A MAP-modulated fluid flow model with multiple vacations

Jung Woo Baek · Ho Woo Lee · Se Won Lee ·
Soohan Ahn

Abstract We consider a MAP-modulated fluid flow queueing model with multiple vacations. As soon as the fluid level reaches zero, the server leaves for repeated vacations of random length V until the server finds any fluid in the system. During the vacation period, fluid arrives from outside according to the MAP (Markovian Arrival Process) and the fluid level increases vertically at the arrival instance. We first derive the vector Laplace–Stieltjes transform (LST) of the fluid level at an arbitrary point of time in steady-state and show that the vector LST is decomposed into two parts, one of which the vector LST of the fluid level at an arbitrary point of time during the idle period. Then we present a recursive moments formula and numerical examples.

Keywords Markov-modulated fluid flows · Markovian arrival process · Multiple vacations · Server control policy

---

This paper was prepared and submitted when Jung Woo Baek was a postdoc researcher at Research Inst. of Information and Communication, Sungkyunkwan University, Suwon, Korea and Se Won Lee was a BK-21 postdoc researcher at Dept. of Industrial Engineering, Sungkyunkwan University, Suwon, Korea.

J.W. Baek
Department of Electrical and Computer Engineering, University of Manitoba, Winnipeg, Canada
e-mail: rainbeak@gmail.com

H.W. Lee (✉)
Department of Systems Management Engineering, Sungkyunkwan University, Suwon, Korea
e-mail: hwlee@skku.edu

S.W. Lee
Department of Business Administration, Dongguk University Seoul, Seoul, Korea
e-mail: swlee94@dongguk.edu

S. Ahn
Department of Statistics, The University of Seoul, Seoul, Korea
e-mail: sahn@uos.ac.kr

1 Introduction

We consider a MAP-modulated fluid flow system in which the server leaves for repeated vacations as soon as the fluid level reaches zero. This is an extension of the conventional Markov-modulated fluid flow system to control the server's idle state. We assume that the background process follows the MAP (Markovian Arrival Process) (Lucantoni et al. 1990) with parameters $\boldsymbol{C}$ and $\boldsymbol{D}$.

The queueing systems in which the fluid (workload) increases or decreases linearly according to the phases of the background Markov process have been studied under the name of Markov-modulated fluid flow (MMFF) models. The conventional MMFF model was introduced by Anick et al. (1982) to analyze a data handling system with multi-input sources. More details about the conventional MMFF models can be found in Ahn (2009), Ahn and Ramaswami (2003, 2004, 2005), Ahn et al. (2005), Asmussen (1995) and Bean et al. (2005). Various applications to the real world systems can be found in Aggarwal et al. (2005), Badescu and Landriault (2009), Badescu et al. (2007), Kulkarni and Yan (2007), Mitra (1988), Takada (2001), Tan and Gershwin (2007), Tzenova et al. (2005), Yan and Kulkarni (2008) and Yeralan et al. (1986). Analysis of the MAP-modulated model with D policy can be found in Baek et al. (2011).

The conventional MMFF model has been successfully applied to various real world systems such as telecommunication systems and production/inventory systems. However, wider applications were limited due to the assumption that the system (server) has to begin to process the fluid as soon as the fluid level turns to be positive. This limited range of applications due to the lack of server control can be found, for example, in the computer and communication systems which usually require a maintenance period whenever there is no workload to be processed. Similar example can be found in a production system in which the server needs to delay the production until a certain amount of raw material is accumulated. From the engineering point of view, delaying the production is meaningful when the setup cost is very high. By delaying the production the production cycle becomes larger and it results in a low average setup cost per unit time.

In this paper, we consider a new modification of the conventional MMFF model such that the system has a vacation period whenever the fluid level reaches zero. During the vacation period, no service is rendered by the server and the fluid level only increases by the inflow of the fluid. Fluid models driven by a background server vacation queues were studied by Mao et al. (2010a, 2010b, 2010c) in which the rate of change of the fluid levels is controlled by the background vacation queueing systems. But their models could not be regarded as the server vacation MMFF models in the true sense.

In a queueing context, our model is analogous to the server vacation queueing systems. A server vacation queueing system is in a wide sense a queueing system in which the server is idle even though there are customers (or workload) waiting for service in the system. More details about vacation queues can be found in Doshi (1986), Heyman (1977), Lee and Srinivasan (1989) and Levy and Yechiali (1975). Studies on the MAP(BMAP)/G/1 queue with a server vacation model can be found in Baek et al. (2008), Chang et al. (2002), Lee and Baek (2005) and Lee et al. (2001).

It is known that the vector Laplace Stieltjes transform (LST) of the queue length and the workload distributions for many of the MAP(BMAP)/G/1 vacation queues is factored into two parts (Baek et al. 2008; Chang et al. 2002 and Lee et al. 2001) one of which is the LST of the queue length or the workload at an arbitrary time during the idle period. Readers will see that the vector LST of the fluid level of our system is also factored into a similar form.

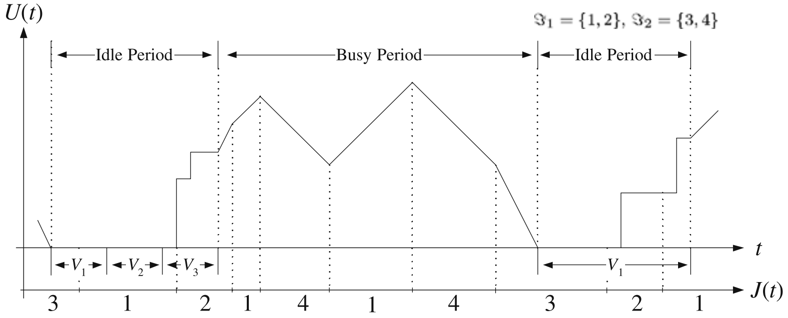


Fig. 1 MAP modulated fluid flow system with multiple vacations

2 The system and the model

2.1 The system

In this paper, we consider a MAP-modulated fluid flow queueing system with multiple vacations with following specifications (readers are advised to see Fig. 1 to understand the system behavior):

(1) The background process follows a Markovian Arrival Process (MAP) with parameter matrices $\boldsymbol{C}$ and $\boldsymbol{D}$. Thus, the background process is a continuous-time Markov chain with infinitesimal generator $\boldsymbol{Q} = \boldsymbol{C} + \boldsymbol{D}$. We will call this background process the underlying Markov chain (UMC). We note that this UMC process determines the arrival of customers during the idle (vacation) period and the input/output rates of fluid during the busy period.

(2) During the busy period (processing period), the rate of change of the fluid level is r_i if the UMC is in phase i. If $r_i > 0$, the fluid level increases linearly. If $r_i < 0$, the fluid level decreases linearly. Thus we have two sets $\{\Im_1, \Im_2\}$ of phases where $\Im_1$ is the set of the UMC phases with increasing rates and $\Im_2$ is the set of the UMC phases with decreasing rates.

(3) As soon as the fluid level reaches zero, the server leaves for repeated vacations of iid random length $\{V_1, V_2, \ldots\}$ with DF $V(x)$ (vacation period or idle period) independently of the UMC process. If there exists any fluid at the end of a vacation, the busy period starts immediately. If not, the server leaves for an another vacation.

(4) During the vacation period, the server does not process the fluid. Customers arrive in the system during the idle period according to the common MAP($\boldsymbol{C}, \boldsymbol{D}$) of the background process, and the fluid level jumps up at the arrival instances. The jump size follows a non-negative random amount S with a general distribution function $S(x)$ $(x > 0)$. We assume finite mean variance of S.

