# An Achievable Region for Double-Unicast Networks with Linear Network Coding

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***Abstract*—In this paper, we present an achievable rate region for double-unicast networks by assuming that the intermediate nodes perform random linear network coding, and the source and sink nodes optimize their strategies to maximize the achievable region. Such a setup can be modeled as a deterministic interference channel, whose capacity region has been obtained in [1]. For the particular class of *linear* deterministic interference channels of our interest, in which the outputs and interference are *linear* deterministic functions of the inputs, we show that the known capacity region can be achieved by *linear* strategies. As a result, for a given set of network coding coefficients chosen by the intermediate nodes, the proposed linear precoding and decoding for the source and sink nodes will give the maximum achievable rate region for double-unicast networks. We further derive a suboptimal but easy-to-compute rate region that is independent of the network coding coefficients used at the intermediate nodes, and is instead specified by the min-cuts of the network. It is found that even this suboptimal region is strictly larger than the existing achievable rate regions in the literature.**

***Index Terms*—Network Coding, achievable rate region, double-unicast networks, linear precoding, deterministic interference channels.**

## I. INTRODUCTION

With network coding, the information-bearing messages are allowed to be coded at the intermediate nodes and hence the achievable rate region of the network can be considerably enlarged [2]–[4]. For single-session networks where all sink nodes want to decode the same message originated from the same source node, the network coding capacity is determined by the min-cuts of the network and it is achieved by simple *random linear network coding* (RLNC), in which the coding coefficients at each intermediate node are chosen randomly and independently [5]. On the other hand, when there are multiple sessions sharing the network, characterizing the network coding capacity region remains an open problem. In particular, it has been shown that linear network coding is insufficient for general multi-session networks [6]. One class of multi-session networks of particular interest is known as multiple-unicast networks, in which each message is generated by one source node and demanded by exactly one sink node [7]. It has been found that for any directed acyclic multi-session network, there exists a corresponding multiple-unicast network that has the same solvability over any alphabet [8]. As the simplest case of general multiple-unicast networks where network coding is relevant, double-unicast networks with two source-sink pairs have received significant research interests recently. For such networks, the achievability of the rate pair $(1, 1)$ has been established with various methods [9]–[11]. On the other hand, since the capacity region for double-unicast networks remains unknown, many works have been devoted to obtaining better outer and inner bounds for the capacity region [12]–[19].

In [12], based on the idea of information dominance, a generalized network sharing outer bound for double-unicast networks was introduced. Later, such a bound was improved in [13] by including some additional weighted bounds. In [14], a linear programming (LP) bound was derived based on solving LP optimization problems. LP bound is the tightest computable outer bound known to date and it was conjectured to actually give the capacity region for double-unicast networks [12]. Recently, this conjecture was proven to be incorrect in [20] by showing that the double-unicast problem is as hard as any general multiple-unicast problem. As for the inner bounds of double-unicast networks, most achievable schemes are based on some suboptimal yet simple-to-implement linear network coding strategies. In [15], the design problem was formulated as an LP optimization problem by packing butterfly structures in the network. In [16], an achievable region was obtained by the so-called "rate-exchange" method, where starting from the single-user rate for one of the users, a non-zero rate for the other user is achieved by sacrificing the single-user rate via interference nulling. In [17]–[19], achievable rate regions were obtained by applying RLNC at each intermediate node, together with some linear/nonlinear precessing at the source and sink nodes.

In this paper, similar to [17]–[19], we consider the achievable rate region for double-unicast networks by assuming that *RLNC* is employed by the intermediate nodes, which, though suboptimal in general, is of practical interest due to its simplicity, distributiveness, and robustness to network dynamics. Under such an assumption, the design problem reduces to finding the optimal strategies at the source and sink nodes so that the achievable rate region is maximized. Different from that in single-session networks, the optimal strategy at each source node for double-unicast networks needs to achieve a good balance between maximizing the rate to its own sink node and minimizing the interference to the other.

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For a given set of network coding coefficients chosen by the intermediate nodes, such a scenario can be modeled as a deterministic interference channel, whose capacity region has been obtained in [1]. For the particular class of *linear* deterministic interference channel of our interest, in which the outputs and the interferences are *linear* deterministic functions of the inputs, we show that the capacity region specified in [1] can be achieved by linear precoding and decoding at the source and sink nodes, respectively. The main techniques used in the proposed linear scheme are Han-Kobayashi rate splitting [21] and concatenated precoding composed of zero-forcing and random block-diagonal precoding matrices. Specifically, the information bearing symbols are first split into a common part which is decodable at both sink nodes, and a private part which is decodable at the designated sink node only. The resulting data symbols are then precoded by concatenated precoding matrices, which are carefully designed such that the interferences caused by the undesired data symbols are minimized at both sinks. For a given set of coding coefficients chosen by the intermediate nodes, the proposed linear scheme for the sink and source nodes thus gives the maximum achievable rate region for double-unicast network. We further derive a suboptimal but easy-to-compute achievable region that is independent of the network coding coefficients used at the intermediate nodes, which is instead specified by the min-cuts of the network. We analytically show that even this suboptimal rate region is strictly larger than the existing regions obtained in the literature [16]–[19].

A similar achievable region is obtain in [18] by directly applying the capacity result of the deterministic interference channel in [1]. However, the achievable scheme given in [1] is based on the information-theoretic superposition coding, which is difficult to be implemented in practice. In contrast, by exploiting the *linear* structure of the resulting deterministic interference channel, our proposed achievable scheme requires simple linear precoding and decoding at the source and sink nodes, respectively, which is easy to be implemented. Moreover, the rate region expressed in terms of the min-cuts of the network derived in this paper is shown to be strictly larger than that obtained in [18].

The rest of this paper is organized as follows. Section II introduces the system model and presents the known capacity result of the resulting deterministic interference channel. Section III proposes a *linear* precoding and decoding scheme that achieves the capacity region of the linear deterministic interference channel, which, together with the RLNC applied at the intermediate nodes, gives an achievable rate region for double-unicast networks. In Section IV, a rate region in terms of the min-cuts of the network is derived and it is analytically compared with the existing achievable regions in the literature. Finally, we conclude this paper in Section V.

*Notations*: Throughout this paper, vectors and matrices are represented by boldface lower- and upper-case letters, respectively. For a matrix $\mathbf{A}$, we use $\mathbf{A}^T$, $\mathbf{A}^{-1}$ and $\mathrm{rank}(\mathbf{A})$ to denote its transpose, inverse and rank, respectively. $\mathbf{0}_{n\times m}$ represents a zero matrix of size $n \times m$ and the subscripts are omitted when no ambiguity will be caused. The range (or column space) and null space of a matrix $\mathbf{A}$ are denoted by $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$, respectively. $H(\cdot)$ is the entropy function, and $(x)^+$ is defined as $(x)^+ \triangleq \max\{0, x\}$.

## II. SYSTEM MODEL

A network can be represented by a directed acyclic graph $G = (V, E)$, where $V$ is the set of nodes and $E$ is the set of edges. We consider double-unicast networks with two source nodes $S = \{s_1, s_2\} \subset V$ and two sink nodes $T = \{t_1, t_2\} \subset V$, where $s_1$ and $s_2$ are intended to send independent messages $\mathbf{d}_1$ and $\mathbf{d}_2$ to $t_1$ and $t_2$, respectively. Without loss of generality, we assume unit edge capacity, i.e., each edge is capable of carrying one symbol per time slot. By Merger's theorem, the minimum cut between the source subset $S_{N_1} \subseteq S$ and the sink subset $T_{N_2} \subseteq T$ equals to the number of edge disjoint paths from $S_{N_1}$ to $T_{N_2}$, denoted as $k_{N_1-N_2}$, where $N_1, N_2 \subseteq \{1, 2\}$. It then follows that the number of outgoing edges from the source node $s_i$, denoted as $m_i$, is no smaller than $k_{i-12}$. Furthermore, if $m_i > k_{i-12}$, we may add an imaginary source $s'_i$ connecting to $s_i$ via $k_{i-12}$ edges without affecting the network capacity. Therefore, without loss of generality, we may assume that $m_i = k_{i-12}$, i.e., there are $k_{i-12}$ outgoing edges from source $s_i$. Similarly, the number of incoming edges to sink $t_i$, denoted as $n_i$, is assumed equal to $k_{12-i}$, $i \in \{1, 2\}$ [19] [1]

With RLNC performed at each intermediate node, the input-output relation between the source and sink nodes of double-unicast networks is given as [22][2]

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{H}_{11}\mathbf{x}_1 + \mathbf{H}_{12}\mathbf{x}_2 \\ \mathbf{y}_2 &= \mathbf{H}_{21}\mathbf{x}_1 + \mathbf{H}_{22}\mathbf{x}_2, \end{aligned} \tag{1}$$

where $\mathbf{y}_i \in \mathbb{F}_q^{k_{12-i}}$ denotes the symbols received by sink $t_i$; $\mathbf{x}_i \in \mathbb{F}_q^{k_{i-12}}$ consists of the symbols sent over the $k_{i-12}$ outgoing edges of $s_i$ and it is generated from the information bearing symbols $\mathbf{d}_i$; and $\mathbf{H}_{ji} \in \mathbb{F}_q^{k_{12-j}\times k_{i-12}}$ represents the transition matrix from $s_i$ to $t_j$, which is determined by the network topology and the coding coefficients used at the intermediate nodes [22].

It is observed that the input-output relation given in (1) belongs to a special class of the deterministic interference channel considered in [1], in which the outputs $\mathbf{y}_1$ and $\mathbf{y}_2$ and the interferences $\mathbf{z}_1 = \mathbf{H}_{21}\mathbf{x}_1$ and $\mathbf{z}_2 = \mathbf{H}_{12}\mathbf{x}_2$ are *linear* deterministic functions of the inputs $\mathbf{x}_1$ and $\mathbf{x}_2$. By applying the result in [1], the capacity region of (1) is given by the union of the set of all rate pairs $(R_1, R_2)$ satisfying

$$R_1 \le H(\mathbf{y}_1|\mathbf{z}_2) \tag{2}$$

$$R_2 \le H(\mathbf{y}_2|\mathbf{z}_1) \tag{3}$$

$$R_1 + R_2 \le H(\mathbf{y}_1) + H(\mathbf{y}_2|\mathbf{z}_1, \mathbf{z}_2) \tag{4}$$

$$R_1 + R_2 \le H(\mathbf{y}_1|\mathbf{z}_1, \mathbf{z}_2) + H(\mathbf{y}_2) \tag{5}$$

$$R_1 + R_2 \le H(\mathbf{y}_1|\mathbf{z}_1) + H(\mathbf{y}_2|\mathbf{z}_2) \tag{6}$$

$$2R_1 + R_2 \le H(\mathbf{y}_1) + H(\mathbf{y}_1|\mathbf{z}_1, \mathbf{z}_2) + H(\mathbf{y}_2|\mathbf{z}_2) \tag{7}$$

$$R_1 + 2R_2 \le H(\mathbf{y}_1|\mathbf{z}_1) + H(\mathbf{y}_2) + H(\mathbf{y}_2|\mathbf{z}_1, \mathbf{z}_2), \tag{8}$$

---

[1] The techniques presented in this paper can be extended to the cases with arbitrary $m_i$'s and $n_i$'s (see Remark 2 in Section III).

