\alpha > 0,\ \beta > 2,\ k_3 > 0, \tag{4.32}$$

where $\lambda_{\min}(H)$ denotes the minimum eigenvalue of H, $\lambda_{\max}(H)$ denotes the maximum eigenvalue of H, $\mu = \alpha + m_{n(n-2)} + \beta$, and h_j, $j \in \{1, ..., n\}$ is a constant as described in (4.24).

Proof. Define a vector $y(t) = [r^T(t), E_f^T(t), E_1^T(t), ..., E_{n-1}^T(t), (k_4\mathbf{1}-K_2(t))^T, \sqrt{P(t)}, \sqrt{Q(t)}]^T$. Select a Lyapunov function candidate as

$$V(t, y) = \frac{1}{2}r^T Hr + \frac{1}{2}E_f^T E_f + \frac{1}{2}E_1^T E_1 + \cdots + \frac{1}{2}E_{n-1}^T E_{n-1}$$
$$+ \frac{1}{2}(k_4\mathbf{1}-K_2)^T(k_4\mathbf{1}-K_2) + P + Q. \tag{4.33}$$

According to Lemma 4.1 and the Rayleigh-Ritz theorem, we have $\lambda_{\min}(H)\left\|r\right\|^2 \leq r^THr \leq \lambda_{\max}(H)\left\|r\right\|^2$. Thus, there exist two positive definite functions $W_1(y)$ and $W_2(y)$ such that $W_1(y) \leq V(t,y) \leq W_2(y)$. Using (4.10), (4.14)-(4.16), (4.18) and (4.23) leads to

$$
\begin{aligned}
\dot{V} &= r^TH\dot{r} + E_f^T\dot{E}_f + E_1^T\dot{E}_1 + \cdots + (E_{n-1})^T\dot{E}_{n-1} + \dot{P} + \dot{Q} - (k_4\mathbf{1}{-}K_2)^T\dot{K}_2 \\
&= r^TH\{-k_1Hr - \mathrm{sgn}(E_f)\,(K_2 + k_3\mathbf{1}) + \mu r - \mu\beta H^{-1}E_f + \sum_{j=1}^{n} h_jH^{-1}E_j + \tilde{N} + N_d\} \\
&+ E_f^T(Hr - \beta E_f) + E_1^T(E_2 - E_1) + E_2^T(E_3 - E_2 - E_1) + \cdots \\
&+ E_{n-1}^T(E_n - E_{n-1} - E_{n-2}) - (k_4\mathbf{1}{-}K_2)^T\mathrm{sgn}(E_f)Hr + \dot{P} + \dot{Q}.
\end{aligned}
$$

Using (4.29) and (4.31), we have

$$
\begin{aligned}
\dot{V} &= -k_1r^TH^2r + \mu r^THr - \beta E_f^TE_f - E_1^TE_1 - E_2^TE_2 - \cdots - E_{n-1}^TE_{n-1} + E_{n-1}^TE_n \\
&- \mu\beta r^TE_f + \sum_{j=1}^{n-1} h_jr^TE_j + h_nr^TE_n + r^TH\tilde{N} + r^THN_d + E_f^THr - r^TH\mathrm{sgn}(E_f)(K_2 + k_3\mathbf{1}) \\
&- r^TH(N_d - k_4\mathrm{sgn}(E_f)\mathbf{1}) + k_3(Hr - \beta E_f)^T\mathrm{sgn}(E_f)\mathbf{1} - (k_4\mathbf{1}{-}K_2)^T\mathrm{sgn}(E_f)Hr \\
&= -k_1r^TH^2r + \mu r^THr - \beta E_f^TE_f - E_1^TE_1 - E_2^TE_2 - \cdots - E_{n-1}^TE_{n-1} + E_{n-1}^TE_f \\
&- \alpha E_{n-1}^TE_{n-1} - \mu\beta r^TE_f + \sum_{j=1}^{n-1} h_jr^TE_j + h_nr^T(E_f - \alpha E_{n-1}) + r^TH\tilde{N} + E_f^THr \\
&- k_3\beta E_f^T\mathrm{sgn}(E_f)\mathbf{1}.
\end{aligned}
$$

Based on Young's Inequality, we have

$$
\begin{aligned}
-\mu\beta r^TE_f &\leq \frac{\mu^2\beta^2}{2}||r||^2 + \frac{1}{2}||E_f||^2, \\
h_jr^TE_j &\leq \frac{h_j^2}{2}||r||^2 + \frac{1}{2}||E_j||^2, \\
E_{n-1}^TE_f &\leq \frac{1}{2}||E_{n-1}||^2 + \frac{1}{2}||E_f||^2,
\end{aligned}
$$

$$h_n r^T E_f \leq \frac{h_n^2}{2}||r||^2 + \frac{1}{2}||E_f||^2,$$
$$-h_n \alpha r^T E_{n-1} \leq \frac{h_n^2 \alpha^2}{2}||r||^2 + \frac{1}{2}||E_{n-1}||^2,$$
$$E_f^T H r \leq \frac{\|H\|^2}{2}||r||^2 + \frac{1}{2}||E_f||^2.$$

Thus, $\dot{V}(t, y)$ can be upper bounded as

$$\begin{aligned}
\dot{V} \leq & -(k_{1a}\lambda_{\min}^2(H) - \mu\lambda_{\max}(H) - \frac{\mu^2\beta^2}{2} - \sum_{j=1}^{n}\frac{h_j^2}{2} - \frac{h_n^2\alpha^2}{2} - \frac{\lambda_{\max}^2(H)}{2})||r||^2 \\
& - (\beta - 2)||E_f||^2 - \frac{1}{2}||E_1||^2 - \frac{1}{2}||E_2||^2 - \cdots - \frac{1}{2}||E_{n-2}||^2 - \alpha||E_{n-1}||^2 - k_3\beta \|E_f\|_1 \\
& - (k_{1b}\lambda_{\min}^2(H)||r||^2 - ||r|| \, \|H\| \, \rho(||z||)||z||) \\
& \leq -\gamma||z||^2 + \frac{\lambda_{\max}^2(H)\rho^2(||z||)}{4k_{1b}\lambda_{\min}^2(H)}||z||^2 = -W(y),
\end{aligned} \tag{4.34}$$

where $W(y) = (\gamma - \frac{\lambda_{\max}^2(H)\rho^2(||z||)}{4k_{1b}\lambda_{\min}^2(H)})||z||^2$ and γ is defined as

$$\gamma = \min\{k_{1a}\lambda_{\min}^2(H) - \mu\lambda_{\max}(H) - \frac{\mu^2\beta^2}{2} - \sum_{j=1}^{n}\frac{h_j^2}{2} - \frac{h_n^2\alpha^2}{2} - \frac{\lambda_{\max}^2(H)}{2}, \ \beta - 2, \ \frac{1}{2}, \ \alpha\}.$$

If the control gains k_{1a}, k_3, α, β are selected to satisfy the conditions in (4.32), we have $\gamma > 0$.

$W(y)$ is a continuous positive semi-definite function defined in the following domain:

$$\mathcal{D} \triangleq \{y(t) \in \mathbb{R}^{N(n+2)+2}\}| \, \|y\| \leq \rho^{-1}(\frac{2\lambda_{\min}(H)}{\lambda_{\max}(H)}\sqrt{\gamma k_{1b}})\}.$$

The size of the domain can be increased by increasing k_{1b}.

Under the conditions in (4.32), $\dot{V}(t, y)$ is negative semidefinite. Based on (4.34), we have $V \in \mathcal{L}_\infty$ in $\mathcal{D}$. Thus, $E_1(t)$, ..., $E_{n-1}(t)$, $E_f(t)$, $r(t) \in \mathcal{L}_\infty$. The closed-loop error system can be used to conclude that the remaining signals are bounded in $\mathcal{D}$. The definitions of $W(y)$ and z can be used to prove that $W(y)$ is uniformly continuous in $\mathcal{D}$. Based on Lemma 2.5, the tracking error $E_i(t)$ will approach zero as $t \to \infty$. From Lemma 4.2, semi-global asymptotic consensus tracking of the desired trajectory is achieved. □

4.4 Robust Output Feedback Consensus Tracking

With the tracking errors defined in (4.3) and (4.4), for the purpose of simplifying the analysis, we select $\alpha = 1$, then we have

$$e_{fi} = e_i^n + e_i^{n-1}. \tag{4.35}$$

Since $x_i^1(t)$ is the output of the systems, and only $x_i^1(t)$ and $x_d^1(t)$ are measurable states, we have that $e_i^1(t)$ is measurable and $e_i^2(t)$, $\cdots$, $e_i^n(t)$ are unmeasurable. Thus we define the following distributed high-gain observer

$$\begin{aligned}
\dot{\hat{e}}_i^1 &= \hat{e}_i^2 - \hat{e}_i^1 + \frac{l_1}{\epsilon}(e_i^1 - \hat{e}_i^1),\\
\dot{\hat{e}}_i^2 &= \hat{e}_i^3 - \hat{e}_i^2 - \hat{e}_i^1 + \frac{l_2}{\epsilon^2}(e_i^1 - \hat{e}_i^1),\\
&\vdots\\
\dot{\hat{e}}_i^{n-1} &= \hat{e}_{fi} - 2\hat{e}_i^{n-1} - \hat{e}_i^{n-2} + \frac{l_{n-1}}{\epsilon^{n-1}}(e_i^1 - \hat{e}_i^1),\\
\dot{\hat{e}}_{fi} &= \frac{l_n}{\epsilon^n}(e_i^1 - \hat{e}_i^1),
\end{aligned} \tag{4.36}$$

where $\hat{e}_i^1(t)$, $\hat{e}_i^2(t)$, $\cdots$, $\hat{e}_i^{n-1}(t)$ and $\hat{e}_{fi}(t)$ are the observation error signals, and l_1, $\cdots$, l_n and ϵ are constant scalars that will be determined later.

The distributed output feedback control law is designed as

$$u_i = -k\hat{e}_{fi}. \tag{4.37}$$

In order to facilitate the subsequent analysis, we define the concatenated vectors

$\hat{E}_i(t)$, $E_f(t)$, $\hat{E}_f(t) \in \mathbb{R}^N$ as

$$\hat{E}_i = [\hat{e}_1^i, \hat{e}_2^i, \cdots, \hat{e}_N^i]^T,$$
$$E_f = [e_{f1}, e_{f2}, \cdots, e_{fN}]^T,$$
$$\hat{E}_f = [\hat{e}_{f1}, \hat{e}_{f2}, \cdots, \hat{e}_{fN}]^T.$$

Then, based on (4.35) and (4.36), we have

$$\begin{aligned}
E_f &= E_n + E_{n-1}, \qquad (4.38)\\
\dot{\hat{E}}_1 &= \hat{E}_2 - \hat{E}_1 + \frac{l_1}{\epsilon}(E_1 - \hat{E}_1),\\
\dot{\hat{E}}_2 &= \hat{E}_3 - \hat{E}_2 - \hat{E}_1 + \frac{l_2}{\epsilon^2}(E_1 - \hat{E}_1),\\
&\vdots\\
\dot{\hat{E}}_{n-1} &= \hat{E}_f - 2\hat{E}_{n-1} - \hat{E}_{n-2} + \frac{l_{n-1}}{\epsilon^{n-1}}(E_1 - \hat{E}_1),\\
\dot{\hat{E}}_f &= \frac{l_n}{\epsilon^n}(E_1 - \hat{E}_1). \qquad (4.39)
\end{aligned}$$

To facilitate the following analysis, we define a concatenated vector $\Psi(t) = [\psi_1^T(t),$ $\psi_2^T(t)$, $\psi_3^T(t)$, $\cdots$, $\psi_n^T(t)]^T$, where

$$\begin{aligned}
\psi_i &= \frac{1}{\varepsilon^{n-i}}(E_i - \hat{E}_i), i \in \{1, ..., n-1\},\\
\psi_n &= E_f - \hat{E}_f. \qquad (4.40)
\end{aligned}$$

Taking the derivative of (4.40), we get

$$\begin{aligned}
\epsilon\dot{\psi}_1 &= -l_1\psi_1 + \psi_2 - \epsilon\psi_1,\\
\epsilon\dot{\psi}_2 &= -l_2\psi_1 + \psi_3 - \epsilon^2\psi_1 - \epsilon\psi_2,\\
&\vdots
\end{aligned}$$

$$\epsilon\dot{\psi}_{n-1} = -l_{n-1}\psi_1 + \psi_n - \epsilon^2\psi_{n-2} - 2\epsilon\psi_{n-1},$$
$$\epsilon\dot{\psi}_n = -l_n\psi_1 + \epsilon\dot{E}_f,$$

and equivalently,

$$\epsilon\dot{\Psi} = \tilde{A}_0\Psi + \epsilon\Phi, \tag{4.41}$$

where $\Phi(t)$ is defined as $\Phi = -[\psi_1^T(t), \epsilon\psi_1^T + \psi_2^T, \cdots, \epsilon\psi_{n-2}^T + 2\psi_{n-1}^T, -\dot{E}_f^T]^T \in \mathbb{R}^{Nn}$, and $\tilde{A}_0$ is defined as

$$\tilde{A}_0 = A_0 \otimes I_N,$$
$$A_0 = \begin{bmatrix} -l_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_{n-1} & 0 & \cdots & 1 \\ -l_n & 0 & \cdots & 0 \end{bmatrix}. \tag{4.42}$$

From (4.42), we have that if $l_1, \cdots, l_n$ are selected to make A_0 Hurwitz, then there exists a positive definite and symmetric matrix P, so that

$$PA_0 + A_0^T P = -I_n. \tag{4.43}$$

Thus we have

$$\tilde{P}\tilde{A}_0 + \tilde{A}_0^T\tilde{P} = -I_{Nn}, \tag{4.44}$$

where $\tilde{P} = P \otimes I_N$.

By using (4.12) and (4.40), taking the derivative of (4.38), and premultiplying it by H^{-1}, we have

$$H^{-1}\dot{E}_f = H^{-1}(\dot{E}_n + \dot{E}_{n-1}) = -k\hat{E}_f + \Omega + \Omega_d - E_{n-1} = -kE_f + k\psi_n + \Omega + \Omega_d - E_{n-1}, \tag{4.45}$$

where

$$\Omega = H^{-1}(E_n - E_{n-1} - E_{n-2}) + H^{-1}\sum_{j=0}^{n-2} m_{nj}E_1^{(j+1)} + E_{n-1} + F - F(x_d),$$

$$\Omega_d = -H^{-1}B\dot{x}_d^n\mathbf{1} + D + F(x_d).$$

Based on Assumptions 4.2-4.4, there exists a positive constant c_3, such that

$$\|\Omega_d\| \leq c_3.$$

Using Mean Value Theorem, Ω can be upper bounded as

$$||\Omega|| \leq \rho(||z||)||z||, \tag{4.46}$$

where $z(E_1, \cdots, E_{n-1}, E_f)$ is defined as

$$z \triangleq [E_1^T, \cdots, E_{n-1}^T, E_f^T]^T, \tag{4.47}$$

and $\rho(\cdot)$ is a positive, globally invertible, nondecreasing function.

Based on the definition of $\Phi(t)$, we know that if ϵ is selected to satisfy $0 < \epsilon < 1$, $\|\Phi(t)\|$ can be upper bounded as

$$\begin{aligned}\|\Phi\| &\leq \sigma_1 \|\Psi\| + \|H(-kE_f + k\psi_n + \Omega + \Omega_d - E_{n-1})\| \\ &\leq (\sigma_1 + k\|H\|)\|\Psi\| + (k+1)\|H\|\|z\| + \|H\|(||\Omega|| + \|\Omega_d\|),\end{aligned} \tag{4.48}$$

where σ_1 is a constant regardless of ϵ and k.

In order to facilitate the stability analysis, let k and $\frac{1}{\epsilon}$ be selected as $k = k_1+k_2+k_3$ and $\frac{1}{\epsilon} = \epsilon_1 + \epsilon_2 + \epsilon_3$, respectively. Next, we propose sufficient conditions for the system in (4.1) to guarantee robust UUB consensus tracking.

Theorem 4.2. *Consider the high-order multi-agent system (4.1). If Assumptions 4.1-4.4 hold, then the output feedback control algorithm described in (4.36) and (4.37) can ensure robust UUB consensus tracking, provided that the control gains l_1, $\cdots$, l_n are selected such that A_0 defined in (4.42) is Hurwitz, ϵ_1, ϵ_2, ϵ_3, ϵ, k_1, k_2, k_3 and k are selected to satisfy the following conditions:*

$$
\begin{aligned}
&\epsilon_1 > \frac{k^2\delta_1^2}{4} + 2(\sigma_1 + k\|H\|)\|P\| + \delta_1^2(k+1)^2\|P\|^2\|H\|^2, \\
&0 < \epsilon < 1,\ k_1 > \frac{2}{\delta_1^2},\ \delta_1^2 > 2,
\end{aligned} \tag{4.49}
$$

where σ_1 is a positive constant defined in (4.48).

