A DISTRIBUTED MULTI-AGENT SYSTEM APPROACH FOR SOLVING CONSTRAINED OPTIMIZATION PROBLEMS USING PROBABILITY COLLECTIVES

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A thesis submitted to the Nanyang Technological University
in partial fulfilment of the requirement for the degree of
Doctor of Philosophy

2012

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This thesis is dedicated to the memory of
my loving brother 'Vishal'

ACKNOWLEDGEMENTS

"A Journey of Thousand Miles Starts with a Single Step"

It was indeed a great pleasure to work with Dr. Tai Kang, my supervisor at Nanyang Technological University (NTU) towards my PhD research work. His guidance, support, and friendliness left an incredible mark on me. He was always around whenever I needed him, and helped me focus in the right direction. His motivation and persistence helped me professionally and I am sure it has helped many others as well. He is a benchmark for me.

I would like to express a special mention about the financial assistance provided by NTU. I am thankful to the School of MAE for giving me the opportunity to work as Teaching Assistant.

I would like to acknowledge Dr. Rodney Teo and Ye Chuan Yeo (DSO National Laboratories, Singapore) for their useful discussions on the research topic, as well as Dr. David H. Wolpert (NASA Ames Research Center, Moffett Field, CA, USA) and Dr. Stefan R. Bieniawski (Boeing Phantom Works, Seal Beach, CA, USA) for their help with the ongoing research concepts.

I would like to express my gratitude towards Dr. R. C. Mehta, Dr. Francis Nickols, Dr. Yeo Song Huat and Dr. Sunil Joshi for their help at various moments during the PhD study. I am grateful to my best friends Fan Zhihua, Hassan Mirzahosseinian, Salah Haridy, Sheng Lingling, Bhone Myint Kyaw, Ou Yanjing and Alvin Ang for their incredible support in all respect. I also thank Mr. Teo Hai Beng for his friendliness and help in the lab matters. Thanks to Mr. Kelkar and Surjit for their help in the crucial time at the end of my stay in Singapore. Also, thanks to Prof. Rohit and Prof. Barpande from University of Pune for their supportive role at various moments.

I wish to thank my loving wife Prajakta and my son Nityay for their great support who were always there for all the hard times. I also wish to thank my mother and father who motivated me and gave me spirit, and were thousands of miles away but never made me feel so. Without them this success was not possible.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS ..... **II**
TABLE OF CONTENTS ..... **IV**
LIST OF FIGURES ..... **VII**
LIST OF TABLES ..... **IX**
ABSTRACT ..... **X**
NOMENCLATURE ..... **XII**
CHAPTER 1 ..... **1**
INTRODUCTION **1**
1.1 An overview of PC 2
1.2 Background and Motivation 3
1.3 Research Objectives 6
1.4 Original Contributions Arising from this Work 7
1.5 Scope of the Research Work 8
1.6 Organization of the Thesis 10
CHAPTER 2 ..... **11**
DISTRIBUTED, DECENTRALIZED AND COOPERATIVE APPROACH **11**
2.1 Distributed, Decentralized and Cooperative Approach 11
2.2 Advantages of the Distributed, Decentralized and Cooperative Approach 15
2.3 Applications of the Distributed, Decentralized and Cooperative Approach 16
CHAPTER 3 ..... **23**
COLLECTIVE INTELLIGENCE USING PC **23**
3.1 Literature Review of PC 24
3.2 Characteristics of PC 27
3.3 Conceptual Framework of PC 29
3.4 Formulation of Unconstrained PC 31
3.5 Nash Equilibrium 40
3.6 Validation of the Unconstrained PC 42
3.6.1 Results and Discussion 43
CHAPTER 4 ..... **46**
CONSTRAINED PC (APPROACH 1) **46**
4.1 Heuristic Approach 46
4.2 The Traveling Salesman Problem (TSP) 46

4.2.1 The Multiple Traveling Salesmen Problem (MTSP) 47
4.2.1.1 Variations of the MTSP 48
4.2.2 The Vehicle Routing Problem (VRP) 49
4.2.2.1 Variations of the VRP 51
4.2.3 Algorithms and Local Improvement Techniques for solving the MTSP 52
4.2.3.1 Local Improvement Techniques 53
4.2.3.2 Various Algorithms for Solving the MTSP 57
4.2.3.3 Solving Multiple Unmanned Aerial Vehicles (MUAVs) Path Planning Problem as a MTSP/VRP 60
4.3 Solution to the Multiple Depot MTSP (MDMTSP) using PC 62
4.3.1 Sampling 62
4.3.2 Formation of Intermediate Combined Route Set 64
4.3.3 Node Insertion and Elimination Heuristic 66
4.3.4 Neighboring Approach for Updating the Sample Space 69
4.3.5 Node Swapping Heuristic 70
4.4 Test Cases of the Multiple Depot MTSP (MDMTSP) 71
4.5 Test Cases of the Single Depot MTSP (SDMTSP) with Randomly Located Nodes 75
4.6 Comparison and Discussion 77
CHAPTER 5 81
CONSTRAINED PC (APPROACH 2) 81
5.1 Penalty Function Approach 81
5.2 Solutions to Constrained Test Problems 85
5.2.1 Test Problem 1 86
5.2.2 Test Problem 2 89
5.2.3 Test Problem 3 91
5.3 Discussion 93
CHAPTER 6 95
CONSTRAINED PC (APPROACH 3) 95
6.1 Feasibility-based Rule I 96
6.1.1 Modifications to Step 5 of the Unconstrained PC Approach 97
6.1.2 Modifications to Step 7 of the Unconstrained PC Approach 98
6.1.3 Modifications to Step 6 of the Unconstrained PC approach 100

6.2 The Circle Packing Problem (CPP) 103
6.2.1 Formulation of the CPP 104
6.2.2 Case 1: CPP with Circles Randomly Initialized Inside the Square 105
6.2.3 Case 2: CPP with Circles Randomly Initialized 110
6.2.4 Voting Heuristic 114
6.2.5 Agent Failure Case 115
6.3 Discussion 120
6.4 Feasibility Based Rule II 122
6.4.1 Modifications to Step 5 of the Unconstrained PC Approach 122
6.4.2 Modifications to Step 7 of the Unconstrained PC Approach 124
6.4.3 Modifications to Step 6 of the Unconstrained PC Approach 125
6.5 The Sensor Network Coverage Problem (SNCP) 127
6.5.1 Formulation of the SNCP 128
6.5.2 Variations of the SNCP Solved 131
6.5.2.1 Variation 1 132
6.5.2.2 Variation 2 133
6.6 Discussion 141
CHAPTER 7 144
CONCLUSIONS AND RECOMMENDATIONS 144
7.1 Conclusions 144
7.2 Recommendations for Future Work 147
7.2.1 Multi-Objective Probability Collectives (MOPC) 148
7.2.2 PC for the Urban Traffic Control 149
APPENDIX A 151
Analogy of Homotopy Function to Helmholtz Free Energy 151
APPENDIX B 153
Nearest Newton Descent Scheme 153
APPENDIX C 156
Broyden-Fletcher-Goldfarb-Shanno (BFGS) Method for Minimizing the Homotopy Function 156
APPENDIX D 160
Individual Sensor Coverage Calculation 160
Reference 165

LIST OF FIGURES

Figure 2.1: A Centralized System……..11
Figure 2.2: Biological MASs……....12
Figure 3.1: Unconstrained PC Algorithm Flowchart……...35
Figure 3.2: Probability Distribution of Agent i……...38
Figure 3.3: Convergence Plot for Trial 5……..44
Figure 4.1: Illustration of Converged Probability Plots……...65
Figure 4.2: Insertion Heuristic……67
Figure 4.3: Elimination Heuristic……68
Figure 4.4: Neighboring Approach……..70
Figure 4.5: Swapping Heuristic……....71
Figure 4.6: Test Case 1……72
Figure 4.7: Convergence Plot for Test Case 1……...73
Figure 4.8: Test Case 2……74
Figure 4.9: Convergence Plot for Test Case 2……..74
Figure 4.10: Randomly Located Nodes (Sample Case 1)…..……...76
Figure 4.11: Randomly Located Nodes (Sample Case 2)……...77
Figure 5.1: Constrained PC Algorithm Flowchart (Penalty Function Approach)……...84
Figure 5.2: Tension/Compression Spring……87
Figure 5.3: Convergence Plot for Problem 1……....88
Figure 5.4: Convergence Plot for Problem 2……....90
Figure 5.5: Convergence Plot for Problem 3……....92
Figure 6.1: Constrained PC Algorithm Flowchart (Feasibility-based Rule I)……..102
Figure 6.2: Solution History for Case 1……..107
Figure 6.3: Convergence of the Objective Function for Case 1……108
Figure 6.4: Convergence of the Strategies for Case 1……...109
Figure 6.5: Solution History for Case 2……111
Figure 6.6: Solution Convergence Plot for Case 2……112
Figure 6.7: Convergence of the Strategies for Case 2……113
Figure 6.8: Voting Heuristic……...114
Figure 6.9: Solution History for Agent Failure Case……..118
Figure 6.10: Convergence of the Objective Function for Agent Failure Case……..118
Figure 6.11: Convergence of the Strategies for Agent Failure Case……..119