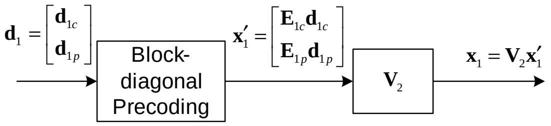
[2] Throughout this paper, the input and output symbols are represented by column vectors.

over all input distributions $\mathbf{x}_1 \in \mathbb{F}_q^{k_{1-12}}$ and $\mathbf{x}_2 \in \mathbb{F}_q^{k_{2-12}}$.

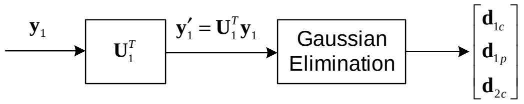
Note that the capacity-achieving scheme given in [1] is based on information-theoretic superposition coding, which is difficult to be implemented in practice. In the next section, we show that for the particular *linear* deterministic interference channel given by (1), the capacity region specified in (2)-(8) can be achieved with simple *linear* precoding and decoding at the source and the sink nodes, which thus provides a rate region achievable by linear schemes for double-unicast networks.

## III. A LINEAR CAPACITY-ACHIEVING SCHEME

Fig. 1 provides a schematic overview for our proposed linear capacity-achieving scheme, which consists of rate-splitting, block-diagonal precoding, and invertible linear transformations at the source nodes, and invertible linear transformations and Gaussian eliminations at the sink nodes.



(a) Rate splitting and linear precoding at $s_1$.



(b) Decoding process at $t_1$

Fig. 1: The proposed achievability scheme with linear precoding and decoding at $s_1$ and $t_1$, respectively.

Before presenting the detailed designs for each block in Fig. 1, we first reformulate the capacity region specified in (2)-(8) in terms of the channel ranks. By applying Lemma 7 given in Appendix B, the region given in (2)-(8) can be equivalently written as[3]

$$R_1 \leq \text{rank}(\mathbf{H}_{11}) \tag{9}$$

$$R_2 \leq \text{rank}(\mathbf{H}_{22}) \tag{10}$$

$$R_1 + R_2 \leq \text{rank}\left(\begin{bmatrix}\mathbf{H}_{11} & \mathbf{H}_{12}\end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix}\mathbf{H}_{12}\\ \mathbf{H}_{22}\end{bmatrix}\right) - \text{rank}(\mathbf{H}_{12}) \tag{11}$$

$$R_1 + R_2 \leq \text{rank}\left(\begin{bmatrix}\mathbf{H}_{21} & \mathbf{H}_{22}\end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix}\mathbf{H}_{11}\\ \mathbf{H}_{21}\end{bmatrix}\right) - \text{rank}(\mathbf{H}_{21}) \tag{12}$$

$$\begin{aligned} R_1 + R_2 \leq{}& \text{rank}\left(\begin{bmatrix}\mathbf{H}_{11} & \mathbf{H}_{12}\\ \mathbf{H}_{21} & \mathbf{0}\end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix}\mathbf{H}_{21} & \mathbf{H}_{22}\\ \mathbf{0} & \mathbf{H}_{12}\end{bmatrix}\right)\\ &- \text{rank}(\mathbf{H}_{21}) - \text{rank}(\mathbf{H}_{12}) \end{aligned} \tag{13}$$

$$\begin{aligned} 2R_1 + R_2 \leq{}& \text{rank}\left(\begin{bmatrix}\mathbf{H}_{11}\\ \mathbf{H}_{21}\end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix}\mathbf{H}_{21} & \mathbf{H}_{22}\\ \mathbf{0} & \mathbf{H}_{12}\end{bmatrix}\right)\\ &+ \text{rank}\left(\begin{bmatrix}\mathbf{H}_{11} & \mathbf{H}_{12}\end{bmatrix}\right) - \text{rank}(\mathbf{H}_{21}) - \text{rank}(\mathbf{H}_{12}) \end{aligned} \tag{14}$$

$$\begin{aligned} R_1 + 2R_2 \leq{}& \text{rank}\left(\begin{bmatrix}\mathbf{H}_{12}\\ \mathbf{H}_{22}\end{bmatrix}\right) + \text{rank}\left(\begin{bmatrix}\mathbf{H}_{11} & \mathbf{H}_{12}\\ \mathbf{H}_{21} & \mathbf{0}\end{bmatrix}\right)\\ &+ \text{rank}\left(\begin{bmatrix}\mathbf{H}_{21} & \mathbf{H}_{22}\end{bmatrix}\right) - \text{rank}(\mathbf{H}_{21}) - \text{rank}(\mathbf{H}_{12}). \end{aligned} \tag{15}$$

Furthermore, with RLNC employed at each intermediate node over $\mathbb{F}_q$, with probability 1 as $q \to \infty$, we have [19]:

$$\begin{aligned} &\text{rank}(\mathbf{H}_{11}) = k_{1-1}; \quad \text{rank}(\mathbf{H}_{12}) = k_{2-1};\\ &\text{rank}(\mathbf{H}_{21}) = k_{1-2}; \quad \text{rank}(\mathbf{H}_{22}) = k_{2-2};\\ &\text{rank}\left(\begin{bmatrix}\mathbf{H}_{11}\\ \mathbf{H}_{21}\end{bmatrix}\right) = k_{1-12}; \quad \text{rank}\left(\begin{bmatrix}\mathbf{H}_{12}\\ \mathbf{H}_{22}\end{bmatrix}\right) = k_{2-12};\\ &\text{rank}\left(\begin{bmatrix}\mathbf{H}_{11} & \mathbf{H}_{12}\end{bmatrix}\right) = k_{12-1}; \text{rank}\left(\begin{bmatrix}\mathbf{H}_{21} & \mathbf{H}_{22}\end{bmatrix}\right) = k_{12-2}. \end{aligned} \tag{16}$$

The remaining matrix ranks on the right hand sides of (9)-(15) cannot be explicitly represented in terms of the min-cuts of the network. However, they can be reformulated into equivalent forms involving the min-cuts by introducing a decomposition of the interfering transition matrices. Taking the transition matrix $\mathbf{H}_{12}$ with size $k_{12-1} \times k_{2-12}$ and rank $k_{2-1}$ as an example, we let

- $\mathbf{V}_{10} \in \mathbb{F}_q^{k_{2-12} \times (k_{2-12} - k_{2-1})}$ be a matrix whose columns span the null space of $\mathbf{H}_{12}$, denoted as $\mathcal{N}(\mathbf{H}_{12})$,
- $\mathbf{V}_{11} \in \mathbb{F}_q^{k_{2-12} \times k_{2-1}}$ be a matrix such that $\mathcal{R}(\mathbf{V}_{11})$ and $\mathcal{R}(\mathbf{V}_{10})$ are complementary subspaces ( [23], p.90).
- $\mathbf{U}_{10} \in \mathbb{F}_q^{k_{12-1} \times (k_{12-1} - k_{2-1})}$ be a matrix whose columns span the left null space of $\mathbf{H}_{12}$, denoted as $\mathcal{N}(\mathbf{H}_{12}^T)$, and
- $\mathbf{U}_{11} \in \mathbb{F}_q^{k_{12-1} \times k_{2-1}}$ be a matrix such that $\mathcal{R}(\mathbf{U}_{11})$ and $\mathcal{R}(\mathbf{U}_{10})$ are complementary subspaces,

Let $\mathbf{U}_1 = \begin{bmatrix}\mathbf{U}_{11} & \mathbf{U}_{10}\end{bmatrix}$ and $\mathbf{V}_1 = \begin{bmatrix}\mathbf{V}_{11} & \mathbf{V}_{10}\end{bmatrix}$. Following from the definition of the complementary subspaces ( [23], p.90), we have $\dim(\mathcal{R}(\mathbf{U}_{10}) + \mathcal{R}(\mathbf{U}_{11})) = k_{12-1}$ and $\dim(\mathcal{R}(\mathbf{V}_{10}) + \mathcal{R}(\mathbf{V}_{11})) = k_{2-12}$. $\mathbf{U}_1$ and $\mathbf{V}_1$ are full rank square matrices and hence are invertible. Let $\mathbf{\Lambda}_1 = \mathbf{U}_1^T\mathbf{H}_{12}\mathbf{V}_1$, we then have

$$\begin{aligned} \mathbf{\Lambda}_1 &= \begin{bmatrix}\mathbf{U}_{11}^T\\ \mathbf{U}_{10}^T\end{bmatrix}\mathbf{H}_{12}\begin{bmatrix}\mathbf{V}_{11} & \mathbf{V}_{10}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{U}_{11}^T\mathbf{H}_{12}\mathbf{V}_{11} & \mathbf{U}_{11}^T\mathbf{H}_{12}\mathbf{V}_{10}\\ \mathbf{U}_{10}^T\mathbf{H}_{12}\mathbf{V}_{11} & \mathbf{U}_{10}^T\mathbf{H}_{12}\mathbf{V}_{10}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{U}_{11}^T\mathbf{H}_{12}\mathbf{V}_{11} & \mathbf{0}_{k_{2-1}\times(k_{2-12}-k_{2-1})}\\ \mathbf{0}_{(k_{12-1}-k_{2-1})\times k_{2-1}} & \mathbf{0}_{(k_{12-1}-k_{2-1})\times(k_{2-12}-k_{2-1})}\end{bmatrix}. \end{aligned}$$