(5) S, V_i and UMC (background) process are independent.

Figure 1 depicts our system. For the system, we say that the system idle if it is in the vacation period. Thus an idle period is the length that consists of the successive vacations. By a cycle we mean an interval between two successive busy period termination points. A cycle consists of an idle period and a busy period. We note that the busy period of our system behaves identically to the conventional MMFF system with the fluid level at the end of the vacation (idle) period.

The examples of discrete increase during the idle period and continuous increase during the busy period can be found in many real-world systems. Chemical processing systems and water purification systems are good examples.

2.2 Preliminaries

In this section, we review the conventional MMFF system and important theoretical results for later use. Results in this section can be found in Ahn and Ramaswami (2004, 2005).

The conventional MMFF model is a two dimensional stochastic process $\{U(t), J(t), t\}$ where $U(t)$ is the fluid level at time t and $J(t)$ is the phase of the UMC at time t. For the conventional MMFF model, let us divide the UMC phases into two sets $\{\Im_1, \Im_2\}$ where $\Im_1$ is the set of UMC phases with increasing rates and $\Im_2$ is the set of UMC phases with decreasing rates. According to the set of phases that belong to $\Im_1$ and $\Im_2$, the infinitesimal generator can be expressed as

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix},$$

which is partitioned according to $\Im_i$ $(i = 1, 2)$.

During the sojourn time the UMC is in $i \in \Im_1$, the fluid level increases with rate (slope) $r_i > 0$ and decreases with slope $r_i < 0$ during $i \in \Im_2$.

Let us define $\boldsymbol{\Gamma}$ as the diagonal matrix of $\gamma_i = |r_i|$, and $\boldsymbol{P}$ as follows,

$$\boldsymbol{P} = \frac{\boldsymbol{\Gamma}^{-1}\boldsymbol{Q}}{\lambda} + \boldsymbol{I}, \tag{2.1}$$

where λ is a positive number with

$$\lambda \ge \max_{i \in \Im} \left[-\boldsymbol{\Gamma}^{-1}\boldsymbol{Q}\right]_{ii}.$$

We partition $\boldsymbol{P}$ and $\boldsymbol{\Gamma}$ according to $\Im_1$ and $\Im_2$ as

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{pmatrix}, \qquad \boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{\Gamma}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Gamma}_2 \end{pmatrix}.$$

$\boldsymbol{P}$ is used to compute the performances by using the uniformization technique.

In the analysis of MMFF related systems, the first passage times play important roles. Let $\tau = \inf\{t > 0, U(t) = 0\}$ be the first passage time to level 0. Let us define the following Laplace–Stieltjes transforms (LST),

$$\begin{aligned} \left[\boldsymbol{\Psi}^*(\theta)\right]_{ij} &= E\left[e^{-\theta\tau}, J(\tau) = j | U(0) = 0, J(0) = i\right] \quad (i \in \Im_1, j \in \Im_2), \\ \left[\boldsymbol{G}^*_{12}(\theta|x)\right]_{ij} &= E\left[e^{-\theta\tau}, J(\tau) = j | U(0) = x, J(0) = i\right] \quad (i \in \Im_1, j \in \Im_2), \\ \left[\boldsymbol{G}^*_{22}(\theta|x)\right]_{ij} &= E\left[e^{-\theta\tau}, J(\tau) = j | U(0) = x, J(0) = i\right] \quad (i \in \Im_2, j \in \Im_2). \end{aligned}$$

$\boldsymbol{\Psi}^*(\theta)$ represents the length of the busy period of the simple MMFF system. $\boldsymbol{G}^*_{12}(\theta|x)$ represents the first depletion time of the simple MMFF that starts with level x and an increasing slope. $\boldsymbol{G}^*_{22}(\theta|x)$ represents the first depletion time of the simple MMFF that starts with level x and a decreasing slope. Then, we have

$$\boldsymbol{\Psi}^*(\theta) = \left[\left(\boldsymbol{P}_{11} - \frac{\theta}{\lambda}\boldsymbol{\Gamma}_1^{-1}\right)\boldsymbol{\Psi}^*(\theta) + \boldsymbol{P}_{12}\right]\left[\boldsymbol{I} - \frac{\boldsymbol{H}^*(\theta)}{\lambda}\right]^{-1}, \tag{2.2}$$

$$\boldsymbol{G}^*_{12}(\theta|x) = \boldsymbol{\Psi}^*(\theta)\boldsymbol{G}^*_{22}(\theta|x) \quad (x > 0), \tag{2.3}$$

and

$$\boldsymbol{G}_{22}^{*}(\theta|x) = e^{\boldsymbol{H}^{*}(\theta)x} \quad (x > 0), \tag{2.4}$$

in which

$$\boldsymbol{H}^{*}(\theta) = \boldsymbol{\Gamma}_{2}^{-1}\big[\boldsymbol{Q}_{22} - \theta \boldsymbol{I} + \boldsymbol{Q}_{21}\boldsymbol{\Psi}^{*}(\theta)\big]. \tag{2.5}$$

3 Analysis of the system

In this section, we analyze our system. Let us define the following notation and probabilities:

- r_i: rate of the change of the fluid level when UMC phase is i,
- $\gamma_i = |r_i|$,
- $\Im_1$: set of UMC phases with $r_i > 0$,
- $\Im_2$: set of UMC phases with $r_i < 0$,
- $\Im = \Im_1 \cup \Im_2$,
- m_i: number of UMC phases in $\Im_i$ $(i = 1, 2)$,
- m: number of UMC phases $(= m_1 + m_2)$,
- $V(x)$, $v(x)$: distribution function (DF) and probability density function (pdf) of V,
- $V^{*}(\theta)$: LST of $V(x)$,
- S : amount of fluid brought in by an arrival during the vacation period (random variable),
- $S(x)$, $s(x)$: DF and pdf of S,
- $S^{*}(\theta)$: LST of $S(x)$,
- $\boldsymbol{R}_i$: diagonal matrix of rates in $\Im_i$ $(i = 1, 2)$,
- $\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_2 \end{pmatrix}$, $\boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{\Gamma}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Gamma}_2 \end{pmatrix}$,
- $\pi_i = \lim_{t\to\infty} \Pr[J(t) = i]$ $(1 \le i \le m)$,
- $\boldsymbol{\pi} = \{\pi_1, \pi_2, \ldots, \pi_m\}$: steady-state phase probability vector of the UMC process,
- $\xi(t) = \begin{cases} 0, & \text{system is idle at time } t, \\ 1, & \text{system is busy at time } t, \end{cases}$
- $\mathbf{e}$: column vector of 1's.