Figure 6.12: Constrained PC Algorithm Flowchart (Feasibility-based Rule II)……...126
Figure 6.13: An Example of a Sensor Network Coverage Problem……..129
Figure 6.14: Solution History for Variation 1……134
Figure 6.15: Convergence of the Area of the Enclosing Rectangle and Collective Coverage for Variation 1……...……..135
Figure 6.16: Solution History for Case 1 of Variation 2……...136
Figure 6.17: Convergence of the Area of the Enclosing Rectangle and Collective Coverage for Case 1 of Variation 2……...……..137
Figure 6.18: Solution History for Case 2 of Variation 2……138
Figure 6.19: Convergence of the Area of the Enclosing Rectangle and Collective Coverage for Case 2 of Variation 2……139
Figure 6.20: Solution History for Case 3 of Variation 2……140
Figure 6.21: Convergence of the Area of the Enclosing Rectangle and Collective Coverage for Case 3 of Variation 2……....141
Figure B.1: Solution History of Probability Distribution using Nearest Newton Descent Schem.155
Figure C.1: Convergence of the Homotopy Functions for the SNCP using BFGS Method…….158
Figure C.2: Solution History of Probability Distribution using BFGS Method……159
Figure D.1: Case 1……...160
Figure D.2: Case 2……...161
Figure D.3: Case 3……...162
Figure D.4: Case 4……...163

LIST OF TABLES

Table 3.1: Performance using PC Approach……....45
Table 3.2: Performance Comparison of Various Algorithms solving Rosenbrock Function……...45
Table 4.1: Initial Sampling using a 3 × 7 Grid…….63
Table 4.2: Performance Comparison of Various Algorithms solving the MTSP……79
Table 5.1: Performance comparison of various Algorithms solving Problem 1…….88
Table 5.2: Performance comparison of various Algorithms solving Problem 2…….....90
Table 5.3: Performance Comparison of Various Algorithms Solving Problem 3……...92
Table 6.1: Summary of Results……..134

ABSTRACT

Complex systems generally have many components and it is difficult to understand the whole system only by knowing each component and its individual behavior. This is because any move by a component affects the further decisions/moves by the other components and so on. As the number of components grows, complexity may grow exponentially, making the entire system too cumbersome to be treated in a centralized way. The best option to deal with such a system is to decompose it into a number of sub-systems and treat it as a collection of sub-systems or a Multi-Agent System (MAS). The major challenge is to make these agents work in a coordinated way, optimizing their local goals and contributing the maximum towards optimization of the global objective. The theory of Collective Intelligence (COIN) using the distributed, decentralized, multi-agent optimization approach referred to as Probability Collectives (PC) is presented in this thesis. In PC, the self-interested agents optimize their local goals which contribute in optimizing the global goal.

In the current work, the original PC approach is modified by reducing the computational complexity and improving the convergence and efficiency. In order to further extend the PC approach and make it more generic and powerful, a number of constraint handling techniques are incorporated into the overall framework to develop the capability for solving constrained problems since real-world practical problems are inevitably constrained problems. In the course of these modifications, various inherent characteristics of the PC methodology are thoroughly explored, investigated and validated. The thesis demonstrated the validation of the modified PC approach by successfully optimizing the

Rosenbrock Function. The first constrained PC approach exploits various problem specific heuristics for successfully solving two test cases of the Multi-Depot Multiple Traveling Salesmen Problem (MDMTSP) and several cases of the Single Depot MTSP (SDMTSP). The second constrained PC approach incorporating penalty functions into the PC framework is tested by solving a number of constrained test problems. In addition, two variations of the Feasibility-based Rule for handling constraints are proposed. The Feasibility-based Rule I produced excellent results solving two cases of the Circle Packing Problem (CPP). The Feasibility-based Rule II is tested by successfully solving various cases of the Sensor Network Coverage Problem (SNCP). The results highlighted robustness of the PC algorithm solving all the cases of the SNCP.

NOMENCLATURE

ACO	:	Ant Colony Optimization
ACS	:	Ant Colony System
ANN	:	Artificial Neural Networks
ADOPT	:	Asynchronous Distributed OPTimization
B & B	:	Branch and Bound
BFGS	:	Broyden-Fletcher-Goldfarb-Shanno
CGA	:	Chaos Genetic Algorithm
CPP	:	Circle Packing Problem
COIN	:	Collective Intelligence
CDE	:	Cultural Differential Evolution
CX	:	Cycle Crossover
DFS	:	Depth First Search
DA	:	Deterministic Annealing
DE	:	Differential Evolution
DPSO	:	Discrete Particle Swarm Optimization
DCPSC	:	Distributed Coordination Framework for Project Schedule Changes
FoI	:	Field of Interest
FSA	:	Filter Simulated Annealing
GA	:	Genetic Algorithm
GENOCOP	:	GEnetic algorithm for Numerical Optimization of Constrained Problems
HAGA	:	Heuristic Adaptive Genetic Algorithm
HACO	:	Heuristic Ant Colony Optimization

HPSO	:	Heuristic Particle Swarm Optimization
LCGA	:	Loosely Coupled Genetic Algorithm
MGA	:	Modified Genetic Algorithm
MARL	:	Multi-Agent Reinforcement Learning
MAS	:	Multi-Agent System
MDMTSP	:	Multiple Depot Multiple Traveling Salesmen Problem
MDVRP	:	Multiple Depots Vehicle Routing Problem
MDVRPI	:	Multiple Depot Vehicle Routing Problem with Inter-Depot Routes
MOPC	:	Multi-Objective Probability Collectives
MTSP	:	Multiple Traveling Salesmen Problem
MTSPTW	:	Multiple Traveling Salesmen Problem with Time Window
MTSPTD	:	Multiple Traveling Salesmen Problem with Time Deadlines
PSO	:	Particle Swarm Optimization
PDP	:	Pick-up and Delivery Problem
PC	:	Probability Collectives
PAL	:	Punctuated Anytime Learning
RFID	:	Radio Frequency Identification
RL	:	Reinforcement Learning
Retsina	:	Reusable Task Structure-based Intelligent Network Agents
SOM	:	Self-Organizing Map
SNCP	:	Sensor Network Coverage Problem
SPC	:	Sequentially Updated PC
SA	:	Simulated Annealing
SDMTSP	:	Single Depot Multiple Traveling Salesmen Problem

TDVRP	:	Time-Dependent Vehicle Routing Problem
TSP	:	Traveling Salesman Problem
UAV	:	Unmanned Aerial Vehicles
UCAV	:	Unmanned Combat Aerial Vehicle
UV	:	Unmanned Vehicles
VRP	:	Vehicle Routing Problem
VRPTW	:	Vehicle Routing Problem with Time Window
VRPB	:	Vehicle Routing Problem with Backhauls
VRPM	:	Vehicle Routing Problem with Multiple Use of Vehicles
VRPTD	:	Vehicle Routing Problem with Time Deadlines

CHAPTER 1

INTRODUCTION

A Complex System is a broad term encompassing a research approach to problems in the diverse disciplines such as neurosciences, social sciences, meteorology, chemistry, physics, computer science, psychology, artificial life, evolutionary computation, economics, earthquake prediction, molecular biology, etc. Generally it includes many components that not only interact but also compete with one another to deliver the best they can to reach the desired system objective. Moreover, it is difficult to understand the whole system only by knowing each component and its individual behavior. This is because any move by a component affects the further decisions/moves by other components and so on. There are many complex systems in engineering such as Internet Search, Engineering Design, Manufacturing and Scheduling, Logistics, Sensor Networks, Vehicle Routing, Aerospace Systems, etc.

Traditionally, such complex systems were seen as centralized systems, but as the complexity grew, it became necessary to handle the systems using a distributed and decentralized optimization approach. In a distributed and decentralized approach, the system is divided into smaller subsystems and optimized individually to get the system level optimum. Such subsystems together can be seen as a collective which is a group of self-interested learning agents. Such a group can be referred to as a Multi-Agent System (MAS). In a distributed MAS, the rational and self-interested behavior of the agents is very important to achieve the best possible local goal/reward/payoff, but it is not trivial to make such agents work collectively to achieve the best possible global or system objective.

An emerging Artificial Intelligence tool in the framework of Collective Intelligence (COIN) for modeling and controlling distributed MAS referred to as Probability Collectives (PC) was first proposed by Dr. David Wolpert in 1999 in a technical report presented to NASA [1] and was further elaborated by his PhD student Stefan R. Bieniawski [2] in 2005. It is inspired from a sociophysics viewpoint with deep connections to Game Theory, Statistical Physics, and Optimization [3, 4]. A brief idea of PC and the related work are discussed in the following sections.

1.1 An overview of PC

PC is a distributed and decentralized approach, in which the individual variables of the system represent autonomous computational agents. Being self-interested these agents play a social game producing the output in some definite direction to iteratively optimize their local goals and simultaneously optimizing the system objective. This system objective can be seen as a measure of performance of the whole system.

These individual agents working on a distributed, decentralized platform iteratively perform the local computations and make decisions based on the collective goal being achieved. They collectively reach the equilibrium when no further improvement in the individual reward is possible by changing their actions further. This equilibrium can be referred to as Nash Equilibrium [5-8]. In order to avoid the conflict among the agents over the individual interests (also referred to as Tragedy of Commons [1, 2, 7, 9, 10]), cooperation among agents is also required.

Furthermore, unlike stochastic approaches such as Genetic Algorithms (GA) or Swarm Optimization, rather than deciding over the agent's moves/set of actions, PC allocates

probability values to each agent's moves. In each iteration, every agent independently updates its own probability distribution to select a particular action out of its strategy set (which is formulated from the particular interval of allowable states/values) having the highest probability of optimizing its local goal and which also results in optimizing the system objective [1-3]. This is based on the cooperation among the agents, or in other words, the prior knowledge communicated among the agents about the set of actions/strategies of all other agents.

The path planning of a group of Unmanned Aerial Vehicles (UAVs), deployment of a group sensors over a certain Field of Interest (FoI) covering the maximum possible area as well as solving the ever-growing urban traffic control problem on a truly distributed and decentralized platform can be seen as possible real life applications where PC can be useful.