**Lemma 1.** *The* $k_{2-1} \times k_{2-1}$ *square matrix* $\mathbf{D}_{12} \triangleq \mathbf{U}_{11}^T\mathbf{H}_{12}\mathbf{V}_{11}$ *is nonsingular.*

*Proof:* Please refer to Appendix C. ■

Therefore, $\mathbf{H}_{12}$ can then be decomposed as

$$\mathbf{H}_{12} = (\mathbf{U}_1^T)^{-1}\mathbf{\Lambda}_1\mathbf{V}_1^{-1}. \tag{17}$$

Similarly, $\mathbf{H}_{21}$ can be decomposed as

$$\begin{aligned} \mathbf{H}_{21} &= (\mathbf{U}_2^T)^{-1}\mathbf{\Lambda}_2\mathbf{V}_2^{-1}\\ &= \begin{bmatrix}\mathbf{U}_{21}^T\\ \mathbf{U}_{20}^T\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{D}_{21} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{V}_{21} & \mathbf{V}_{20}\end{bmatrix}^{-1}. \end{aligned} \tag{18}$$

**Lemma 2.** *With the above decompositions of* $\mathbf{H}_{12}$ *and* $\mathbf{H}_{21}$*, we have*

$$\text{rank}\left(\begin{bmatrix}\mathbf{H}_{11} & \mathbf{H}_{12}\\ \mathbf{H}_{21} & \mathbf{0}\end{bmatrix}\right) = \text{rank}(\mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{20}) + k_{1-2} + k_{2-1}$$

$$\text{rank}\left(\begin{bmatrix}\mathbf{H}_{21} & \mathbf{H}_{22}\\ \mathbf{0} & \mathbf{H}_{12}\end{bmatrix}\right) = \text{rank}(\mathbf{U}_{20}^T\mathbf{H}_{22}\mathbf{V}_{10}) + k_{1-2} + k_{2-1}.$$

*Proof:* Please refer to Appendix D. ■

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[3] As the analysis given in the following is based on the finite field of order $q$, the data rate in the following context is given in terms of symbols, which is equivalent to the conventional rate expressed in bits normalized by $\log(q)$.

With (16) and Lemma 2, (9)-(15) can be equivalently expressed as

$$
\begin{aligned}
R_1 \leq& k_{1-1} && (19)\\
R_2 \leq& k_{2-2} && (20)\\
R_1 + R_2 \leq& k_{12-1} + k_{2-12} - k_{2-1} && (21)\\
R_1 + R_2 \leq& k_{12-2} + k_{1-12} - k_{1-2} && (22)\\
R_1 + R_2 \leq& \mathrm{rank}(\mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{20}) + \mathrm{rank}(\mathbf{U}_{20}^T\mathbf{H}_{22}\mathbf{V}_{10})\\
&+ k_{1-2} + k_{2-1} && (23)\\
2R_1 + R_2 \leq& k_{12-1} + k_{1-12} + \mathrm{rank}(\mathbf{U}_{20}^T\mathbf{H}_{22}\mathbf{V}_{10}) && (24)\\
R_1 + 2R_2 \leq& k_{12-2} + k_{2-12} + \mathrm{rank}(\mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{20}) && (25)
\end{aligned}
$$

Given a rate pair $(R_1, R_2)$ that satisfies the inequalities in (19)-(25), we will next show that it can be achieved by applying the linear scheme depicted in Fig. 1. First, with $\mathbf{H}_{12}$ and $\mathbf{H}_{21}$ decomposed as in (17) and (18), respectively, and by absorbing $\mathbf{V}_j^{-1}$ into the input vector $\mathbf{x}_i$ and multiplying the output vector $\mathbf{y}_i$ with $\mathbf{U}_i^T$, $i, j \in \{1, 2\}, i \neq j$, the channel model given in (1) can be equivalently written as

$$
\begin{aligned}
\mathbf{y}'_1 &= \mathbf{U}_1^T\mathbf{y}_1 = \mathbf{H}'_{11}\mathbf{x}'_1 + \mathbf{\Lambda}_1\mathbf{x}'_2\\
\mathbf{y}'_2 &= \mathbf{U}_2^T\mathbf{y}_2 = \mathbf{H}'_{22}\mathbf{x}'_2 + \mathbf{\Lambda}_2\mathbf{x}'_1,
\end{aligned}
\qquad (26)
$$

where $\mathbf{x}'_1 = \mathbf{V}_2^{-1}\mathbf{x}_1$, $\mathbf{x}'_2 = \mathbf{V}_1^{-1}\mathbf{x}_2$, $\mathbf{H}'_{11} = \mathbf{U}_1^T\mathbf{H}_{11}\mathbf{V}_2$ and $\mathbf{H}'_{22} = \mathbf{U}_2^T\mathbf{H}_{22}\mathbf{V}_1$. The advantage of the equivalent channel model (26) is that it results in block-diagonal interfering channel matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$, which are easier to deal with. Similar transformations have been used in [24] for deriving the optimal degrees of freedom (DoF) of the two-user wireless MIMO-IC. To determine the input signal vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, it is then sufficient to obtain $\mathbf{x}'_1$ and $\mathbf{x}'_2$ since they are related by the invertible linear transformations $\mathbf{x}_1 = \mathbf{V}_2\mathbf{x}'_1$ and $\mathbf{x}_2 = \mathbf{V}_1\mathbf{x}'_2$.

Motivated by the rate-splitting technique used in the celebrated Han-Kobayashi schemes in interference channels [21], we decompose the $R_1$ symbols in $\mathbf{d}_1$ into two parts: the common information $\mathbf{d}_{1c} \in \mathbb{F}_q^{R_{1c}}$, which is decodable at both $t_1$ and $t_2$, and the private information $\mathbf{d}_{1p} \in \mathbb{F}_q^{R_{1p}}$, which is decodable at $t_1$ only. We then have $\mathbf{d}_1 = \begin{bmatrix}\mathbf{d}_{1c}^T & \mathbf{d}_{1p}^T\end{bmatrix}^T$ and $R_1 = R_{1c} + R_{1p}$. To map $\mathbf{d}_1$ to the $k_{1-12}$-dimensional input vector $\mathbf{x}'_1$, a block-diagonal linear precoding is applied as

$$
\mathbf{x}'_1 = \begin{bmatrix}\mathbf{E}_{1c} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}_{1p}\end{bmatrix}\mathbf{d}_1 = \begin{bmatrix}\mathbf{E}_{1c}\mathbf{d}_{1c}\\ \mathbf{E}_{1p}\mathbf{d}_{1p}\end{bmatrix}, \qquad (27)
$$

where $\mathbf{E}_{1c} \in \mathbb{F}_q^{k_{1-2}\times R_{1c}}$ and $\mathbf{E}_{1p} \in \mathbb{F}_q^{(k_{1-12}-k_{1-2})\times R_{1p}}$ are randomly and independently generated. Note that with the above block-diagonal precoding, the common and private symbols $\mathbf{d}_{1c}$ and $\mathbf{d}_{1p}$ are constrained to the first $k_{1-2}$ and the last $(k_{1-12} - k_{1-2})$ components of $\mathbf{x}'_1$, respectively. As such, the private symbols $\mathbf{d}_{1p}$ will not interfere the received signal vector $\mathbf{y}'_2$ at $t_2$, and at the same time, the common symbols $\mathbf{d}_{1c}$ can be decoded at both $t_1$ and $t_2$ if the rate pair $(R_1, R_2)$ satisfies the inequalities given in (19)-(25), as will become clear later.

Similarly, $\mathbf{x}'_2$ can be obtained as $\mathbf{x}'_2 = \begin{bmatrix}\mathbf{E}_{2c}\mathbf{d}_{2c}\\ \mathbf{E}_{2p}\mathbf{d}_{2p}\end{bmatrix}$, where $\mathbf{d}_{2c} \in \mathbb{F}_q^{R_{2c}}$, $\mathbf{d}_{2p} \in \mathbb{F}_q^{R_{2p}}$, $\mathbf{E}_{2c} \in \mathbb{F}_q^{k_{2-1}\times R_{2c}}$, and $\mathbf{E}_{2p} \in \mathbb{F}_q^{(k_{2-12}-k_{2-1})\times R_{2p}}$. $\mathbf{E}_{2c}$ and $\mathbf{E}_{2p}$ are randomly and independently generated.

The output $\mathbf{y}'_1$ at $t_1$ given in (26) can then be written as

$$
\begin{aligned}
\mathbf{y}'_1 &= \mathbf{U}_1^T\mathbf{H}_{11}\mathbf{V}_2\mathbf{x}'_1 + \mathbf{\Lambda}_1\mathbf{x}'_2\\
&= \begin{bmatrix}\mathbf{U}_{11}^T\mathbf{H}_{11}\mathbf{V}_{21} & \mathbf{U}_{11}^T\mathbf{H}_{11}\mathbf{V}_{20} & \mathbf{D}_{12}\\ \mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{21} & \mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{20} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{E}_{1c}\mathbf{d}_{1c}\\ \mathbf{E}_{1p}\mathbf{d}_{1p}\\ \mathbf{E}_{2c}\mathbf{d}_{2c}\end{bmatrix}.
\end{aligned}
\qquad (28)
$$

Equation (28) clearly shows that the private symbols in vector $\mathbf{d}_{2p}$ transmitted by $s_2$ does not interfere $\mathbf{y}'_1$. Furthermore, the desired symbols $\mathbf{d}_{1c}$ and $\mathbf{d}_{1p}$ can be decoded at $t_1$ if the linear system equations given in (28) is uniquely solvable.

Due to symmetry, at $t_2$ we have

$$
\mathbf{y}'_2 = \begin{bmatrix}\mathbf{U}_{21}^T\mathbf{H}_{22}\mathbf{V}_{11} & \mathbf{U}_{21}^T\mathbf{H}_{22}\mathbf{V}_{10} & \mathbf{D}_{21}\\ \mathbf{U}_{20}^T\mathbf{H}_{22}\mathbf{V}_{11} & \mathbf{U}_{20}^T\mathbf{H}_{22}\mathbf{V}_{10} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{E}_{2c}\mathbf{d}_{2c}\\ \mathbf{E}_{2p}\mathbf{d}_{2p}\\ \mathbf{E}_{1c}\mathbf{d}_{1c}\end{bmatrix}. \qquad (29)
$$

To find a sufficient condition on $R_{1p}$, $R_{1c}$, $R_{2p}$, and $R_{2c}$ such that the system of linear equations given by (28) and (29) are uniquely solvable, the following results will be used.