Let us define the following probabilities,

$$U_{idle,i}(x,t) = \Pr\big[U(t) \le x,\, J(t) = i, \xi(t) = 0\big] \quad (x \ge 0),$$
$$U_{busy,i}(x,t) = \Pr\big[U(t) \le x,\, J(t) = i, \xi(t) = 1\big] \quad (x > 0),$$

and steady-state quantities as

$$U_{idle,i}(x) = \lim_{t\to\infty} U_{idle,i}(x,t),$$
$$U_{busy,i}(x) = \lim_{t\to\infty} U_{busy,i}(x,t).$$

Let us define the vectors and vector LSTs as follows:

$$\boldsymbol{U}_{idle}(x) = \big\{U_{idle,1}(x), U_{idle,2}(x), \ldots, U_{idle,m}(x)\big\},$$
$$\boldsymbol{U}_{busy}(x) = \big\{U_{busy,1}(x), U_{busy,2}(x), \ldots, U_{busy,m}(x)\big\},$$
$$\boldsymbol{u}_{idle}^{*}(\theta) = \int_0^{\infty} e^{-\theta x} d\boldsymbol{U}_{idle}(x), \qquad \boldsymbol{u}_{busy}^{*}(\theta) = \int_0^{\infty} e^{-\theta x} d\boldsymbol{U}_{busy}(x).$$

Then, the vector LST $\boldsymbol{u}^{*}(\theta)$ of the fluid level at an arbitrary time in steady state can be obtained from

$$\boldsymbol{u}^{*}(\theta) = \boldsymbol{u}_{idle}^{*}(\theta) + \boldsymbol{u}_{busy}^{*}(\theta). \tag{3.1}$$

We note that the system becomes stable if and only if the average increase of the fluid is less than zero, or, equivalently, average outflow rate is higher than the average inflow rate. We thus have the following stability condition,

$$\boldsymbol{\pi}_{idle}\boldsymbol{D}\mathbf{e}E(S) + \boldsymbol{\pi}_{busy}\boldsymbol{R}\mathbf{e} < 0$$

where, $\boldsymbol{\pi}_{idle}$ and $\boldsymbol{\pi}_{busy}$ are steady-state phase probability vectors of the UMC process during the idle period and the busy period respectively. It is easy to confirm that above stability condition is reduced to $\boldsymbol{\pi}\boldsymbol{R}\mathbf{e} < 0$ for the ordinary MMFF system.

3.1 Analysis of the idle period

In this section, we derive the vector LST $\boldsymbol{u}^*_{idle}(\theta)$ of the fluid level at an arbitrary time during the idle period. Let us define the following probability.

$\left[\tilde{\boldsymbol{V}}_n(x)\right]_{ij}$ = Pr(n jumps (arrivals) occur during a vacation, the length of the vacation is less than or equal to x and the UMC phase at the end of the vacation is j under the condition that the vacation starts with UMC phase i) ($n \geq 0, x \geq 0$).

Let us define the following matrix transforms,

$$\boldsymbol{V}^*_n(\theta) = \int_0^\infty e^{-\theta x}\, d\tilde{\boldsymbol{V}}_n(x), \tag{3.2}$$

$$\boldsymbol{V}^*(z,\theta) = \sum_{n=0}^\infty \boldsymbol{V}^*_n(\theta) z^n = \int_0^\infty e^{-[\theta \boldsymbol{I} - (\boldsymbol{C}+\boldsymbol{D}z)]x}\, dV(x), \tag{3.3}$$

$$\boldsymbol{V}(z) = \boldsymbol{V}^*(z,\theta)|_{\theta=0} = \sum_{n=0}^\infty \boldsymbol{V}_n z^n = \int_0^\infty e^{(\boldsymbol{C}+\boldsymbol{D}z)x}\, dV(x), \tag{3.4}$$

where $\tilde{\boldsymbol{V}}_n(x)$ is the matrix of $[\tilde{\boldsymbol{V}}_n(x)]_{ij}$. Then we have the following theorem.

Theorem 3.1 *Let $\boldsymbol{\kappa}$ be the stationary probability vector of the UMC phase at the start of a cycle (i.e., at the start of an idle period). We then have*

$$\boldsymbol{u}^*_{idle}(\theta) = (1-\rho)\left.\frac{\boldsymbol{\kappa}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}[\boldsymbol{V}(z)-\boldsymbol{I}](\boldsymbol{C}+\boldsymbol{D}z)^{-1}}{E(V)\boldsymbol{\kappa}[\boldsymbol{I}-\boldsymbol{V}_0]^{-1}\mathbf{e}}\right|_{z=S^*(\theta)} \tag{3.5}$$

where, $\boldsymbol{V}_0 = \boldsymbol{V}(z)|_{z=0}$ is the probability that no jump occurs during a vacation and ρ is the probability that the server is busy at an arbitrary time.

Proof We note that the fluid level process during the idle period is identical to the workload process of the MAP/G/1 queue with multiple vacations. Let $\boldsymbol{p}_{idle}(z)$ be the vector generating function (GF) of the number of the customers (jumps) at an arbitrary idle time point. Using $N = 1$ in equation (3.49) of Lee et al. (2001), we get

$$\boldsymbol{p}_{idle}(z) = \frac{\boldsymbol{\kappa}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}[\boldsymbol{V}(z)-\boldsymbol{I}](\boldsymbol{C}+\boldsymbol{D}z)^{-1}}{E(V)\boldsymbol{\kappa}[\boldsymbol{I}-\boldsymbol{V}_0]^{-1}\mathbf{e}}.$$

Since each arriving customer brings in amount S of fluid into the system, using $S^*(\theta)$ in place of z after multiplying by $(1-\rho)$ proves the theorem. □

$\boldsymbol{\kappa}$ and ρ will be obtained later.

3.2 Analysis of the busy period

To derive the vector LST $\boldsymbol{u}^*_{busy}(\theta)$ of the fluid level at an arbitrary time during the busy period, we need to derive the fluid level distribution at the start of the busy period.

Let us define the following probability,

$U_{B,ij}(x) =$ Pr(at the start of a busy period, the fluid level is less than or equal to x and the UMC phase is j under the condition that the UMC phase is i at the start of the idle period) $\quad (x > 0)$.

Let $\boldsymbol{U}_B(x)$ be the matrix of $U_{B,ij}(x)$ and $\boldsymbol{U}^*_B(\theta) = \int_0^\infty e^{-\theta x}\, d\boldsymbol{U}_B(x)$ be its LST. From Lucantoni et al. (1990), we have

$$\boldsymbol{U}_B(x) = (\boldsymbol{I} - \boldsymbol{V}_0)^{-1} \left[\sum_{n=1}^{\infty} \boldsymbol{V}_n S^{(n)}(x) \right] \quad (x \geq 0), \tag{3.6}$$

$$\boldsymbol{U}^*_B(\theta) = (\boldsymbol{I} - \boldsymbol{V}_0)^{-1} \big[\boldsymbol{V}(z) - \boldsymbol{V}_0 \big]_{z=S^*(\theta)}, \tag{3.7}$$

where, $\boldsymbol{V}_n = \boldsymbol{V}^*_n(\theta)|_{\theta=0}$ and $S^{(n)}$ is the DF of the n-fold convolution of S with itself.

To derive $\boldsymbol{u}^*_{busy}(\theta)$, we first set up the system equations that represent the level changes during the infinitesimal time Δt. Let q_{ij} be the (i, j)-element of the infinitesimal generator $\boldsymbol{Q}$. Then we have the following infinitesimal system equation,

$$\begin{aligned} U_{busy,i}(x, t + \Delta t) = U_{busy,i}(x - r_i \Delta t, t)(1 + q_{ii} \Delta t) \\ + \sum_{\substack{j=1 \\ (j \neq i)}}^{m} U_{busy,j}(x - r_j \Delta t, t) q_{ji} \Delta t + \Phi_{B,i}(x, t) \Delta t, \end{aligned} \tag{3.8}$$

where $\Phi_{B,i}(x, t)$ is the rate, with respect to time, at which the system turns to be busy with the probability that the fluid level is less than or equal to x and the UMC phase is i at time t.