1.2 Background and Motivation

There are many population based algorithms such as GA, Particle Swarm Optimization (PSO), Ant Colony Optimization (ACO), Wasp Colony System, Swarm-bot, etc. which have been used for solving complex problems. As mentioned previously, as the complexity grew these problems become quite tedious to be handled using the above algorithms. It needs the problem to be treated with a distributed and decentralized approach. The PC methodology is found to be an obvious tool that can deal with this increasing complexity because of its ability to treat the problem by decomposing it into sub-problems. This is an interesting characteristic of PC which lets the user deal with a variety of large problems more efficiently and in a more simplified way.

PC has been applied in variegated areas solving complex problems such as optimization of mechanical structures [11]; airplane fleet assignment to a group of airports minimizing the number of flights [12], collision avoidance [13], wireless sensor networks [14-20], university course scheduling [21], etc. Moreover, the comparison between GA and PC in [22] solving a variety of benchmark instances indicated the superiority of PC in dealing with functions with important characteristics such as multimodality, nonlinearity and non-separability.

Furthermore, Dr. Rodney Teo and Mr. Ye Chuan Yeo from the Cooperative Machines Lab of DSO National Laboratories, Singapore were interested to solve path planning problems for Multiple UAVs (MUAVs) by modeling them as Multiple Traveling Salesmen Problems (MTSPs), with a possible extension towards a decentralized optimization and control of these autonomous vehicles. This motivated the author of this thesis to solve the MTSP by applying PC as a distributed and decentralized optimization methodology.

In addition, the author of this thesis noticed the challenging area of the MTSP as one where PC can be quite useful and show its potential. This is because the MTSP is a NP-hard combinatorial optimization problem that, although easy to understand, remains a challenge for researchers over the years. The MTSP is important because of its large number of possible practical application areas such as logistics and manufacturing scheduling [23, 24], delivery and pick-up scheduling [25-27], school bus delivery routing [28, 29], satellite navigation systems [30], overnight security problem [31], etc. Furthermore, the tools used for solving the MTSP such as PSO, ACO, GA, etc. are found

to be computationally expensive and have slow convergence. The related discussion can be found in Chapter 4 of this thesis. This makes the PC to be seen as a promising option.

Most importantly, in the entire literature on PC, no generic constraint handling technique was found to have been developed for incorporating into the overall PC framework. Furthermore, having understood its potential in variegated areas and its ability to deal with real world complex problems, a generic and powerful constraint handling technique is required. This motivated the author to develop different constraint handling techniques to incorporate into PC and further test them for solving a variety of constrained problems.

In order to exploit this opportunity of testing the constraint handling techniques by solving a variety of problems, the Circle Packing Problem (CPP) was chosen as it has a great practical importance in production and packing for the textile, apparel, naval, automobile, aerospace, food industries, etc. [32]. In addition, the CPP has received considerable attention in the 'pure' mathematics literature but only limited attention in the operations research literature [33].

Furthermore, the Sensor Network Coverage Problem (SNCP) was noticed to be one of the most appropriate practical problems where the potential of constrained PC can be tested. This is because the sensor network plays a significant role in various strategic applications such as hostile and hazardous environmental and habitat exploration and surveillance, critical infrastructure monitoring and protection, situational awareness of battlefield and target detection, natural disaster relief, industrial sensing and diagnosis, biomedical health monitoring, seismic sensing, etc. [34-42]. The deployment or positioning of the individual sensor directly affects the coverage, detection capability,

connectivity and associated communication cost and resource management of the entire network [36, 37, 43, 44]. According to [45], coverage is the important performance metric that quantifies the quality and effectiveness of the surveillance/monitoring provided by the sensor network. This highlighted the requirement of an effective sensor deployment algorithm quantifying the coverage of the entire network [38, 43, 44].

In addition, the author of this thesis realizes the opportunity to explore various inherent characteristics of PC methodology when solving different variations of the above mentioned problems.

The next few subsections discuss the research objectives, the scope of the work as well as the original contributions arising from this work while achieving the objectives.

1.3 Research Objectives

The following are the objectives of the current research work.

1. Extend the PC approach further in order to investigate and exploit its inherent and desirable characteristics as well as key benefits of being a distributed, decentralized and cooperative approach. This includes modifying the PC approach to make it more efficient and faster.
2. Develop a more general and powerful approach of PC by incorporating constraint handling techniques necessary for solving constrained optimization problems, and further test and validate these techniques by solving a variety of challenging constrained problems.
3. Solve the path planning problem of Multiple UAVs (MUAVs) by modeling it as a MTSP and solving it by the PC approach.

1.4 Original Contributions Arising from this Work

Some of the objectives mentioned were accomplished and efforts are being made to achieve those which are proposed. The original contributions arising from this work are listed as follows:

1. Improvements to the original PC approach. The original PC approach was improved with a reduction in the computational complexity. A scheme for updating the solution space was developed which contributed to faster convergence and improved efficiency of the overall algorithm. In addition, PC was assumed to be converged when there was no further improvement in the final goal and/or when a predefined number of iterations was exceeded. Moreover, in order to make the solution jump out of possible local minima, a perturbation approach was developed and incorporated into the PC approach.
2. Furthermore, in the efforts to make PC more generic and powerful, a number of constraint handling techniques in the form of penalty function and feasibility-based rules were developed for incorporating into the overall algorithm. This allowed PC to solve practical problems which inevitably are constrained problems.
3. In the first attempt to develop a constraint handling technique, different problem specific heuristics such as the node insertion heuristic and the node elimination heuristic were proposed and incorporated into the PC algorithm for solving the NP-hard problem such as the Multiple Traveling Salesmen Problem (MTSP). The implementation of these techniques helped the algorithm to converge faster as well as

to jump out of local minima. In addition, in order to avoid the repeated sampling of the nodes by vehicles, a simple sampling procedure was also developed and implemented.

4. For the first time, the MTSP was solved using a distributed, decentralized and cooperative approach such as PC.
5. For the first time, the Circle Packing Problem (CPP) was solved using a distributed, decentralized approach such as PC. It also helped to demonstrate the desirable and key characteristic of a distributed approach to avoid the tragedy of commons.
6. For the first time, the important ability of PC to deal with the practically significant subsystem failure or agent failure problem was demonstrated by solving a specially designed case of the CPP incorporating agent failure.
7. Similar to the CPP and the MTSP, for the first time the Sensor Network Coverage Problem (SNCP) was solved using a distributed, decentralized approach such as PC.

1.5 Scope of the Research Work

The thorough literature review on PC was done to understand and investigate the basic concept of PC. The PC was modified and formulated. The modifications were done to increase the efficiency as well as the performance.

The Rosenbrock Function (with limited number of variables) was solved to validate the modified PC.

The further focus of the modification work was towards making the PC approach more generic and powerful by developing different constraint handling techniques to incorporate into the overall framework. In order to test the performance and viability of the constrained PC, a variety of challenging and rigorous constrained problems were

solved. This further includes developing necessary techniques assisting the constraint handling techniques.

The constrained PC was applied to two specially developed test problems (with limited number of nodes and vehicles) for the Multiple Depot MTSP (MDMTSP) as well as several randomly generated cases of the Single Depot MTSP (SDMTSP). To validate the performance and efficiency of the approach, the results were compared with some other algorithms solving the MTSP with approximately the same number of nodes and vehicles.

Furthermore, the constrained PC was applied to solve several constrained test problems available in the literature. The performance and efficiency of the solutions were compared with the contemporary algorithms solving these problems.

Moreover, two specially developed cases of the Circle Packing Problem (CPP) with limited number of circles were solved using the constrained PC approach. The performance and efficiency of the solution were validated by solving these cases several times with different initial configurations.

In addition to above, the constrained PC was applied to solve three specially developed cases of the Sensor Network Coverage Problem (SNCP) with limited number of sensors. Similar to the CPP, the performance and efficiency of the solution were validated by solving the cases several times with different initial configurations.

Although PC is a distributed and decentralized approach, all the computations and simulations were conducted on a single workstation.

1.6 Organization of the Thesis

The remainder of this thesis is organized as follows:

Chapter 2 provides the related literature, advantages and applications of the distributed, decentralized and cooperative approach. It also highlights its importance in the field of UAVs. **Chapter 3** gives details of using PC in the COIN framework. It includes the related work on PC, its characteristics, detailed formulation of modified unconstrained PC approach and the formulation of the Nash equilibrium. It also provides the testing and validation of the modified PC approach solving the Rosenbrock function**. Chapter 4** discusses the early effort of incorporating the constraint handling technique into the PC algorithm using problem specific heuristic techniques solving various cases of the combinatorial optimization problem such as the MTSP. This chapter also includes detailed discussed literature on MTSP as well as the literature on MUAVs path planning using the MTSP approach. The effort of handling constraints by incorporating the penalty function approach into the PC and solving a number of constrained test problems is described in **Chapter 5**. Two variations of the constraint handling technique of Feasibility-based Rule incorporated into the PC approach and solving several cases of the challenging Circle Packing Problem (CPP) and Sensor Network Coverage Problem (SNCP) are described in **Chapter 6.** Concluding remarks and recommendations for future work including other potential constraint handling approaches as well as possible practical applications of PC are discussed in **Chapter 7**. The necessary **Appendices (A to D)** are provided at the end of the thesis.

CHAPTER 2

DISTRIBUTED, DECENTRALIZED AND COOPERATIVE APPROACH

2.1 Distributed, Decentralized and Cooperative Approach

As mentioned previously, complex systems can be handled by decomposing them into smaller subsystems. These systems can be optimized individually to attain the system level optimum. This section discusses the literature related to the distributed, decentralized and cooperative approach in order to underscore its superiority over the centralized approach.

In a centralized system as shown in Figure 2.1, a single agent is supposed to have all the capabilities, such as problem solving, in order to alleviate the user's cognitive load. The agent is provided with the general knowledge which is useful to do a wide variety of tasks [46].