**Lemma 3.** *( [23], p.100) Given the linear relationship* $\mathbf{y} = \mathbf{A}\mathbf{x}$*, where* $\mathbf{A} \in \mathbb{F}_q^{p\times l}$*,* $\mathbf{x} \in \mathbb{F}_q^l$*, and* $\mathbf{y} \in \mathbb{F}_q^p$*,* $\mathbf{x}$ *can be uniquely determined from* $\mathbf{y}$ *if and only if* $\mathbf{A}$ *is of full column rank, i.e.,* $\mathrm{rank}(\mathbf{A}) = l$*.*

For notational convenience, let

$$
\mathbf{M}_1 \triangleq \begin{bmatrix}\mathbf{U}_{11}^T\mathbf{H}_{11}\mathbf{V}_{21} & \mathbf{U}_{11}^T\mathbf{H}_{11}\mathbf{V}_{20} & \mathbf{D}_{12}\\ \mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{21} & \mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{20} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{E}_{1c} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}_{1p} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{E}_{2c}\end{bmatrix}. \qquad (30)
$$

Hence, (28) can be written as $\mathbf{y}'_1 = \mathbf{M}_1\begin{bmatrix}\mathbf{d}_{1c}^T & \mathbf{d}_{1p}^T & \mathbf{d}_{2c}^T\end{bmatrix}^T$. According to Lemma 3, the sink node $t_1$ is able to decode $\mathbf{d}_{1c}$, $\mathbf{d}_{1p}$, and $\mathbf{d}_{2c}$ if $\mathbf{M}_1$ is of full column rank. Next, we will find a sufficient condition on the data rates $R_{1c}$, $R_{1p}$, and $R_{2c}$ such that $\mathbf{M}_1$ has full column rank.

**Lemma 4.** *Let* $\mathbf{A}_1 \in \mathbb{F}_q^{p\times l_1}$*,* $\mathbf{A}_2 \in \mathbb{F}_q^{p\times l_2}$ *and* $\mathbf{A}_3 \in \mathbb{F}_q^{p\times l_3}$ *be given matrices, and* $\mathbf{E}_1 \in \mathbb{F}_q^{l_1\times k_1}$*,* $\mathbf{E}_2 \in \mathbb{F}_q^{l_2\times k_2}$ *and* $\mathbf{E}_3 \in \mathbb{F}_q^{l_3\times k_3}$ *be random matrices whose entries are uniformly generated from* $\mathbb{F}_q$*. Then as* $q \to \infty$*,* $\mathrm{rank}([\mathbf{A}_1\mathbf{E}_1 \quad \mathbf{A}_2\mathbf{E}_2 \quad \mathbf{A}_3\mathbf{E}_3]) = k_1 + k_2 + k_3$ *holds with high probability if the following conditions are satisfied:*

- $k_1 \leq \mathrm{rank}(\mathbf{A}_1)$
- $k_2 \leq \mathrm{rank}(\mathbf{A}_2)$
- $k_3 \leq \mathrm{rank}(\mathbf{A}_3)$
- $k_1 + k_2 \leq \mathrm{rank}([\mathbf{A}_1 \quad \mathbf{A}_2])$
- $k_1 + k_3 \leq \mathrm{rank}([\mathbf{A}_1 \quad \mathbf{A}_3])$
- $k_2 + k_3 \leq \mathrm{rank}([\mathbf{A}_2 \quad \mathbf{A}_3])$
- $k_1 + k_2 + k_3 \leq \mathrm{rank}([\mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_3])$

*Proof:* Please refer to Appendix E. ■

By applying Lemma 4 to (30), a sufficient condition for $\mathbf{M}_1$ to be of full column rank, and hence (28) is uniquely solvable, is given by

$$
R_{1c} \leq \mathrm{rank}(\mathbf{U}_1^T\mathbf{H}_{11}\mathbf{V}_{21}) \overset{(a)}{=} \mathrm{rank}(\mathbf{H}_{11}\mathbf{V}_{21}) \qquad (31)
$$

$$
R_{1p} \leq \mathrm{rank}(\mathbf{U}_1^T\mathbf{H}_{11}\mathbf{V}_{20}) \overset{(b)}{=} k_{1-12} - k_{1-2} \qquad (32)
$$

$$
R_{2c} \leq \mathrm{rank}\left(\begin{bmatrix}\mathbf{D}_{12}\\ \mathbf{0}\end{bmatrix}\right) = k_{2-1} \qquad (33)
$$

$$R_{1c} + R_{1p} \leq \mathrm{rank}(\mathbf{U}_1^T \mathbf{H}_{11} \mathbf{V}_2) = k_{1-1} \tag{34}$$

$$\begin{aligned} R_{1c} + R_{2c} &\leq \mathrm{rank}\left(\begin{bmatrix} \mathbf{U}_{11}^T \mathbf{H}_{11} \mathbf{V}_{21} & \mathbf{D}_{12} \\ \mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_{21} & \mathbf{0} \end{bmatrix}\right) \\ &\overset{(c)}{=} k_{2-1} + \mathrm{rank}(\mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_{21}) \end{aligned} \tag{35}$$

$$\begin{aligned} R_{1p} + R_{2c} &\leq \mathrm{rank}\left(\begin{bmatrix} \mathbf{U}_{11}^T \mathbf{H}_{11} \mathbf{V}_{20} & \mathbf{D}_{12} \\ \mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_{20} & \mathbf{0} \end{bmatrix}\right) \\ &\overset{(d)}{=} k_{2-1} + \mathrm{rank}(\mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_{20}) \end{aligned} \tag{36}$$

$$\begin{aligned} R_{1c} + R_{1p} + R_{2c} &\leq \mathrm{rank}\left(\begin{bmatrix} \mathbf{U}_{11}^T \mathbf{H}_{11} \mathbf{V}_{21} & \mathbf{U}_{11}^T \mathbf{H}_{11} \mathbf{V}_{20} & \mathbf{D}_{12} \\ \mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_{21} & \mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_{20} & \mathbf{0} \end{bmatrix}\right) \\ &\overset{(e)}{=} k_{12-1}, \end{aligned} \tag{37}$$

where $(a)$ holds since $\mathbf{U}_1$ is invertible, and $(b)$ follows from Lemma 6 in Appendix A since

$$\begin{aligned} &\mathrm{rank}(\mathbf{U}_1^T \mathbf{H}_{11} \mathbf{V}_{20}) = \mathrm{rank}(\mathbf{H}_{11} \mathbf{V}_{20}) \\ &= \mathrm{rank}(\mathbf{V}_{20}) - \dim(\mathcal{N}(\mathbf{H}_{11}) \cap \mathcal{R}(\mathbf{V}_{20})) \\ &= \mathrm{rank}(\mathbf{V}_{20}) - \dim(\mathcal{N}(\mathbf{H}_{11}) \cap \mathcal{N}(\mathbf{H}_{21})) \\ &\overset{(f)}{=} \mathrm{rank}(\mathbf{V}_{20}) = k_{1-12} - k_{1-2}, \end{aligned} \tag{38}$$

where $(f)$ follows from (16), i.e., $\begin{bmatrix} \mathbf{H}_{11}^T & \mathbf{H}_{21}^T \end{bmatrix}^T$ is of full column rank. Moreover, $(c)$ and $(d)$ can be obtained by applying elementary column operations since $\mathbf{D}_{12}$ is nonsingular as given in Lemma 1; and $(e)$ can be shown with elementary column operations together with a similar proof as that for $(b)$, i.e.,

$$\begin{aligned} &\mathrm{rank}\left(\begin{bmatrix} \mathbf{U}_{11}^T \mathbf{H}_{11} \mathbf{V}_{21} & \mathbf{U}_{11}^T \mathbf{H}_{11} \mathbf{V}_{20} & \mathbf{D}_{12} \\ \mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_{21} & \mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_{20} & \mathbf{0} \end{bmatrix}\right) \\ &= \mathrm{rank}(\mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_2) + \mathrm{rank}(\mathbf{D}_{12}) \\ &= \mathrm{rank}(\mathbf{U}_{10}^T \mathbf{H}_{11}) + k_{2-1} \\ &= \mathrm{rank}(\mathbf{U}_{10}) - \dim(\mathcal{R}(\mathbf{U}_{10}) \cap \mathcal{N}(\mathbf{H}_{11}^T)) + k_{2-1} \\ &= k_{12-1} - \dim(\mathcal{N}(\mathbf{H}_{12}^T) \cap \mathcal{N}(\mathbf{H}_{11}^T)) \\ &= k_{12-1} \end{aligned} \tag{39}$$

By symmetry, a sufficient condition for sink $t_2$ to successfully decode $\mathbf{d}_{2c}$, $\mathbf{d}_{2p}$ and $\mathbf{d}_{1c}$ is given by

$$R_{2c} \leq \mathrm{rank}(\mathbf{H}_{22} \mathbf{V}_{11}) \tag{40}$$

$$R_{2p} \leq k_{2-12} - k_{2-1} \tag{41}$$

$$R_{1c} \leq k_{1-2} \tag{42}$$

$$R_{2c} + R_{2p} \leq k_{2-2} \tag{43}$$

$$R_{2c} + R_{1c} \leq k_{1-2} + \mathrm{rank}(\mathbf{U}_{20}^T \mathbf{H}_{22} \mathbf{V}_{11}) \tag{44}$$

$$R_{2p} + R_{1c} \leq k_{1-2} + \mathrm{rank}(\mathbf{U}_{20}^T \mathbf{H}_{22} \mathbf{V}_{10}) \tag{45}$$

$$R_{2c} + R_{2p} + R_{1c} \leq k_{12-2} \tag{46}$$

By substituting with $R_1 = R_{1c} + R_{1p}$ and $R_2 = R_{2c} + R_{2p}$, the sufficient conditions on the data rate $R_1$ and $R_2$ to ensure full decodability at the respective destinations can be obtained using the standard Fourier-Motzkin elimination over (31)-(37) and (40)-(46). The detailed steps involving the Fourier-Motzkin elimination can be found in [25] and the resulting achievable rate region is given by (19)-(25). Furthermore, given any valid rate pair $(R_1, R_2)$ satisfying (19)-(25), there exists at least one valid common-private rate splitting $(R_{1c}, R_{1p}, R_{2c}, R_{2p})$ that can be found by solving $R_{1p} = R_1 - R_{1c}$ and $R_{2p} = R_2 - R_{2c}$ subject to the constraints specified in (31)-(37) and (40)-(46).