Defining $\Phi_{B,i}(x) = \lim_{t\to\infty} \Phi_{B,i}(x, t)$ and a vector $\boldsymbol{\Phi}_B(x) = \{\Phi_{B,1}(x), \ldots, \Phi_{B,m}(x)\}$, (3.8) can be written as

$$\frac{d}{dx} \boldsymbol{U}_{busy}(x) \boldsymbol{R} = \boldsymbol{U}_{busy}(x) \boldsymbol{Q} + \boldsymbol{\Phi}_B(x). \tag{3.9}$$

Let $E(C)$ be the mean length of an arbitrary cycle, i.e., the length from the time point at which an idle period starts to the next such time point. Note that $\kappa_i = 0$ for $i \in \Im_1$ because the busy period can not end when the UMC phase is in an increasing state. Since the system becomes idle and busy only once in a cycle, we now have

$$\Phi_{B,i}(x) = \sum_{j=1}^{m} \frac{\kappa_j}{E(C)} U_{B,ji}(x) \quad (x \geq 0), \tag{3.10}$$

or in vector notation,

$$\boldsymbol{\Phi}_B(x) = \frac{\boldsymbol{\kappa}}{E(C)} \boldsymbol{U}_B(x). \tag{3.11}$$

$E(C)$ will be obtained later.

Remark 3.1 In (3.10) $\frac{\kappa_j}{E(C)}$ is the rate at which an idle period starts with UMC phase j and $U_{B,ji}(x)$ is the probability that the fluid level is less than or equal to x and the UMC phase is i at the end of an idle period under the condition that the idle period starts with UMC phase j. Thus (3.10) can be easily obtained from the meaning of $\Phi_{B,i}(x)$.

To solve (3.9), we need to get $u_{busy,i}(0) = \lim_{x\to 0}\{\frac{d}{dx}U_{busy,i}(x)\}$, which is none other than the rate at which the busy period ends with UMC phase i. Since the fluid level decreases with slope r_i when the UMC phase is in $i \in \Im_2$, we obviously have

$$u_{busy,i}(0)(-r_i) = \frac{\kappa_i}{E(C)} \quad (i \in \Im_2). \tag{3.12}$$

We also have

$$u_{busy,i}(0) = 0 \quad (i \in \Im_1), \tag{3.13}$$

because the busy period can not end when the UMC phase is in an increasing state.

Defining a vector $\boldsymbol{u}_{busy}(0) = \{u_{busy,1}(0), \ldots, u_{busy,m}(0)\}$, we can summarize (3.12) and (3.13) in the next theorem.

Theorem 3.2 *We have*

$$\boldsymbol{u}_{busy}(0)(-\boldsymbol{R}) = \frac{\boldsymbol{\kappa}}{E(C)}. \tag{3.14}$$

To solve (3.9), let us take the Laplace transforms of both sides of (3.9) and use (3.14). Then, we have

$$\theta \boldsymbol{u}^*_{busy}(\theta)\boldsymbol{R} - \boldsymbol{u}_{busy}(0)\boldsymbol{R} = \boldsymbol{u}^*_{busy}(\theta)\boldsymbol{Q} + \frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}^*_B(\theta). \tag{3.15}$$

Rearranging the terms, we get

$$\boldsymbol{u}^*_{busy}(\theta) = \frac{\boldsymbol{\kappa}}{E(C)}\left[-\boldsymbol{I} + \boldsymbol{U}^*_B(\theta)\right][\theta\boldsymbol{R} - \boldsymbol{Q}]^{-1}. \tag{3.16}$$

Then, using (3.7), (3.16) becomes

$$\boldsymbol{u}^*_{busy}(\theta) = \frac{\boldsymbol{\kappa}}{E(C)}\left[-\boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{V}_0)^{-1}\left\{\boldsymbol{V}\left[S^*(\theta)\right] - \boldsymbol{V}_0\right\}\right][\theta\boldsymbol{R} - \boldsymbol{Q}]^{-1}, \tag{3.17}$$

where $\boldsymbol{V}[S^*(\theta)] = \boldsymbol{V}(z)|_{z=S^*(\theta)}$.

Now, (3.17) can be completely determined once $\boldsymbol{\kappa}$ and $E(C)$ are determined.

3.3 Determining $\boldsymbol{\kappa}$ and $E(C)$

Let $\boldsymbol{K}^*(\theta)$ be the matrix LST of the length of a cycle. We then have the following theorem.

Theorem 3.3 *We have*

$$\boldsymbol{K}^*(\theta) = \sum_{n=1}^{\infty}\int_{x=0}^{\infty}\left[\boldsymbol{I} - \boldsymbol{V}^*_0(\theta)\right]^{-1}\boldsymbol{V}^*_n(\theta)\boldsymbol{G}^*(\theta|x)\,dS^{(n)}(x), \tag{3.18}$$

where

$$\boldsymbol{G}^*(\theta|x) = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{G}^*_{12}(\theta|x) \\ \boldsymbol{0} & \boldsymbol{G}^*_{22}(\theta|x) \end{pmatrix}. \tag{3.19}$$

Proof Note that $\boldsymbol{V}^*_n(\theta)$ is the matrix LST, with respect to time, of a vacation length with exactly n customer arrivals. Then, the LST of the length of the idle period and the density of the amount of fluid at the start of a busy period can be represented by

$$\sum_{n=1}^{\infty}\left[\boldsymbol{I} - \boldsymbol{V}^*_0(\theta)\right]^{-1}\boldsymbol{V}^*_n(\theta)\,dS^{(n)}(x). \tag{3.20}$$

Since the fluid level process during the busy period is stochastically equivalent to the conventional MMFF process, under the condition that the busy period is initiated with fluid level x, the LST of the length of the busy period can be denoted by $\boldsymbol{G}^*(\theta|x)$. Thus, integrating with respect to all possible x, after postmultiplying (3.20) by $\boldsymbol{G}^*(\theta|x)$, finishes the proof. □

We note that $\boldsymbol{K} = \boldsymbol{K}^*(\theta)|_{\theta=0}$ represents the UMC phase shift probability during a cycle and becomes

$$\begin{aligned}\boldsymbol{K} &= \sum_{n=1}^{\infty}\int_{x=0}^{\infty}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}\boldsymbol{V}_n\big[\boldsymbol{G}^*(\theta|x)|_{\theta=0}\big]\,dS^{(n)}(x)\\ &= \sum_{n=1}^{\infty}\int_{x=0}^{\infty}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}\boldsymbol{V}_n\begin{pmatrix}\mathbf{0} & \boldsymbol{G}^*_{12}(\theta|x)|_{\theta=0}\\ \mathbf{0} & \boldsymbol{G}^*_{22}(\theta|x)|_{\theta=0}\end{pmatrix}dS^{(n)}(x)\\ &= \sum_{n=1}^{\infty}\int_{x=0}^{\infty}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}\boldsymbol{V}_n\begin{pmatrix}\mathbf{0} & \boldsymbol{\Psi}e^{\boldsymbol{H}x}\\ \mathbf{0} & e^{\boldsymbol{H}x}\end{pmatrix}dS^{(n)}(x),\end{aligned} \tag{3.21}$$

where $\boldsymbol{H} = \boldsymbol{H}^*(\theta)|_{\theta=0} = \boldsymbol{\Gamma}_2^{-1}[\boldsymbol{Q}_{22} + \boldsymbol{Q}_{21}\boldsymbol{\Psi}]$, and $\boldsymbol{\Psi} = \boldsymbol{\Psi}^*(\theta)|_{\theta=0}$ can be obtained from the algorithm of Ahn and Ramaswami (2005). The algorithm for $\boldsymbol{V}_n$ can be obtained from Lucantoni et al. (1990).

The probability vector $\boldsymbol{\kappa}$ can now be obtained from

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}\boldsymbol{K}, \qquad \boldsymbol{\kappa}\mathbf{e} = 1. \tag{3.22}$$

The mean length $E(C)$ of a cycle is given by the following theorem.