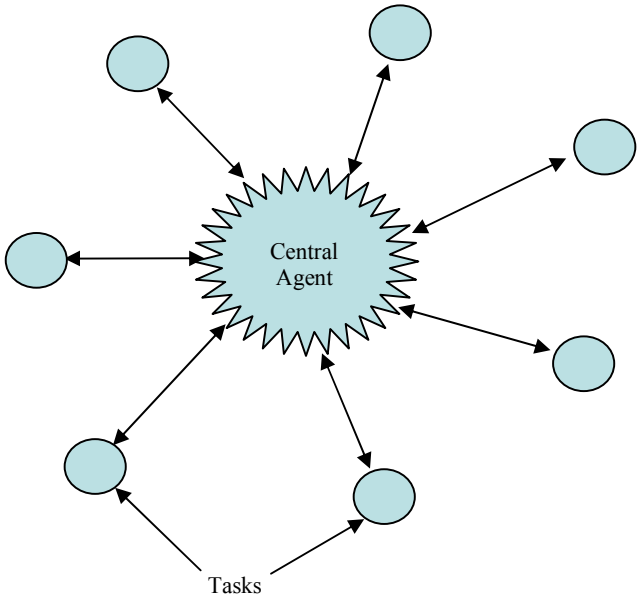


Figure 2.1: A Centralized System

This single agent needs enormous knowledge to deal effectively with the user requests; it needs to do required computations and also needs enough storage space, etc. Such a system may have a processing bottleneck. It also cannot be a robust system because of the possibility of single point failure which may have serious impact. For example, the process of information search over internet, information finding, filtering, etc. may overwhelm a centralized system [47, 48]. On the other hand, if the work is divided or decomposed into different tasks (e.g. information search, filtering, evaluation and integration) by classifying the expertise needed in the system, the potential bottleneck can be avoided. The classification of expertise here refers to the decomposition of the system into various agents representing experts of the particular area in the system. This expertise is the set of knowledge or actions the agent is supposed to have under the particular circumstance. It is therefore natural to have a distributed Multi-Agent System (MAS) having expertise for different heterogeneous information sources [46]. Working ants [49] and social bees [50] shown in Figure 2.2 are examples of natural biological MASs. In these biological MASs the bees and ants work independently with a global goal/system objective such as the collection of food.

Figure 2.2: Biological MASs [49, 50]

Furthermore, it is evident that such distributed, decentralized MAS is intended for open-ended design problems where there is no well-defined solution available and requires an in-depth search of the design space and also requires adaptability to user preferences [48]. Designing a MAS to perform well on a collective platform is non-trivial. Moreover, the straightforward agent learning in a MAS on collective platform cannot be implemented as it can lead to a suboptimal solution and also may interfere with individual interests [10]. This is referred to as 'Tragedy of Commons' in which the rational and self-interested independent individuals deplete the shared limited resource in a greedy way, even if it is well understood that it may not be beneficial for long term interest collectively for all, i.e. an individual may receive a benefit but on the other hand the loss will be shared among all [9]. This conflict may further lead to total system collapse. It also highlights the inadequacy of the classic Reinforcement Learning (RL) approach. An attractive option is to devise a distributed system in which different parts are referred to as agents; each having local control, having cooperation among one another and contributing towards a common aim. The cooperation among such agents is also important in order to avoid the replication of the work and/or information and reduction in the computational load [46].

In the case of such cooperative distributed and decentralized system, there is also a need for the right to share the information among the agents. It is important when dealing with security/military systems, banking systems, construction systems, etc. When these systems are used as centralized systems, the information from all the individual processes need to be communicated to the centralized controller, planner, scheduler, etc. This can be a hindrance when there may be some subsystems not willing to share the information with

the third party (or a centralized system). Such systems are very difficult to be optimized centrally unless the right to share the information or right to cooperate is clearly defined.

The centralized approach is not suitable for the modern highly dynamic environment as by the time the re-computation and re-distribution is done the environment is already changed. This makes it clear that a centralized system may add latency in the overall processing. On the other hand, the sub-problems can naturally be distributed and decentralized into agents and on the basis of the inter-agent-cooperation, quicker decisions can be taken and executed. According to [46, 47] autonomous agents perform computations independently from other agents, and contribute their results in a parallel and distributed fashion. Such collaborating agents make the system robust and flexible to adapt easily to the user preference changes. It is also believed that the cooperating agents in a distributed system collectively show better performance than a centralized system or a distributed system with agents having absolutely no cooperation.

Furthermore, the major challenges addressed in the collective behavior of the autonomous agents are how to enable the distributed agents to dynamically acquire their goal-directed cooperative behavior in performing a certain task, and how to apply the collective learning when addressing an ill-defined problem [51]. According to [46, 51-54], the crucial step is to formulate and develop the structure and organize the agents in a MAS ensuring achievement of the global optimum using the local knowledge. The other challenges are how to ensure the robustness, how to impart adaptability, how to maintain privacy, etc [55].

2.2 Advantages of the Distributed, Decentralized and Cooperative Approach

The above approach is used in variegated applications because of its various advantages. Some of the important advantages are discussed here.

1. It reduces the dependence on a single central system, thus reducing the chance of single point failure. This imparts the important characteristic referred to as robustness. This is essential in the field of UAV as failure of the centralized controller can be devastating and may result in collision. This is addressed in the related work on UAVs [2, 56-62].
2. It reduces computational overhead on a centralized system.
3. It reduces communication delays between resource agents and the coordinator as the information is transferred to the corresponding agents rather than the central system/agent [63]. It also solves the security issue such as sharing of information with a centralized system [46] as the exchange of local information is allowed between the entitled sub-systems/agents.
4. It allows the planning to be done on a shorter time scale, allowing additional planning/coordination rounds to take place.
5. It reduces complexity of the coordination problem making the controllers simpler and cheaper.
6. It reduces overall complexity of the problem formulation.
7. It imparts modularity and flexibility as one can reconfigure or expand the system by adding new components, sub-systems in the realm of the problem and still the

computations can be carried out locally and no single system overload occurs [58, 64]. This feature is referred to as scalability.

8. As the computations are done at the local level, the system can be made more flexible by dividing it into different problem solvers based on the abilities. This way the decision making becomes faster by exploiting parallelism [65].
9. It enhances the local communication and subsequently reduces the overwhelming energy consumption. According to [66], a sensor network communicating the raw data to the central point uses more energy as compared to local communication in the distributed approach.
10. The large and complex systems can be built into individual and simpler modules which are easier to debug and maintain [63, 54].
11. It imparts reliability through redundancy [51].
12. It expedites the decision-making process by reducing latency [51, 54].
13. It helps to improve the real time response and behavior [31].

Some of the important applications exploiting the above advantages are discussed below.

2.3 Applications of the Distributed, Decentralized and Cooperative Approach

The complex adaptive systems which work on distributed, decentralized and cooperative MASs, because of their various advantages, have been successfully applied to various engineering applications such as internet search, engineering design, manufacturing, scheduling, logistics, sensor networks, vehicle routing, UAV path

planning and many more are currently being researched. Some of the applications are discussed below.

Meta-heuristics is the field in which several search processes interchange the information while searching the optimal solution. To have a successful search process, the information exchange is very important. In case of a centralized system there is a central agent that carries out the interchange of the information between the various processes. In the decentralized cooperative system, each process has its own rules to decide when, what and how to interchange the relevant information with the relevant processes [53]. This approach of local communication reduces computational overload for the system, thus speeding up further decision making. According to [53], logistics problems such as the Vehicle Routing Problem (VRP), loading problem (Bin Packing) and location problem can be solved using a decentralized cooperative way.

The information sources available online are inherently distributed and are having different modalities. It is therefore natural to have a distributed MAS having expertise in different heterogeneous information sources. The robustness of such a system is most important because online information services are dynamic and unstable in nature. A multi-agent computational infrastructure referred to as Reusable Task Structure-based Intelligent Network Agents (Retsina) was proposed in [46]. This MAS searches information on the internet, filters out irrelevant information, integrates information from the heterogeneous sources and also updates the stored information in a distributed manner.

It is true that a sensor network communicating the raw data to the central point uses more energy as compared to the distributed approach [67]. This forces undesirable limits on the amount of information to be collected. This also imposes constraints on the amount

of information to be communicated as it affects communication overload. In the case of distributed and decentralized sensor networks, each node is an individual solving the problems independently and cooperatively. Similarly, the Radio Frequency Identification (RFID) system proposed in [68] is a distributed and decentralized system in which a subsystem or a sensor is not only a node collecting and transferring the data but also does the processing and computing job in order to take decisions locally. These subsystems, nodes or sensors are equipped with as much knowledge, logic, and rights as possible to make it a better alternative to the centralized system.

In the approach presented in [69], the agents in MASs handle the pre- and post-processing of various computational analysis tools such as spreadsheets or CAD systems in order to have a common communication between them. These agents communicate through a common framework where they act as experts and communicate their results to the centralized design process. As the number of design components increase, the number of agents and the complexity also increases. This results in the growing need for communication, making it computationally cumbersome for a centralized system. This is one of the reasons that the centralized approach is becoming insignificant in the concurrent design.

The construction industries are becoming more complex and involving many subcontractors. In a project, subcontractors perform most of the work independently and manage their own resources. Their work affects the other subcontractors and eventually the entire project. This makes it clear that the centralized system or control of such process is inadequate to handle such situations. This suggests that the control needs to be

distributed into individual sub-contractors in order to reschedule their projects dynamically [52].

A very different distributed MAS approach illustrated in [52] is the Distributed Coordination Framework for Project Schedule Changes (DCPSC). The objective is the minimization of the total extra cost each sub-contractor has to incur because of the abrupt project schedule change. Every sub-contractor is assumed to be a software agent to enhance communication using the internet. The agents in this distributed system compete and take socially rational decisions to maintain a logical sequence of the network. As the project needs to be rescheduled dynamically, extensive communication/negotiation among the subcontractor agents is required. In this approach these agents interact with one another to evaluate the impact of changes, simulate decisions in order to reduce the possible individual loss.