Proof. Define a vector $y(t) = [z^T(t),\ \Psi^T(t)]^T \in \mathbb{R}^{2Nn}$. Select a Lyapunov function candidate $V(t,y)$ as $V = \frac{1}{2}E_1^TE_1 + \cdots + \frac{1}{2}E_{n-1}^TE_{n-1} + \frac{1}{2}E_f^TH^{-1}E_f + \Psi^T\tilde{P}\Psi$. Through similar analysis as in Theorem 1, we can find positive definite functions $W_1(y)$ and $W_2(y)$ such that $W_1(y) \le V(t,y) \le W_2(y)$. Taking the derivative of V gives

$$
\begin{aligned}
\dot{V} = &-E_1^TE_1 - \cdots - E_{n-1}^TE_{n-1} + E_{n-1}^T(E_f - E_{n-1}) + E_f^T(-kE_f + k\psi_n + \Omega + \Omega_d - E_{n-1}) \\
&+ \frac{1}{\epsilon}\Psi^T(\tilde{P}\tilde{A}_0 + \tilde{A}_0^T\tilde{P})\Psi + 2\Psi^T\tilde{P}\Phi.
\end{aligned}
$$

By using (4.41) and (4.48), we have

$$
\begin{aligned}
\dot{V} \le &-E_1^TE_1 - \cdots - 2E_{n-1}^TE_{n-1} - k_1E_f^TE_f + E_f^T(k\psi_n + \Omega) - k_2E_f^TE_f - k_3E_f^TE_f + E_f^T\Omega_d \\
&- \frac{1}{\epsilon}\|\Psi\|^2 + 2\|\Psi\|\|P\|\|\Phi\| \\
\le &-\gamma_1\|z\|^2 + k\|z\|\|\Psi\| + \|E_f\|\rho(||z||)||z|| - k_2E_f^TE_f - k_3E_f^TE_f + E_f^T\Omega_d \\
&- \frac{1}{\epsilon}\|\Psi\|^2 + 2\|\Psi\|\|P\|\{(\sigma_1 + k\|H\|)\|\Psi\| + (k+1)\|H\|\|z\| + \|H\|(||\Omega|| + \|\Omega_d\|)\},
\end{aligned}
$$

where $\gamma_1 = \min\{1, k_1\}$. Based on Young's Inequality, we have

$$
\dot{V} \le -(\gamma_1 - \frac{2}{\delta_1^2})\|z\|^2 - \{\epsilon_1 - \frac{k^2\delta_1^2}{4} - 2(\sigma_1 + k\|H\|)\|P\| - \delta_1^2(k+1)^2\|P\|^2\|H\|^2\}\|\Psi\|^2
$$

$$
\begin{aligned}
&-(k_2\left\|E_f\right\|^2-\left\|E_f\right\|\left\|\Omega_d\right\|)-(k_3\left\|E_f\right\|^2-\left\|E_f\right\|\rho(||z||)||z||)-(\epsilon_2\left\|\Psi\right\|^2\\
&-2\left\|\Psi\right\|\left\|P\right\|\left\|H\right\|\left\|\Omega_d\right\|)\\
&-(\epsilon_3\left\|\Psi\right\|^2-2\left\|\Psi\right\|\left\|P\right\|\left\|H\right\|\rho(||z||)||z||)\\
&\leq-\gamma_2\left\|y\right\|^2+(\frac{\rho^2(||z||)}{4k_3}+\frac{\rho^2(||z||)\left\|P\right\|^2\left\|H\right\|^2}{\epsilon_3})||y||^2+(\frac{1}{4k_2}+\frac{1}{\epsilon_2}\left\|P\right\|^2\left\|H\right\|^2)\left\|\Omega_d\right\|^2\\
&\leq-W(y)+(\frac{1}{4k_2}+\frac{1}{\epsilon_2}\left\|P\right\|^2\left\|H\right\|^2)\left\|\Omega_d\right\|^2
\end{aligned}
$$

where γ_2 is defined as $\gamma_2=\min\{\gamma_1-\frac{2}{\delta_1^2},\ \epsilon_1-\frac{k^2\delta_1^2}{4}-2(\sigma_1+k\left\|H\right\|)\left\|P\right\|-\delta_1^2(k+1)^2\left\|P\right\|^2\left\|H\right\|^2\}$ and $W(y)$ is defined as $W(y)=-(\gamma_2-\frac{\rho^2(||z||)}{4k_3}-\frac{\rho^2(||z||)\|P\|^2\|H\|^2}{\epsilon_3})||y||^2$. Under the conditions in (4.49), $W(y)$ is a positive semidefinite function defined in the following domain

$$
\mathcal{D}\triangleq\{y(t)\in\mathbb{R}^{2Nn}\}|\ \|y\|\leq\rho^{-1}(\frac{\sqrt{\gamma_2}}{\sqrt{\frac{1}{4k_3}+\frac{\|P\|^2\|H\|^2}{\epsilon_3}}}\}.
$$

The size of the domain can be increased by increasing k_3 and ϵ_3. There exist two positive constants λ_1 and λ_2 such that $\lambda_1 W(y)\leq V(t,y)\leq\lambda_2 W(y)$. Then we get

$$
\dot{V}\leq-\frac{1}{\lambda_2}V+(\frac{1}{4k_2}+\frac{1}{\epsilon_2}\left\|P\right\|^2\left\|H\right\|^2)\left\|\Omega_d\right\|^2. \tag{4.50}
$$

Linear system analysis methods (Lemma A. 19 in [129]) can be used to prove that the solution of the differential equation (4.50) satisfies

$$
V(t)\leq V(0)e^{-\frac{1}{\lambda_2}t}+\lambda_3\left\|\Omega_d\right\|^2(1-e^{-\frac{1}{\lambda_2}t}), \tag{4.51}
$$

where λ_3 is defined as $\lambda_3=\lambda_2(\frac{1}{4k_3}+\frac{1}{\epsilon_2}\left\|P\right\|^2\left\|H\right\|^2)$.

Thus, the Lyapunov function V is bounded. Moreover, the bound can be arbitrarily small through properly selecting k_3 and ϵ_2. Through a similar analysis as in Theorem 4.1, we conclude that the conditions in (4.49) can ensure robust UUB consensus tracking of the desired trajectory.

Remark 4.7. *For the consensus tracking problem of high-order systems, the interaction between the structure of the communication topology and the node dynamics is more complex than first-order and second-order cases. This is indicated in the construction of the Lyapunov function in Theorem 4.2. Specially, H represents the properties of the graph and (4.43) represents the individual node controller design. Since both of them are involved in the Lyapunov analysis, it brings more challenges.*

□

4.5 Simulation

Consider a multi-agent system with 4 agents. Each agent has 3 states denoted by $x_i^1(t)$, $x_i^2(t)$, $x_i^3(t)$. The system dynamics of the agents are described by

$$\begin{aligned}\dot{x}_i^1 &= x_i^2,\\ \dot{x}_i^2 &= x_i^3,\\ \dot{x}_i^3 &= u_i + f_i(x_i^1, x_i^2, x_i^3) + d_i(t),\end{aligned} \tag{4.52}$$

where

$$\begin{aligned}f_1 &= 0.1\sin(x_1^1) + 0.2\cos(x_1^2) + 0.1\cos(x_1^3),\\ f_2 &= 0.2\sin(2x_2^1) + 0.2\cos(x_2^2) + 0.2\cos(x_2^3),\\ f_3 &= 0.3\sin(3x_3^1) + 0.2\cos(x_3^2) + 0.3\cos(x_3^3),\\ f_4 &= 0.4\sin(4x_4^1) + 0.2\cos(x_4^2) + 0.4\cos(x_4^3),\end{aligned}$$

and

$$d_1 = 0.1\sin(0.1t), d_2 = 0.2\sin(0.2t),$$

$$d_3 = 0.3\sin(0.3t), d_4 = 0.4\sin(0.4t)$$

are the unmodelled dynamics and unknown disturbances, respectively.

The initial states are given by

$$\begin{aligned} x^1(0) &= [0.5, 1, 1.5, 2], \\ x^2(0) &= [1, 2, 3, 4], \\ x^3(0) &= [0.5, 0.5, 1.5, 1]. \end{aligned}$$

The dynamics of the desired tracking signal is given by

$$\begin{aligned} &\dot{x}_d^1 = x_d^2, \\ &\dot{x}_d^2 = x_d^3, \\ &\dot{x}_d^3 = \sin(2x_d^3 + 1) + \cos(t), \\ &x_d^1(0) = 1, x_d^2(0) = 2, x_d^3(0) = 1. \end{aligned} \tag{4.53}$$

The communication topology of the 4 agents is shown in Fig. 4.1. The adjacency matrix A associated with this group of agents is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The diagonal matrix which represents the access of the 4 agents to the desired trajectory (4.53) is given by $B = \text{diag}\{0, 0, 1, 1\}$.

4.5.1 Robust State Feedback Consensus Tracking

Based on Theorem 4.1, the control gains are chosen as

$$\begin{aligned} k_1 &= 15, k_3 = 1, \\ \alpha &= 2, \beta = 3. \end{aligned}$$

As shown in Figs. 4.2-4.6, the state feedback consensus tracking is achieved under the proposed algorithm (4.3)-(4.7).

4.5.2 Robust Output Feedback Consensus Tracking

Based on Theorem 4.2, the control gains are chosen as

$$\begin{aligned} l_1 &= 12, l_2 = 44, l_3 = 48, \\ \epsilon &= 0.08, k = 4.5. \end{aligned}$$

As shown in Figs. 4.7-4.10, the output feedback consensus tracking is achieved under the proposed algorithm (4.36) and (4.37).

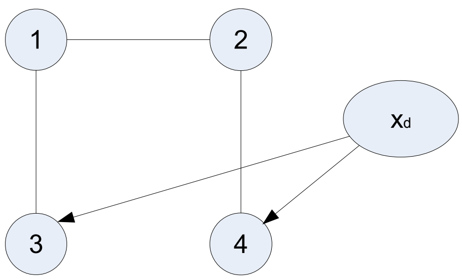

Figure 4.1: The communication graph.

4.6 Conclusions

In this chapter, we solved the robust consensus tracking problem for a class of high-order multi-agent systems with unmodelled dynamics and unknown disturbances. We first considered the case when all the states are measurable. A time-varying gain was designed to eliminate the dependence on prior knowledge of the upper bound of the uncertainties. Based on this design, a continuous robust control algorithm was developed for the agents of the high-order multi-agent system to achieve robust consensus tracking of a desired trajectory signal. Lyapunov analysis and an invariance-like theorem were used to derive sufficient conditions for semi-global asymptotic consensus tracking. A robust output feedback control algorithm was designed to obtain the UUB consensus tracking result. The tracking errors can be arbitrarily small through properly selecting control gains.

In this part, we discussed the consensus tracking problems for second-order and high-order multi-agent systems, respectively. In addition to consensus problems, distributed optimization and distributed equilibrium seeking also have wide applications in multi-agent networks. On the other hand, many consensus techniques can be applied to these problems (e.g., average consensus [130], finite-time consensus [56], etc.). In the next part, we will focus on the distributed optimization and distributed equilibrium seeking problems.

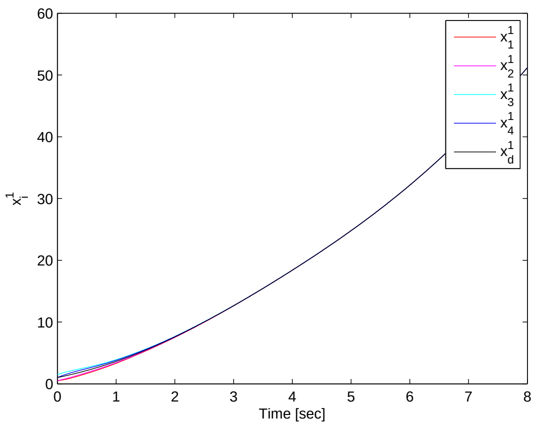

Figure 4.2: Under the state feedback control law (4.3)-(4.7), the position states $x_i^1(t), i \in \{1, \ldots, 4\}$ of the agents reach consensus asymptotically and track the desired position information $x_d^1(t)$.

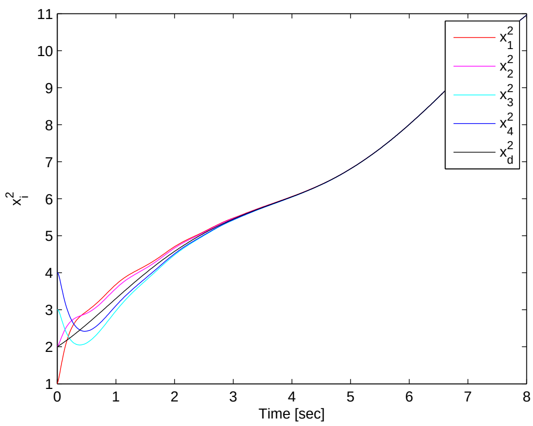

Figure 4.3: Under the state feedback control law (4.3)-(4.7), the velocity states $x_i^2(t), i \in \{1, \ldots, 4\}$ of the agents reach consensus asymptotically and track the desired velocity information $x_d^2(t)$.

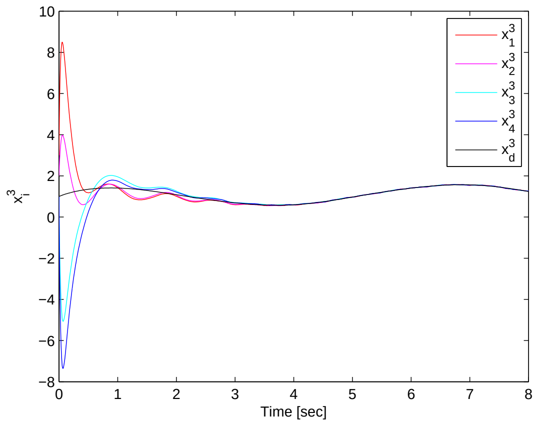

Figure 4.4: Under the state feedback control law (4.3)-(4.7), the acceleration states $x_i^3(t), i \in \{1, \ldots, 4\}$ of the agents reach consensus asymptotically and track the desired acceleration information $x_d^3(t)$.

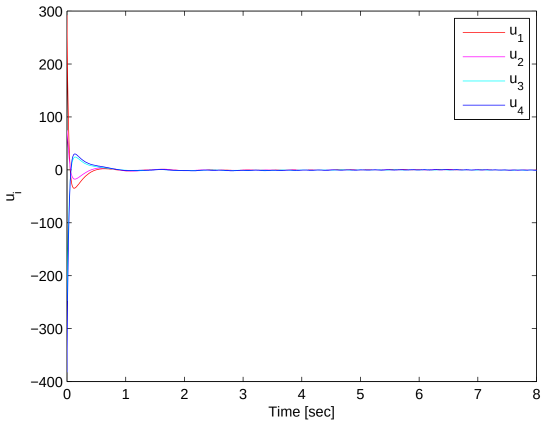

Figure 4.5: Trajectories of the control inputs $u_i(t)$ ($i \in \{1, \ldots, 4\}$) under the state feedback control law (4.3)-(4.7).

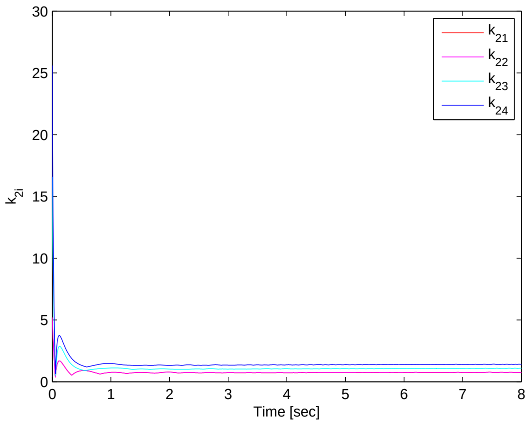

Figure 4.6: Trajectories of the time-varying gains $k_{2i}(t)$ $(i \in \{1, \ldots, 4\})$ under the state feedback control law (4.3)-(4.7).

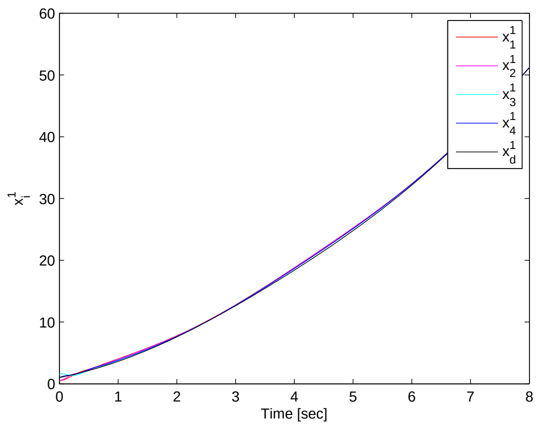

Figure 4.7: Under the output feedback control law (4.36) and (4.37), the position states $x_i^1(t), i \in \{1, \ldots, 4\}$ of the agents reach UUB consensus and track the desired position information $x_d^1(t)$.

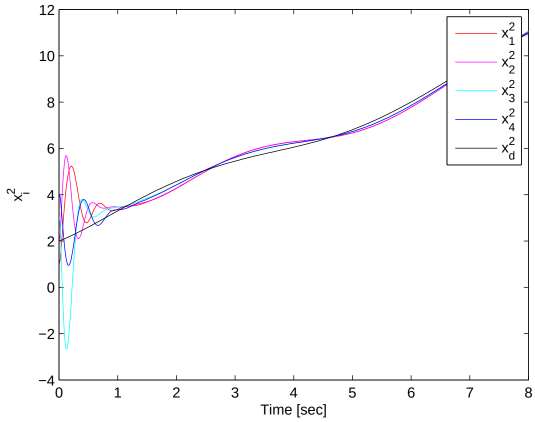

Figure 4.8: Under the output feedback control law (4.36) and (4.37), the velocity states $x_i^2(t), i \in \{1, \ldots, 4\}$ of the agents reach UUB consensus and track the desired velocity information $x_d^2(t)$.

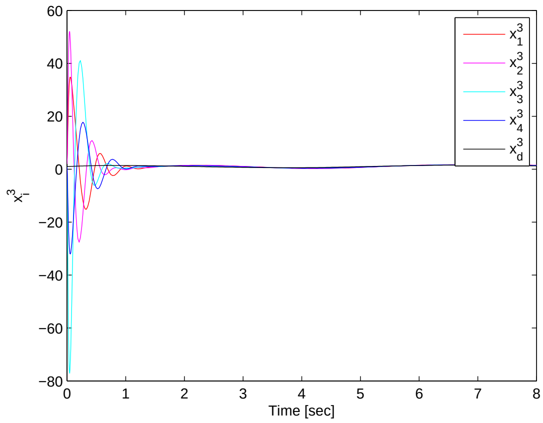

Figure 4.9: Under the output feedback control law (4.36) and (4.37), the acceleration states $x_i^3(t), i \in \{1, \ldots, 4\}$ of the agents reach UUB consensus and track the desired position information $x_d^3(t)$.

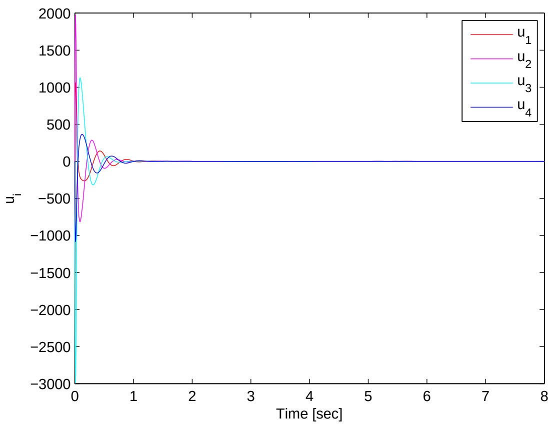

Figure 4.10: Trajectories of the control inputs $u_i(t)$ ($i \in \{1, \dots, 4\}$) under the output feedback control law (4.36) and (4.37).

Part II

Distributed Optimization and Nash Equilibrium Seeking

Chapter 5

Distributed Time-Varying Quadratic Optimization for Multiple Agents under Undirected Graphs

5.1 Introduction

Distributed optimization refers that a group of distributed agents, each having access to a local objective function, collaborate with each other to optimize a global objective function [30]. When there exist constraints that correspond to a convex set, projected gradient method [23, 26, 27, 131] can be used to iteratively seek the optimal solution. Most of the existing literature on distributed optimization consider static objective functions. In this chapter, we consider a distributed optimization problem with time-varying objective functions. Each agent is subject to a compact and convex constraint set. In this case, the optimal solutions will change with time and generate a trajectory. The objective is to design a distributed projected gradient algorithm such that the algorithm tracks the trajectory of the optimal solution of the time-varying objective function. In order to simplify the analysis, we restrict the objective functions to be quadratic. The contributions of this chapter can be summarized as follows:

1. The optimization problem studied in this chapter is time-varying, which is more complicated than the time-invariant optimization problem considered in the existing literature [30, 132, 133]. By using Lyapunov methods, it is proven that the tracking errors are uniformly ultimately bounded with arbitrarily small bound.

2. Projected gradient based methods are proposed to deal with the compact and convex constraint set. Each agent projects the local information into its own constraint set and updates its own state. While the existing work [74, 75] didn't consider the constraints.

The rest of this chapter can be organized as follows: in Section 5.2, the problem is formulated mathematically, with Section 5.2.1 considering neighboring coupled objectives and Section 5.2.2 considering neighboring coupled objectives. In Section 5.3, two numerical examples are given to show the effectiveness of the proposed methods. Finally, Section 5.4 concludes this chapter.