Theorem 3.4 *We have*

$$E(C) = E(V)\boldsymbol{\kappa}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}\mathbf{e} + \sum_{n=1}^{\infty}\int_{x=0}^{\infty}\boldsymbol{\kappa}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}\boldsymbol{V}_n\boldsymbol{g}(x)\,dS^{(n)}(x), \tag{3.23}$$

where

$$\boldsymbol{g}(x) = \begin{pmatrix}\boldsymbol{g}_1(x)\\ \boldsymbol{g}_2(x)\end{pmatrix} = \begin{pmatrix}-\frac{d}{d\theta}\boldsymbol{G}^*_{12}(\theta|x)\big|_{\theta=0}\mathbf{e}\\ -\frac{d}{d\theta}\boldsymbol{G}^*_{22}(\theta|x)\big|_{\theta=0}\mathbf{e}\end{pmatrix}.$$

Proof We note that

$$E(C) = \boldsymbol{\kappa}\left[-\frac{d}{d\theta}\boldsymbol{K}^*(\theta)\right]_{\theta=0}\mathbf{e}. \tag{3.24}$$

Taking a derivative of (3.18), (3.24) becomes

$$\begin{aligned}E(C) = {}& \boldsymbol{\kappa}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}\left[-\frac{d}{d\theta}\boldsymbol{V}_0^*(\theta)\right]_{\theta=0}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}\sum_{n=1}^{\infty}\boldsymbol{V}_n\int_{x=0}^{\infty}\big[\boldsymbol{G}^*(\theta|x)\big]_{\theta=0}dS^{(n)}(x)\mathbf{e}\\ &+ \boldsymbol{\kappa}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}\sum_{n=1}^{\infty}\left[-\frac{d}{d\theta}\boldsymbol{V}_n^*(\theta)\right]_{\theta=0}\int_{x=0}^{\infty}\big[\boldsymbol{G}^*(\theta|x)\big]_{\theta=0}dS^{(n)}(x)\mathbf{e}\\ &+ \boldsymbol{\kappa}\sum_{n=1}^{\infty}\int_{x=0}^{\infty}(\boldsymbol{I}-\boldsymbol{V}_0)^{-1}\boldsymbol{V}_n\boldsymbol{g}(x)\,dS^{(n)}(x).\end{aligned} \tag{3.25}$$

Note that

$$\int_{x=0}^{\infty} \left[\boldsymbol{G}^*(\theta|x)\right]_{\theta=0} dS^{(n)}(x)\mathbf{e} = \mathbf{e}, \tag{3.26}$$

and

$$(\boldsymbol{I} - \boldsymbol{V}_0)^{-1} \sum_{n=1}^{\infty} \boldsymbol{V}_n \mathbf{e} = \mathbf{e}. \tag{3.27}$$

Using (3.26) and (3.27) in (3.25), we get

$$\begin{aligned} E(C) &= \boldsymbol{\kappa}(\boldsymbol{I} - \boldsymbol{V}_0)^{-1} \sum_{n=0}^{\infty} \left[-\frac{d}{d\theta} \boldsymbol{V}_n^*(\theta)\right]_{\theta=0} \mathbf{e} \\ &\quad + \boldsymbol{\kappa} \sum_{n=1}^{\infty} \int_{x=0}^{\infty} (\boldsymbol{I} - \boldsymbol{V}_0)^{-1} \boldsymbol{V}_n \boldsymbol{g}(x)\, dS^{(n)}(x). \end{aligned} \tag{3.28}$$

Then using $\sum_{n=0}^{\infty}[-\frac{d}{d\theta}\boldsymbol{V}_n^*(\theta)]_{\theta=0}\mathbf{e} = E(V)\mathbf{e}$ in (3.28) finishes the proof. □

Now, we need to obtain $\boldsymbol{g}(x) = \binom{\boldsymbol{g}_1(x)}{\boldsymbol{g}_2(x)}$ used in (3.23). We have the following theorem.

Theorem 3.5 *We have*

$$\boldsymbol{g}_1(x) = -\boldsymbol{\Psi}^{(1)}\mathbf{e} + \boldsymbol{\Psi}\boldsymbol{g}_2(x) \quad (x > 0), \tag{3.29}$$

where

$$\boldsymbol{g}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[\boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{22} + \boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{21}\boldsymbol{\Psi}\right]^{(n-1)} \left[\boldsymbol{\Gamma}_2^{-1}\mathbf{e} + \boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{21}\left(-\boldsymbol{\Psi}^{(1)}\mathbf{e}\right)\right] \quad (x > 0), \tag{3.30}$$

and

$$-\boldsymbol{\Psi}^{(1)}\mathbf{e} = -\frac{d}{d\theta}\boldsymbol{\Psi}^*(\theta)\bigg|_{\theta=0} \mathbf{e} = -[\boldsymbol{\Psi}\boldsymbol{P}_{21} + \boldsymbol{P}_{11} - \boldsymbol{I}]^{-1}\left(\frac{\boldsymbol{\Gamma}_1^{-1}\boldsymbol{\Psi}}{\lambda} + \frac{\boldsymbol{\Psi}\boldsymbol{\Gamma}_2^{-1}}{\lambda}\right)\mathbf{e}. \tag{3.31}$$

Proof Let us rewrite (2.2) in the following way

$$\boldsymbol{\Psi}^*(\theta)\left[\boldsymbol{I} - \frac{\boldsymbol{H}^*(\theta)}{\lambda}\right] = \left[\boldsymbol{P}_{11}\boldsymbol{\Psi}^*(\theta) + \boldsymbol{P}_{12}\right] - \frac{\theta}{\lambda}\boldsymbol{\Gamma}_1^{-1}\boldsymbol{\Psi}^*(\theta). \tag{3.32}$$

Taking a derivative of (3.32) with respect to θ and taking $\theta = 0$, we get

$$\begin{aligned} &\boldsymbol{\Psi}^{(1)}\left[\boldsymbol{I} - \frac{\boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{22}}{\lambda} - \frac{\boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{21}\boldsymbol{\Psi}}{\lambda}\right] + \boldsymbol{\Psi}\left[\frac{\boldsymbol{\Gamma}_2^{-1}}{\lambda} - \frac{\boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{21}}{\lambda}\boldsymbol{\Psi}^{(1)}\right] \\ &\quad = \boldsymbol{P}_{11}\boldsymbol{\Psi}^{(1)} - \frac{\boldsymbol{\Gamma}_1^{-1}}{\lambda}\boldsymbol{\Psi}. \end{aligned} \tag{3.33}$$

Using $\boldsymbol{P}_{21} = \frac{\boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{21}}{\lambda}$, $\boldsymbol{P}_{22} = \boldsymbol{I} + \frac{\boldsymbol{\Gamma}_2^{-1}\boldsymbol{Q}_{22}}{\lambda}$ (see (2.1)), we have

$$-\boldsymbol{\Psi}^{(1)}[\boldsymbol{P}_{22} + \boldsymbol{P}_{21}\boldsymbol{\Psi} - 2\boldsymbol{I}] - [\boldsymbol{\Psi}\boldsymbol{P}_{21} + \boldsymbol{P}_{11}]\boldsymbol{\Psi}^{(1)} = -\left[\frac{\boldsymbol{\Gamma}_1^{-1}}{\lambda}\boldsymbol{\Psi} + \boldsymbol{\Psi}\frac{\boldsymbol{\Gamma}_2^{-1}}{\lambda}\right]. \tag{3.34}$$

Postmultiplying both sides of (3.34) by $\mathbf{e}$ and using $[\boldsymbol{P}_{22} + \boldsymbol{P}_{21}\boldsymbol{\Psi}]\mathbf{e} = \mathbf{e}$ yields (3.31).