There is a growing trend in many countries to generate and distribute power/energy locally using renewable and non-conventional sources [70]. They are 'distributed' because they are placed at or near the point of energy consumption, unlike traditional 'centralized' systems where electricity/energy is generated at a remotely located, large-scale power plant and then transmitted down power lines to the consumer [71]. The same is true for the distribution of heating and cooling systems [70]. Depending on the peak and non-peak periods, the supply of energy has to be changed dynamically to accommodate the unforeseen behavior and current demand characteristics.

The distributed and decentralized systems related to robotic systems need very high robustness and flexibility. Examples of such applications are semi-automatic space exploration [72], rescue [73], and underwater exploration [74]. The most common

technique to ensure the robustness in robotic systems is to decompose the complex system into a distributed and decentralized system and also to introduce redundancy like that of wireless networks [55, 63, 67] in which even though some nodes fail, the network continues to operate [75]. Based on the experimentation, it is claimed that robotic hardware redundancy is necessary along with distributed and decentralized control [76]. It is worth to highlight that the self-reconfigurable robots such as MTRN [77] and PolyBot [78] use centralized control despite a good hardware flexibility. As these robots are less robust to failures, this disadvantage overwhelms flexibility.

Unmanned Vehicles (UVs) or Unmanned Aerial Vehicles (UAVs) are seen as flexible and useful for many applications such as searching targets, mapping a given area, traffic surveillance, fire monitoring, etc. [79]. They are very useful in the environment where the use of manned airplane mission may be dangerous or impossible. UV/UAV is a strong potential field in which decentralized and distributed approach can be applied for conflict resolution, collision avoidance, better utilization of air-space and mapping efficient trajectory. Some of these applications are discussed below.

The work demonstrated in [58] highlights the advantages of the distributed and decentralized approach when applied to Air/Ground Traffic Management. Based on the specialized job, the system is decomposed into three types of agents: airplanes/flight deck, air traffic service provider and airline operational control. In the traditional approach of Air/Ground Traffic Management the focus typically is on smooth, stable and orderly flow of traffic at the expense of efficiency. The modern concept of free flight presented in [58], avoids the latency in decision making in highly dense traffic conditions. If the planned trajectories are in conflict zone with one another the corresponding airplanes change the

trajectories without communicating to the centralized system, avoiding communication overhead and further delay in the decision. This makes the centralized system insignificant keeping the communication to local level. Such decentralized control strategy reduces conflict alerts (crisis) as well.

In the dynamic environment of a war field, the risk or potential threats change their positions with respect to time. The objective of the UAVs is to reach a particular point at the same time following the shortest path. The Probabilistic Map approach was proposed in [80], in which based on the local information and/or positions of the threats the local updating of the terrain map was done and was directly communicated with the neighboring UAVs. This cooperation helped real time updating of the map and also decentralized and distributed control by each airplane was possible. A similar approach could also be seen in the sensor networks to find out the location of the airplane [54] in which different nodes positioned topographically at different locations monitor the same aircraft but the perception of each node is different. This does not necessarily provide information about the aircraft movement/location unless the information gathered by all these sensors is integrated and communicated to neighboring nodes to produce an overall picture of aircraft movements, i.e. the overall solution.

The problem of UAVs formation following a stationary/moving target was addressed in [60]. In the formation, each UAV tries to keep a desired distance from the neighboring one. It was claimed that forming the objective function controlling all the UAVs in the flock showing the formation instincts such as collision avoidance, obstacle avoidance, formation keeping, etc. is non-trivial. Also, dealing with such single objective function is computationally expensive. This becomes worse when the formation tries to add more

UAVs. In [60], each aircraft was modeled separately for collision avoidance, formation keeping, target seeking, obstacle avoidance, etc. A similar approach was implemented by [61] for a fleet of Multiple UAVs (MUAVs). In this model, gyroscopic force was used to change the direction of individual airplanes to avoid possible collisions. In both of these proposed works, as each vehicle has local control and computations, the system shows high scalability to include a large number of vehicles in the fleet. Similar to this, computationally Distributed, Decentralized Optimization for Cooperative Control of Multi-agent Swarm-like Systems was proposed in [81].

The number of potential conflicts grows exponentially as the number of vehicles grows. This highlights the insignificance of centralized approach and shows the need for a distributed and decentralized control approach. The space under consideration may get added with more vehicles, increasing the possibility of conflict. This problem was handled by developing a model of field sensing approach having attraction towards the target and repulsion to the obstacle or neighboring vehicle in [66]. Each UAV was modeled as a magnetic dipole and was provided with a sensor to detect the magnetic field generated by other UAVs estimating the gradient. This estimate was used to go in the opposite direction to avoid conflict. There was absolutely no cooperation among the UAVs. The complex system such as airplane fleet assignment was also implemented successfully in [12]. It is discussed in the next chapter on Probability Collectives (PC).

The PC methodology in COIN framework through which the distributed, decentralized and cooperative optimization can be implemented is discussed in detail in the following chapter.

CHAPTER 3

COLLECTIVE INTELLIGENCE USING PC

An emerging Artificial Intelligence tool in the framework of Collective Intelligence (COIN) for modeling and controlling distributed MAS referred to as Probability Collectives (PC) was first proposed by Dr. David Wolpert in 1999 in a technical report presented to NASA [1]. It is inspired from a sociophysics viewpoint with deep connections to Game Theory, Statistical Physics, and Optimization [3, 4]. From another viewpoint, the method of PC theory is an efficient way of sampling the joint probability space, converting the problem into the convex space of probability distribution. PC considers the variables in the system as individual agents/players in a game being played iteratively [2, 12, 11]. Unlike stochastic approaches such as Genetic Algorithm (GA), Swarm Optimization or Simulated Annealing (SA), rather than deciding on the agent's moves/set of actions, PC allocates the probability values for selecting each of the agent's moves. At each iteration, every agent independently updates its own probability distribution over a strategy set which is the set of moves/actions affecting its local goal which in turn also affects the global or system objective [3]. The process continues and reaches equilibrium when no further increase in reward is possible for the individual agent by changing its actions further. This equilibrium concept is referred to as Nash equilibrium [5]. The concept is successfully formalized and implemented through the PC methodology. The approach works on probability distributions, directly incorporating uncertainty and is based on prior knowledge of the actions/strategies of all the other agents. The literature on PC is discussed in the following section.

3.1 Literature Review of PC

The approach of PC has been implemented for solving the problems from variegated areas [6-8, 10-22, 82-88]. The associated literature is discussed in the following few paragraphs.

It was demonstrated in [86] optimizing the Schaffer's function that the search process in PC is more robust/reproducible as compared to GA. In addition, PC also outperformed GA in the rate of descent, trapping in false minima and long term optimization when tested and compared for the multimodality, nonlinearity and non-separability in solving other benchmark problems such as Schaffer's function, Rosenbrock function, Ackley Path function and Michalewicz Epistatic function. Some of the fundamental differences between GA and PC were also discussed in [22]. At the core of the GA optimization algorithm is the population of solutions. In every iteration, each individual solution from the population is tested for its fitness to the problem at hand [22] and the population is updated accordingly. GA plots the best-so-far curve showing the fitness of the best individual in the last preset generations. In PC, on the other hand, the probability distribution of the possible solutions is updated iteratively. After a predefined number of iterations, the probability distribution of the available strategies across the variable space is plotted in PC when optimizing an associated Homotopy function. It also directly incorporates uncertainty due to both imperfect sampling and the stochastic independence of the agents' actions [22]. Furthermore, PC outperformed the Heuristic Adaptive GA (HAGA) [89], Heuristic ACO (HACO) [90] and Heuristic PSO (HPSO) [91] in stability and robustness solving the air combat decision-making for coordinated multiple target

assignment problem [87]. The above comparison indicated that PC can potentially be applied to wide application areas.

The superiority of the decentralized PC architecture over a centralized one was underlined in [10] when solving the 8-queens problem. Both the approaches differ from each other because of the distributed sample generation and updating of the probabilities in the former approach. In addition, PC was also compared with the backtracking algorithm referred to as Asynchronous Distributed OPTimization (ADOPT) [92]. Although the ADOPT algorithm is a distributed approach, the communication and computational load was not equally distributed among the agents. It was also demonstrated that although ADOPT was guaranteed to find the solution in each run, communication and computations required were more than for the same problem solved using PC.

The approach of PC was successfully applied in solving the complex combinatorial optimization problem of airplane fleet assignment having the goal of minimization of the number of flights with 129 variables and 184 constraints. Applying a centralized approach to this problem may increase the communication and computational load. Furthermore, it may add latency in the system and result in the growing possibility of conflict in schedules and continuity. Using PC, the goal was collectively achieved by exploiting the advantages of a distributed and decentralized approach by the airplanes selecting their own schedules depending upon the individual payoffs for the possible routes [12].

The potential of PC in mechanical design was demonstrated for optimizing the cross-sections of individual bars and segments of a 10 bar truss [11]. The problem was formulated as a discrete problem and the solution was feasible but was worse than those

obtained by other methods [93-95]. The approach of PC [11, 12] was also tested on the discrete problem of university course scheduling [21], but the implementation failed to generate any feasible solution. It is important to note that although the problems in [11, 12, 21] are constrained problems, the constraints were not explicitly treated or incorporated in the PC algorithms.