**Remark 1.** *Note that since* (9)-(15) *is the capacity region for the linear deterministic interference channel given by* (1), *it gives the maximum achievable rate region for double-unicast networks with any given transition matrices* $\{\mathbf{H}_{ji}\}$. *In other words, for any given coding coefficients chosen by the intermediate nodes, there are no other source and sink strategies, linear or non-linear, can outperform the proposed linear scheme described above.*

**Remark 2.** *While the proposed linear achievability scheme has been presented mainly based on the rate region specified in* (19)-(25) *under the assumption that RLNC over sufficiently large field is applied at the intermediate nodes, the same techniques can be applied to show the achievability of the rate region specified in the original form in* (9)-(15) *with a given set of transition matrices resulting from any (not necessarily random) linear network coding schemes. Specifically, for general transition matrices* $\mathbf{H}_{12} \in \mathbb{F}_q^{n_1 \times m_2}$ *and* $\mathbf{H}_{21} \in \mathbb{F}_q^{n_2 \times m_1}$ *with ranks* $r_{12}$ *and* $r_{21}$, *respectively, the corresponding precoding matrices in Fig. 1 can be similarly obtained as*

- $\mathbf{V}_2 = \begin{bmatrix} \mathbf{V}_{21} & \mathbf{V}_{20} \end{bmatrix} \in \mathbb{F}_q^{m_1 \times m_1}$, *where the columns of* $\mathbf{V}_{20} \in \mathbb{F}_q^{m_1 \times (m_1 - r_{21})}$ *span* $\mathcal{N}(\mathbf{H}_{21})$ *and* $\mathcal{R}(\mathbf{V}_{21})$ *and* $\mathcal{R}(\mathbf{V}_{20})$ *are complementary subspaces.*
- $\mathbf{U}_1 = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{10} \end{bmatrix} \in \mathbb{F}_q^{n_1 \times n_1}$, *where the columns of* $\mathbf{U}_{10} \in \mathbb{F}_q^{n_1 \times (n_1 - r_{12})}$ *span* $\mathcal{N}(\mathbf{H}_{12}^T)$ *and* $\mathcal{R}(\mathbf{U}_{11})$ *and* $\mathcal{R}(\mathbf{U}_{10})$ *are complementary subspaces.*
- $\mathbf{E}_{1c} \in \mathbb{F}_q^{r_{21} \times R_{1c}}$ *and* $\mathbf{E}_{1p} \in \mathbb{F}_q^{(m_1 - r_{21}) \times R_{1p}}$ *are generated randomly.*

The proposed linear achievability scheme presented above is illustrated with the following simple example.

**Example 1.** *Consider the network shown in Fig. 2(a) with unit edge capacity. The min-cut values of this network are respectively given by* $k_{1-1} = 2$, $k_{2-2} = 3$, $k_{2-1} = k_{1-2} = 2$, $k_{1-12} = k_{12-1} = 2$ *and* $k_{2-12} = k_{12-2} = 3$. *Assume that random network coding coefficients have been chosen in the filed of size*[4] $q = 7$ *with the coding coefficients at each intermediate node indicated in the figure, and the resulting transition matrices are given by:*

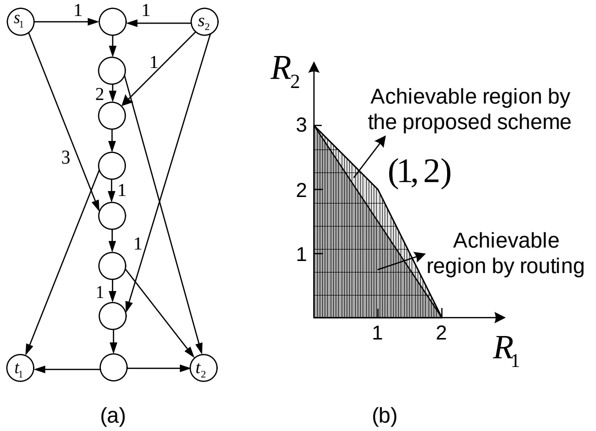
$$\mathbf{H}_{11} = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}, \mathbf{H}_{12} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix},$$

$$\mathbf{H}_{21} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 2 & 3 \end{bmatrix}, \mathbf{H}_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

*Then the achievable region (after removing redundant inequalities) specified in* (9)-(15) *for this particular example is plotted in Fig. 2(b). As a comparison, the achievable rate region with routing is also plotted, which is shown to be strictly smaller than that achieved with network coding. By applying above proposed scheme, we next give the specific designs that achieve the outmost rate pair* $(R_1, R_2) = (1, 2)$. *Let* $\mathbf{d}_1 = d_{11}$ *and* $\mathbf{d}_2 = \begin{bmatrix} d_{21} & d_{22} \end{bmatrix}^T$ *be the information-bearing symbols.*

---

[4] One commonly used field size for RLNC is $2^8$. Here, a small field size is used for illustration purposes.



(a) (b)

Fig. 2: (a) An example of double-unicast networks. (b) The achievable rate regions.

*Then, following* (31)-(37) *and* (40)-(46)*, the private-common rate splitting is given by* $\mathbf{d}_{1c} = d_{11}, \mathbf{d}_{1p} = \emptyset, \mathbf{d}_{2c} = d_{21}$ *and* $\mathbf{d}_{2p} = d_{22}$*, where* $\emptyset$ *denotes an empty vector/matrix.*

*Based on* $\mathbf{H}_{12}$ *and* $\mathbf{H}_{21}$ *given in* (47)*, the following component matrices can be obtained:*

$$\mathbf{U}_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \mathbf{U}_{10} = \emptyset, \mathbf{V}_{11} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{V}_{10} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix},$$

$$\mathbf{U}_{21} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 2 & 3 \end{bmatrix}, \mathbf{U}_{20} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \mathbf{V}_{21} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \mathbf{V}_{20} = \emptyset.$$

*Assume that the following matrices are randomly generated, which will be applied as the linear precoding matrices to map the data symbols to the transmit vectors:*

$$\mathbf{E}_{1c} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \mathbf{E}_{1p} = \emptyset, \mathbf{E}_{2c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{E}_{2p} = \begin{bmatrix} 3 \end{bmatrix}.$$

*The transmit vector by* $s_1$ *and* $s_2$ *can then be obtained as*

$$\mathbf{x}_1 = \mathbf{V}_2 \mathbf{x}_1' = \mathbf{V}_{21} \mathbf{E}_{1c} \mathbf{d}_{1c} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} d_{11} = \begin{bmatrix} 3d_{11} \\ 2d_{11} \end{bmatrix}$$

$$\mathbf{x}_2 = \mathbf{V}_1 \mathbf{x}_2' = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{10} \end{bmatrix} \begin{bmatrix} \mathbf{E}_{2c} \mathbf{d}_{2c} \\ \mathbf{E}_{2p} \mathbf{d}_{2p} \end{bmatrix} = \begin{bmatrix} 3d_{21} + 3d_{22} \\ 5d_{21} + d_{22} \\ 3d_{21} \end{bmatrix}$$

*Therefore, the received symbol vectors at* $t_1$ *and* $t_2$ *are respectively given by*

$$\mathbf{y}_1 = \mathbf{H}_{11} \mathbf{x}_1 + \mathbf{H}_{12} \mathbf{x}_2 = \begin{bmatrix} 6d_{11} + 4d_{21} \\ 5d_{11} \end{bmatrix} \quad (47)$$

$$\mathbf{y}_2 = \mathbf{H}_{22} \mathbf{x}_2 + \mathbf{H}_{21} \mathbf{x}_1 = \begin{bmatrix} 3d_{21} + 3d_{22} + 3d_{11} \\ 4d_{21} + 5d_{11} \\ 5d_{11} \end{bmatrix}. \quad (48)$$

*It can be verified that both* $t_1$ *and* $t_2$ *can recover their desired symbols by solving the system of linear equations given by* (47) *and* (48)*, respectively.*

## IV. AN ACHIEVABLE RATE REGION IN TERMS OF MIN-CUTS

The achievable rate region for double-unicast networks specified in (19)-(25) is given in terms of the min-cuts of the network, as well as the ranks of the matrices $\mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_{20}$ and $\mathbf{U}_{20}^T \mathbf{H}_{22} \mathbf{V}_{10}$. In this section, we derive another rate region for (1) only in terms of the min-cuts of the network, which is sub-optimal but can be more easily computed. The following result will be used:

**Lemma 5.** *If the coding coefficients at the intermediate nodes are randomly selected from a sufficiently large field, then we have the following inequalities:*

$$\text{rank}\left(\mathbf{U}_{10}^T \mathbf{H}_{11} \mathbf{V}_{20}\right) \geq \left(k_{12-1} + k_{1-12} - k_{1-1} - k_{1-2} - k_{2-1}\right)^+, \quad (49)$$

$$\text{rank}\left(\mathbf{U}_{20}^T \mathbf{H}_{22} \mathbf{V}_{10}\right) \geq \left(k_{12-2} + k_{2-12} - k_{2-2} - k_{2-1} - k_{1-2}\right)^+. \quad (50)$$

*Proof:* Please refer to Appendix F. ■

By substituting (49) and (50) into (19)-(25), the following achievable rate region only in terms of min-cuts of the network can be obtained:

$$\begin{aligned} R_1 \leq & k_{1-1} & (51) \\ R_2 \leq & k_{2-2} & (52) \\ R_1 + R_2 \leq & k_{12-1} + k_{2-12} - k_{2-1} & (53) \\ R_1 + R_2 \leq & k_{12-2} + k_{1-12} - k_{1-2} & (54) \\ R_1 + R_2 \leq & \left(k_{12-2} + k_{2-12} - k_{2-2} - k_{1-2} - k_{2-1}\right)^+ & \\ & + \left(k_{12-1} + k_{1-12} - k_{1-1} - k_{1-2} - k_{2-1}\right)^+ & \\ & + k_{1-2} + k_{2-1} & (55) \\ 2R_1 + R_2 \leq & \left(k_{12-2} + k_{2-12} - k_{2-2} - k_{1-2} - k_{2-1}\right)^+ & (56) \\ & + k_{12-1} + k_{1-12} & (57) \\ R_1 + 2R_2 \leq & \left(k_{12-1} + k_{1-12} - k_{1-1} - k_{1-2} - k_{2-1}\right)^+ & (58) \\ & + k_{12-2} + k_{2-12}. & (59) \end{aligned}$$

### A. Comparison with Existing Rate Regions

As discussed in Remark 1, the region specified in (9)-(15) gives the largest possible achievable rate region of double-unicast networks for any given set of network coding coefficients, which cannot be outperformed by any other schemes. In this subsection, we will show that even the rate region specified in (51)-(59) is strictly larger than those achievable regions obtained in the literature [16]–[19].