To derive (3.30), we take a derivative of (2.4) with respect to θ and take $\theta = 0$ to get

$$\begin{aligned} \boldsymbol{g}_2(x) &= -\left[\frac{d}{d\theta}\boldsymbol{G}^*_{22}(\theta|x)\right]_{\theta=0}\mathbf{e} = \left[-\frac{d}{d\theta}e^{\boldsymbol{H}^*(\theta)x}\right]_{\theta=0}\mathbf{e} \\ &= -\frac{d}{d\theta}\sum_{n=0}^{\infty}\left[\frac{[\boldsymbol{H}^*(\theta)]^n x^n}{n!}\right]_{\theta=0}\mathbf{e} \\ &= \sum_{n=1}^{\infty}\frac{x^n}{n!}\left[\boldsymbol{H}^*(\theta)|_{\theta=0}\right]^{n-1}\left[-\frac{d}{d\theta}\boldsymbol{H}^*(\theta)\right]_{\theta=0}\mathbf{e}. \end{aligned} \tag{3.35}$$

Using $\boldsymbol{H}^*(\theta) = \boldsymbol{\Gamma}_2^{-1}[\boldsymbol{Q}_{22} - \theta\boldsymbol{I} + \boldsymbol{Q}_{21}\boldsymbol{\Psi}^*(\theta)]$ in (3.35) yields (3.30).

To derive (3.29), let us take a derivative of (2.3). Then we have

$$\boldsymbol{g}_1(x) = -\frac{d}{d\theta}\left[\boldsymbol{\Psi}^*(\theta)\boldsymbol{G}^*_{22}(\theta|x)\right]_{\theta=0}\mathbf{e} = -\boldsymbol{\Psi}^{(1)}\boldsymbol{G}^*_{22}(\theta|x)|_{\theta=0}\mathbf{e} + \boldsymbol{\Psi}\boldsymbol{g}_2(x). \tag{3.36}$$

Using $\boldsymbol{G}^*_{22}(\theta|x)|_{\theta=0}\mathbf{e} = \mathbf{e}$ yields (3.29). □

Let $E(I)$ be the mean length of the idle period. Noting that $\boldsymbol{V}_0$ represents the probability that a vacation ends without any arrival of customers (jumps), $\boldsymbol{\kappa}(\boldsymbol{I} - \boldsymbol{V}_0)^{-1}\mathbf{e}$ is the mean number of the vacations during an arbitrary idle period. Then, we easily have

$$E(I) = E(V)\boldsymbol{\kappa}(\boldsymbol{I} - \boldsymbol{V}_0)^{-1}\mathbf{e}. \tag{3.37}$$

Thus, the probability ρ that the system is busy at an arbitrary time point can be obtained from

$$\rho = \frac{E(C) - E(I)}{E(C)} = 1 - \frac{E(I)}{E(C)}. \tag{3.38}$$

3.4 The fluid level at an arbitrary time point

In this section, we derive the vector LST $\boldsymbol{u}^*(\theta)$ of the fluid level at an arbitrary time. Using (3.5) and (3.16) in (3.1), we have the following theorem.

Theorem 3.6 *We have*

$$\begin{aligned} \boldsymbol{u}^*(\theta) &= \boldsymbol{u}^*_{idle}(\theta)\left[\theta\boldsymbol{R} - \boldsymbol{D} + \boldsymbol{D}S^*(\theta)\right](\theta\boldsymbol{R} - \boldsymbol{Q})^{-1} \\ &= (1-\rho)\frac{\boldsymbol{\kappa}(\boldsymbol{I} - \boldsymbol{V}_0)^{-1}[\boldsymbol{V}[S^*(\theta)] - \boldsymbol{I}]\cdot[\boldsymbol{C} + \boldsymbol{D}S^*(\theta)]^{-1}}{E(V)\boldsymbol{\kappa}[\boldsymbol{I} - \boldsymbol{V}_0]^{-1}\mathbf{e}} \\ &\quad \times \left[\theta\boldsymbol{R} - \boldsymbol{D} + \boldsymbol{D}S^*(\theta)\right](\theta\boldsymbol{R} - \boldsymbol{Q})^{-1}. \end{aligned} \tag{3.39}$$

Proof Let us rewrite (3.5) in the following way

$$\boldsymbol{u}^*_{idle}(\theta)\left[\boldsymbol{C} + \boldsymbol{D}S^*(\theta)\right] = (1-\rho)\frac{\boldsymbol{\kappa}(\boldsymbol{I} - \boldsymbol{V}_0)^{-1}[\boldsymbol{V}[S^*(\theta)] - \boldsymbol{I}]}{E(V)\boldsymbol{\kappa}[\boldsymbol{I} - \boldsymbol{V}_0]^{-1}\mathbf{e}}. \tag{3.40}$$

Using (3.37) and (3.38) in (3.40), we then have

$$\begin{aligned} &\boldsymbol{u}^*_{idle}(\theta)\left[\boldsymbol{C} + \boldsymbol{D}S^*(\theta)\right] \\ &\quad = \frac{\boldsymbol{\kappa}(\boldsymbol{I} - \boldsymbol{V}_0)^{-1}[\boldsymbol{V}[S^*(\theta)] - \boldsymbol{V}_0]}{E(C)} + \frac{\boldsymbol{\kappa}(\boldsymbol{I} - \boldsymbol{V}_0)^{-1}(\boldsymbol{V}_0 - \boldsymbol{I})}{E(C)} \\ &\quad = \frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}^*_B(\theta) - \frac{\boldsymbol{\kappa}}{E(C)}. \end{aligned} \tag{3.41}$$

Using (3.14) in (3.15) yields

$$\theta \boldsymbol{u}^*_{busy}(\theta)\boldsymbol{R} - \boldsymbol{u}^*_{busy}(\theta)\boldsymbol{C} - \boldsymbol{u}^*_{busy}(\theta)\boldsymbol{D} = \frac{\boldsymbol{\kappa}}{E(C)}\boldsymbol{U}^*_B(\theta) - \frac{\boldsymbol{\kappa}}{E(C)}. \tag{3.42}$$

Now, direct subtracting (3.42) from (3.41) finishes the proof. □

Remark 3.2 Equation (3.39) shows that the vector LST $\boldsymbol{u}^*(\theta)$ of the fluid level at an arbitrary time point is factored into two parts, one of which is the vector LST $\boldsymbol{u}^*_{idle}(\theta)$ of the fluid level at an arbitrary idle time point. This factorization is very similar to the factorization of the BMAP/G/1 queue with generalized vacations (Baek et al. 2008 and Chang et al. 2002).

Remark 3.3 The importance of (3.39) is that it shows the possibility that any other MAP-modulated fluid flow with server control policies can be represented in the same form. For example, it is shown that the LST of the fluid level at an arbitrary time of the MAP-modulated fluid flow model under D-policy can be expressed in the same decomposed form (Baek et al. 2011).

Remark 3.4 Another importance is that the moment of fluid level at an arbitrary time can be represented as a function of its lower moments and the moment of fluid level at an arbitrary time during the idle period (It will be shown in detail in the following section). Thus the entire analysis of our model can be reduced to characterizing $\boldsymbol{u}^*_{idle}(\theta)$.