Very recently, two different PC approaches were proposed in [13] solving an autonomous conflict resolution problem for cooperating airplanes to avoid the mid-air collision. In the first approach, every airplane was assumed to be an autonomous agent. These agents selected their individual paths and avoided collision with other airplanes traveling in the neighborhood. In order to implement this approach, a complex negotiation mechanism was required for the airplanes to communicate and cooperate with one another. In the second approach which is a semicentralized approach, every airplane was given a chance to become a host airplane which computed and distributed the solution to all other airplanes. It is important to mention that the host airplane computed the solution based on the independent solution shared by previous host airplane. This process continued in a sequence until all the airplanes selected their individual paths. Both the approaches were validated solving an interesting airplane conflict resolution problem in which the airplanes were equidistantly arranged on the periphery of a circle. The targets of the individual airplanes were set as the opposite points on the periphery of the circle, setting the center point of the circle as a point of conflict. In both the approaches, collision constraints were molded into special penalty functions which were further integrated into the objective function using suitable balancing factors. Similar to the weighted sum multi-objective

approach [96, 97], the balancing factors were assigned based on the importance of the corresponding constraints.

A variation of the original PC approach in [1-4, 11, 12] referred to as Sequentially Updated PC (SPC) was proposed in [88]. The variation was achieved by changing the sampling criterion and the method for estimating the sampling space in every iteration. The SPC was tested by optimizing the unconstrained Hartman's functions and the vehicle target assignment type of game. The SPC performed better with higher dimension Hartman's functions only but failed to converge in the target assignment game. The approach of PC was also successfully applied in solving combinatorial optimization problems such as the joint optimization of the routing and resource allocation in wireless networks [14-20].

3.2 Characteristics of PC

The PC approach has the following key characteristics that make it a competitive choice over other algorithms for optimizing collectives.

1. PC is a distributed solution approach in which each agent independently updates its probability distribution at any time instance and can be applied to continuous, discrete or mixed variables, etc. [3, 1, 11, 12]. Since the probability distribution of the strategy set is always a vector of real numbers regardless of the type of variable under consideration, conventional techniques of optimization for Euclidean vectors, such as gradient descent, can be exploited.
2. It is robust in the sense that the cost function (global/system objective) can be irregular or noisy, i.e. it can accommodate noisy and poorly modeled problems [3, 10].

3. The failed agent can just be considered as one that does not update its probability distribution, without affecting the other agents. On the other hand, it may severely hamper the performance of other techniques [10].
4. It provides the sensitivity information about the problem in the sense that a variable with a peaky distribution (having highest probability value) is more important in the solution than a variable with a broad distribution, i.e. peaky distribution provides the best choice of action that can optimize the global goal [3].
5. The formation of the Homotopy function for each agent (variable) helps the algorithm to jump out of the possible local minima and further reach the global minima [12, 7, 8, 85].
6. It can successfully avoid the tragedy of commons, skipping the local minima and further reach the true global minima [10].
7. The computational and communication load is marginally less and equally distributed among all the agents [10].
8. It can efficiently handle problems with a large number of variables [12].

With PC solving optimization problems as a MAS, it is worth discussing some of its characteristics to compare the similarities and differences with Multi-Agent Reinforcement Learning (MARL) methods. Most MARL methods such as fully cooperative, fully competitive and mixed (neither cooperative nor competitive) are based on Game Theory, Optimization and Evolutionary Computation [98]. According to [98], most of these types of methods possess less scalability and are sensitive to imperfect observations. Any uncertainty or incomplete information may lead to unexpected behavior of the agents. However, the scalability of the fully cooperative methods such as

coordination free methods can be enhanced by explicitly using the communication and/or uncertainty techniques [98-101]. On the other hand, PC is scalable and can handle uncertainty in terms of probability. Moreover, the random strategies selected by any agent can be coordinated or negotiated with the other agents based on the social conventions, right to communication, etc. This social aspect makes PC a cooperative approach. Furthermore, indirect coordination based methods work on the concept of biasing the selection towards the likelihood of the good strategies. This concept is similar to the one used in the PC algorithm presented here, in which agents choose the strategy sets only in the neighborhood of the best strategy identified in the previous iteration. In the case of mixed MARL algorithms, the agents have no constraints imposed on their rewards. It is similar to the PC algorithm in which the agents respond or select the strategies and exhibit self-interested behavior. However, the mixed MARL algorithms may encounter multiple Nash Equilibria while in PC a unique Nash equilibrium can be achieved.

3.3 Conceptual Framework of PC

PC treats the variables in an optimization problem as individual self interested learning agents/players of a game being played iteratively [11, 12]. While working in some definite direction, these agents select actions over a particular interval and receive some local rewards on the basis of the system objective achieved because of those actions. In other words, these agents optimize their local rewards or payoffs, which also optimize the system level performance. The process iterates and reaches equilibrium (referred to as Nash equilibrium) when no further increase in the reward is possible for the individual agent through changing its actions further. Moreover, the method of PC theory is an

efficient way of sampling the joint probability space, converting the problem into the convex space of probability distribution. PC allocates probability values to each agent's moves, and hence directly incorporates uncertainty. This is based on prior knowledge of the recent action or behavior selected by all other agents. In short, the agents in the PC framework need to have knowledge of the environment along with every other agent's recent action or behavior.

In every iteration, each agent randomly samples from within its own strategy set as well as from within other agents' strategy sets and computes the corresponding system objectives. The other agents' strategy sets are modeled by each agent based on their recent actions or behavior only, i.e. based on partial knowledge. By minimizing the collection of system objectives, every agent identifies the possible strategy which contributes the most towards the minimization of the collection of system objectives. Such a collection of functions is computationally expensive to minimize and also may lead to local minima [2]. In order to avoid this difficulty, the collection of system objectives is deformed into another topological space forming the Homotopy function parameterized by computational temperature T [102-105]. Due to its analogy to Helmholtz free energy [2, 4, 102-106], the approach of Deterministic Annealing (DA) converting the discrete variable space into continuous variable space of probability distribution is applied in minimizing the Homotopy function. At every successive temperature drop, the minimization of the Homotopy function is carried out using a second order optimization scheme such as the Nearest Newton Descent Scheme [1-4, 6, 10-22, 82-89] or BFGS Scheme [7, 8], etc.

At the end of every iteration, each agent i converges to a probability distribution clearly distinguishing the contribution of its every corresponding strategy value. For every agent, the strategy value with the maximum probability value is referred to as the favorable strategy and is used to compute the system objective. The solution is accepted only if the associated system objective and corresponding strategy values are not worse than the previous iterations solution. In this way, the algorithm continues until convergence by selecting the samples from the neighborhood of the recent favorable strategies.

In some of the applications, the agents are also needed to provide the knowledge of the inter-agent-relationship. It is one of the information/strategy sets which every other entitled agent is supposed to know. There is also global information that every agent is supposed to know. This allows agents to know the right to model other agents' actions or behavior. All of the decisions are taken autonomously by each agent considering the available information in order to optimize the local goals and hence to achieve the optimum global goal or system objective.

The following section describes the detailed formulation of the PC procedure as an unconstrained approach.

3.4 Formulation of Unconstrained PC

Consider a general unconstrained problem (in minimization sense) comprising N variables and objective function G. In the context of PC, the variables of the problem are considered as computational agents/players of a social game being played iteratively [1, 2].

Each variable, i.e. each agent i is given a predefined sampling interval referred to as $\Psi_i \in \left[\Psi_i^{lower}, \Psi_i^{upper}\right]$. As a general case, the interval can also be referred to as the sampling space. The lower limit Ψ_i^{lower} and upper limit Ψ_i^{upper} of the interval Ψ_i may be updated iteratively as the algorithm progresses.

Each agent i randomly samples $X_i^{[r]}$, $r = 1, 2, ..., m_i$ strategies from within the corresponding sampling interval Ψ_i forming a strategy set $\mathbf{X}_i$ represented as

$$\mathbf{X}_i = \{X_i^{[1]}, X_i^{[2]}, X_i^{[3]}, ..., X_i^{[m_i]}\} \quad , \quad i = 1, 2, ..., N \tag{3.1}$$

Every agent is assumed to have an equal number of strategies, i.e. $m_1 = m_2 = ... = m_i = ... = m_{N-1} = m_N$. The procedure of modified PC theory is explained below in detail with the algorithm flowchart in Figure 3.1.

The procedure begins with the initialization of the sampling interval Ψ_i for each agent i, temperature $T >> 0$ or $T = T_{initial}$ or $T \to \infty$ (simply high enough), the temperature step size α_T ($0 < \alpha_T \leq 1$), convergence parameter $\varepsilon = 0.0001$, interval factor λ_{down} and algorithm iteration counter $n = 1$. The value of α_T, m_i and λ_{down} are chosen based on preliminary trials of the algorithm.

Step 1. Agent *i* selects its first strategy $X_i^{[1]}$. Agent *i* then samples randomly from other agents' strategies as well. These are random guesses by agent *i* about which strategies have been chosen by the other agents. All these form a 'combined strategy set' $\mathbf{Y}_i^{[1]}$ given by

$$\mathbf{Y}_i^{[1]} = \left\{X_1^{[?]}, X_2^{[?]}, ..., X_i^{[1]}, ..., X_{N-1}^{[?]}, X_N^{[?]}\right\} \tag{3.2}$$

The superscript [?] indicates that it is a 'random guess' and not known in advance. In addition, agent i forms one combined strategy set for each of the remaining strategies of its strategy set $\mathbf{X}_i$ (i.e. $r = 2,3,...,m_i$), as shown below.

$$\begin{aligned}
\mathbf{Y}_i^{[2]} &= \left\{ X_1^{[?]}, X_2^{[?]}, ..., X_i^{[2]}, ..., X_{N-1}^{[?]}, X_N^{[?]} \right\} \\
\mathbf{Y}_i^{[3]} &= \left\{ X_1^{[?]}, X_2^{[?]}, ..., X_i^{[3]}, ..., X_{N-1}^{[?]}, X_N^{[?]} \right\} \\
&\vdots \\
\mathbf{Y}_i^{[r]} &= \left\{ X_1^{[?]}, X_2^{[?]}, ..., X_i^{[r]}, ..., X_{N-1}^{[?]}, X_N^{[?]} \right\} \\
&\vdots \\
\mathbf{Y}_i^{[m_i]} &= \left\{ X_1^{[?]}, X_2^{[?]}, ..., X_i^{[m_i]}, ..., X_{N-1}^{[?]}, X_N^{[?]} \right\}
\end{aligned} \tag{3.3}$$

Similarly, all the remaining agents form their combined strategy sets.