5.2 Main Results

The problem studied in this chapter can be formulated as follows.

Problem 5.1. (Distributed Time-Varying Quadratic Optimization) *Consider a multi-agent system with N agents which is denoted by $V = \{1, 2, \cdots, N\}$. The agents communicate with each other via an undirected and connected graph topology denoted by $G = (V, E)$ where E is the edge set with $E \subset V \times V$. Each agent i has a* *local, quadratic objective function $f_i(x,t)$. The agents coordinate with each other to minimize the global objective function $f(x,t) = \sum_{i=1}^{N} f_i(x,t)$ with a nonempty compact convex constraint set Ω. The optimal solution $x^*(t)$ is unknown, time-varying, bounded and $f_i(x^*(t),t)$, for all $i \in \{1,2,\cdots,N\}$ are bounded. Design an updating law for the agents such that they can track $x^*(t)$.*

In this chapter, we say that a time-varying matrix $R(t)$ is positive definite if there exists a positive constant q_1 such that $q_1 I \le R(t)$. The positive definite matrix is said to be bounded if there exist a positive constant $q_2 \ge q_1$ such that $R(t) \le q_2 I$. Let L be a diagonal constant matrix. Then, $\lambda_{\min}(L)$, $\lambda_{\max}(L)$ denote the minimum and maximum eigenvalues of L, respectively. Let X be a set in a real vector space and $f(x,t) : (X,t) \to \mathbb{R}$ be a function with $x \in X$. f is called convex for x in X if and only if for any $x_1, x_2 \in X$, $t > 0$ and $0 < c < 1$, $f(cx_1 + (1-c)x_2, t) \le cf(x_1,t) + (1-c)f(x_2,t)$. f is called strongly convex for x in X with parameter $m > 0$ if and only if for any $x_1, x_2 \in X$, $t > 0$, $f(x_2,t) - f(x_1,t) \ge \nabla f(x_1,t))^T(x_2 - x_1) \ge \frac{m}{2}\left\|x_1 - x_2\right\|^2$.

5.2.1 Distributed Quadratic Optimization with Neighboring Coupled Objective Functions

In this section, we solve Problem 5.1 by using the projected gradient method. The result of this section is based on the assumption that $f(x,t)$ is quadratic and strongly convex for the variable x. Let Ω_i be the constraint set for the decision variable x_i and assume that for any vector $z = \left[z_1^T, \cdots, z_N^T\right]^T$, $z \in \Omega$ if and only if $z_i \in \Omega_i$ for all $i \in \{1, \cdots N\}$, where Ω is the global constraint set.

Given a nonempty compact convex set $K \subset \mathbb{R}^N$, the projection of a variable $y \in \mathbb{R}^N$ into K is the mapping $P : \mathbb{R}^N \to K$ defined by $P_K(y) = \arg_{z \in K} \min \left\|y - z\right\|$. Motivated by the global projection dynamics for single-agent, time-invariant optimization problems proposed in [134], we design the following distributed updating law to search the solution of the time-varying optimization problem

$$ \tag{5.1} $$

where k, α are positive constants to be determined.

Suppose that $f(x,t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i^T R_{ij}(t) x_j + \sum_{i=1}^{N} s_i(t)^T x_i + h(t)$, where $R_{ij}(t) \in \mathbb{R}^{m_i} \times \mathbb{R}^{m_j}$, $s_i(t) \in \mathbb{R}^{m_i}$ and $h(t) \in \mathbb{R}$ are real, time-varying sufficient smooth coefficients and $R_{ij}(t) = R_{ji}^T(t)$.

Let $[s_i(t)] = [s_1(t)^T, s_2(t)^T, \cdots, s_N(t)^T]^T$ and $R(t) = \begin{bmatrix} R_{11}(t) & R_{12}(t) & \cdots & R_{1N}(t) \\ R_{21}(t) & & & R_{2N}(t) \\ \vdots & & \ddots & \\ R_{N1}(t) & \cdots & & R_{NN}(t) \end{bmatrix}$.

By the strong convexity of $f(x,t)$, the matrix $R(t)$ is symmetric positive definite.

If the agents' objective functions are coupled with their neighbors' decision variables only, then $\frac{\partial}{\partial x_i}(f_i(x,t) + \sum_{j=1}^{N} a_{ij} f_j(x,t)) = \frac{\partial f(x,t)}{\partial x_i}$ holds for all agents. Hence, (5.1) can be rewritten as

$$\dot{x}_i = kP_{\Omega_i}(-\alpha\frac{\partial f(x,t)}{\partial x_i} + x_i) - kx_i. \tag{5.2}$$

The objective of this section is to provide sufficient conditions for the global convergence of (5.1) and stability of the dynamical system. The following lemmas will be used in the subsequent stability analysis.

Lemma 5.1. *[135] For all $x, y \in \mathbb{R}^N$ and a nonempty compact convex set $K \subset \mathbb{R}^N$, $\|P_K(x) - P_K(y)\| \le \|x - y\|$.*

Lemma 5.2. *Let $x = [x_1^T, \cdots, x_N^T]^T$. Then, $P_\Omega(x) = [P_{\Omega_1}^T(x_1), \cdots, P_{\Omega_N}^T(x_N)]^T$.*

Proof.

$$\begin{aligned} P_\Omega(x) &= \arg_{z\in\Omega}\min\|x-z\| = \arg_{z\in\Omega}\min\|x-z\|^2 \\ &= \arg_{[z_1^T,\cdots,z_N^T]^T\in\Omega}\min\sum_i\|x_i-z_i\|^2 \\ &= [(\arg_{z_1\in\Omega_1}\min\|x_1 - z_1\|^2)^T,\cdots,(\arg_{z_N\in\Omega_N}\min\|x_N - z_N\|^2)^T]^T \\ &= [P_{\Omega_1}^T(x_1),\cdots,P_{\Omega_N}^T(x_N)]^T. \end{aligned}$$

□

Lemma 5.3. *A trajectory $x^*(t)$ is an optimal solution of Problem 5.1 if and only if for any fixed parameter $\alpha > 0$, $P_\Omega(-\alpha\frac{\partial f(x^*(t),t)}{\partial x} + x^*(t)) = x^*(t)$ for all $t \geq t_0$.*

Proof. The proof is similar to the time-invariant case in [135]. Let $x^*(t)$ be a trajectory satisfying $x^*(t) \in \Omega$ and $f(x^*(t),t) \leq f(x,t)$, $\forall x \in \Omega$. Then if $x \in \Omega$, $z(t) = x^*(t) + \nu(x - x^*(t)) \in \Omega$ for $0 \leq \nu \leq 1$. Therefore, the function $\varphi(\nu,t) = f(x^*(t) + \nu(x - x^*(t)),t)$ attains its minimum at time t when $\nu = 0$. Then it can be obtained that $\frac{\partial f(x^*(t),t)}{\partial x}^T(x - x^*(t)) \geq 0$, $\forall x \in \Omega, t \geq t_0$. Since $f(x,t)$ is convex for x, $\frac{\partial f(x^*(t),t)}{\partial x}^T(x - x^*(t)) \geq 0$, $\forall x \in \Omega, t \geq t_0$ is also a sufficient condition for the optimality of $x^*(t)$.

According to a similar analysis as Theorem 2.3 in [135], $x^*(t)$ is a solution satisfying $\frac{\partial f(x^*(t),t)}{\partial x}^T(x - x^*(t)) \geq 0$, $\forall x \in \Omega$ if and only if for any fixed parameter $\alpha > 0$, $x^*(t)$ satisfies $P_\Omega(-\alpha\frac{\partial f(x^*(t),t)}{\partial x} + x^*(t)) = x^*(t)$, which completes the proof. □

Based on Lemma 5.2, the concatenated form of the system (5.2) can be written as

$$\dot{x} = kP_\Omega(-\alpha\frac{\partial f(x,t)}{\partial x} + x) - kx. \tag{5.3}$$

The main result of this section can be stated in the following theorem.

Theorem 5.1. *The updating law in (5.1) guarantees $\|x(t) - x^*(t)\|$ uniformly ultimately bounded as $t \to \infty$ with ultimate bound $\|x(t) - x^*(t)\| \leq \sqrt{\frac{1}{2k(1-\varepsilon)-1}}\varpi$, where*

$0 < \varepsilon < 1$ is a positive constant, ϖ is the upper bound of $\|\dot{x}^*(t)\|$, provided that there exist positive constants q_1 and $q_2 \geq q_1$ such that $q_1 I \leq R(t) \leq q_2 I$, $\dot{x}^*(t)$ exists, the elements in $R(t)$ and $x^*(t)$, $\dot{x}^*(t)$ are bounded, and the parameters ε, k, α are chosen such that

$$\begin{aligned} \frac{q_2 - q_1}{q_2 + q_1} &< \varepsilon < 1, \ k > \frac{1}{2(1-\varepsilon)}, \\ \frac{1-\varepsilon}{q_1} &< \alpha < \frac{1+\varepsilon}{q_2}. \end{aligned} \tag{5.4}$$

Proof. Define a Lyapunov candidate as $V = \frac{1}{2} e^T e$, where $e = x - x^*(t)$, and $x^*(t)$ is the optimal solution satisfying $P_\Omega(-\alpha R(t) x^*(t) - \alpha[s_i(t)] + x^*(t)) = x^*(t)$.

Hence,

$$\begin{aligned} & P_\Omega(-\alpha \tfrac{\partial f(x,t)}{\partial x} + x) - x = P_\Omega(-\alpha R(t) x - \alpha[s_i(t)] + x) - x \\ = \quad & P_\Omega(-\alpha R(t) e - \alpha R(t) x^*(t) - \alpha[s_i(t)] + x) - x \\ = \quad & P_\Omega(-\alpha R(t) e - \alpha R(t) x^*(t) - \alpha[s_i(t)] + e + x^*(t)) \\ & - P_\Omega(-\alpha R(t) x^*(t) - \alpha[s_i(t)] + x^*(t)) - e. \end{aligned}$$

It follows that

$$\begin{aligned} \dot{V} = \quad & e^T(k P_\Omega(-\alpha \tfrac{\partial f(x,t)}{\partial x} + x) - kx - \dot{x}^*(t)) \\ = \quad & e^T(k P_\Omega(-\alpha R(t) e - \alpha R(t) x^*(t) - \alpha[s_i(t)] + e + x^*(t)) \\ & - k P_\Omega(-\alpha R(t) x^*(t) - \alpha[s_i(t)] + x^*(t)) - ke - \dot{x}^*(t)) \\ & \leq -k e^T e + k \|e\| \, || P_\Omega(-\alpha R(t) e - \alpha R(t) x^*(t) - \alpha[s_i(t)] + e + x^*(t)) \\ & - P_\Omega(-\alpha R(t) x^*(t) - \alpha[s_i(t)] + x^*(t)) || + \|e\| \, \|\dot{x}^*(t)\| . \end{aligned}$$

Using Lemma 5.1, we have

$$\dot{V} \quad \leq -k e^T e + k \|e\| \, \|\alpha R(t) e - e\| + \|e\| \, \|\dot{x}^*(t)\|$$

$$\leq -k(1 - \frac{1}{2k} - \|\alpha R(t) - I\|)(\|e\|^2 - \frac{1}{2k(1 - \|\alpha R(t) - I\|) - 1}\|\dot{x}^*(t)\|^2).$$

Since $\frac{1-\varepsilon}{q_1} < \alpha < \frac{1+\varepsilon}{q_2}$, it can be derived that $-\varepsilon I < \alpha R(t) - I < \varepsilon I$. Thus, $(\alpha R(t) - I)^T(\alpha R(t) - I) < \varepsilon^2 I$, which implies $\|\alpha R(t) - I\| < \varepsilon$. Thus, $\dot{V} \leq -k(1 - \frac{1}{2k} - \varepsilon)(\|e\|^2 - \frac{1}{2k(1-\varepsilon)-1}\varpi^2)$ and $\dot{V} < 0$, $\forall \|e\|^2 > \frac{1}{2k(1-\varepsilon)-1}\varpi^2 = \delta$. Hence, the solutions are uniformly ultimately bounded with the ultimate bound $\|e\|^2 < \frac{1}{2k(1-\varepsilon)-1}\varpi^2$. □

Remark 5.1. *The conditions in the theorem imply that the time-related parameters in the objective functions are bounded. These conditions are required due to the existence of penalty parameters and the subsequent stability analysis.*

Remark 5.2. *In (5.4), if the knowledge of q_1 and q_2 is unknown, one may select a sufficient small α and a sufficient large k to guarantee convergence.*

5.2.2 Distributed Quadratic Optimization with Generally Coupled Objective Functions

In this section, a generally coupled constrained distributed optimization problem, in which the local objective functions $f_i(x,t)$ are quadratic and strongly convex for x, is considered.

The following problem is formulated to seek the optimal solution of Problem 5.1.

$$\begin{aligned} \min\ f(\mathbf{x},t) &= \sum\nolimits_{i=1}^{N} f_i(\mathbf{x}_i,t), \text{subject to } \mathbf{L}\mathbf{x} = \mathbf{0}, \\ \mathbf{x}_i &\in \Omega_i, i \in \{1,\cdots,N\}, \end{aligned} \tag{5.5}$$

where $\mathbf{x}_i \in \mathbb{R}^{\sum_{i=1}^{N} m_i}$ denotes agent i's estimate on the optimal solution $x^*(t)$, Ω_i is the local constraint set for agent i which satisfies $\cap_{i=1}^{N}\Omega_i = \Omega$ with Ω being the global constraint set.

In this section, we use a penalty function based method to solve (5.5). Define

$$F(\mathbf{x},t)=\sum\nolimits_{i=1}^{N} f_i(\mathbf{x}_i,t)+\tfrac{1}{2}\beta\mathbf{x}^T\mathbf{L}\mathbf{x}, \tag{5.6}$$

where β is a positive constant penalty parameter and $\frac{1}{2}\beta\mathbf{x}^T\mathbf{L}\mathbf{x}$ denotes the penalty for the violation of the constraint $\mathbf{L}\mathbf{x}=\mathbf{0}$. Then, in order to estimate the optimal solution, agent $i, i\in\{1,2,\cdots,N\}$ can coordinate with its neighbors to solve the following problem:

$$\begin{aligned} \min\ F(\mathbf{x},t) &= \sum\nolimits_{i=1}^{N} f_i(\mathbf{x}_i,t)+\tfrac{1}{2}\beta\mathbf{x}^T\mathbf{L}\mathbf{x}, \\ \mathbf{x}_i &\in \Omega_i, i\in\{1,\cdots,N\}, \end{aligned} \tag{5.7}$$

where β is the penalty parameter.

The updating law for agent $i, i\in\{1,2,\cdots,N\}$ is designed as follows

$$\begin{aligned} \dot{\mathbf{x}}_i &= kP_{\Omega_i}(-\alpha\tfrac{\partial f_i(\mathbf{x}_i,t)}{\partial \mathbf{x}_i}-\alpha\beta\sum\nolimits_{j=1}^{N} a_{ij}(\mathbf{x}_i-\mathbf{x}_j) \\ &\quad +\mathbf{x}_i)-k\mathbf{x}_i, \end{aligned} \tag{5.8}$$

where α, k are constant parameters to be determined later.

Similar to Lemma 5.3, we can obtain the following conclusion.

Lemma 5.4. $\mathbf{x}^*(t)$ *is an optimal solution of (5.7) if and only if for some fixed parameter* $\alpha>0$, $P_{\Omega'}(-\alpha\frac{\partial F(\mathbf{x}^*(t),t)}{\partial \mathbf{x}}+\mathbf{x}^*(t))=\mathbf{x}^*(t)$, $\forall\mathbf{x}\in\Omega'$, $t\geq t_0$, *where* Ω' *is the Cartesian product defined as* $\Omega'=\Omega_1\times\cdots\times\Omega_N$.

The main result of this section can be summarized in the following theorem.

Theorem 5.2. *The updating law in (5.8) guarantees* $\|\mathbf{x}(t)-\mathbf{x}^*(t)\|$ *uniformly ultimately bounded as* $t\to\infty$ *with ultimate bound* $\|\mathbf{x}(t)-\mathbf{x}^*(t)\|\leq\sqrt{\frac{1}{2k(1-\varepsilon_L)-1}}\varpi_L$, *where* $0<\varepsilon_L<1$ *is a positive constant,* ϖ_L *is the upper bound of* $\|\dot{\mathbf{x}}^*(t)\|$, *provided* *that there exist positive constants q_1 and $q_2 \geq q_1$ such that $q_1 I \leq R(t) \leq q_2 I$, $\dot{\mathbf{x}}^*(t)$ exists, the elements in $R(t)$ and $\mathbf{x}^*(t)$, $\dot{\mathbf{x}}^*(t)$ are bounded and the control parameters ε_L, α, β, k are chosen such that*

$$\begin{aligned} \frac{q_2 + \beta\lambda_{\max}(\mathbf{L}) - q_1}{q_2 + \beta\lambda_{\max}(\mathbf{L}) + q_1} &< \varepsilon_L < 1, \ k > \frac{1}{2(1-\varepsilon_L)}, \\ \frac{1-\varepsilon_L}{q_1} &< \alpha < \frac{1+\varepsilon_L}{q_2 + \beta\lambda_{\max}(\mathbf{L})}. \end{aligned} \tag{5.9}$$

Proof. The proof is similar to that of Theorem 5.1, and thus is omitted here. □

5.3 Numerical Example

In this section, numerical examples are provided to verify the effectiveness of the proposed methods.

Example 5.1. *A network of 3 agents coordinate with each other to solve $f(x,t) = \sum_{i=1}^{3} f_i(x,t)$, where $f_1(x,t) = x_1^2 + (x_1 - x_2 - 3\sin(t))^2$, $f_2(x,t) = (x_1 - x_2)^2 + (x_2 - x_3)^2$, $f_3(x,t) = x_3^2 + (x_2 - x_3)^2$ are the local objective functions for agents 1-3, respectively. Let $\Omega_i = \{x_i \in \mathbb{R} | i-1 \leq x_i \leq i+1\}$ be the local constraint set for agent i. The communication graph for the three agents is shown in Fig. 5.1. Let x_i be the state of agent i and (5.1) be the updating law. Fig. 5.2 shows the simulation result of the estimate on the optimal solutions $x_1^*(t), x_2^*(t)$ and $x_3^*(t)$.*

Example 5.2. *A network of 3 agents coordinate with each other to solve $f(x,t) = \sum_{i=1}^{3} f_i(x,t)$ with $f_1(x,t) = \frac{1}{6}x^T Rx + \sin(t)x_1$, $f_2(x,t) = \frac{1}{6}x^T Rx + 2x_2$, and $f_3(x,t) = \frac{1}{6}x^T Rx + \cos(t)x_3$ being the local objective functions for agents 1-3, respectively, where*

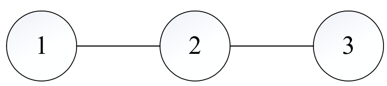

Figure 5.1: The communication graph for the networks in Examples 5.1 and 5.2.