With "rate-exchange" method, the authors in [16] have obtained an achievable region for double-unicast networks that is given by the convex hull of the following two regions:

| $\mathcal{R}_1$ | $\mathcal{R}_2$ |
|---|---|
| $R_1 \leq k_{1-1}$<br>$2R_1 + R_2 \leq k_{2-2}$ | $R_1 + 2R_2 \leq k_{1-1}$<br>$R_2 \leq k_{2-2}$ |

**Proposition 1.** *The achievable rate region specified in* (51)-(59) *for double-unicast networks is strictly larger than the convex hull of* $\mathcal{R}_1$ *and* $\mathcal{R}_2$*.*

*Proof:* Since the region specified in (51)-(59) is convex, it is sufficient to show that it includes both $\mathcal{R}_1$ and $\mathcal{R}_2$. First, we show that any rate pair $(R_1', R_2')$ in $\mathcal{R}_1$ satisfies the inequalities (51)-(59). With $R_1' \leq k_{1-1}$ and $2R_1' + R_2' \leq k_{2-2}$, we have:

$$
\begin{aligned}
&R_1' \leq k_{1-1}\\
&R_2' \leq k_{2-2} - 2R_1' \leq k_{2-2}\\
&R_1' + R_2' \leq k_{2-2} \leq k_{2-12} \leq k_{2-12} + k_{12-1} - k_{2-1}\\
&R_1' + R_2' \leq k_{2-2} \leq k_{12-2} \leq k_{12-2} + k_{1-12} - k_{1-2}\\
&R_1' + R_2' \leq k_{2-12} \leq k_{2-12} + k_{12-2} - k_{2-2}\\
&\quad \leq (k_{12-2} + k_{2-12} - k_{2-2} - k_{1-2} - k_{2-1})^+\\
&\quad + (k_{12-1} + k_{1-12} - k_{1-1} - k_{1-2} - k_{2-1})^+ + k_{1-2} + k_{2-1}\\
&2R_1' + R_2' \leq k_{2-2} \leq k_{12-2}\\
&\quad \leq k_{12-1} + k_{1-12} + (k_{12-2} + k_{2-12} - k_{2-2} - k_{1-2} - k_{2-1})^+\\
&R_1' + 2R_2' \leq 2k_{2-2} \leq k_{2-12} + k_{12-2}\\
&\leq k_{12-2} + k_{2-12} + (k_{12-1} + k_{1-12} - k_{1-1} - k_{1-2} - k_{2-1})^+ .
\end{aligned}
$$

By symmetry, any rate pair $(R_1', R_2')$ in $\mathcal{R}_2$ can also be shown to satisfy (51)-(59). Thus, the achievable rate region specified in (51)-(59) is strictly larger than the convex hull of $\mathcal{R}_1$ and $\mathcal{R}_2$. ■

Another achievable rate region for double-unicast networks has been obtained in [18], which is given by

$$
\begin{aligned}
R_1 &\leq k_{1-1} &(60)\\
R_2 &\leq k_{2-2} &(61)\\
R_1 + R_2 &\leq k_{12-1} + k_{2-12} - k_{2-1} &(62)\\
R_1 + R_2 &\leq k_{12-2} + k_{1-12} - k_{1-2} &(63)\\
R_1 + R_2 &\leq k_{1-12} + k_{2-12} - k_{1-2} - k_{2-1} &(64)\\
2R_1 + R_2 &\leq k_{12-12} + k_{1-12} - k_{1-2} - k_{2-1} &(65)\\
R_1 + 2R_2 &\leq k_{12-12} + k_{2-12} - k_{1-2} - k_{2-1}. &(66)
\end{aligned}
$$

**Proposition 2.** *The achievable rate region specified in* (51)-(59) *for double-unicast networks is strictly larger than that given in* (60)-(66).

*Proof:* Since the two regions are specified with similar set of inequalities and (60)-(63) are identical to (51)-(54), we only need to show that the right hand sides (RHS) of (55)-(59) are larger than the RHS of (64)-(66). The details are omitted for brevity. ■

In [19], another achievable rate region for double-unicast networks has been obtained by separately treating the low-interference case with $k_{1-2} + k_{2-1} \leq \min(k_{12-1}, k_{12-2})$ and the high-interference case with $k_{1-2} + k_{2-1} \geq \min(k_{12-1}, k_{12-2})$. The expression for the resulting rate region is complicated by the relationship among the min-cuts of the network. It can also be shown that the region given by (51)-(59) is strictly larger than that obtained in [19]. The details are omitted for brevity.

To provide a numerical example for the analytical rate comparisons given above, we consider the network shown in Fig. 2(a). It can be verified that the rate pair $(R_1, R_2) = (1, 2)$ is not within the achievable regions obtained in [16]–[19], but it is in the region given by (51)-(59).

## V. CONCLUSION

In this paper, we have proposed a linear precoding and decoding scheme for the source and sink nodes of double-unicast networks by assuming that RLNC is performed at each intermediate node. The main idea of the proposed scheme is rate splitting and the concatenated precoding composed of zero-forcing and random block-diagonal precoding matrices. Since the proposed scheme achieves the capacity region of the resulting linear deterministic interference channel, it gives the maximum achievable rate region for double-unicast network for any given set of coding coefficients used at the intermediate nodes. A suboptimal rate region given only in terms of the min-cuts of the network has also been obtained. Analytical comparisons have been provided to show the strict rate improvement over the existing schemes.

Since RLNC is in general a suboptimal strategy for the intermediate nodes, there exists a gap between the obtained optimal achievable rate region with RLNC and the capacity region of double-unicat networks. One possible future working direction is to find a better network coding strategy at the intermediate nodes by optimizing the transition matrices to further improve the achievable rate region.

## APPENDIX A
## A USEFUL LEMMA

**Lemma 6.** *[23] Let* $\mathbf{A} \in \mathbb{F}_q^{p \times l}$ *and* $\mathbf{B} \in \mathbb{F}_q^{l \times k}$*, then we have* $\text{rank}(\mathbf{AB}) = \text{rank}(\mathbf{A}) - \dim(\mathcal{R}(\mathbf{A}^T) \cap \mathcal{N}(\mathbf{B}^T)) = \text{rank}(\mathbf{B}) - \dim(\mathcal{N}(\mathbf{A}) \cap \mathcal{R}(\mathbf{B}))$.

*Proof:* Please refer to page 116 in [23]. ■

## APPENDIX B
## A USEFUL LEMMA RELATING ENTROPY AND MATRIX RANK

**Lemma 7.** *Let* $\mathbf{x}$ *be a random vector of dimension* $l \times 1$ *and* $\mathbf{A} \in \mathbb{F}_q^{p_1 \times l}$ *and* $\mathbf{B} \in \mathbb{F}_q^{p_2 \times l}$ *are two given matrices. We then have* $H(\mathbf{Ax}|\mathbf{Bx}) \leq \text{rank}\left(\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}\right) - \text{rank}(\mathbf{B})$*, where the equality holds when the entries of* $\mathbf{x}$ *are independently and uniformly chosen from the finite field* $\mathbb{F}_q$.

*Proof:* Lemma 7 can be proved by applying similar techniques as that in Section III of [26]. For notational convenience, denote by $r_A$, $r_B$ and $r_{AB}$ the ranks of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}$, respectively. Let $\mathbf{N}_1 \in \mathbb{F}_q^{l \times (l - r_{AB})}$ be a matrix whose columns form a basis for $\mathcal{N}\left(\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}\right)$. Then we can find a matrix $\mathbf{N}_2$ of size $l \times (r_{AB} - r_B)$ such that the columns of $\mathbf{N}_1$ and $\mathbf{N}_2$ form a basis for $\mathcal{N}(\mathbf{B})$. Moreover, let $\mathbf{N}_3 \in \mathbb{F}_q^{l \times r_B}$ be the basis of $\mathcal{R}(\mathbf{B}^T)$. Therefore, the columns in $\begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3\end{bmatrix}$ span the $l$-dimensional space. . As a consequence, any given $\mathbf{x} \in \mathbb{F}_q^{l \times 1}$ can be represented as $\mathbf{x} = \begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3\end{bmatrix}\mathbf{x}'$. Thus,

$$
\begin{aligned}
H(\mathbf{Ax}|\mathbf{Bx}) &= H\left(\mathbf{AN}_2\mathbf{x}_2' + \mathbf{AN}_3\mathbf{x}_3' \mid \mathbf{BN}_3\mathbf{x}_3'\right)\\
&\overset{(a)}{=} H\left(\mathbf{AN}_2\mathbf{x}_2' \mid \mathbf{BN}_3\mathbf{x}_3', \mathbf{x}_3'\right)\\
&\leq H(\mathbf{AN}_2\mathbf{x}_2') \leq \text{rank}(\mathbf{AN}_2)\\
&= \text{rank}(\mathbf{N}_2) = r_{AB} - r_B.
\end{aligned}
$$

where $(a)$ follows since $\mathbf{x}_3'$ can be uniquely determined from $\mathbf{BN}_3\mathbf{x}_3'$.

When all the entries of $\mathbf{x}$ are independently and uniformly distributed in $\mathbb{F}_q$, so are those in $\mathbf{x}'$. In this case, all the inequalities above become equalities.