Furthermore, every agent i computes m_i associated objective function values as follows:

$$\left[G\left(\mathbf{Y}_i^{[1]}\right), G\left(\mathbf{Y}_i^{[2]}\right), ..., G\left(\mathbf{Y}_i^{[r]}\right), ..., G\left(\mathbf{Y}_i^{[m_i]}\right) \right] \tag{3.4}$$

The ultimate goal of every agent i is to identify its strategy value which contributes the most towards the minimization of the sum of these system objective values, i.e. $\sum_{r=1}^{m_i} G(\mathbf{Y}_i^{[r]})$, hereafter referred to as the collection of system objectives.

Step 2. The minimum of the function $\sum_{r=1}^{m_i} G(\mathbf{Y}_i^{[r]})$ is very hard to achieve as the function may have many possible local minima. Moreover, directly minimizing this function is quite cumbersome as it may need excessive computational effort [2]. One of the ways to deal with this difficulty is to deform the function into another topological space by constructing a related and 'easier' function $f\left(\mathbf{X}_i\right)$. Such a

method is referred to as the Homotopy method [102-105]. The function $f(\mathbf{X}_i)$ can be referred to as 'easier' because it is easy to compute, the (global) minimum of such a function is known and easy to locate [102-104]. The deformed function can also be referred to as Homotopy function J parameterized by computational temperature T represented as follows:

$$J_i\left(q(\mathbf{X}_i),T\right)=\sum_{r=1}^{m_i}G(\mathbf{Y}_i^{[r]})-Tf(\mathbf{X}_i)\ ,\qquad T\in[0,\infty) \tag{3.5}$$

The approach of Deterministic Annealing (DA) is applied to minimize the Homotopy function in equation (3.5). The motivation behind this is its analogy to the Helmholtz free energy [6-8, 85, 106]. It suggests the conversion of discrete variables into random real valued variables such as probabilities. This converts the original collection of system objectives $\sum_{r=1}^{m_i}G(\mathbf{Y}_i^{[r]})$ into the 'expected collection of system objectives' $\sum_{r=1}^{m_i}E\left(G(\mathbf{Y}_i^{[r]})\right)$. Furthermore, a suitable function for $f(\mathbf{X}_i)$ is chosen. The general choice is to use the entropy function $\mathrm{S}_i=-\sum_{r=1}^{m_i}q(X_i^{[r]})\log_2 q(X_i^{[r]})$ [101-103]. The following steps of DA are formulated based on the analogy of the Homotopy function to the Helmholtz free energy [6-8, 85, 106] discussed in Appendix A.

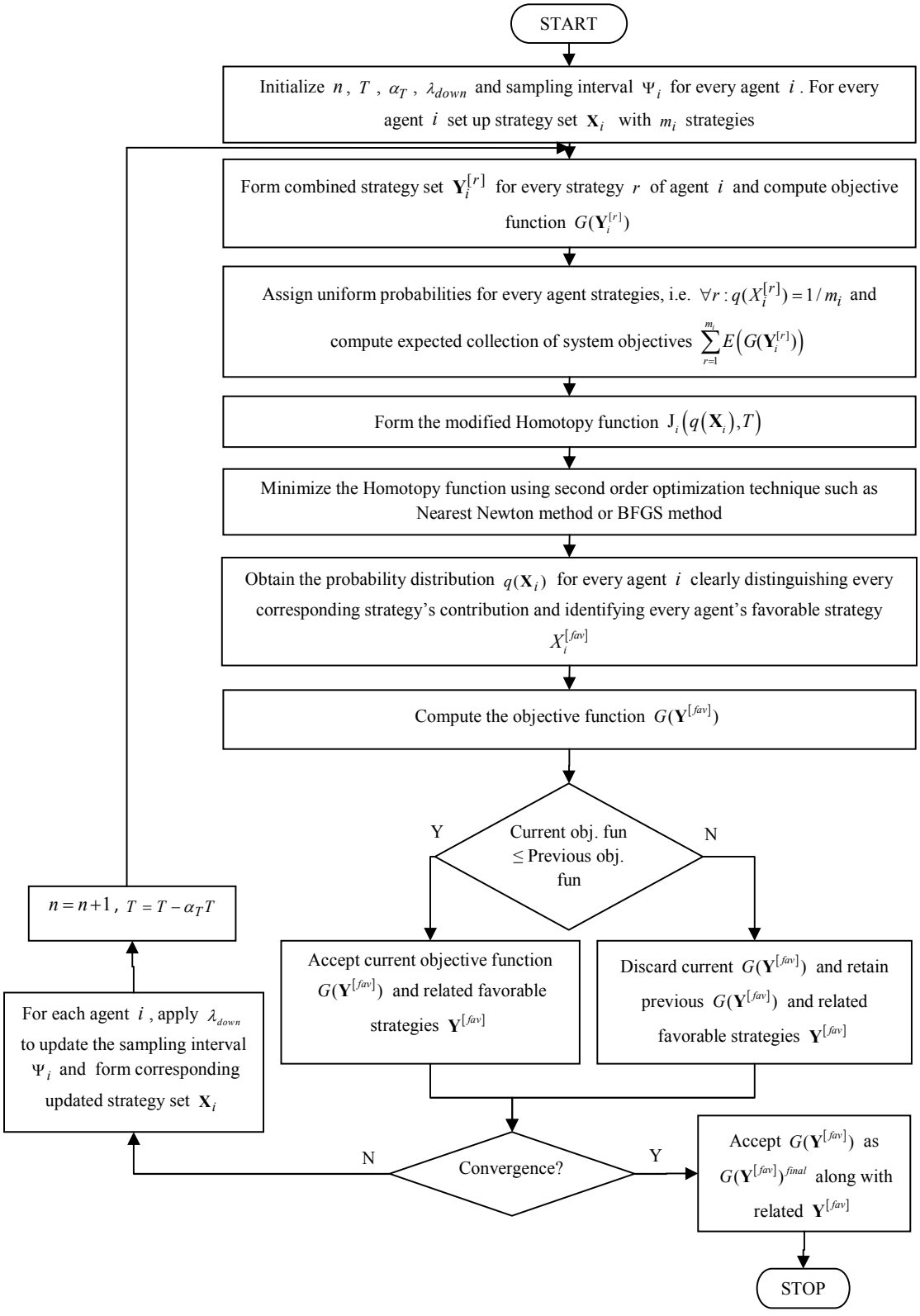


Figure 3.1: Unconstrained PC Algorithm Flowchart

(a) Agent i assigns uniform probabilities to its strategies. This is because, at the beginning, the least information is available (the largest uncertainty and highest entropy) about which strategy is favorable for the minimization of the collection of system objectives $\sum_{r=1}^{m_i} G(\mathbf{Y}_i^{[r]})$. Therefore, at the beginning of the 'game', each agent's every strategy has probability $1/m_i$ of being most favorable. Therefore, probability of strategy r of agent i is

$$q(X_i^{[r]}) = 1/m_i \quad , \quad r = 1, 2, ..., m_i \tag{3.6}$$

Each agent i, from its every combined strategy set $\mathbf{Y}_i^{[r]}$ and corresponding system objective $G(\mathbf{Y}_i^{[r]})$ computed previously, further computes m_i corresponding expected system objective values $E\left(G(\mathbf{Y}_i^{[r]})\right)$ as follows [1-4, 11, 12]:

$$E\left(G(\mathbf{Y}_i^{[r]})\right) = G(\mathbf{Y}_i^{[r]})\, q(X_i^{[r]}) \prod_{(i)} q(X_{(i)}^{[?]}) \tag{3.7}$$

where (i) represents every agent other than i. Every agent i then computes the expected collection of system objectives denoted by $\sum_{r=1}^{m_i} E\left(G(\mathbf{Y}_i^{[r]})\right)$. It also means that the PC approach can convert any discrete variables into continuous variable values in the form of probabilities corresponding to these discrete variables. As mentioned earlier, the problem now becomes continuous but still not easier to solve.

(b) Thus the Homotopy function to be minimized by each agent i in equation (3.5) is developed as follows:

$$
\begin{aligned}
J_i\left(q\left(\mathbf{X}_i\right),T\right) &= \sum_{r=1}^{m_i} E\left(G(\mathbf{Y}_i^{[r]})\right) - T\,\mathrm{S}_i \\
&= \sum_{r=1}^{m_i}\left(G(\mathbf{Y}_i^{[r]})\, q(X_i^{[r]}) \prod_{(i)} q(X_{(i)}^{[?]}) \right) - T\left(-\sum_{r=1}^{m_i} q(X_i^{[r]}) \log_2 q(X_i^{[r]}) \right) \\
&= G(\mathbf{Y}_i^{[1]})\, q(X_i^{[1]}) \prod_{(i)} q(X_{(i)}^{[?]}) + G(\mathrm{Y}_i^{[2]})\, q(X_i^{[2]}) \prod_{(i)} q(X_{(i)}^{[?]}) + \ldots \\
&\quad \ldots + G(\mathbf{Y}_i^{[m_i-1]})\, q(X_i^{[m_i-1]}) \prod_{(i)} q(X_{(i)}^{[?]}) + G(\mathrm{Y}_i^{[m_i]})\, q(X_i^{[m_i]}) \prod_{(i)} q(X_{(i)}^{[?]}) \\
&\quad - T\left(-\sum_{r=1}^{m_i} q(X_i^{[r]}) \log_2 q(X_i^{[r]}) \right)
\end{aligned}
\tag{3.8}
$$

where $T \in [0,\infty)$. When the temperature T is high enough, the entropy term dominates the expected collection of system objectives and the problem becomes very easy to be solved.