$R = [3, -1, -1\ ;\ -1, 2, 0\ ;\ -1, 0, 3]$. Let $\Omega_i = \{x \in \mathbb{R}^3 | -2 \leq x_1 \leq 2,\ 0 \leq x_2 \leq 1,$ $-3 \leq x_3 \leq 1\}$ be the local constraint set for agent i. The communication graph for the three agents is the same as in Example 5.1. Let $\mathbf{x}_i = [x_{i1}, x_{i2}, x_{i3}]^T$ be the state of agent i and (5.8) be the updating law. Fig. 5.3 shows the simulation results of the three agents' estimate on the optimal solutions $x_1^*(t), x_2^*(t)$ and $x_3^*(t)$, respectively.

5.4 Conclusions

In this chapter, we dealt with the time-varying distributed optimization problem with compact convex local constraint sets. Both neighboring coupled quadratic objective functions and generally coupled quadratic objective functions were considered, using projected gradient methods. Uniformly ultimately bounded tracking results were obtained and the tracking errors can be arbitrarily small by tuning the control gains. Note that there are many applications that can be modelled as not only optimization problems but also game problems, e.g., demand response in smart grids [25, 33]. The Nash equilibrium seeking problems can be viewed as an extension of distributed optimization problems and some distributed optimization problems can be solved using game-theoretic methods [136]. In the next chapter, we will develop consensus based Nash equilibrium seeking algorithms for the player such that the players distributively seek a Nash equilibrium in a generalized convex game.

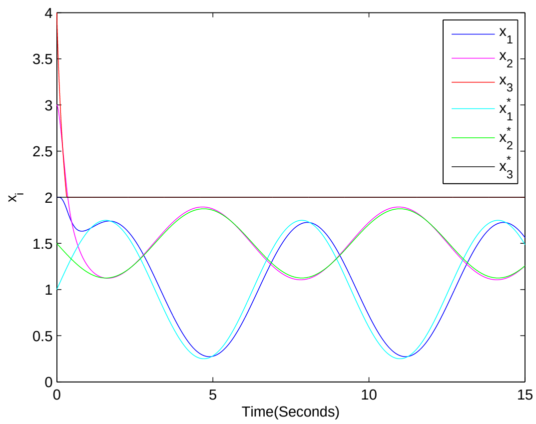

Figure 5.2: The estimate on the optimal solutions $x_1^*(t), x_2^*(t)$ and $x_3^*(t)$ when the objective functions are neighboring coupled.

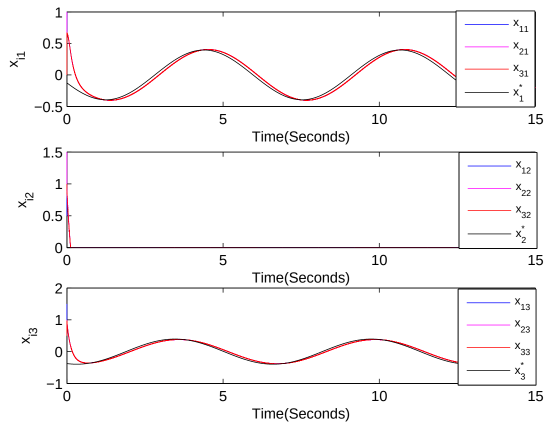

Figure 5.3: The estimate on the optimal solutions $x_1^*(t), x_2^*(t)$, and $x_3^*(t)$ when the objective functions are generally coupled.

Chapter 6

Distributed Nash Equilibrium Seeking for Noncooperative Games with Shared Constraints and Nonsmooth Objective Functions

6.1 Introduction

As stated in Chapter 1, Nash equilibrium seeking is an interesting and important problem with wide applications. A point is a Nash equilibrium if no unilateral deviation in strategy by any single player is profitable for that player [137]. Let $f_i(x)$ be player i's objective function with $x = [x_1, \cdots, x_N]^T$ being the strategy of all the players and C_i being the strategy space of player i. A point $x^* = [x_1^*, \cdots, x_N^*]^T$ is said to be a Nash equilibrium of the game if for all i, $f_i(x_i^*, x_{-i}^*) \leq f_i(x_i, x_{-i}^*)$ for all $i \in C_i$ with $x_{-i} = [x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N]^T$.

The generalized Nash games assume that each player's strategy space C_i may be related to the rival players' strategies [84–86]. Mathematically, the generalized Nash game with inequality constraints can be described as: $\min_{x_i} \ f_i(x),$ such that $g_{ik}(x) \leq 0, k \in \{1, \cdots, \kappa\}, i \in \{1, \cdots, N\}$ where $g_{ik}(x)$ is the k-th constraint of player i. In this chapter, we consider the generalized Nash games with shared constraints. The strategy space of the game can be described as $C = \{x | g_k(x) \leq 0, k \in \{1, \cdots, \kappa\}\}$, where $g_k(x)$ is the k-th constraint that shared by all the players. There are many applications that can be modelled as such a kind of game, e.g., market liberalization of electricity, natural gas, femto-cell power allocation, and environmental pollution control (for more details on these applications, see the survey paper [86]). Recently, distributed computation algorithms for optimization and game problems have attracted much attention. In [74], the authors solved a distributed Nash equilibrium seeking problem for unconstrained non-cooperative games. In [90], discrete-time adaptive algorithms were presented to solve GNEPs for neighboring coupled objective functions and constraints. In [138], differentiable penalties were used to estimate the Nash equilibrium of games in the environment of unknown statistical distribution.

Nonsmooth objective functions have attracted much interest in recent years due to the applications in many areas such as circuits and resource allocation [117]. The differentiability assumption might not hold in some problems. For example, players face congestion costs that are piecewise smooth [139]. Some research have been conducted related to nonsmooth problems. In [140], a subgradient based discrete-time algorithm was proposed to compute the Nash equilibrium under a time-varying multi-agent network. However, the games are restricted to zero-sum game. Nonsmooth analysis theory [115] can be used to analyze continuous-time set-valued dynamical systems. [141] proposed a projected gradient based algorithm to solve the distributed optimization problem of a sum of nonsmooth convex cost functions with local constraints. [32] studied saddle point dynamics for a class of nonsmooth convex optimization problems with either equality or inequality constraints. For noncooperative games, singular perturbation can be used to estimate global state information [74], however, this method cannot be directly applied to nonsmooth games due to the differentiablity assumption.

In this chapter, we present a continuous-time distributed algorithm to seek a Nash equilibrium for a generalized convex game with nonsmooth objective functions, including class-C^2 and locally Lipschitz objective functions. The main contributions of this chapter are summarized as follows:

1. For distributed Nash equilibrium seeking problems, singular perturbation can be used to analyze the convergence of the algorithms [74]. However, this method cannot be directly applied to nonsmooth games due to the differentiability assumption. In this paper, based on nonsmooth analysis theory and converse Lyapunov theorems for differential inclusions, we provide convergence analysis for the singular perturbed nonsmooth systems. Moreover, compared with [74], constraints relying on other players' strategies are considered.

2. Compared with the existing works [90] and [140], the games in this paper are more general. In [90], the objective functions were required to be smooth and neighboring coupled. In [140], zero-sum games were studied. 3) This paper provides distributed differential equation and differential inclusion based methods to solve generalized Nash equilibrium problems. As continuous-time algorithms, they have advantages over continuous-time physical control systems and use well-designed continuous-time control techniques [31].

The rest of this chapter is organized as follows: In Section 6.2, the distributed Nash equilibrium seeking problem is formulated mathematically. In Section 6.3 and Section 6.4, Class-C^2 and locally Lipschitz objective functions are considered, respectively. Section 6.5 gives two numerical examples to illustrate the effectiveness of the proposed algorithms. Finally, Section 6.6 concludes this chapter.

6.2 Problem Formulation

Consider the set of players $\mathcal{V} \triangleq \{1, \cdots, N\}$ where the action of player i is denoted as $x_i \in \mathbb{R}$. Each player i minimizes its objective function $f_i(x) : \mathbb{R}^N \to \mathbb{R}$ where $x = [x_1, \cdots, x_N]^T \in \mathbb{R}^N$ is the action vector of N players. Suppose that each player is subject to κ shared constraints that depend on the players' actions, i.e., $g_j(x) \leq 0$, $j \in \mathcal{K} \triangleq \{1, ..., \kappa\}$. If agent j is not a neighbor of agent i, then player i has no access to player j's action directly. Otherwise, player i can get the information of player j via a connected undirected graph topology. Our objective is to design a distributed Nash equilibrium seeking law for the players such that their actions converge to a Nash equilibrium of the game. Denote $x_{-i} = [x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N]^T \in \mathbb{R}^{N-1}$. Then, for player $i \in \mathcal{V}$, its optimization problem can be described as

$$\min_{x_i} \ f_i(x_i, x_{-i}), \text{ such that } g_j(x_i, x_{-i}) \leq 0, \ j \in \mathcal{K}. \tag{6.1}$$

Throughout this chapter, suppose that the following assumptions on the constraints always hold.

Assumption 6.1. $g_j(x) : \mathbb{R}^N \to \mathbb{R}$ *is a class-*C^2 *function and is convex with respect to* x *for every* $j \in \mathcal{K}$.

Assumption 6.2. *The constraint set* $\mathcal{C} = \{x \in \mathbb{R}^N \mid g_j(x) \leq 0, \ j \in \mathcal{K}\}$ *is nonvoid and bounded. Furthermore, the Slater's conditions hold.*

6.3 Distributed Nash Equilibrium Seeking for Games with Class-C^2 Objective Functions

In this section, we aim to find a Nash equilibrium of a generalized convex game, where the objective functions satisfy the following assumption.

Assumption 6.3. *$f_i(x_i, x_{-i}) : \mathbb{R}^N \to \mathbb{R}$ is a class-C^2 function and is convex with respect to x_i for every $i \in \mathcal{V}$.*

6.3.1 Existence and Uniqueness of the Normalized Nash Equilibrium

Under Assumptions 6.1-6.3, the players' optimization problems are convex and Nash equilibria exist [84]. Moreover, a point $x^* = [x_1^*, \cdots, x_N^*]^T \in \mathbb{R}^N$ is a Nash equilibrium of the game if and only if the following KKT conditions hold for all $i \in \mathcal{V}$ and some $\lambda_{ij}^* \geq 0$, $j \in \mathcal{K}$:

$$\nabla_{x_i} f_i(x^*) + \sum\nolimits_{j \in \mathcal{K}} \lambda_{ij}^* \nabla_{x_i} g_j(x^*) = 0,$$

$$\lambda_{ij}^* g_j(x^*) = 0, \ g_j(x^*) \leq 0, \ j \in \mathcal{K}. \tag{6.2}$$

Usually, the problem in (6.1) has multiple Nash equilibria. In this chapter, we do not assume the uniqueness of the Nash equilibrium. Instead, we consider a special kind of Nash equilibrium, namely the normalized equilibrium point [84], which is defined according to the KKT condition (6.2), such that for all $i \in \mathcal{V}$,

$$\nabla_{x_i} f_i(x^*) + \sum\nolimits_{j \in \mathcal{K}} \lambda_j^* \nabla_{x_i} g_j(x^*) = 0,$$

$$\lambda_j^* g_j(x^*) = 0, \ g_j(x^*) \leq 0, \ j \in \mathcal{K}, \tag{6.3}$$

where $\lambda_j^* \geq 0$ is the common Lagrange multiplier for each shared constraint $g_j(x)$.

Before presenting the algorithm, we show that under some assumptions, the normalized Nash equilibrium satisfying (6.3) exists and is unique. Firstly, we introduce the concept of monotonicity for a single-valued mapping $F : \mathbb{R}^N \to \mathbb{R}^N$.

Definition 6.1 (On montonicity of a single-valued mapping)**.** *[90] (1) A mapping $F : \mathbb{R}^N \to \mathbb{R}^N$ is said to be monotone on $\mathbb{R}^N$ if for all $x, \check{x} \in \mathbb{R}^N, x \neq \check{x}, (F(x) -$*

$F(\check{x}))^T(x - \check{x}) \geq 0$.

(2) A mapping $F : \mathbb{R}^N \to \mathbb{R}^N$ *is said to be strictly monotone on* $\mathbb{R}^N$ *if for all* $x, \check{x} \in \mathbb{R}^N, x \neq \check{x}, (F(x) - F(\check{x}))^T(x - \check{x}) > 0$.

Assumption 6.4. *[84] The function* $\nabla_x f(x) = [\nabla_{x_1} f_1(x), \cdots, \nabla_{x_N} f_N(x)]^T$ *is strictly monotone on* $\mathbb{R}^N$.

Lemma 6.1. *[84] Under Assumptions 6.1-6.4, the normalized Nash equilibrium* x^* *satisfying (6.3) exists and is unique.*

The following assumption will be used in the stability analysis.

Assumption 6.5. *Under Assumptions 6.1-6.4, the pair* $(x^*, \lambda_1^*, \cdots, \lambda_\kappa^*)$ *satisfying (6.3) is unique.*

Remark 6.1. *Assumption 6.5 is equivalent to the following assumption: Let* $g_{r_1}, \cdots, g_{r_p}$ *with* $r_1, \cdots, r_p \in \mathcal{K}$ *be the active constraints at the normalized Nash equilibrium, i.e.,* $g_{r_1}(x^*) = \cdots = g_{r_p}(x^*) = 0$. *If there exist multiple active constraints, the matrix* $[\nabla_x g_{r_1}(x^*), \cdots, \nabla_x g_{r_p}(x^*)]$ *should be of full column rank.*

Remark 6.2. *Assumption 6.5 is used to guarantee the uniqueness of the pair* $(x^*, \lambda_1^*, \cdots, \lambda_\kappa^*)$. *Without Assumption 6.5, only the uniqueness of* x^* *can be guaranteed. For example, suppose that* $g_1(x) = x_1 + x_2 + 2$ *and* $g_2(x) = 2x_1 + 2x_2 + 4$ *are two active constraints at the normalized Nash equilibrium, i.e.,* $g_1(x^*) = g_2(x^*) = 0$. *According to (3), the solutions of* λ_1^*, λ_2^* *are not unique. Assumption 6.5 holds for the following two cases: 1) The constraints can't be active at the same time with* $g_1(x) = x_1 + x_2 - 2$ *and* $g_2(x) = -x_1 - x_2 - 2$; *2) The gradients of multiple active constraints are linearly independent with* $g_1(x) = x_1 + x_2 + 2$ *and* $g_2(x) = 2x_1 + 3x_2 + 4$.

6.3.2 Control Design and Stability Analysis

We now propose a distributed control law using the estimation of neighboring actions. Let $Y_i = [y_{i1}, \cdots, y_{iN}]^T$ be player i's estimation on all the players' actions, which is produced by the following consensus based protocol

$$
\begin{aligned}
y_{ii} &= x_i, i \in \mathcal{V} \\
\dot{y}_{ij} &= -w_{ij} \sum_{k=1}^{N} a_{ik}(y_{ij} - y_{kj}), j \in \mathcal{V}/\{i\},
\end{aligned} \tag{6.4}
$$

with w_{ij} being a positive constant.

Based on (6.4), the updating law for play i is designed as

$$
\begin{aligned}
\dot{x}_i &= -\bar{k}_i(\nabla_{x_i} f_i(Y_i) + \sum\nolimits_{j\in\mathcal{K}} \nabla_{x_i}\lambda_{ij} g_j(Y_i)), \\
\dot{\lambda}_{1j} &= \bar{k}_{1j}\lambda_{1j} g_j(Y_1), \\
\dot{\lambda}_{ij} &= -\gamma_{ij} \sum_{k=1}^{N} a_{ik}(\lambda_{ij} - \lambda_{kj}),\ j \in \mathcal{K}, i \neq 1,
\end{aligned} \tag{6.5}
$$

where player 1 is selected to calculate the common multipliers λ_{1j} and consensus based control laws are used to broadcast them to all the other players. In (6.5), $\lambda_{1j}(0) > 0$, $\gamma_{ij} > 0$, and $\bar{k}_i = \varepsilon k_i$, where ε is a small positive constant and $k_i > 0$ is a positive constant.

The distributed Nash equilibrium seeking algorithm in (6.4) and (6.5) uses leader-following consensus to estimate unknown information and saddle point dynamics to achieve convergence. In the following analysis, we use singular perturbation theory to prove the stability of the system. First, an auxiliary system is designed. Then, based on the stability of this auxiliary system, we prove the convergence of the original system using Lyapunov based methods.

For agent i, define a subgraph $\mathcal{G}_i = \{\mathcal{V}_i, \mathcal{E}_i\}$, where $\mathcal{V}_i = \mathcal{V}/\{i\}$ and $\mathcal{E}_i \subset \mathcal{V}_i \times \mathcal{V}_i$ indicate the set of vertices and edges, respectively. Let L_i be the Laplacian matrix of $\mathcal{G}_i$ and B_i =diag$\{a_{1i}, \cdots, a_{(i-1)i}, a_{(i+1)i}, \cdots, a_{Ni}\} \in \mathbb{R}^{(N-1)\times(N-1)}$ is the information exchange matrix of agent i. Then, we can rewrite (6.4) and (6.5) as:

$$\begin{aligned}
\dot{x} &= -\bar{\mathbf{k}}(\nabla_x f(Y) + \sum\nolimits_{j\in\mathcal{K}} \mathrm{diag}\{\lambda_{1j},\, \lambda_j\}\nabla_x g_j(Y)),\\
\dot{\lambda}_{1j} &= \bar{k}_{1j}\lambda_{1j} g_j(Y_1),\\
\dot{\lambda}_j &= -\Gamma_j(L_1 + B_1)(\lambda_j - \lambda_{1j}\mathbf{1}),\ j \in \mathcal{K},\\
\dot{\bar{Y}}_i &= -W_i(L_i + B_i)(\bar{Y}_i - x_i\mathbf{1}), i \in \mathcal{V},
\end{aligned} \tag{6.6}$$

where $Y = [Y_1^T, \cdots, Y_N^T]^T \in \mathbb{R}^{N^2}$, $\nabla_x f(Y) = [\nabla_{x_1} f_1(Y_1), \cdots, \nabla_{x_N} f_N(Y_N)]^T$, $\nabla_x g(Y) = [\nabla_{x_1} g(Y_1),\, \cdots,\, \nabla_{x_N} g(Y_N)]^T \in \mathbb{R}^N$, $\lambda_j = [\lambda_{2j},\, \cdots,\, \lambda_{Nj}]^T$, $\bar{Y}_i = [y_{1i},\, \cdots,\, y_{(i-1)i},\, y_{(i+1)i},\, \cdots,\, y_{Ni}]^T \in \mathbb{R}^{N-1}$, $\bar{\mathbf{k}}$ $=\mathrm{diag}\{\bar{k}_1, \cdots, \bar{k}_N\} \in \mathbb{R}^{N\times N}$, Γ_j $=\mathrm{diag}\{\gamma_{2j}, \cdots, \gamma_{Nj}\}$, W_i $=\mathrm{diag}\{w_{1i}, \cdots, w_{(i-1)i},\ w_{(i+1)i}, \cdots, w_{Ni}\}\in \mathbb{R}^{(N-1)\times(N-1)}$. Let $[\zeta_j]$ be a column vector defined as $[\zeta_1^T, \cdots, \zeta_\kappa^T]^T$ and $[\eta_i]$ be a column vector defined as $[\eta_1^T, \cdots, \eta_N^T]^T$. Define an error variable $\Delta(t) = [(x(t) - x^*)^T,\ [\lambda_{1j}(t) - \lambda_{1j}^*]^T,\ [\lambda_j(t) - \lambda_{1j}^*\mathbf{1}]^T,\ [\bar{Y}_i(t) - x_i^*\mathbf{1}]^T]^T$. The following theorem presents the main result of this section.