This thus completes the proof of Lemma 7. ■

## APPENDIX C
## PROOF OF LEMMA 1

To prove Lemma 1, we need to show that $\mathrm{rank}(\mathbf{D}_{12}) = k_{2-1}$. First, according to Lemma 6, $\mathrm{rank}(\mathbf{U}_{11}^T\mathbf{H}_{12})$ can be computed as

$$\begin{aligned}\mathrm{rank}(\mathbf{U}_{11}^T\mathbf{H}_{12}) &= \mathrm{rank}(\mathbf{U}_{11}^T) - \dim(\mathcal{R}(\mathbf{U}_{11}) \cap \mathcal{N}(\mathbf{H}_{12}^T)) \\ &= \mathrm{rank}(\mathbf{U}_{11}) - \dim(\mathcal{R}(\mathbf{U}_{11}) \cap \mathcal{R}(\mathbf{U}_{10})) \\ &\overset{(a)}{=} \mathrm{rank}(\mathbf{U}_{11}) = k_{2-1}.\end{aligned}$$

where $(a)$ follows since $\mathbf{U}_{10}$ and $\mathbf{U}_{11}$ are complementary subspaces. Therefore, we have $\dim(\mathcal{N}(\mathbf{U}_{11}^T\mathbf{H}_{12})) = k_{2-12} - \mathrm{rank}(\mathbf{U}_{11}^T\mathbf{H}_{12}) = k_{2-12} - k_{2-1}$. According to Lemma 2.4.1 on page 94 of [23], we have $\mathcal{N}(\mathbf{H}_{\mathbf{12}}) \subseteq \mathcal{N}(\mathbf{U}_{11}^T\mathbf{H}_{12})$. Since $\dim(\mathcal{N}(\mathbf{H}_{12})) = \dim(\mathcal{N}(\mathbf{U}_{11}^T\mathbf{H}_{12})) = k_{2-12} - k_{2-1}$, we must have $\mathcal{N}(\mathbf{H}_{\mathbf{12}}) = \mathcal{N}(\mathbf{U}_{11}^T\mathbf{H}_{12})$. Hence, by applying Lemma 6, we have

$$\begin{aligned}\mathrm{rank}(\mathbf{D}_{12}) &= \mathrm{rank}(\mathbf{U}_{11}^T\mathbf{H}_{12}\mathbf{V}_{11}) \\ &= \mathrm{rank}(\mathbf{V}_{11}) - \dim(\mathcal{N}(\mathbf{U}_{11}^T\mathbf{H}_{12}) \cap \mathcal{R}(\mathbf{V}_{11})) \\ &= k_{2-1} - \dim(\mathcal{N}(\mathbf{H}_{12}) \cap \mathcal{R}(\mathbf{V}_{11})) \\ &= k_{2-1} - \dim(\mathcal{R}(\mathbf{V}_{10}) \cap \mathcal{R}(\mathbf{V}_{11})) \\ &= k_{2-1},\end{aligned}$$

where the last equality follows since $\mathcal{R}(\mathbf{V}_{10})$ and $\mathcal{R}(\mathbf{V}_{11})$ are complementary subspaces. This thus completes the proof of Lemma 1.

## APPENDIX D
## PROOF OF LEMMA 2

Since the matrices $\mathbf{U}_1, \mathbf{U}_2, \mathbf{V}_1$, and $\mathbf{V}_2$ are non-singular, the block-diagonal matrices $\begin{bmatrix}\mathbf{U}_1^T & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_2^T\end{bmatrix}$ and $\begin{bmatrix}\mathbf{V}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_1\end{bmatrix}$ are non-singular as well. With the fact that the matrix rank is unchanged after multiplying a non-singular matrix, we have

$$\begin{aligned}&\mathrm{rank}\begin{bmatrix}\mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{0}\end{bmatrix} \\ &= \mathrm{rank}\left(\begin{bmatrix}\mathbf{U}_1^T & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_2^T\end{bmatrix}\begin{bmatrix}\mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{V}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_1\end{bmatrix}\right) \\ &= \mathrm{rank}\begin{bmatrix}\mathbf{U}_1^T\mathbf{H}_{11}\mathbf{V}_2 & \mathbf{\Lambda}_1 \\ \mathbf{\Lambda}_2 & \mathbf{0}\end{bmatrix} \\ &= \mathrm{rank}\begin{bmatrix}\mathbf{U}_{11}^T\mathbf{H}_{11}\mathbf{V}_{21} & \mathbf{U}_{11}^T\mathbf{H}_{11}\mathbf{V}_{20} & \mathbf{D}_{12} & \mathbf{0} \\ \mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{21} & \mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{20} & \mathbf{0} & \mathbf{0} \\ \mathbf{D}_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} \\ &\overset{(a)}{=} \mathrm{rank}\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{D}_{12} \\ \mathbf{0} & \mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{20} & \mathbf{0} \\ \mathbf{D}_{21} & \mathbf{0} & \mathbf{0}\end{bmatrix} \\ &= \mathrm{rank}(\mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{20}) + k_{1-2} + k_{2-1},\end{aligned}$$

where (a) can be obtained by elementary row and column operations with the fact that $\mathbf{D}_{12}$ and $\mathbf{D}_{21}$ are non-singular square matrices. With similar techniques, $\mathrm{rank}\left(\begin{bmatrix}\mathbf{H}_{21} & \mathbf{H}_{22} \\ \mathbf{0} & \mathbf{H}_{12}\end{bmatrix}\right)$ can also be obtained.

## APPENDIX E
## PROOF OF LEMMA 4

To prove Lemma 4, we need to first show the following results.

**Fact 1.** *Let* $\mathbf{G} \in \mathbb{F}_q^{l\times u}$ *be a given matrix with* $\mathrm{rank}(\mathbf{G}) = r$, *and* $\mathbf{B} \in \mathbb{F}_q^{k\times l}$ *be a random matrix whose entries are uniformly generated from* $\mathbb{F}_q$. *Then, we have* $\mathrm{rank}(\mathbf{BG}) = \min(k, r)$ *with high probability as* $q \to \infty$.

*Proof:* Fact 1 can be shown by separately considering the two cases with $k \le r$ and $k > r$.

**Case I:** $k \le r$. In this case, we can construct a submatrix $\mathbf{G}^s \in \mathbb{F}_q^{l\times k}$ from $\mathbf{G}$ by selecting its $k$ independent columns. Then we have

$$\mathrm{rank}(\mathbf{BG}) \ge \mathrm{rank}(\mathbf{BG}^s) = k, \text{as } q \to \infty, \tag{67}$$

where the last equality can be shown by considering the determinant of the square matrix $\mathbf{BG}^s \in \mathbb{F}_q^{k\times k}$. Specifically, as $\mathbf{B}$ is randomly generated, the determinant of $\mathbf{BG}^s$ is a polynomial on the entries of $\mathbf{B}$. If this polynomial is not identically zero, then with high probability as $q \to \infty$, the determinant of $\mathbf{BG}^s$ is nonzero, and hence $\mathrm{rank}(\mathbf{BG}^s) = k$. To show that the polynomial is not identically zero, it is sufficient to choose one particular matrix for $\mathbf{B}$, e.g., $\mathbf{B} = (\mathbf{G}^s)^T$, so that $\mathrm{rank}(\mathbf{BG}^s) = \mathrm{rank}((\mathbf{G}^s)^T\mathbf{G}^s) = k$. Therefore, the determinant of $\mathbf{BG}^s$ is not identically zero and we have $\mathrm{rank}(\mathbf{BG}^s) = k$ with high probability as $q \to \infty$.

Besides, by considering the matrix dimensions of $\mathbf{BG}$, we obviously have $\mathrm{rank}(\mathbf{BG}) \le k$. Together with (67), we have the desired result: $\mathrm{rank}(\mathbf{BG}) = k = \min\{k, r\}$ for $k \le r$.

**Case 2:** $k > r$**.** In this case, $\mathbf{B}$ can be decomposed as $\mathbf{B} = \begin{bmatrix}\mathbf{B}_1 \\ \mathbf{B}_2\end{bmatrix}$, where $\mathbf{B}_1 \in \mathbb{F}_q^{r\times l}$ and $\mathbf{B}_2 \in \mathbb{F}_q^{(k-r)\times l}$. Following similar arguments as those in Case 1, we can show that $\mathrm{rank}(\mathbf{BG}) \ge \mathrm{rank}(\mathbf{B}_1\mathbf{G}) = r$. Together with the fact that $\mathrm{rank}(\mathbf{BG}) \le \mathrm{rank}(\mathbf{G}) = r$, we have $\mathrm{rank}(\mathbf{BG}) = \min\{k, r\}$ for $k > r$ as well.

This completes the proof of Fact 1. ■

**Fact 2.** *Let* $\mathbf{G}_1 \in \mathbb{F}_q^{l\times u_1}$ *and* $\mathbf{G}_2 \in \mathbb{F}_q^{l\times u_2}$ *be two given matrices with* $\mathrm{rank}(\mathbf{G}_1) = r_1$, $\mathrm{rank}(\mathbf{G}_2) = r_2$ *and* $\mathrm{rank}\left(\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\end{bmatrix}\right) = r_{12}$. *Let* $\mathbf{B} \in \mathbb{F}_q^{u_1\times k}, k \le r_1$, *be a random matrix whose entries are uniformly generated from* $\mathbb{F}_q$. *Then we have* $\mathrm{rank}\left(\begin{bmatrix}\mathbf{G}_1\mathbf{B} & \mathbf{G}_2\end{bmatrix}\right) = \min\{r_{12}, k + r_2\}$ *with high probability as* $q \to \infty$.

*Proof:* Fact 2 can be shown by separately considering the two cases with $k \le r_{12} - r_2$ and $k > r_{12} - r_2$.