Remark 3.5 The complete interpretation of $[\theta\boldsymbol{R} - \boldsymbol{D} + \boldsymbol{D}S^*(\theta)](\theta\boldsymbol{R} - \boldsymbol{Q})^{-1}$ contained in (3.39) has not been successful so far.

3.5 Moments

In this section, we derive the recursive formula for moments from (3.39). For notational simplicity, let us define, for a matrix (or vector) LST $\boldsymbol{M}^*(\theta)$,

$$\boldsymbol{M}^{(n)} = \frac{d^n}{d\theta^n}\boldsymbol{M}^*(\theta)\bigg|_{\theta=0}, \qquad \boldsymbol{M}^{(0)} = \boldsymbol{M} = \boldsymbol{M}^*(\theta)|_{\theta=0}.$$

We then have the following theorem.

Theorem 3.7 *We have*

$$\begin{aligned}\boldsymbol{u}^{(n)}\mathbf{e} = \frac{1}{\boldsymbol{\pi}\boldsymbol{R}\mathbf{e}}\Bigg\{ & \boldsymbol{u}^{(n)}_{idle}\big[\boldsymbol{R} - \boldsymbol{D}E(S)\big]\mathbf{e} - n\boldsymbol{u}^{(n-1)}_{idle}\big[\boldsymbol{R} - \boldsymbol{D}E(S)\big](\mathbf{e}\boldsymbol{\pi} - \boldsymbol{Q})^{-1}\boldsymbol{R}\mathbf{e} \\ & + \frac{1}{1+n}\sum_{k=2}^{n+1}\binom{n+1}{k}(-1)^k E\big(S^k\big)\boldsymbol{u}^{(n+1-k)}_{idle}\boldsymbol{D}\mathbf{e} \\ & + n\boldsymbol{u}^{(n-1)}\boldsymbol{R}(\mathbf{e}\boldsymbol{\pi} - \boldsymbol{Q})^{-1}\boldsymbol{R}\mathbf{e} \\ & - \sum_{k=2}^{n}\binom{n}{k}(-1)^k E\big(S^k\big)\boldsymbol{u}^{(n-k)}_{idle}\boldsymbol{D}(\mathbf{e}\boldsymbol{\pi} - \boldsymbol{Q})^{-1}\boldsymbol{R}\mathbf{e}\Bigg\}.\end{aligned} \tag{3.43}$$

Proof Taking nth derivative of (3.39) yields

$$\begin{aligned}\boldsymbol{u}^{(n)}(-\boldsymbol{Q})&+n\boldsymbol{u}^{(n-1)}\boldsymbol{R}\\&=n\boldsymbol{u}_{idle}^{(n-1)}\big[\boldsymbol{R}-\boldsymbol{D}E(S)\big]+\sum_{k=2}^{n}(-1)^{k}E\big(S^{k}\big)\binom{n}{k}\boldsymbol{u}_{idle}^{(n-k)}\boldsymbol{D}.\end{aligned}\tag{3.44}$$

Using $\boldsymbol{\pi}(\mathbf{e}\boldsymbol{\pi}-\boldsymbol{Q})^{-1}=\boldsymbol{\pi}$ in (3.44), we get

$$\begin{aligned}\boldsymbol{u}^{(n)}+n\boldsymbol{u}^{(n-1)}\boldsymbol{R}(\mathbf{e}\boldsymbol{\pi}-\boldsymbol{Q})^{-1}=\boldsymbol{u}^{(n)}\mathbf{e}\boldsymbol{\pi}+n\boldsymbol{u}_{idle}^{(n-1)}\big[\boldsymbol{R}-\boldsymbol{D}E(S)\big](\mathbf{e}\boldsymbol{\pi}-\boldsymbol{Q})^{-1}\\-\sum_{k=2}^{n}(-1)^{k}E\big(S^{k}\big)\binom{n}{k}\boldsymbol{u}_{idle}^{(n-k)}\boldsymbol{D}(\mathbf{e}\boldsymbol{\pi}-\boldsymbol{Q})^{-1}.\end{aligned}\tag{3.45}$$

Postmultiplying (3.44) by $\mathbf{e}$ and using $(n+1)$ in place of n, we get

$$\boldsymbol{u}^{(n)}\boldsymbol{R}\mathbf{e}=\boldsymbol{u}_{idle}^{(n)}\big[\boldsymbol{R}-\boldsymbol{D}E(S)\big]\mathbf{e}+\frac{1}{n+1}\sum_{k=2}^{n+1}(-1)^{k}E\big(S^{k}\big)\binom{n+1}{k}\boldsymbol{u}_{idle}^{(n+1-k)}\boldsymbol{D}\mathbf{e}.\tag{3.46}$$

Now, postmultiplying (3.45) by $\boldsymbol{R}\mathbf{e}$ and using (3.46) finishes the proof. □

4 Numerical examples

In this section, we present numerical examples. We consider two cases. The length of a vacation and jump size of the fluid level follow respectively

Case 1: (exponential, exponential);
Case 2: (Erlang of order 2, Erlang of order 2).

4.1 Exponential vacation and jump size

We assume that the vacation length and jump size follow the exponential distributions. The mean $E(V)$ of a vacation length is assumed to be 0.02 for all cases. We vary the mean $E(S)$ of a jump size from 0.1 to 0.9 to accommodate wide range of traffic intensities. We consider two set of parameter matrices $\boldsymbol{C}$, $\boldsymbol{D}$ and rate matrix $\boldsymbol{R}$ as follows:

- Case 1-1

$$\boldsymbol{C}=\begin{pmatrix}-10&1&3&1\\1&-7&1&1\\3&1&-13&2\\2&1&1&-12\end{pmatrix},\qquad \boldsymbol{D}=\begin{pmatrix}2&1&1&1\\1&1&1&1\\1&1&3&2\\1&2&3&2\end{pmatrix},$$

$$\boldsymbol{R}=\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&-2&0\\0&0&0&-3\end{pmatrix}.$$

- Case 1-2

$$\boldsymbol{C}=\begin{pmatrix}-8&1&2&2\\1&-9.5&1&3\\1&2&-10&2\\1&1&1&-10\end{pmatrix},\qquad \boldsymbol{D}=\begin{pmatrix}1&0.5&0.5&1\\2&0.5&1&1\\1&2&1&1\\3&2&1&1\end{pmatrix},$$

Table 1 Mean fluid levels of Case 1

$E(S)$	0.1	0.3	0.5	0.7	0.9
Case 1-1	1.17189	1.42469	1.66095	1.892	2.12074
Case 1-2	1.07135	1.29421	1.51507	1.735	1.9544

Table 2 Mean fluid levels of Case 2

$E(S)$	0.1	0.3	0.5	0.7	0.9
Case 2-1	1.16262	1.37116	1.55652	1.73443	1.90899
Case 2-2	1.0719	1.26424	1.44487	1.61974	1.79135

Table 3 Mean fluid levels with the changes of the outflow rate and the jump size

ρ	0.77	0.797	0.828	0.866	0.911	0.965
$E(U_1)$	1.0719	1.2580	1.5539	2.0903	3.3319	8.9724
$E(U_2)$	1.0719	1.1495	1.2642	1.4626	1.9018	3.8669

$$\boldsymbol{R} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix}.$$

Table 1 shows the mean fluid level for each case.

4.2 Erlang vacation and jump size

We carry out the same experimentation with Erlang vacation and jump size distributions. Table 2 shows the mean fluid level for each case.