Theorem 6.1. *Suppose that Assumptions 6.1-6.5 hold and let (6.4) and (6.5) be the updating law. Then, for each pair of positive constants (r, ϱ), there exists a positive constant $\varepsilon^*(r, \varrho)$ such that for every $0 < \varepsilon < \varepsilon^*(r, \varrho)$, there exists a time T such that $\|\Delta(t)\| \leq \varrho$, $\forall t \geq T$ for every $\|\Delta(0)\| \leq r$.*

Proof. Firstly, introduce the following auxiliary system

$$\begin{aligned}
\dot{x} &= -\mathbf{k}(\nabla_x f(x) + \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}\nabla_x g_j(x)),\\
\dot{\lambda}_{1j} &= k_{1j}\lambda_{1j} g_j(x), j \in \mathcal{K},
\end{aligned} \tag{6.7}$$

where $\mathbf{k}$ $=\mathrm{diag}\{k_1, \cdots, k_N\} \in \mathbb{R}^{N\times N}$.

For (6.7), define a Lyapunov candidate as $V_0 = \frac{1}{2}(x-x^*)^T\mathbf{k}^{-1}(x-x^*)+\sum\limits_{j\in\mathcal{K}}\frac{1}{k_{1j}}(\lambda_{1j}-\lambda_{1j}^* - \lambda_{1j}^*\log(\lambda_{1j}) + \lambda_{1j}^*\log(\lambda_{1j}^*))$, where $[x^{*T}, \lambda_{11}^*, \cdots, \lambda_{1\kappa}^*]^T$ is the point satisfying

(6.3). Taking the derivative of V along (6.7), we have

$$\begin{aligned}\dot{V}_0 &= (x-x^*)^T(-\nabla_x f(x) - \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}\nabla_x g_j(x)) \\ &\quad + \sum\nolimits_{j\in\mathcal{K}} (\lambda_{1j}-\lambda_{1j}^*) g_j(x). \end{aligned} \tag{6.8}$$

Based on (6.3), if $\lambda_{1j}^* > 0$, we have $(\lambda_{1j}-\lambda_{1j}^*)g_j(x^*) = 0$; otherwise $\lambda_{1j}^* = 0$, which implies that $(\lambda_{1j}-\lambda_{1j}^*)g_j(x^*) \leq 0$. Under Assumption 6.4, we have $(x-x^*)^T(\nabla_x f(x) - \nabla_x f(x^*)) \geq 0$. Using (6.3), equation (6.8) can be rewritten as

$$\begin{aligned}\dot{V}_0 &\leq -(x-x^*)^T\nabla_x f(x^*) - \sum\nolimits_{j\in\mathcal{K}} (x-x^*)^T \lambda_{1j}\nabla_x g_j(x) \\ &+ \sum\nolimits_{j\in\mathcal{K}} (\lambda_{1j}-\lambda_{1j}^*)(g_j(x)-g_j(x^*)) \\ &\leq \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}^*(g_j(x)-g_j(x^*)) + \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}(g_j(x^*)-g_j(x)) \\ &+ \sum\nolimits_{j\in\mathcal{K}} (\lambda_{1j}-\lambda_{1j}^*)(g_j(x)-g_j(x^*)) \\ &= 0.\end{aligned}$$

Based on Assumption 6.4, letting $\dot{V}_0 \equiv 0$ gives $x = x^*$. Then, the largest invariant set can be written as

$$\begin{aligned}&\mathcal{S} = \{(x, \lambda_{11}, \cdots, \lambda_{1\kappa}) \in \mathbb{R}^N \times \mathbb{R}^{\geq 0} \times \cdots \times \mathbb{R}^{\geq 0} : \\ &\mathbf{0} = \nabla_x f(x) + \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}\nabla_x g_j(x), \\ &x = x^*,\ \lambda_{1j}g_j(x^*) = 0,\ j \in \mathcal{K}\}.\end{aligned}$$

It can be seen that all the points in $\mathcal{S}$ satisfy the condition in (6.3). According to Lemma 6.1 and Assumption 6.5, such a point is unique. Define $z = [(x-x^*)^T, [\lambda_{1j}-\lambda_{1j}^*]^T]^T$. Thus, for each $z \neq \mathbf{0}$, there exists a positive definite function $\beta(z)$

such that

$$(x - x^*)^T(-\nabla_x f(x) - \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}\nabla_x g_j(x)) \\ + \sum\nolimits_{j\in\mathcal{K}}(\lambda_{1j} - \lambda_{1j}^*)g_j(x) < -\beta(z). \tag{6.9}$$

For notational convenience, denote $\hat{Y}_i = \bar{Y}_i - x_i\mathbf{1}$ and $\hat{\lambda}_j = \lambda_j - \lambda_{1j}\mathbf{1}$. Define a Lyapunov candidate $V(z, [\hat{Y}_i], [\hat{\lambda}_j]) = cV_0 + (1-c)V_1$, where c is a positive constant and $V_1 = \frac{1-c}{2}(\sum\limits_{i=1}^{N}\hat{Y}_i^T\hat{Y}_i + \sum_{j\in\mathcal{K}}\hat{\lambda}_j^T\hat{\lambda}_j)$. It can be obtained that there exist two class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that $\alpha_1(z, [\hat{Y}_i], [\hat{\lambda}_j]) \leq V(z, [\hat{Y}_i], [\hat{\lambda}_j]) \leq \alpha_2(z, [\hat{Y}_i], [\hat{\lambda}_j])$. Thus, taking the derivative of V along (6.6) gives

$$\begin{aligned}
\dot{V} = &-c\varepsilon(x - x^*)^T(\nabla_x f(Y) - \nabla_x f(x)) \\
&- c\varepsilon(x - x^*)^T\nabla_x f(x) - c\varepsilon(x - x^*)^T \\
&\times (\sum\nolimits_{j\in\mathcal{K}} \operatorname{diag}\{\lambda_{1j},\, \lambda_j\}\nabla_x g_j(Y) \\
&- \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}\nabla_x g_j(Y)) - c\varepsilon(x - x^*)^T \\
&\times (\sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}\nabla_x g_j(Y) - \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}\nabla_x g_j(x)) \\
&- c\varepsilon(x - x^*)^T \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}\nabla_x g_j(x) \\
&- c\varepsilon \sum\nolimits_{j\in\mathcal{K}}(\lambda_{1j} - \lambda_{1j}^*)(g_j(Y) - g_j(x)) \\
&- c\varepsilon \sum\nolimits_{j\in\mathcal{K}}(\lambda_{1j} - \lambda_{1j}^*)g_j(x) \\
&- (1-c)\sum_{i=1}^{N}\hat{Y}_i^T W_i(L_i + B_i)\hat{Y}_i \\
&- (1-c)\sum_{i=1}^{N}\hat{Y}_i^T\frac{dx_i}{dt}\mathbf{1} - (1-c)\sum\nolimits_{j\in\mathcal{K}}\hat{\lambda}_j^T\Gamma_j \\
&\times (L_1 + B_1)\hat{\lambda}_j - (1-c)\sum\nolimits_{j\in\mathcal{K}}\hat{\lambda}_j^T\frac{d\lambda_{1j}}{dt}\mathbf{1}.
\end{aligned}$$

Furthermore, according to Assumptions 6.1 and 6.3, for every Δ belonging to a compact set,

$$
\begin{aligned}
\dot{V} \leq & -c\varepsilon\beta(z) + cl_1\varepsilon \|x - x^*\| (\sum_{i=1}^{N} \left\|\hat{Y}_i\right\|^2)^{\frac{1}{2}} \\
& + c\varepsilon \|x - x^*\| \sum\nolimits_{j\in\mathcal{K}} (\left\|\hat{\lambda}_j\right\| \|\nabla_x g_j(Y)\|) \\
& + c\varepsilon l_2 \kappa \|x - x^*\| (\sum_{i=1}^{N} \left\|\hat{Y}_i\right\|^2)^{\frac{1}{2}} \max\{\lambda_{1j}\} \\
& + c\varepsilon l_3 \sum\nolimits_{j\in\mathcal{K}} \left\|\lambda_{1j} - \lambda_{1j}^*\right\| (\sum_{i=1}^{N} \left\|\hat{Y}_i\right\|^2)^{\frac{1}{2}} \\
& - (1-c) l_4 \sum_{i=1}^{N} \left\|\hat{Y}_i\right\|^2 - (1-c) \sum_{i=1}^{N} \hat{Y}_i^T \frac{dx_i}{dt} \mathbf{1} \\
& - (1-c) l_5 \sum\nolimits_{j\in\mathcal{K}} \left\|\hat{\lambda}_j\right\|^2 \\
& - (1-c) \sum\nolimits_{j\in\mathcal{K}} \hat{\lambda}_j^T \frac{d\lambda_{1j}}{dt} \mathbf{1},
\end{aligned}
$$

where $l_1, \cdots, l_5$ are some positive constants.

Let $\delta_1, \cdots, \delta_8$ be positive constants that can be arbitrarily chosen. Using Young's Inequality gives

$$
\begin{aligned}
\dot{V} \leq & -c\varepsilon\beta(z) - [(1-c)l_4 - \frac{\varepsilon}{\delta_1} - \frac{\varepsilon}{\delta_3} - \frac{\varepsilon}{\delta_4} \\
& - \frac{\varepsilon}{\delta_5}] \sum_{i=1}^{N} \left\|\hat{Y}_i\right\|^2 - [l_5(1-c) - \frac{\varepsilon}{\delta_2} - \frac{\varepsilon}{\delta_6}] \\
& \times \sum\nolimits_{j\in\mathcal{K}} \left\|\hat{\lambda}_j\right\|^2 + \frac{\varepsilon c^2 l_1^2 \delta_1}{4} \|x - x^*\|^2 \\
& + \frac{\varepsilon c^2 \kappa^2 \delta_2}{4} \|x - x^*\|^2 \max\{\|\nabla_x g_j(Y)\|^2\} \\
& + \frac{\varepsilon c^2 l_2^2 \kappa^2 \delta_3}{4} \|x - x^*\|^2 \max\{\lambda_{1j}^2\} \\
& + \frac{\varepsilon \delta_4 c^2 l_3^2}{4} (\sum\nolimits_{j\in\mathcal{K}} \left\|\lambda_{1j} - \lambda_{1j}^*\right\|)^2 \\
& + \frac{\varepsilon \delta_5 (1-c)^2}{4} \sum_{i=1}^{N} \left\|\frac{1}{\varepsilon} \frac{dx_i}{dt} \mathbf{1}\right\|^2 \\
& + \frac{\varepsilon \delta_6 (1-c)^2}{4} \sum\nolimits_{j\in\mathcal{K}} \left\|\frac{1}{\varepsilon} \frac{d\lambda_{1j}}{dt} \mathbf{1}\right\|^2 .
\end{aligned} \tag{6.10}
$$

Let ε be chosen such that $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, where $\varepsilon_1 = (\frac{1}{\delta_1} + \frac{1}{\delta_3} + \frac{1}{\delta_4} + \frac{1}{\delta_5} + \frac{1}{\delta_7})^{-1}(1-c)l_4$, $\varepsilon_2 = (\frac{1}{\delta_2} + \frac{1}{\delta_6} + \frac{1}{\delta_8})^{-1}(1-c)l_5$. Then (6.10) can be rewritten as

$$\begin{aligned}\dot{V} &\leq -\varepsilon[c\beta(z) + \frac{1}{\delta_7}\sum_{i=1}^{N}\left\|\hat{Y}_i\right\|^2 + \frac{1}{\delta_8}\sum\nolimits_{j\in\mathcal{K}}\left\|\hat{\lambda}_j\right\|^2] \\ &\quad + o(\varepsilon\delta_1) + o(\varepsilon\delta_2) + o(\varepsilon\delta_3) + o(\varepsilon\delta_4) \\ &\quad + o(\varepsilon\delta_5) + o(\varepsilon\delta_6) \\ &\leq -\varepsilon c_\rho \rho(\|\Delta\|) + \varepsilon\sum_{i=1}^{6} o(\delta_i),\end{aligned}$$

where $c_\rho = \min\{c, \frac{1}{\delta_7}, \frac{1}{\delta_8}\}$ and $\rho(\|\Delta\|) = \beta(z) + \sum\limits_{i=1}^{N}\left\|\hat{Y}_i\right\|^2 + \sum\nolimits_{j\in\mathcal{K}}\left\|\hat{\lambda}_j\right\|^2$.

Therefore, for each $\|\Delta(t)\| \geq \rho^{-1}(\frac{2}{c_\rho}(\sum\limits_{i=1}^{6} o(\delta_i)))$, $\dot{V} \leq -\frac{\varepsilon}{2}c_\rho\rho(\|\Delta\|)$, based on Theorem 4.18 of [142], there exists δ_i^* such that for any $0 < \delta_i < \delta_i^*$, $i = 1, \cdots, 6$, there exists ε^* such that for any $0 < \varepsilon < \varepsilon^*$, there exists $T > 0$ and class-$\mathcal{KL}$ function ρ_0 [142] such that

$$\begin{aligned}&\|\Delta(t)\| \leq \rho_0(\|\Delta(0)\|, t), t \leq T, \\ &\|\Delta(t)\| \leq \alpha_1^{-1}(\alpha_2(\rho^{-1}(\frac{2}{c_\rho}(\sum_{i=1}^{6} o(\delta_i))))), t \geq T.\end{aligned}$$

□

6.4 Distributed Nash Equilibrium Seeking for Games with Locally Lipschitz Objective Functions

Compared with class-C^2 objective functions studied in the last section, locally Lipschitz objective functions are more complicated due to the following three reasons:

1) The functions are not differentiable at some points. To deal with this problem, subgradients are used for control design, which leads to nonsmooth dynamical systems; 2) For a locally Lipschitz objective function, its subdifferential is not locally Lipschitz but upper semicontinuous (see Lemma 2.2), which brings challenges to singular perturbation analysis; 3) The existence and uniqueness of the normalized Nash equilibrium for games with locally Lipschitz objective functions have to be analyzed. The development of this section can be summarized as follows: Firstly, motivated by [84], we present existence and uniqueness conditions of the normalized Nash equilibrium based on nonsmooth analysis. Then, differential inclusion theory is used to prove the stability of an auxiliary system. Noting that traditional singular perturbation theory cannot be directly applied to nonsmooth systems, we use converse Lyapunov theorems for different inclusions to prove the stability of the original system by utilizing the outer semicontinuity of the set-valued mapping.

In this section, Assumption 6.3 is replaced by the following one.

Assumption 6.6. $f_i(x_i, x_{-i}) : \mathbb{R}^N \to \mathbb{R}$ *is locally Lipschitz and is convex with respect to* x_i *for every* $i \in \mathcal{V}$.

6.4.1 Control Design

Similar to Section 6.3.2, the updating law for player i is designed as

$$\begin{aligned}
\dot{x}_i &= -\bar{k}_i(h_i(Y_i) + \textstyle\sum_{j\in\mathcal{K}} \lambda_{ij} \nabla_{x_i} g_j(Y_i)), \\
\dot{\lambda}_{1j} &= \bar{k}_{1j}\lambda_{1j} g_j(Y_1), \\
\dot{\lambda}_{ij} &= -\gamma_{ij} \sum_{k=1}^{N} a_{ik}(\lambda_{ij} - \lambda_{kj}),\ j \in \mathcal{K}, i \neq 1,
\end{aligned} \tag{6.11}$$

where Y_i is produced by (6.4). In (6.11), $h_i(Y_i)$ is the subgradient of $f_i(Y_i)$, defined as $h_i(Y_i) \in \partial_{x_i} f_i(Y_i)$, where $\partial_{x_i} f_i(Y_i)$ is the subdifferential of $f_i(x)$ at Y_i, and $\bar{k}_i$ and

γ_{ij} are defined the same as in (6.5), $\lambda_{1j}(0) > 0$.

Let $\partial_x f(Y)$ be the Cartesian product of the sets $\partial_{x_i} f_i(Y_i)$. Then, (6.11) can be rewritten as

$$\begin{aligned}
\dot{x} &\in -\bar{\mathbf{k}}(\partial_x f(Y) + \sum\nolimits_{j\in\mathcal{K}} \mathrm{diag}\{\lambda_{1j},\, \lambda_j\}\nabla_x g_j(Y)),\\
\dot{\lambda}_{1j} &= \bar{k}_{1j}\lambda_{1j} g_j(Y_1), j \in \mathcal{K},\\
\dot{\lambda}_j &= -\Gamma_j(L_1 + B_1)(\lambda_j - \lambda_{1j}\mathbf{1}), j \in \mathcal{K},\\
\dot{\bar{Y}}_i &= -W_i(L_i + B_i)(\bar{Y}_i - x_i\mathbf{1}), i \in \mathcal{V},
\end{aligned} \tag{6.12}$$

where Y, λ_j, $\bar{Y}_i$, $\bar{\mathbf{k}}$, Γ_j, W_i are defined the same as in (6.6).

The following definition generalizes monotonicity of a single-valued mapping (see Definition 6.1) to a set-valued mapping $F(x)$.

Definition 6.2 (On montonicity of a set-valued mapping)**.** *[143] (1) $F(x)$ is said to be monotone on $\mathbb{R}^N$ if for all $x, \breve{x} \in \mathbb{R}^N$, $x \neq \breve{x}$, $g(x) \in F(x)$, $g(\breve{x}) \in F(\breve{x})$, $(g(x) - g(\breve{x}))^T(x - \breve{x}) \geq 0$.*

(2) $F(x)$ is said to be strictly monotone on $\mathbb{R}^N$ if for all $x, \breve{x} \in \mathbb{R}^N$, $x \neq \breve{x}$, $g(x) \in F(x)$, $g(\breve{x}) \in F(\breve{x})$, $(g(x) - g(\breve{x}))^T(x - \breve{x}) > 0$.

Assumption 6.7. *The subdifferential $\partial_x f(x)$ is strictly monotone on $\mathbb{R}^N$.*

6.4.2 Existence and Uniqueness of the Normalized Nash Equilibrium

According to [84], Nash equilibria for the game satisfying Assumptions 6.1, 6.2 and 6.6 exist. Furthermore, the following lemma holds.

Lemma 6.2. *A point $x^* = [x_1^*, \cdots, x_N^*]^T \in \mathbb{R}^N$ is a Nash equilibrium of the game satisfying Assumptions 6.1, 6.2 and 6.6 if and only if the following KKT conditions* *hold for all* $i \in \mathcal{V}$ *and some* $\lambda_{ij}^* \geq 0$*:*

$$0 \in \partial_{x_i} f_i(x^*) + \sum\nolimits_{j\in\mathcal{K}} \lambda_{ij}^* \nabla_{x_i} g_j(x^*),$$

$$\lambda_{ij}^* g_j(x^*) = 0,\ \ g_j(x^*) \leq 0,\ \ j \in \mathcal{K},$$

where $\lambda_{ij}^* \geq 0$ *is the Lagrange multiplier of player* i *for each shared constraint.*

Proof. The proof follows Theorem 2.2.5 of [144] for optimization problems. □

Motivated by [84], Lemma 6.1 can be extended to games with locally Lipschitz objectives.