**Case 1:** $k \le r_{12} - r_2$. Since $\mathrm{rank}\left(\begin{bmatrix}\mathbf{G}_1\mathbf{B} & \mathbf{G}_2\end{bmatrix}\right) \le \mathrm{rank}(\mathbf{G}_1\mathbf{B}) + \mathrm{rank}(\mathbf{G_2}) = k + r_2$, we only need to show that $\mathrm{rank}\left(\begin{bmatrix}\mathbf{G}_1\mathbf{B} & \mathbf{G}_2\end{bmatrix}\right) \ge k + r_2$. To this end, let $\mathbf{G}_2^s$ be a submatrix of $\mathbf{G}_2$ consisting of its $r_2$ independent columns. Since $\mathrm{rank}(\mathbf{G}_2) = r_2$, the columns in $\mathbf{G}_2^s$ span the same subspace as those in $\mathbf{G}_2$. Thus, we have $\mathrm{rank}\left(\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2^s\end{bmatrix}\right) = \mathrm{rank}\left(\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\end{bmatrix}\right) = r_{12}$. Let $\mathbf{U}$ be a random matrix uniformly generated from $\mathbb{F}_q^{(k+r_2)\times l}$. According to Fact 1, we have

$$\mathrm{rank}\left(\mathbf{U}\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2^s\end{bmatrix}\right) = \min\{k + r_2, r_{12}\} = k + r_2. \tag{68}$$

Let $\tilde{\mathbf{G}}_1 = \mathbf{U}\mathbf{G}_1$ and $\tilde{\mathbf{G}}_2^s = \mathbf{U}\mathbf{G}_2^s$. Then we have

$$\begin{aligned} &\operatorname{rank}\left(\begin{bmatrix}\mathbf{G}_1\mathbf{B} & \mathbf{G}_2\end{bmatrix}\right) \geq \operatorname{rank}\left(\begin{bmatrix}\mathbf{G}_1\mathbf{B} & \mathbf{G}_2^s\end{bmatrix}\right) \\ &\geq \operatorname{rank}\left(\mathbf{U}\begin{bmatrix}\mathbf{G}_1\mathbf{B} & \mathbf{G}_2^s\end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix}\tilde{\mathbf{G}}_1\mathbf{B} & \tilde{\mathbf{G}}_2^s\end{bmatrix}\right) = k + r_2, \end{aligned} \tag{69}$$

where the last equality can be shown by verifying that the determinant of the square matrix $\begin{bmatrix}\tilde{\mathbf{G}}_1\mathbf{B} & \tilde{\mathbf{G}}_2^s\end{bmatrix}$ as a polynomial of entries in $\mathbf{B}$ is not identically zero. To this end, we can choose one particular $\mathbf{B}$ such that $\operatorname{rank}\left(\begin{bmatrix}\tilde{\mathbf{G}}_1\mathbf{B} & \tilde{\mathbf{G}}_2^s\end{bmatrix}\right) = k + r_2$. According to (68), there are at least $k$ independent columns in $\tilde{\mathbf{G}}_1$ that are not in the subspace spanned by $\tilde{\mathbf{G}}_2^s$. Denote the indices of these columns by $i_1, i_2, ..., i_k$. If $\mathbf{B}$ is chosen such that the $j$th column is a unit vector with non-zero value only at the $i_j$th entry, $j = 1, ..., k$, then $\tilde{\mathbf{G}}_1\mathbf{B}$ is a submatrix of $\tilde{\mathbf{G}}_1$ with columns not in the subspace spanned by $\tilde{\mathbf{G}}_2^s$; hence $\begin{bmatrix}\tilde{\mathbf{G}}_1\mathbf{B} & \tilde{\mathbf{G}}_2^s\end{bmatrix}$ is a full-rank matrix. Therefore, we can conclude that the determinant of $\begin{bmatrix}\tilde{\mathbf{G}}_1\mathbf{B} & \tilde{\mathbf{G}}_2^s\end{bmatrix}$ is not an identically zero polynomial and $\operatorname{rank}\left(\begin{bmatrix}\tilde{\mathbf{G}}_1\mathbf{B} & \tilde{\mathbf{G}}_2^s\end{bmatrix}\right) = k + r_2$ with high probability as $q \to \infty$.

**Case 2:** $k > r_{12} - r_2$**.** Since $\operatorname{rank}\left(\begin{bmatrix}\mathbf{G}_1\mathbf{B} & \mathbf{G}_2\end{bmatrix}\right) \leq \operatorname{rank}\left(\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\end{bmatrix}\right) = r_{12}$, we only need to show that $\operatorname{rank}\left(\begin{bmatrix}\mathbf{G}_1\mathbf{B} & \mathbf{G}_2\end{bmatrix}\right) \geq r_{12}$. Let $\mathbf{V} \in \mathbb{F}_q^{(r_{12} \times l)}$ be a random matrix with entries uniformly generated from $\mathbb{F}_q$, and let $\bar{\mathbf{G}}_1 = \mathbf{V}\mathbf{G}_1$ and $\bar{\mathbf{G}}_2 = \mathbf{V}\mathbf{G}_2$. According to Fact 1, we have $\operatorname{rank}(\bar{\mathbf{G}}_1) = r_1$, $\operatorname{rank}(\bar{\mathbf{G}}_2) = r_2$, and $\operatorname{rank}\left(\begin{bmatrix}\bar{\mathbf{G}}_1 & \bar{\mathbf{G}}_2\end{bmatrix}\right) = r_{12}$. Therefore, we can construct a matrix $\mathbf{G}_2^{s_1}$ by choosing $(r_{12} - r_1)$ independent columns from $\bar{\mathbf{G}}_2$ that are not in the subspace spanned by $\bar{\mathbf{G}}_1$. Furthermore, let $\mathbf{G}_2^{s_2}$ be a submatrix of $\bar{\mathbf{G}}_2$ consisting of $(r_1 - k)$ columns that are not in the subspace spanned by $\mathbf{G}_2^{s_1}$. Such a choice is feasible since $(r_{12} - r_1) + (r_1 - k) = r_{12} - k \leq r_2$. Then, it follows that $\operatorname{rank}\left(\begin{bmatrix}\bar{\mathbf{G}}_1 & \bar{\mathbf{G}}_2^{s_1} & \bar{\mathbf{G}}_2^{s_2}\end{bmatrix}\right) = r_{12}$. Following similar arguments as those for case 1, we can show that $\begin{bmatrix}\bar{\mathbf{G}}_1\mathbf{B} & \bar{\mathbf{G}}_2^{s_1} & \bar{\mathbf{G}}_2^{s_2}\end{bmatrix}$ is a full-rank square matrix with high probability as $q \to \infty$. Hence, we have the desired result:

$$\begin{aligned} &\operatorname{rank}\left(\begin{bmatrix}\mathbf{G}_1\mathbf{B} & \mathbf{G}_2\end{bmatrix}\right) \geq \operatorname{rank}\left(\begin{bmatrix}\bar{\mathbf{G}}_1\mathbf{B} & \bar{\mathbf{G}}_2\end{bmatrix}\right) \\ &\geq \operatorname{rank}\left(\begin{bmatrix}\bar{\mathbf{G}}_1\mathbf{B} & \bar{\mathbf{G}}_2^{s_1} & \bar{\mathbf{G}}_2^{s_2}\end{bmatrix}\right) = r_{12}. \end{aligned} \tag{70}$$

This completes the proof of Fact 2. ■

Now, we can prove Lemma 4 by recursively applying Fact 2 as follows:

$$\begin{aligned} \gamma_1 &\triangleq \operatorname{rank}\left(\begin{bmatrix}\mathbf{A}_1\mathbf{E}_1 & \mathbf{A}_2 & \mathbf{A}_3\end{bmatrix}\right) \\ &= \min\left\{\operatorname{rank}([\mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_3]), k_1 + \operatorname{rank}([\mathbf{A}_2 \quad \mathbf{A}_3])\right\}. \end{aligned} \tag{71}$$

$$\begin{aligned} \gamma_2 &\triangleq \operatorname{rank}\left(\begin{bmatrix}\mathbf{A}_1\mathbf{E}_1 & \mathbf{A}_2\mathbf{E}_2 & \mathbf{A}_3\end{bmatrix}\right) \\ &= \min\left\{\gamma_1, k_2 + \operatorname{rank}([\mathbf{A}_1\mathbf{E}_1 \quad \mathbf{A}_3])\right\} \\ &= \min\left\{\gamma_1, k_2 + \operatorname{rank}([\mathbf{A}_1 \quad \mathbf{A}_3], k_1 + k_2 + \operatorname{rank}(\mathbf{A}_3))\right\} \end{aligned} \tag{72}$$

$$\begin{aligned} \gamma_3 &\triangleq \operatorname{rank}([\mathbf{A}_1\mathbf{E}_1 \quad \mathbf{A}_2\mathbf{E}_2 \quad \mathbf{A}_3\mathbf{E}_3]) \\ &= \min\left\{\gamma_2, k_3 + \operatorname{rank}([\mathbf{A}_1\mathbf{E}_1 \quad \mathbf{A}_2\mathbf{E}_2])\right\} \\ &= \min\left\{\gamma_2, k_3 + \operatorname{rank}([\mathbf{A}_1\mathbf{E}_1 \quad \mathbf{A}_2]), k_2 + k_3 + \operatorname{rank}([\mathbf{A}_1\mathbf{E}_1])\right\} \\ &= \min\left\{\begin{array}{l}\gamma_2, k_3 + \operatorname{rank}([\mathbf{A}_1 \quad \mathbf{A}_2]), k_1 + k_3 + \operatorname{rank}(\mathbf{A}_2), \\ k_2 + k_3 + \operatorname{rank}(\mathbf{A}_1), k_1 + k_2 + k_3\end{array}\right\} \\ &= \min\left\{\begin{array}{l}\operatorname{rank}([\mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_3]), k_1 + \operatorname{rank}([\mathbf{A}_2 \quad \mathbf{A}_3]), \\ k_2 + \operatorname{rank}([\mathbf{A}_1 \quad \mathbf{A}_3]), k_3 + \operatorname{rank}([\mathbf{A}_1 \quad \mathbf{A}_2]), \\ k_1 + k_2 + \operatorname{rank}(\mathbf{A}_3), k_1 + k_3 + \operatorname{rank}(\mathbf{A}_2), \\ k_2 + k_3 + \operatorname{rank}(\mathbf{A}_1), k_1 + k_2 + k_3\end{array}\right\} \\ &= k_1 + k_2 + k_3, \end{aligned} \tag{73}$$

where the last equality in (73) follows from the conditions specified in Lemma 4. This completes the proof of Lemma 4.

## APPENDIX F
## PROOF OF LEMMA 5

With Frobenius inequality, we have

$$\begin{aligned} &\operatorname{rank}(\mathbf{U}_{10}^T\mathbf{H}_{11}\mathbf{V}_{20}) \\ &\geq \left(\operatorname{rank}\left(\mathbf{U}_{10}^T\mathbf{H}_{11}\right) + \operatorname{rank}\left(\mathbf{H}_{11}\mathbf{V}_{20}\right) - \operatorname{rank}\left(\mathbf{H}_{11}\right)\right)^+ \\ &\overset{(a)}{=} (k_{12-1} - k_{2-1} + k_{1-12} - k_{1-2} - k_{1-1})^+, \end{aligned}$$

where $(a)$ follows from Lemma 6. Similarly arguments apply for $\operatorname{rank}\left(\mathbf{U}_{20}^T\mathbf{H}_{22}\mathbf{V}_{10}\right)$.

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