4.3 Effects of the jump size and outflow rate

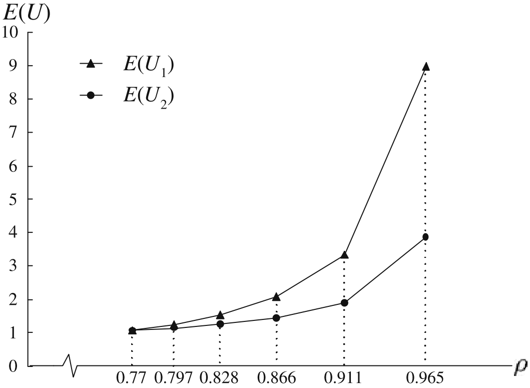
In our system, the traffic intensity ρ is affected by two factors: one is the change of the outflow rate during the busy period, and the other is the change of the jump size (offered load) during the idle period. In this section, we explore the effects of these factors on the system performance.

We use the same parameter matrices and probability distributions as in Case 2-2 of Sect. 4.2. To see the effect of the outflow rate, we change r_4 with all other parameters fixed (system 1). To see the effect of the jump size we change the mean $E(S)$ of the jump size.

Let $E(U_1)$ and $E(U_2)$ be the mean fluid levels at an arbitrary time point in system 1 and system 2 respectively. Table 3 shows the mean fluid levels. Figure 2 depicts the table.

Figure 2 shows that under the same traffic intensity the change of the service rate results in a higher mean fluid level than the change of the mean jump size does. This implies that control of the service rate is more important than the control of the offered load during idle period in our system.

Fig. 2 Mean fluid levels with the changes of the outflow rate and the jump size



5 Conclusion

In this paper, we analyzed the MAP-modulated fluid flow queueing model with multiple vacations. We derived the vector LST of the fluid level at an arbitrary time in steady-state and showed that the vector LST is expressed in a factored form. We also derived recursive formula for moments. We presented numerical examples and explored the effects of outflow rate and jump size.

Acknowledgements Authors are thankful to the referees for their valuable comments.

This paper was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (grant number: 2010-0010023).

Soohan Ahn was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (grant number 20100021831) and also by the University of Seoul 2010 Research Fund.

References

Aggarwal, V., Gautam, N., Kumara, S. R. T., & Greaves, M. (2005). Stochastic fluid flow models for determining optimal switching thresholds. *Performance Evaluation*, *59*(1), 19–46.

Ahn, S. (2009). A transient analysis of Markov fluid models with jumps. *Journal of the Korean Statistical Society*, *38*(4), 351–366.

Ahn, S., & Ramaswami, V. (2003). Fluid flow models and queues: a connection by stochastic coupling. *Stochastic Models*, *19*(3), 325–348.

Ahn, S., & Ramaswami, V. (2004). Transient analysis of fluid flow models via stochastic coupling to a queue. *Stochastic Models*, *20*(1), 71–101.

Ahn, S., & Ramaswami, V. (2005). Efficient algorithms for transient analysis of stochastic fluid flow models. *Journal of Applied Probability*, *42*, 531–549.

Ahn, S., Jeon, J., & Ramaswami, V. (2005). Steady state analysis for finite fluid flow models using finite QBDs. *Queueing Systems*, *49*, 223–259.

Anick, D., Mitra, D., & Sondhi, M. (1982). Stochastic theory of a data handling system with multiple sources. *The Bell System Technical Journal*, *61*, 1871–1894.

Asmussen, S. (1995). Stationary distributions for fluid flow models with or without Brownian noise. *Stochastic Models*, *11*(1), 21–49.

Badescu, A. L., & Landriault, D. (2009). Applications of fluid flow matrix analytic methods in ruin theory—a review. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Madr.*, *103*(2), 353–372.

Badescu, A. L., Drekic, S., & Landriault, D. (2007). On the analysis of a multi-threshold Markovian risk model, *Scandinavian Actuarial Journal*, *2007*(4), 248–260.

Baek, J. W., Lee, H. W., Lee, S. W., & Ahn, S. (2008). A factorization property for BMAP/G/1 vacation queues under variable service speed. *Annals of Operations Research*, *160*, 19–29.

Baek, J. W., Lee, H. W., Lee, S. W., & Ahn, S. (2011). A MAP-modulated fluid flow queueing model under workload control. *Numerical Linear Algebra with Applications*, *18*(6), 897–1083.

Bean, N., O'Reilly, M., & Taylor, P. (2005). Algorithms for return probabilities for stochastic fluid flows. *Stochastic Models*, *21*(1), 149–184.

Chang, S. H., Takine, T., Chae, K. C., & Lee, H. W. (2002). A unified queue length formula for BMAP/G/1 queue with generalized vacations. *Stochastic Models*, *18*(3), 369–386.

Doshi, B. T. (1986). Queueing systems with vacations: survey. *Queueing Systems*, *1*(1), 29–66.

Heyman, D. P. (1977). T-policy for the M/G/1 queue. *Management Science*, *23*(7), 775–778.

Kulkarni, V. G., & Yan, K. (2007). A fluid model with upward jumps at the boundary. *Queueing Systems*, *56*(2), 103–117.

Lee, H. W., & Baek, J. W. (2005). BMAP/G/1 queue under D-policy: queue length analysis. *Stochastic Models*, *21*(2&3), 1–21.

Lee, H. S., & Srinivasan, M. M. (1989). Control policies for the queueing system. *Management Science*, *35*(6), 708–721.

Lee, H. W., Ahn, B. Y., & Park, N. I. (2001). Decompositions of the queue length distributions in the MAP/G/1 queue under multiple and single vacations with N-policy. *Stochastic Models*, *17*(2), 157–190.

Levy, H., & Yechiali, U. (1975). Utilization of idle time in an M/G/1 queueing system. *Management Science*, *22*(2), 202–211.

Lucantoni, D. M., Meier-Hellstern, K. S., & Neuts, M.F. (1990). A single server queue with server vacations and a class of non-renewal arrival processes. *Advances in Applied Probability*, *22*(3), 676–709.

Mao, B., Wang, F., & Tian, N. (2010a). Fluid model driven by an M/M/1 queue with exponential vacation. In *Proceedings of the second international conference on information technology and computer science* (pp. 539–542).

Mao, B., Wang, F., & Tian, N. (2010b). Fluid model driven by an M/M/1/N queue with single. *International Journal of Information and Management Sciences*, *21*, 29–40.

Mao, B., Wang, F., & Tian, N. (2010c) Fluid model driven by an M/M/1 queue with multiple exponential vacations. In *Proceedings of the second international conference on advanced computer control* (Vol. 3, pp. 112–115).

Mitra, D. (1988). Stochastic theory of a fluid model of producers and consumers coupled by a buffer. *Advances in Applied Probability*, *20*(3), 646–676.

Takada, H. (2001). Markov modulated fluid queues with batch fluid arrivals. *Journal of the Operations Research Society of Japan*, *44*(4), 334–365.

Tan, B., & Gershwin, S. B., (2007). *Modelling and analysis of Markovian continuous flow production systems with a finite buffer: a general methodology and applications* (Technical Report ORC-381-07). Massachusetts Institute of Technology Operations Research Center Working Paper Series.

Tzenova, E., Adan, I., & Kulkarni, V. G. (2005). Fluid models with jumps. *Stochastic Models*, *21*(1), 37–55.

Yan, K., & Kulkarni, V. G. (2008). Optimal inventory policies under stochastic production and demand rates. *Stochastic Models*, *24*(2), 173–190.

Yeralan, S., Franck, W. E., & Quasem, M. A. (1986). A continuous materials flow production line model with station breakdown. *European Journal of Operational Research*, *27*(3), 289–300.