Lemma 6.3. *Under Assumptions 6.1, 6.2, 6.6, and 6.7, the normalized Nash equilibrium* x^* *satisfying the following conditions for all* $i \in \mathcal{V}$ *exists and is unique:*

$$0 \in \partial_{x_i} f_i(x^*) + \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}^* \nabla_{x_i} g_j(x^*),$$

$$\lambda_{1j}^* g_j(x^*) = 0,\ \ g_j(x^*) \leq 0,\ \ j \in \mathcal{K}, \tag{6.13}$$

where $\lambda_{1j}^* \geq 0$ *is the common Lagrange multiplier for each shared constraint.*

Proof. (Existence) Let $\theta(x, y) = \sum_{i=1}^N f_i(x_1, \cdots, y_i, \cdots, x_N)$. According to the fixed point theorem, there exists a point x^* such that

$$\theta(x^*, x^*) = \min_y \{\theta(x^*, y) | g_j(y) \leq 0, j \in \mathcal{K}\}.$$

Since $\theta(x, y)$ is convex in y, according to KKT conditions in Theorem 2.2.5 of [144], there exist $\lambda_{1j}^* \geq 0$, $\lambda_{1j}^* g_j(x^*) = 0$ such that

$$0 \in \partial_{x_i} f_i(x^*) + \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}^* \nabla_{x_i} g_j(x^*),$$

for all $i \in \mathcal{V}$, which proves the existence of the normalized equilibrium.

(Uniqueness) Assume that there are two normalized equilibrium points x^0 and x^1. Then, according to Lemma 6.2, there exist $h_i(x^0) \in \partial_{x_i} f_i(x^0)$, $h_i(x^1) \in \partial_{x_i} f_i(x^1)$, $\lambda_{1j}^0 \geq 0$ and $\lambda_{1j}^1 \geq 0$ such that

$$
\begin{aligned}
&0 = h_i(x^0) + \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}^0 \nabla_{x_i} g_j(x^0), \\
&\lambda_{1j}^0 g_j(x^0) = 0, \ g_j(x^0) \leq 0, \ j \in \mathcal{K},
\end{aligned} \tag{6.14}
$$

and

$$
\begin{aligned}
&0 = h_i(x^1) + \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}^1 \nabla_{x_i} g_j(x^1), \\
&\lambda_{1j}^1 g_j(x^1) = 0, \ g_j(x^1) \leq 0, \ j \in \mathcal{K},
\end{aligned} \tag{6.15}
$$

hold for all $i \in \mathcal{V}$.

Denote $h(x) = [h_1(x), \cdots, h_N(x)]^T$. Multiplying $x_i^1 - x_i^0$ for (6.14) and $x_i^0 - x_i^1$ for (6.15), we have $\alpha_0 + \beta_0 = 0$, where

$$
\begin{aligned}
\alpha_0 &= (x^1 - x^0)^T h(x^0) + (x^0 - x^1)^T h(x^1), \\
\beta_0 &= \sum\nolimits_{i,j} (x_i^0 - x_i^1) \lambda_{1j}^1 \nabla_{x_i} g_j(x^1) \\
&\quad + \sum\nolimits_{i,j} (x_i^1 - x_i^0) \lambda_{1j}^0 \nabla_{x_i} g_j(x^0).
\end{aligned}
$$

It follows from Assumption 6.1 that

$$
\begin{aligned}
\beta_0 &\leq \sum\nolimits_j \lambda_{1j}^1 (g_j(x^0) - g_j(x^1)) \\
&\quad + \sum\nolimits_j \lambda_{1j}^0 (g_j(x^1) - g_j(x^0)) \\
&= \sum\nolimits_j \lambda_{1j}^1 g_j(x^0) + \sum\nolimits_j \lambda_{1j}^0 g_j(x^1) \\
&\leq 0.
\end{aligned}
$$

According to Assumption 6.7, $\alpha_0 < 0$, which contradicts to $\alpha_0 + \beta_0 = 0$, thus, the normalized Nash equilibrium is unique. □

Similar to Section 6.3, suppose that the following assumption holds in this section.

Assumption 6.8. *Under Assumptions 6.1, 6.2, 6.6, and 6.7, the vector* $(x^*, \lambda_1^*, \cdots, \lambda_\kappa^*)$ *satisfying (6.13) is unique.*

Remark 6.3. *The following two assumptions are special cases of the conditions: 1) The nondifferentiable points are not the normalized Nash equilibrium of the game; 2) There are no active constraints at the normalized Nash equilibrium. The introduction of this assumption is due to the requirement of the uniqueness of the Lagrangian multipliers.*

6.4.3 Stability Analysis

Firstly, similar to (6.7), we introduce the following auxiliary system

$$\begin{aligned} \dot{x} &\in -\mathbf{k}(\partial_x f(x) + \sum_{j\in\mathcal{K}} \lambda_{1j} \nabla_x g_j(x)), \\ \dot{\lambda}_{1j} &= k_{1j}\lambda_{1j} g_j(x), j \in \mathcal{K}. \end{aligned} \tag{6.16}$$

Lemma 6.4. *The auxiliary system in (6.16) is globally asymptotically stable, provided that* $\lambda_{1j}(0) > 0$.

Proof. Define a Lyapunov candidate as $V_0 = \frac{1}{2}(x - x^*)^T \mathbf{k}^{-1}(x - x^*) + \sum_{j\in\mathcal{K}} \frac{1}{k_{1j}}(\lambda_{1j} - \lambda_{1j}^* - \lambda_{1j}^* \log(\lambda_{1j}) + \lambda_{1j}^* \log(\lambda_{1j}^*))$ where $[x^{*T}, \lambda_{11}^*, \cdots, \lambda_{1\kappa}^*]^T$ is the point satisfying the KKT condition (6.13). Taking the Lie derivative (see [116] for details) of V_0 along (6.16) gives

$$\begin{aligned} L_f V_0 = &(x - x^*)^T(-\partial_x f(x) - \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j} \nabla_x g_j(x)) \\ &+ \sum\nolimits_{j\in\mathcal{K}} (\lambda_{1j} - \lambda_{1j}^*) g_j(x). \end{aligned}$$

Taking $\xi \in L_f V_0$, we have

$$\begin{aligned}&\xi + \sum\nolimits_{j\in\mathcal{K}} (x-x^*)^T \lambda_{1j} \nabla_x g_j(x)\\ &- \sum\nolimits_{j\in\mathcal{K}} (\lambda_{1j} - \lambda_{1j}^*) g_j(x)\\ &\in -(x-x^*)^T \partial_x f(x).\end{aligned}$$

Using Assumption 6.7, the inequality

$$(x-x^*)^T (h(x) - h(x^*)) \geq 0$$

holds for any $h(x) \in \partial_x f(x)$, $h(x^*) \in \partial_x f(x^*)$.

According to Lemma 6.3, there exists a vector $h(x^*) \in \partial_x f(x^*)$ such that

$$\mathbf{0} = h(x^*) + \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}^* \nabla_x g_j(x^*),$$

for some $\lambda_{1j}^* \geq 0$.

Thus,

$$\begin{aligned}\xi &\leq -(x-x^*)^T h(x^*) - \sum\nolimits_{j\in\mathcal{K}} (x-x^*)^T\\ &\times \lambda_{1j} \nabla_x g_j(x) + \sum\nolimits_{j\in\mathcal{K}} (\lambda_{1j} - \lambda_{1j}^*)(g_j(x)\\ &- g_j(x^*))\\ &= -(x-x^*)^T (\sum\nolimits_{j\in\mathcal{K}} \lambda_{1j} \nabla_x g_j(x) - \lambda_{1j}^* \nabla_x g_j(x^*))\\ &+ \sum\nolimits_{j\in\mathcal{K}} (\lambda_{1j} - \lambda_{1j}^*)(g_j(x) - g_j(x^*))\\ &\leq \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j}^* (g_j(x) - g_j(x^*)) + \sum\nolimits_{j\in\mathcal{K}} \lambda_{1j} (g_j(x^*)\\ &- g_j(x)) + \sum\nolimits_{j\in\mathcal{K}} (\lambda_{1j} - \lambda_{1j}^*)(g_j(x) - g_j(x^*))\\ &= 0.\end{aligned}$$

Letting $0 \in L_f V_0$ gives $x = x^*$ due to Assumption 6.7. Then the largest invariant set can be written as

$$\begin{aligned}
&\mathcal{S} = \{(x, \lambda_{11}, \cdots, \lambda_{1\kappa}) \in \mathbb{R}^N \times \mathbb{R}^{\geq 0} \times \cdots \times \mathbb{R}^{\geq 0} : \\
&\mathbf{0} \in \partial_x f(x) + \sum\nolimits_{j \in \mathcal{K}} \lambda_{1j} \nabla_x g_j(x), \\
&x = x^*,\ 0 = \lambda_{1j} g_j(x^*), j \in \mathcal{K}\}.
\end{aligned}$$

It can be seen that all the points in $\mathcal{S}$ satisfy the KKT conditions in (6.13). According to Lemma 6.3 and Assumption 6.8, such a point is unique. Therefore, based on Theorem 3 in [116], for the system in (6.16), the point $[x^{*T}, \lambda_{11}^*, \cdots, \lambda_{1\kappa}^*]^T$ is globally asymptotically stable, provided that $\lambda_{1j}(0) > 0$. □

In the following, we show that the original system is stable.

Let $z = [(x - x^*)^T, [\lambda_{1j} - \lambda_{1j}^*]^T]^T$. According to Lemma 2.3, there exist a smooth converse Lyapunov function V_1 and class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that

$$\alpha_1(\|z\|) \leq V_1(z) \leq \alpha_2(\|z\|),$$

$$\max_{w_0 \in G_z} \langle \nabla V_1(z), w_0 \rangle \leq -V_1(z), \tag{6.17}$$

where $G_z = (-\mathbf{k}(\partial_x f(x) + \sum_{j \in \mathcal{K}} \lambda_{1j} \nabla_x g_j(x)),\ [k_{1j} \lambda_{1j} g_j(x)])$.

Let $\hat{Y}_i = \bar{Y}_i - x_i \mathbf{1}$ and $\hat{\lambda}_j = \lambda_j - \lambda_{1j}\mathbf{1}$. Define the following auxiliary system

$$\begin{aligned}
\dot{\hat{Y}}_i &= -W_i(L_i + B_i)\hat{Y}_i, \\
\dot{\hat{\lambda}}_j &= -\Gamma_j(L_1 + B_1)\hat{\lambda}_j,
\end{aligned}$$

which is globally exponentially stable to the origin. Let $\Psi = [\hat{Y}_1^T, \cdots, \hat{Y}_N^T, \hat{\lambda}_1^T, \cdots, \hat{\lambda}_\kappa^T]^T$. Then, according to the converse Lyapunov theorem, there exists a Lyapunov function $V_2(\Psi)$ such that

$$c_1 \|\Psi\|^2 \le V_2(\Psi) \le c_2 \|\Psi\|^2,$$

$$\langle \nabla V_2(\Psi), w_{\Psi 0}\rangle \le -c_3 V_2(\Psi), \tag{6.18}$$

for some positive constants c_1, c_2, c_3, where $w_{\Psi 0} = [[-W_i(L_i + B_i)\hat{Y}_i]^T,\ [-\Gamma_j(L_1 + B_1)\hat{\lambda}_j]^T]^T$.

For all $z \in C_1, \Psi \in C_2$, where C_1 and C_2 are two compact sets, let $\mu > 0$ be a constant such that $V_1(z) \le \mu$, $V_2(\Psi) \le \mu$, and

$$\langle \nabla V_1(z), w_z\rangle + V_1(z) + |\langle \nabla V_2(\Psi), w_{\Psi 1}\rangle| \le \mu,$$

where $w_z \in H_z \triangleq (-\mathbf{k}(\partial_x f(Y) + \sum_{j\in\mathcal{K}} \mathrm{diag}\{\lambda_{1j},\, \lambda_j\}\nabla_x g_j(Y)),\, [k_{1j}\lambda_{1j}g_j(Y)])$, $w_{\Psi 1} = -\frac{1}{\varepsilon}[\frac{dx_1}{dt},\ \cdots,\ \frac{dx_N}{dt}, \frac{d\lambda_{11}}{dt}, \cdots, \frac{d\lambda_{1\mathcal{K}}}{dt}]^T \otimes \mathbf{1}$. Note that μ is independent of ε. Let $\Delta = [z^T, \Psi^T]^T$, then we can get the following conclusion.

Lemma 6.5. *For an arbitrary constant $\eta > 0$, there exists $v > 0$ such that if $\Delta = [z^T, \Psi^T]^T \in C_1 \times C_2$, $w_z \in H_z$, $\|\Psi\| \le v$, then*

$$\langle \nabla V_1(z), w_z\rangle + |\langle \nabla V_2(\Psi), w_{\Psi 1}\rangle| \le -V_1(z) + \frac{\eta}{2}.$$

Proof. According to Lemma 2.2, the set-valued map H_z is locally bounded and outer semicontinuous[1]. The following proof is similar to Claim 1 of [146]. Suppose that the claim is false. Then, there exists η such that for each positive integer i, there exist $\Delta_i \in C_1 \times C_2$, $w_{zi} \in H_{zi}$, and $w_{\Psi 1i}$ such that $\|\Psi_i\| \le \frac{1}{i}$ and

$$\langle \nabla V_1(z_i), w_{zi}\rangle + |\langle \nabla V_2(\Psi_i), w_{\Psi 1i}\rangle| > -V_1(z_i) + \frac{\eta}{2}. \tag{6.19}$$

[1]A set-valued mapping $F(x)$ is said to be outer semicontinuous if for each point x in its domain, each sequence (x_i, f_i)that satisfies $f_i \in F(x_i)$ for each i and converges to (x, f) has the property that $f \in F(x)$. According to [145], for locally bounded set-valued mappings with closed values, outer semicontinuity agrees with upper semicontinuity.

Since C_1, C_2 are compact sets and H_z is locally bounded, the sequence $(\Delta_i,\, w_{zi})$ has a subsequence converging to $(\Delta_0,\, w_{z0})$, where $\Delta_0 = [z_0^T, \mathbf{0}^T]^T$. Due to the outer semicontinuity of H_z, $w_{z0} \in G_z$. It follows from (6.18) and (6.19) that $\nabla V_2(\mathbf{0}) = \mathbf{0}$ and $\langle \nabla V_1(z_0), w_{z0} \rangle > -V_1(z_0) + \frac{\eta}{2}$, which contradicts to (6.17). □

Theorem 6.2. *Suppose that Assumptions 6.1, 6.2, 6.6, 6.7, 6.8 hold and let (6.4) and (6.11) be the updating law. Then for each pair of positive constants (r, ϱ), there exists a positive constant $\varepsilon^*(r, \varrho)$ such that for every $0 < \varepsilon < \varepsilon^*(r, \varrho)$, there exists a time T such that $\|\Delta(t)\| \leq \varrho$, $\forall t \geq T$ for every $\|\Delta(0)\| \leq r$.*

Proof. Let $0 < \varepsilon < 1$ be a constant satisfying $\varepsilon \leq c_3$ and $\varepsilon(2\mu - \eta) \leq c c_3 c_1 v^2$, where $0 < c < 1$ is a constant. Define $V(\Delta) = V_1(z) + cV_2(\Psi)$. Then, there exist class-$\mathcal{K}_\infty$ functions α_3 and α_4 such that $\alpha_3(\|\Delta\|) \leq V(\Delta) \leq \alpha_4(\|\Delta\|)$.

For all $w = [\varepsilon w_z^T, w_\Psi^T]^T$, where $w_z \in H_z$, $w_\Psi = \dot{Y}_\varepsilon \triangleq w_{\Psi 0} + \varepsilon w_{\Psi 1}$, if $\|\Psi\| \leq v$, then based on Lemma 6.5,

$$
\begin{aligned}
\frac{1}{\varepsilon} \langle \nabla V, w \rangle &= \langle \nabla V_1(z), w_z \rangle + \frac{c}{\varepsilon} \langle \nabla V_2(\Psi), w_\Psi \rangle \\
&= \langle \nabla V_1(z), w_z \rangle + c \langle \nabla V_2(\Psi), w_{\Psi 1} \rangle \\
&+ \frac{c}{\varepsilon} \langle \nabla V_2(\Psi), w_{\Psi 0} \rangle \\
&\leq -V_1(z) + \frac{\eta}{2} - \frac{c}{\varepsilon} c_3 V_2(\Psi) \\
&\leq -V + c(1 - \frac{1}{\varepsilon} c_3) V_2(\Psi) + \frac{\eta}{2} \\
&\leq -V + \eta.
\end{aligned}
$$

If $\|\Psi\| \geq v$, then according to (6.19),

$$
\begin{aligned}
\frac{1}{\varepsilon} \langle \nabla V, w \rangle &\leq -V_1(z) + \mu - \frac{c}{\varepsilon} c_3 V_2(\Psi) \\
&\leq -V + cV_2(\Psi) + \mu - \frac{c}{\varepsilon} c_3 c_1 \|\Psi\|^2
\end{aligned}
$$

$$\leq -V + 2\mu - \frac{c}{\varepsilon} c_3 c_1 v^2$$
$$\leq -V + \eta.$$

Thus, for an arbitrary solution $\Delta(t)$ of the system,

$$\frac{dV(\Delta(t))}{dt} \leq -\varepsilon V(\Delta(t)) + \varepsilon\eta.$$

Therefore, if $\|\Delta(t)\| > \alpha_3^{-1}(2\eta)$, then $\frac{V(\Delta(t))}{2} > \eta$ and $\frac{dV(\Delta(t))}{dt} < -\frac{\varepsilon}{2} V(\Delta(t))$. Since η can be an arbitrary positive constant, the conclusion is proven. $\square$

6.5 Simulation

Example 6.1. *A network of 5 players is shown in Fig. 6.1. Let* $f_1(x,t) = x_1^2 + (x_1 - x_2 - 3)^2$, $f_2(x,t) = (x_1 - x_2)^2 + (x_2 - x_3)^2$, $f_3(x,t) = x_3^2 + (x_2 - x_3)^2$, $f_4(x,t) = (x_4 - x_3 - 1)^2$, *and* $f_5(x,t) = x_5^2 + (x_5 - x_1)^2$ *be the local objective functions for players 1-5, respectively. The constraint set is* $\Omega = \{x \in \mathbb{R}^5 | \ 1 \leq ||x - d|| \leq 3, d = [1, 2, 1, 0, -1]^T\}$. *Let (6.4) and (6.5) be the updating law. Fig. 6.2 shows the simulation result of the estimate on the optimal solutions* $x_1^*(t), x_2^*(t), \cdots, x_5^*(t)$.

Example 6.2. *Consider a network of 5 players with the same communication topology as given in Example 6.1. Let* $f_1(x,t) = \begin{cases} x_1^2 + (x_1 - x_2 - 3)^2, & \textit{if } x_1 \geq 2, \\ 2x_1^2 - 4 + (x_1 - x_2 - 3)^2, & \textit{if } x_1 < 2. \end{cases}$

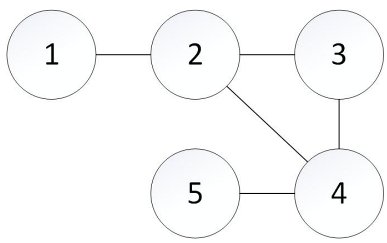

Figure 6.1: The communication graph.

, $f_2(x,t) = (x_1 - x_2)^2 + (x_2 - x_3)^2$*,* $f_3(x,t) = x_3^2 + (x_2 - x_3)^2$*,* $f_4(x,t) = (x_4 - x_3)^2$*, and* $f_5(x,t) = \begin{cases} x_5^2 + (x_5 - x_1)^2, & \textit{if } x_5 \geq 1, \\ 2x_5^2 - 1 + (x_5 - x_1)^2, & \textit{if } x_5 < 1. \end{cases}$ *be the local objective functions for players 1-5, respectively. The constraint set is the same as that in Example 6.1. Let (6.4) and (6.11) be the updating law. Fig. 6.3 shows the simulation result of the estimate on the optimal solutions* $x_1^*(t), x_2^*(t), \cdots, x_5^*(t)$*.*

6.6 Conclusions

In this chapter, we studied the distributed Nash equilibrium seeking problems for generalized convex games with multiple shared constraints. Continuous-time control laws were designed such that each player minimizes its own cost function in a distributed way. Both class-C^2 objective functions and locally Lipschitz objective functions were considered in the chapter.

In the first two parts, we studied the consensus tracking, distributed optimization and distributed Nash equilibrium seeking problems in multi-agent systems. In these problems, to achieve the goals, each individual has to be equipped with a controller, and the control law needs to be designed for each agent. However, in some large-scale networks, e.g., genetic regulatory networks [101], the individuals in the network are affected by each other. People are expected to identify the possibility of controlling the whole network by controlling only a subset of nodes, considering the control efficiency and effectiveness. The controllability problem for multi-agent systems is the topic we will study in the next part.

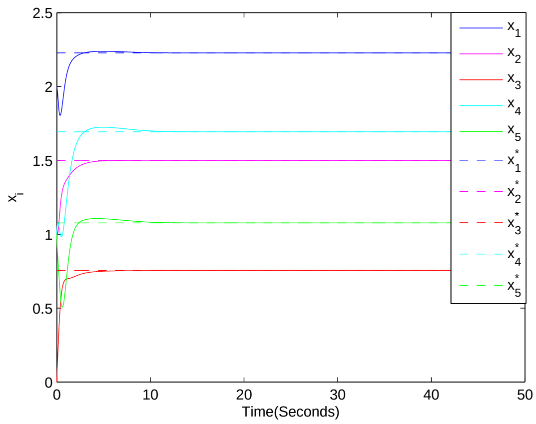

Figure 6.2: The estimate on the optimal solutions $x_1^*(t), x_2^*(t), \cdots, x_5^*(t)$ in Example 6.1.

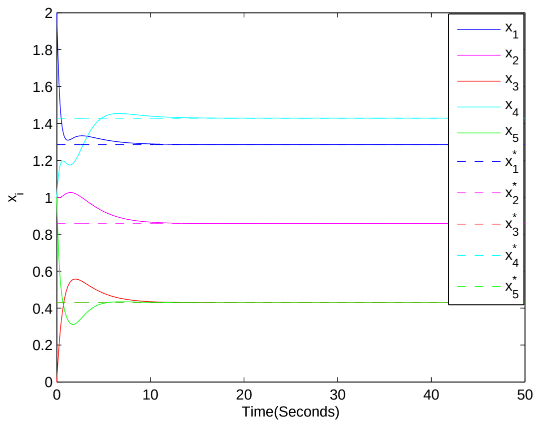

Figure 6.3: The estimate on the optimal solutions $x_1^*(t), x_2^*(t), \cdots, x_5^*(t)$ in Example 6.2.

Part III

Network Controllability

Chapter 7

Controllability of Multi-Agent Networks with Antagonistic Interactions

7.1 Introduction

As mentioned in Chapter 1, most of the existing literature on network controllability aimed to find the relationship between the graph property and the controllability of networks modeled by (1.1), where $x \in \mathbb{R}^N$ is the state vector, $u \in \mathbb{R}^m$ is the control input vector, $A \in \mathbb{R}^{N\times N}$ is a matrix associated with the topological structure, and $B \in \mathbb{R}^{N\times m}$ is the control matrix. For structural controllability based methods, A is a structured matrix. While for state controllability methods, most of the literature view A as the standard Laplacian matrix.

In this chapter, we study the controllability problem of multi-agent networks defined on an undirected signed graph. We consider a similar model as described in (1.1) with A replaced by a signed Laplacian matrix, which means that the network can also have negative links but not only positive ones. In this case, since the row sum of a signed Laplacian matrix may not be equal to zero, many approaches and concepts for the controllability of networks defined on signless graphs are no longer applicable. The main contributions of this work are two-fold:

1. We propose a graph-theoretic characterization of an upper bound on the controllable subspace through the so-called generalized almost equitable partition, and provide a necessary condition for the controllability of the network. We also study how to obtain the partition for a given graph and a set of leaders.

2. We explore the controllability relationship between a structurally balanced network and the corresponding all-positive network, which provides a simpler characterization method for some special graphs since there already have some useful results on the controllability of an all-positive network.

The related topics of this work include the bipartite consensus problems in [108–111], the structural controllability problem in [101, 105], and the graph-theoretical controllability characterization problems in [147, 148]. Firstly, we focus on the controllability problem while [108–111] studied the consensus problem. Secondly, the perspectives and hypotheses for strong structural controllability and standard controllability used in this work are different. In structural controllability framework, the network model depends on the structure of the graph rather than a particular physical realization. By using standard controllability theory, we can get some new results on controllability of networks with antagonistic interactions. We obtain an upper bound on the controllable subspace of the system and a necessary condition for the controllability. Furthermore, we study the relationship between the controllability of a structurally balanced signed graph and the corresponding all-positive graph and give a graph-theoretical characterization method for controllability of some special signed networks. All of these are not included in the existing literature on strong structural controllability. Thirdly, compared with [147, 148], the networks studied in this chapter are more general in the sense that possibly negative weights are considered. As mentioned above, the existence of such weights will bring more technical difficulties in analyzing the graph-theoretical controllability of the networks.

The rest of this chapter is organized as follows. In Section 7.2, the controllability problem of antagonistic networks is formulated mathematically. In Section 7.3, the main results are presented. In Section 7.4, several examples are provided to illustrate the main results. Finally, conclusions are given in Section 7.5.

7.2 System Description

Consider a multi-agent network with node set $\mathcal{V}=\{1,...,N\}$. We suppose that m ($m \leq N$) agents in the network are selected as leaders and each leader is assigned with a control input. The other nodes are followers. Without loss of generality, assume that the first m agents $1,...,m$ correspond to the leaders. Let $\mathcal{V}_m \triangleq \{1,...,m\}$ and $\mathcal{V}_F \triangleq \mathcal{V} \backslash \mathcal{V}_m$ be the sets of leaders and followers, respectively.

Each follower $i \in \mathcal{V}_F$ is governed by the updating law

$$\dot{x}_i = -d_i x_i + \sum\nolimits_{j=1}^{N} a_{ij} x_j, \tag{7.1}$$

where $x_i \in \mathbb{R}$ is the state of each agent, $d_i = \sum_{j=1}^{N} |a_{ij}|$ represents the degree of agent i, and $a_{ij} \in \{1, 0, -1\}$.

Each leader $i \in \mathcal{V}_m$ is governed by the updating law

$$\dot{x}_i = -d_i x_i + \sum\nolimits_{j=1}^{N} a_{ij} x_j + u_i, \tag{7.2}$$

where $u_i \in \mathbb{R}$ is the control input.

Remark 7.1. *Networks with the form (7.1) were studied in the bipartite consensus problems in [108–113], where a node sends the opposite of its true state to its antagonistic neighbors. A similar model was utilized in [149] to model the process of* *opinion forming. State x_i represents the opinion of the individual i. Positive edges represent cooperative interactions (individuals transmit their "true" opinions), while negative edges represent antagonistic interactions (individuals transmit their "false" opinions). Each individual updates its opinion based on the interactions with neighbors and finally forms its own opinion. In addition, some social networks are well known for containing both positive and negative links, such as the Slashdot Zoo[2] [38].*

Let $x = [x_1, \cdots, x_N]^T \in \mathbb{R}^N$ be the aggregated state vector and $u = [u_1, \cdots, u_m]^T \in \mathbb{R}^m$ denote the control input vector. The dynamics of the antagonistic network can be rewritten into a concatenated form as

$$\dot{x} = -Lx + Mu, \tag{7.3}$$

where L is the signed Laplacian matrix defined in Chapter 2.1.2, and M is an $N \times m$ matrix satisfying $M = \begin{bmatrix} I_m \\ 0_{(N-m)\times m} \end{bmatrix}$.

7.3 Controllability of Antagonistic Networks

The objective of this section is to investigate how to characterize the controllability of an antagonistic network. According to the Kalman's controllability condition, (7.3) is controllable from the controlled nodes if and only if the matrix $[M \ LM \ L^2M \ \cdots \ L^{N-1}M]$ has full rank. However, besides the algebraic methods, graphic characterization methods are highly demanded because they can avoid complicated matrix computations.

[2]http://slashdot.org/

7.3.1 An Upper Bound on the Controllable Subspace

In this section, we give a graph-theoretic characterization of an upper bound on the controllable subspace for system (7.3). It is known that the AEP can be used to determine upper bounds on the controllable subspace when the network is all-positive [93], [96]. A partition $\pi = \{V_1, \cdots, V_k\}$ is called an AEP if for every pair of distinct i, $j \in \{1, \cdots, k\}$, there exists a nonnegative number d_{ij} such that any vertex in V_i has d_{ij} neighbors in V_j. However, this partition does not work for a signed graph due to the existence of negative weights, which motivates us to give a generalized definition.

Definition 7.1. *(1) $\pi = \{V_1, \cdots, V_k\}$ is called a GAEP if for every pair of distinct i, $j \in \{1, \cdots, k\}$, there exists a non-negative number d_{ij+} such that any vertex in V_i has d_{ij+} positive neighbors in V_j and for every pair of i, $j \in \{1, \cdots, k\}$ (not necessarily distinct), there exists a non-negative number d_{ij-} such that any vertex in V_i has d_{ij-} negative neighbors in V_j.*

(2) A partition π_L is said to be a leader-isolated GAEP if π_L is a GAEP and each leader is in a singleton class.

Definition 7.2. *A partition π_L^* is called the coarsest leader-isolated GAEP if for any leader-isolated GAEP π_L, each class in π_L is a subclass of some class in π_L^*.*

An example of the coarsest leader-isolated GAEP is given in Fig. 7.1. In the following, we prove the existence and uniqueness of it.

Lemma 7.1. *For each signed graph $\mathcal{G}$, there exists a unique coarsest leader-isolated GAEP.*

Proof. If the graph has only one leader-isolated GAEP, then it has to be the coarsest leader-isolated GAEP. Suppose that the graph has more than one leader-isolated GAEP and the coarsest leader-isolated GAEP does not exist. Then, there exist at least two leader-isolated GAEPs π^*_{L1} and π^*_{L2} such that 1) each class in each leader-isolated GAEP π_L is a subclass of some class in π^*_{L1} or π^*_{L2}; 2) not the all classes in π^*_{L1} (π^*_{L2}) are subclasses of the classes in π^*_{L2} (π^*_{L1}).

Denote $\pi^*_{L1} = \{V_1, V_2, \cdots, V_{l_1}, \bar{\pi}_L\}$ and $\pi^*_{L2} = \{V'_1, V'_2, \cdots, V'_{l_2}, \bar{\pi}_L\}$, where $\bar{\pi}_L$ is the common part of π^*_{L1} and π^*_{L2} and l_1 and l_2 are two positive integers. Since each leader is in a singleton class, $\bar{\pi}_L$ is nonempty. According to Definition 7.1, each node in V_1, $V_2, \cdots, V_{l_1}$, as well as in $V'_1, V'_2, \cdots, V'_{l_2}$, has the same number of positive and negative neighbors in $\bar{\pi}_L$. Then, there exist at least two classes in π^*_{L1}, as well as in π^*_{L2}, that can be grouped into one class such that the newly generated partition has a class which is the union of some classes in π^*_{L1} (or π^*_{L2}). This contradicts to condition 1).

By Definition 7.2, if the coarest leader-isolated GAEP exists, then it must be unique. $\square$

For the system (7.3), the controllable subspace K can be written as

$$K = \mathrm{im}(M) + L \times \mathrm{im}(M) + \cdots + L^{N-1} \times \mathrm{im}(M).$$

which is the smallest L-invariant subspace that contains $\mathrm{im}(M)$ [37, 150]. Here, the operator "+" represents the union of two spaces.

Let $\tilde{D}_+$ and $\tilde{D}_-$ be diagonal matrices with the non-negative diagonal elements $\tilde{d}_{i+}$ and $\tilde{d}_{i-}$ representing the number of positive and negative neighbors of agent

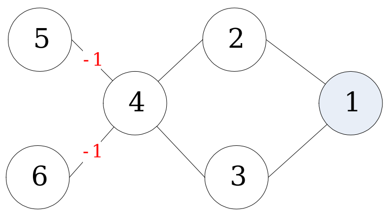

Figure 7.1: A signed graph with a single leader namely 1. The coarsest leader-isolated GAEP is $\pi^*_L = \{\{1\}, \{2, 3\}, \{4\}, \{5, 6\}\}$.

i, respectively. Define $n_{i,j} = d_{ij+} - d_{ij-}$ if $i \neq j$, $i, j \in \{1, \cdots, k\}$, and $n_{i,i} = 0$, where d_{ij+} and d_{ij-} are defined in Definition 7.1. Next, we prove that $\mathrm{im}(P_\pi)$ is L-invariant.

Lemma 7.2. *Let $\pi = \{V_1, \cdots, V_k\}$ be a GAEP of the graph $\mathcal{G}$ and P_π be the characteristic matrix of π. Then, there exists a matrix L_π such that the signed Laplacian L satisfies $(L - 2\tilde{D}_-)P_\pi = P_\pi L_\pi$. Moreover, im($P_\pi$) is L-invariant.*

Proof. Define a matrix $L_\pi \in \mathbb{R}^{k \times k}$ as

$$[L_\pi]_{ij} = \begin{cases} \sum_{s=1}^{k} n_{i,s} & i = j, \\ -n_{i,j} & i \neq j. \end{cases}$$

Without loss of generality, we assume that the nodes belonging to V_1 are indexed by $\{n_0 + 1, n_0 + 2, \cdots, n_0 + |V_1|\}$, where n_0 is some nonnegative integer. Thus, the matrix $P_\pi \in \mathbb{R}^{N \times k}$ can be written as

$$P_\pi = \begin{bmatrix} 0_{n_0} & * \\ 1_{|V_1|} & 0_{|V_1| \times (k-1)} \\ 0_{N-n_0-|V_1|} & * \end{bmatrix}, \tag{7.4}$$

where $|V_1|$ denotes the cardinality of V_1.

Assuming that node p belongs to V_1, the p-th row of $P_\pi L_\pi$ can be written as

$$\begin{aligned} row_p(P_\pi L_\pi) &= row_1(L_\pi) \\ &= \left[\sum_{s=1}^{k} n_{1,s}, \ -n_{1,2}, \cdots, \ -n_{1,k}\right]. \end{aligned}$$

The p-th row of $(L - 2\tilde{D}_-)$ is

$$row_p(L - 2\tilde{D}_-)$$

$$\begin{aligned} &= [-a_{p1}, \cdots, -a_{p(p-1)},\ (\tilde{d}_{p+} - \tilde{d}_{p-}), \\ &\quad -a_{p(p+1)}, \cdots, -a_{pN}]. \end{aligned}$$

According to (7.4) and the definition of a_{ij} in a signed graph, the p-th row of $(L - 2\tilde{D}_-)P_\pi$ can be written as

$$\begin{aligned} & row_p((L - 2\tilde{D}_-)P_\pi) \\ = \ & [(\tilde{d}_{p+} - \tilde{d}_{p-} - (\tilde{d}_{p1+} - \tilde{d}_{p1-})), \\ & -(d_{12+} - d_{12-}), \cdots, -(d_{1k+} - d_{1k-})], \end{aligned}$$

where $\tilde{d}_{p1+}$ denotes the number of positive neighbors of node p in V_1, while $\tilde{d}_{p1-}$ denotes its number of negative neighbors.

Since $\tilde{d}_{p+} - \tilde{d}_{p-} - (\tilde{d}_{p1+} - \tilde{d}_{p1-}) = \sum_{s=1}^{k} n_{1,s}$, it gives $\mathrm{row}_p((L - 2\tilde{D}_-)P_\pi) = \mathrm{row}_p(P_\pi L_\pi)$. Using similar analysis to all the other nodes yields $(L-2\tilde{D}_-)P_\pi = P_\pi L_\pi$.

According to the definition of the GAEP, the nodes in class V_i, $i \in \{1, \cdots, k\}$ have the same number of negative neighbors, denoted by κ_i. Thus, $\tilde{d}_{p1-} = \kappa_1$ and the first column of $\tilde{D}_- P_\pi$ can be written as

$$\mathrm{column}_1(\tilde{D}_- P_\pi) = \begin{bmatrix} 0_{n_0} \\ \kappa_1 * 1_{|V_1|} \\ 0_{N-n_0-|V_1|} \end{bmatrix}.$$

Let $D_\pi \in \mathbb{R}^{k \times k}$ be a diagonal matrix defined as

$$[D_\pi]_{ij} = \begin{cases} 0 & i \neq j, \\ \kappa_i & i = j. \end{cases} \tag{7.5}$$

Thus, $\text{column}_1(\tilde{D}_-P_\pi)=\text{column}_1(P_\pi D_\pi)$. Furthermore, $\tilde{D}_-P_\pi = P_\pi D_\pi$ and $LP_\pi = P_\pi(2D_\pi + L_\pi)$. For a matrix E, the vector space generated by columns of E is L-invariant if and only if there exists a matrix B such that $LE = EB$ [151]. Therefore, $\text{im}(P_\pi)$ is L-invariant. □

Based on Lemma 7.2, the following result can be obtained.

Lemma 7.3. *For any leader-isolated GAEP π_L, the controllable subspace K satisfies $K \subseteq im(P_{\pi_L})$.*

Proof. From (7.3), it can be easily obtained that each column of the matrix M is a column of P_{π_L}, which indicates that $\text{im}(M) \subseteq \text{im}(P_{\pi_L})$. Since $\text{im}(P_{\pi_L})$ is L-invariant, we have

$$\begin{aligned} K &= \text{im}(M) + L \times \text{im}(M) + \cdots + L^{N-1} \times \text{im}(M) \\ &\subseteq \text{im}(P_{\pi_L}) + L \times \text{im}(P_{\pi_L}) + \cdots + L^{N-1} \times \text{im}(P_{\pi_L}) \\ &= \text{im}(P_{\pi_L}). \end{aligned} \tag{7.6}$$

□

Lemma 7.3 provides an upper bound on the controllable subspace based on the leader-isolated GAEP. In the following, we utilize the coarsest leader-isolated GAEP to get a tighter upper bound.

Theorem 7.1. *The controllable subspace K of the system (7.3) satisfies $K \subseteq im(P_{\pi_L^*})$, where π_L^* is the coarsest leader-isolated GAEP.*

Proof. According to Lemma 7.3, $K \subseteq \text{im}(P_{\pi_L^*})$ and $K \subseteq \text{im}(P_{\pi_L})$. Since each class in π_L is a subclass of some class in π_L^*, it can be obtained that $\text{im}(P_{\pi_L^*}) \subseteq \text{im}(P_{\pi_L})$. Thus, $K \subseteq \text{im}(P_{\pi_L^*}) \subseteq \text{im}(P_{\pi_L})$. □

Remark 7.2. *Theorem 7.1 applies when the network has both positive and negative edges. It extends the results in [96] which dealt with an all-positive network.*

Based on Theorem 7.1, we further get the following necessary condition for the controllability of the system (7.3).

Proposition 1: If the system (7.3) is controllable, then each class in the coarsest leader-isolated GAEP π_L^* is a singleton class.

Proof. If a non-singleton class in the coarsest leader-isolated GAEP exists, then the dimension of $\mathrm{im}(P_{\pi_L^*})$ is less than N. Then, based on Theorem 7.1, the dimension of the controllable subspace K is less than N. This contradicts to the supposition that the system is controllable. □

7.3.2 Algorithm to Compute π_L^*

The computation for the upper bound in Theorem 7.1 requires the coarsest leader-isolated GAEP π_L^*. In the following, we propose an algorithm to obtain π_L^*, which is motivated by the algorithms presented in [2] and [148] to find the coarsest leader-isolated AEP. The algorithm is described as follows.

Step 1: Let $\pi_{L0} = \{\{1\}, \{2\}, \cdots, \{m\}, \mathcal{V}_F\}$ be the initial partition.

Step 2: Relabel the classes in the current partition by $V_1, \cdots, V_k$, $k > 1$ and select an arbitrary non-singleton class, e.g., V_i, $1 \leq i \leq k$. Then, for each node in V_i, calculate the number of its positive neighbors in $V_1, \cdots, V_{i-1}, V_{i+1}, \cdots, V_k$ and the number of its negative neighbors in $V_1, \cdots, V_k$. Nodes with the same neighbor sequence are grouped into one class. Replace the old class with the newly created classes.

Step 3: Repeat Step 2 until no class can be split.

Theorem 7.2. *For a signed graph $\mathcal{G}$, the partition obtained via Steps 1-3 is the coarsest leader-isolated GAEP.*

Proof. Based on Definition 7.1, it can be seen that a leader-isolated GAEP can be obtained by Steps 1-3. In the following, we will show that the obtained leader-isolated GAEP π_L^* is the coarsest leader-isolated GAEP of the graph. Suppose that another partition $\pi_{Lr}^* \neq \pi_L^*$ is the coarsest leader-isolated GAEP. Then, each class in π_L^* is a subclass of some class in π_{Lr}^*. Suppose that two nodes p, q, located in two distinct classes V_1, V_2 in π_L^*, belong to the same class V_{1r} in π_{Lr}^*, and they are separated from a class V_0 in a certain iteration. According to Step 2, there exist three cases such that p, q in V_0 are separated: 1) p, q have different numbers of positive neighbors in V_0; 2) p, q have different numbers of positive and/or negative neighbors in another class, e.g., V_3; 3) both 1) and 2) hold. For case 1), V_0 is neither a class in π_{Lr}^* nor a union of classes in π_{Lr}^*. Otherwise, p, q have the same number of positive neighbors in V_0. Then, in this iteration, there must exist another two nodes located in two distinct classes V_0, V_4, which are in the same class in π_{Lr}^*. For case 2), V_3 is neither a class in π_{Lr}^* nor a union of classes in π_{Lr}^*. Otherwise, p, q cannot be separated. Thus, there must exist another two nodes located in two distinct classes V_3, V_5, which are in the same class in π_{Lr}^*. The analysis of case 3) is similar to case 1). Overall, if p, q can be separated, there must exist another two nodes p', q' that belong to two distinct classes in π_L^* and the same class in π_{Lr}^*. Applying this logic iteratively and noticing that there are no such two nodes in the initial partition, we can get the conclusion. □

Example 7.1. *In order to illustrate the algorithm, we consider a signed network shown in Fig. 7.2(a) with agent 1 chosen as the leader. The initial partition is chosen as* $\pi_{L0} = \{\{1\}, \{2,3,4,5,6,7\}\}$. *Figs. 7.2(b)-(d) show the process of finding the coarsest leader-isolated GAEP, where nodes in the same class are denoted by the same color. The coarsest leader-isolated GAEP is* $\pi_L^* = \{\{1\}, \{2,3\}, \{4\}, \{5,6,7\}\}$.

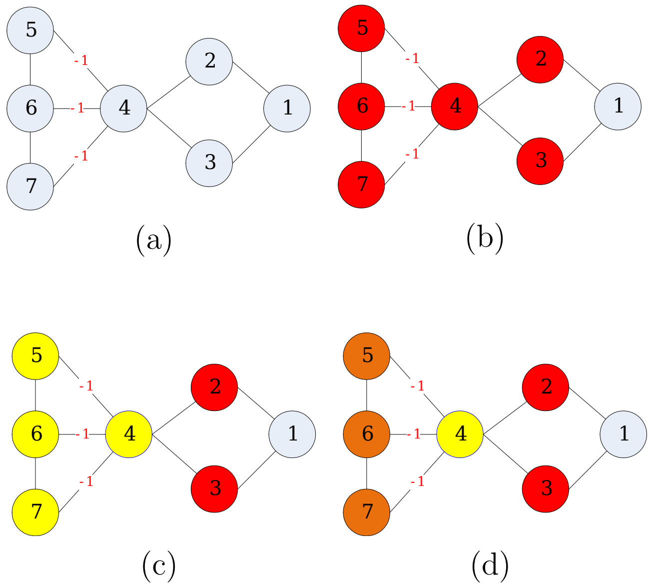

Figure 7.2: (a) A network with 7 nodes, (b) $\pi_{L0} = \{\{1\}, \{2,3,4,5,6,7\}\}$, (c) $\pi_{L1} = \{\{1\}, \{2,3\}, \{4,5,6,7\}\}$, (d) $\pi_L^* = \{\{1\}, \{2,3\}, \{4\}, \{5,6,7\}\}$.

7.3.3 Controllability of Structurally Balanced Signed Networks

Due to the difference between the signed Laplacian and the signless Laplacian, the controllability of an antagonistic network is not necessarily equivalent to the controllability of its corresponding all-positive network (see the counter-example in Example 7.2 of Section 7.4). In this section, we use the structural balance theory to explore the relationship between them.

Structural balance is a basic concept in social network analysis. In a structurally balanced graph, the vertex set $\mathcal{V}$ can be partitioned into two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that every edge between $\mathcal{V}_1$ and $\mathcal{V}_2$ is negative and every edge within $\mathcal{V}_1$ or $\mathcal{V}_2$ is positive [40]. A connected and structurally balanced graph has the following property.

Lemma 7.4. *[108] A connected signed graph $\mathcal{G}(A)$ is structurally balanced if and only if any of the following equivalent conditions holds:*

(1) all cycles of $\mathcal{G}(A)$ are positive;

(2) let $E=$diag(σ) with σ satisfying $\sigma_i = 1$ if $v_i \in \mathcal{V}_1$ and $\sigma_i = -1$ if $v_i \in \mathcal{V}_2$; then EAE (and thus ELE) has all off-diagonal entries nonnegative;

(3) 0 is an eigenvalue of L.

Denote $L_E = ELE$. The entries of L_E satisfy

$$[L_E]_{ij} = \begin{cases} \sum_{s=1}^{N} |a_{i,s}| & i = j, \\ -|a_{i,j}| & i \neq j. \end{cases}$$

Thus, L_E equals to the Laplacian of the network formed by replacing $a_{i,j}$ with $|a_{i,j}|$.

Theorem 7.3. *Suppose that the interconnected antagonistic network in (7.3) is structurally balanced. If the leaders are chosen from the same subset, i.e., $\mathcal{V}_L \subseteq \mathcal{V}_1$ or $\mathcal{V}_L \subseteq \mathcal{V}_2$, then the controllability of (L, M) is equivalent to that of (L_E, M).*

Proof. According to the definition of L_E and the fact that $E = E^{-1}$, we have

$$\begin{aligned}&\text{rank}\{[M\ L_E M\ L_E^2 M\ \cdots\ L_E^{N-1} M]\}\\ &= \text{rank}\{[M\ ELEM\ EL^2EM\ \cdots\ EL^{N-1}EM]\}.\end{aligned} \tag{7.7}$$

If $\mathcal{V}_L \subseteq \mathcal{V}_1$, $EM = M$. If $\mathcal{V}_L \subseteq \mathcal{V}_2$, $EM = -M$. It follows from (7.7) that

$$\begin{aligned}&\text{rank}\{[M\ ELEM\ EL^2EM\ \cdots\ EL^{N-1}EM]\}\\ &= \text{rank}\{E[EM\ LEM\ L^2EM\ \cdots\ L^{N-1}EM]\}\\ &= \text{rank}\{\pm E[M\ LM\ L^2M\ \cdots\ L^{N-1}M]\}\\ &= \text{rank}\{[M\ LM\ L^2M\ \cdots\ L^{N-1}M]\}.\end{aligned}$$

The proof is thus completed. □

Remark 7.3. *Suppose that the interconnected antagonistic network in (7.3) is structurally balanced and only one agent is chosen as the leader, then the system* (L, M) *is controllable if and only if the corresponding* (L_E, M) *is controllable. For tree networks with only one leader, the conclusion always holds since any acyclic signed graph is structurally balanced [108].*

Remark 7.4. *In Theorem 7.3, it is proven that the controllability of a structurally balanced network is equivalent to the controllability of its corresponding all-positive network, if the leaders are chosen from the same subset. In this case, the controllability can be checked using graphic methods for all-positive networks [91–95, 152, 153] by treating negative edges as positive ones.*

7.4 Simulation

Example 7.2. *Consider the all-positive multi-agent network shown in Fig. 7.3.*

If the agents 1, 2, and 3 are chosen as leaders, the Laplacian matrix L and the control matrix M can be written as follows

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}, \ M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.8}$$

The rank of the controllability matrix of system (7.8) is 4. Thus, the system is uncontrollable.

Replacing the weight between agent 2 and agent 4 by -1, we obtain a new network shown in Fig. 7.4, where

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & 1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}, \; M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.9}$$

Through a direct calculation of the rank of the controllability matrix, we can conclude that system (7.9) is controllable. This example shows that the controllability of an antagonistic network is not always equivalent to the controllability of the corresponding all-positive network.

Example 7.3. *Consider the multi-agent network shown in Fig. 7.5.*

If agent 1 is chosen as the leader, then the Laplacian matrix L and the control

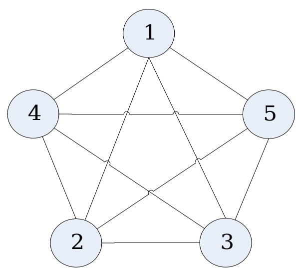

Figure 7.3: The all-positive network is uncontrollable when agents 3, 4, 5 are chosen as leaders.

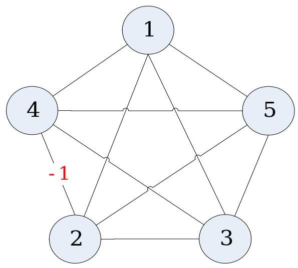

Figure 7.4: If $a_{24} = a_{42} = -1$, the network becomes controllable.

matrix M are

$$L = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \ M = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The rank of the controllability matrix of the system is 3 and thus the system is uncontrollable.

If the agents 1 and 2 are chosen as leaders, then the Laplacian matrix L and the control matrix M are

$$L = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \ M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The dimension of the controllable subspace is 5 and thus the system is controllable.

Using the algorithm presented in Section 7.3.2, we get that the partition $\{\{1\}, \{2\},$

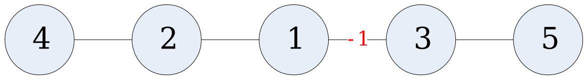

Figure 7.5: A path graph network with 5 agents.

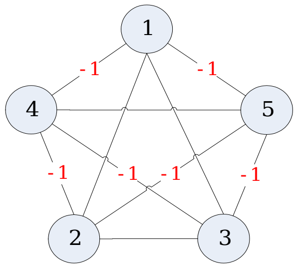

Figure 7.6: A structurally balanced network with 5 agents.

{3}, {4}, {5}} is the coarsest leader-isolated GAEP and each class in this partition is a singleton class. Thus, we can conclude that the condition in Proposition 1 is not sufficient for the controllability of system (7.3).

Example 7.4. *Consider the multi-agent network shown in Fig. 7.6.*

The network is structurally balanced since it can be partitioned into two subsets {1, 2, 3} and {4, 5}. The network shown in Fig. 7.3 is the corresponding all-positive network. If we choose the agents 1, 2 and 3 as leaders, the Laplacian matrix and the control matrix are

$$L = \begin{bmatrix} 4 & -1 & -1 & 1 & 1 \\ -1 & 4 & -1 & 1 & 1 \\ -1 & -1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 4 & -1 \\ 1 & 1 & 1 & -1 & 4 \end{bmatrix}, \; M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.10}$$

The rank of the controllability matrix is 4, which is the same as that of system

(7.8). This validates Theorem 7.3.

7.5 Conclusions

In this chapter, we studied the controllability problem for multi-agent networks described by signed graphs with both positive and negative weights. We provided a graph-theoretic characterization method for an upper bound on the controllable subspace. Based on this upper bound, we presented a necessary condition for the controllability of the network. Furthermore, we studied the relationship between the controllability of a structurally balanced signed graph network and the controllability of the corresponding all-positive network. Studying the effects of adding negative weights to some special graphs will be our future work.

Chapter 8

Conclusions and Future Work

8.1 Conclusions

In this dissertation, several key issues in distributed multi-agent systems were investigated. First of all, distributed cooperative control was studied for multi-agent systems subject to unmodelled nonlinearities and unknown disturbances. Then, distributed optimization and distributed Nash equilibrium seeking algorithms were developed. In addition, considering the control efficiency and network amenability to humans, the network controllability problem was studied based on graph theory and control theory. The problems studied in this dissertation include:

- Robust connectivity preserving consensus tracking and formation control based on integral sliding mode and potential function based design (Chapter 3);
- Distributed robust consensus for a class of high-order multi-agent systems subject to unmodelled nonlinearities and unknown disturbances (Chapter 4);
- Distributed constrained time-varying optimization problems (Chapter 5);
- Distributed Nash equilibrium seeking for nonsmooth non-cooperative games with inequality constraints (Chapter 6);
- Graph-theoretical characterization of controllability of a class of more general multi-agent system models (Chapter 7).

Specifically, the dissertation has the following contributions:

- For the robust connectivity preservation problem, an integral sliding mode based control framework was developed to simultaneously achieve disturbance rejection, finite-time convergence, and connectivity preservation.

- For the robust consensus problem, a distributed identifier was designed to deal with the uncertainties and avoid the discontinuity of the controller. A time-varying gain was designed to eliminate the dependence on prior knowledge of the upper bound of the uncertainties.

- Based on projected gradients and penalty functions, two distributed algorithms were developed to deal with the distributed time-varying constrained optimization problems.

- A distributed algorithm was presented that converges to the normalized Nash equilibrium of a nonsmooth noncooperative game. The objective functions of the game can be of class-C^2 or locally Lipschitz continuous. Singular perturbation and nonsmooth analysis were combined to prove the convergence of the algorithm.

- A generalized form of the tradition concept in graph theory, (almost) equitable partition, was presented for the signed graph. Based on this newly defined partition, a graph-theoretical characterization method for the upper bound of the rank of the controllability matrix was given, with an algorithm to compute it. Controllability for signed and structurally balanced graphs were studied, which brought new characterization methods for some special signed graphs.

8.2 Future Work

The following concerns are considered for future research.

For the distributed robust coordination problems:

(1) The disturbance considered in the robust consensus tracking problem is required to be bounded to its second-order derivative. However, many disturbances in the applications don't satisfy this condition and many of them are even discontinuous. How to model and deal with more practical disturbances is a problem that needs to be solved. (2) There are many cases that the algorithm has to be converted into a discrete-time one. In the proposed algorithms, many of them require the control gains to be large enough, which in discrete-time case is unacceptable or even impossible. Is it possible to solve this problem using adaptive control methods? (3) For the robust connectivity preservation problem, the proposed algorithms are discontinuous by using a signum function. Chattering-free methods need to be developed, e.g., by using the robust integral of the sign of the errors (RISE) [18], or super-twisting algorithms [61]. For the super-twisting based scheme, one may design a similar sliding manifold as developed in (3.6). For the discontinuous control (3.5), the continuous super-twisting ISM consensus protocol proposed in [61] might be used for chattering avoidance. The challenge is to analyze the effect of the disturbances on the connectivity.

For the distributed time-varying optimization and noncooperative Nash equilibrium seeking problems:

(1) The distributed time-varying optimization algorithm is developed for quadratic objective functions and the agents are under box constraints. How to develop distributed algorithms to find the optimal solution of a non-quadratic time-varying objective function? The main difficulty for non-quadratic objective functions is that it has a state-related term that is not easy to be compensated by using gradient methods. One idea is to use the Hessian matrix instead of gradients. How to deal with the constraints relying on neighboring or non-neighboring agents? (2) How to apply the algorithms to more practical problems? For example, for a time-varying

formation control problem, suppose that there is an objective function that depends on the states of the agents. If the desired time-varying formation is formulated as a time-varying constraint, how to develop a distributed control law for the agent to achieve the formation (constraint) and minimize (or maximize) the objective function? (3) The algorithm for the noncooperative Nash equilibrium seeking uses the estimation of states of all the agents, which leads to large communication and computation burden. Is it possible to find a way to solve this problem?

For the network controllability problems:

(1) In Chapter 7, only the upper bound of the rank of the controllability can be found by using the proposed method. How to develop a graph-theoretical method to compute the lower bound of the rank is important, based on which a sufficient condition for the controllability can be established. It is known that distance partition can be used to obtain a lower bound for an all-positive multi-agent network [93]. It is expected to explore whether and how the distance partition could be used to obtain a lower bound of the rank of the controllability matrix. (2) Several conclusions on the controllability characterization are obtained in Chapter 7. As discussed in Chapter 1, the next step is to develop efficient algorithms to get the minimum set of leaders that can be used to control the whole network. (3) Graph-theoretical controllability is useful in large-scale networks, e.g., power networks. As mentioned in Chapter 1, the studied signed network in Chapter 7 is common in social science. A problem is how to model the controllability problem in social networks, especially in the opinion forming problems.

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1. C. Sun, G. Hu, and L. Xie, "Controllability of Multi-agent Networks with Antagonistic Interactions," *IEEE Transactions on Automatic Control*, to appear, DOI: 10.1109/TAC.2017.2697202, 2017.

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