Step 3. The minimization of the Homotopy function in (3.8) can be carried out using Nearest Newton Descent Scheme [1-4, 6, 10-22, 82-89] as well as a suitable second order optimization approach such as Broyden-Fletcher-Goldfarb-Shanno (BFGS) scheme [7, 96, 97]. The Nearest Newton Descent Scheme and the BFGS scheme minimizing equation (3.8) are discussed in Appendix B and Appendix C, respectively.

Step 4. For each agent i, the optimization process converges to a probability variable vector $q(\mathbf{X}_i)$ which can be seen as the individual agent's probability distribution distinguishing every strategy's contribution towards the minimization of the expected collection of system objectives $\sum_{r=1}^{m_i} E\left(G(\mathbf{Y}_i^{[r]})\right)$. In other words, for each agent i, if a particular strategy r contributes the most towards the minimization of the objective compared to other strategies, its corresponding probability certainly increases by some amount more than those for the other strategies'

probability values, and so strategy r is distinguished from the other strategies. Such a strategy is referred to as a favorable strategy $X_i^{[fav]}$. As an illustration, the converged probability distribution for agent i may look like that shown in Figure 3.2 for a case where there are 10 strategies, i.e. $m_i = 10$.

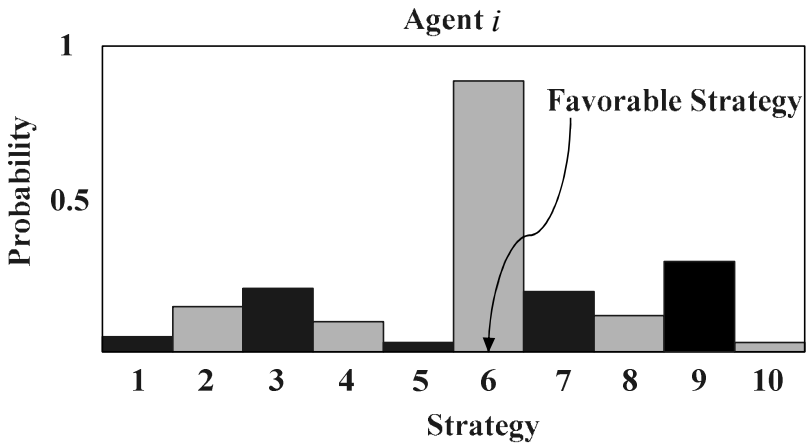


Figure 3.2: Probability Distribution of Agent i

Compute the corresponding system objective $G(\mathbf{Y}^{[fav]})$ where $\mathbf{Y}^{[fav]}$ is given by

$$\mathbf{Y}^{[fav]} = \left\{ X_1^{[fav]}, X_2^{[fav]}, ..., X_{N-1}^{[fav]}, X_N^{[fav]} \right\} \tag{3.9}$$

Step 5. If the current system objective $G(\mathbf{Y}^{[fav]})$ is not worse than that from the previous iteration solution, accept the current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$ as current solution and continue to *step 6*, else discard current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$, and retain the previous iteration solution and continue to *step 6*.

Step 6. If either of the two criteria listed below is valid, accept the current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$ as the final solution referred to as

$G(\mathbf{Y}^{[fav],final})$ and $\mathbf{Y}^{[fav],final} = \left\{ X_1^{[fav],final}, X_2^{[fav],final}, ..., X_{N-1}^{[fav],final}, X_N^{[fav],final} \right\}$,

respectively and stop, else continue to *step 7*.

(a) If temperature $T = T_{final}$ or $T \to 0$.

(b) If there is no significant change in the system objectives for successive considerable number of iterations (i.e. $\left\| G(\mathbf{Y}^{[fav],n}) - G(\mathbf{Y}^{[fav],n-1}) \right\| \le \varepsilon$).

Step 7. Every agent shrinks its sampling interval as follows:

$$\Psi_i \in \left[\left(X_i^{[fav]} - \lambda_{down} \left\| \Psi_i^{upper} - \Psi_i^{lower} \right\| \right), \left(X_i^{[fav]} + \lambda_{down} \left\| \Psi_i^{upper} - \Psi_i^{lower} \right\| \right) \right],\ 0 < \lambda_{down} \le 1 \tag{3.10}$$

where λ_{down} is referred to as the interval factor corresponding to the shrinking of sample space.

Each agent i then samples m_i strategies from within the updated sampling interval Ψ_i and forms the corresponding updated strategy set $\mathbf{X}_i$ represented as follows.

$$\mathbf{X}_i = \{ X_i^{[1]}, X_i^{[2]}, X_i^{[3]}, ..., X_i^{[m_i]} \} \quad , \quad \mathrm{i} = 1, 2, ..., N \tag{3.11}$$

Reduce the temperature $T = T - \alpha_T T$, update the iteration counter $n = n + 1$ and return to *step 1*.

On convergence, the above process reaches the equilibrium referred to as Nash equilibrium. The concept and detailed formulation of the Nash equilibrium achieved is described in the following section.

3.5 Nash Equilibrium

To achieve a Nash equilibrium, every agent in a MAS should have the properties of Rationality and Convergence [98-101]. Rationality refers to the property by which every agent selects (or converges to) the best possible strategy given the strategies of the other agents. The convergence property refers to the stability condition i.e. a policy using which every agent selects (or converges to) the best possible strategy when all the other agents use their policies from a predefined class (preferably same class). The Nash equilibrium is naturally achieved when all the agents in a MAS are convergent and rational. Moreover, a Nash equilibrium is guaranteed when all the agents use stationary policies, i.e. those policies that do not change over time. It is worth to mention here that all the agents in the MAS proposed using PC algorithm exhibit the above mentioned properties. For example, as detailed in the PC algorithm discussed in the previous few paragraphs, there is a fixed framework by which the agents select their own strategies and guess the other agents' strategies.

In any game, there may be a large but finite number of Nash equilibria present, depending on the number of strategies per agent as well as the number of agents. It is essential to choose the best possible combination of the individual strategies selected by each agent. It is computationally prohibitive to go through every possible combination of the individual agent strategies and choose the best out of it that can produce a best possible Nash equilibrium and hence the system objective.

As discussed in the detailed PC algorithm, in each iteration n, every agent i selects the best possible strategy referred to as the favorable strategy $X_i^{[fav],n}$ by guessing the possible strategies of the other agents. This information about its favorable strategy $X_i^{[fav],n}$ is made

known to all the other agents as well. In addition, the corresponding global knowledge such as system objective value $G(\mathbf{Y}^{[fav],n}) = G\left(X_1^{[fav],n}, X_2^{[fav],n}, X_3^{[fav],n}, ..., X_{N-1}^{[fav],n}, X_N^{[fav],n}\right)$ is also available to each agent which clearly helps all the agents take the best possible informed decision in every further iteration. This makes the entire system ignore a considerably large number of Nash equilibria but select the best possible one in each iteration and accept the corresponding system objective $G(\mathbf{Y}^{[fav],n})$. Mathematically the Nash equilibrium solution at any iteration can be represented as follows:

$$\begin{aligned}
G\left(X_1^{[fav],n}, X_2^{[fav],n}, X_3^{[fav],n}, ..., X_{N-1}^{[fav],n}, X_N^{[fav],n}\right) &\le G\left(X_1^{(fav),n}, X_2^{[fav],n}, X_3^{[fav],n}, ..., X_{N-1}^{[fav],n}, X_N^{[fav],n}\right) \\
G\left(X_1^{[fav],n}, X_2^{[fav],n}, X_3^{[fav],n}, ..., X_{N-1}^{[fav],n}, X_N^{[fav],n}\right) &\le G\left(X_1^{[fav],n}, X_2^{(fav),n}, X_3^{[fav],n}, ..., X_{N-1}^{[fav],n}, X_N^{[fav],n}\right) \\
&\vdots \\
G\left(X_1^{[fav],n}, X_2^{[fav],n}, X_3^{[fav],n}, ..., X_{N-1}^{[fav],n}, X_N^{[fav],n}\right) &\le G\left(X_1^{[fav],n}, X_2^{[fav],n}, X_3^{[fav],n}, ..., X_{N-1}^{[fav],n}, X_N^{(fav),n}\right)
\end{aligned} \tag{3.12}$$

where $X_i^{(fav),n}$ represents any strategy other than the favorable strategy $X_i^{[fav],n}$ from the same sample space Ψ_i^n.

Furthermore, from this current Nash equilibrium point with system objective $G(\mathbf{Y}^{[fav],n})$, the algorithm progresses to the next Nash equilibrium point with better system objective $G(\mathbf{Y}^{[fav],n+1})$, i.e. $G(\mathbf{Y}^{[fav],n}) \ge G(\mathbf{Y}^{[fav],n+1})$. As the algorithm progresses, those ignored Nash equilibria as well as the best Nash equilibria selected at previous iterations would be noticed as inferior solutions.

This process continues until there is no change in the current solution $G(\mathbf{Y}^{[fav],n})$, i.e. no new Nash equilibrium has been identified that proves the current Nash equilibrium to

be inferior. Hence the system exhibits stage-wise convergence to a unique Nash equilibrium and the corresponding system objective is accepted as the final solution $G(\mathbf{Y}^{[fav],final})$. As a general case, this progress can be represented as follows:

$$G(\mathbf{Y}^{[fav],1}) \geq G(\mathbf{Y}^{[fav],2}) \geq \ldots \geq G(\mathbf{Y}^{[fav],n}) \geq G(\mathbf{Y}^{[fav],n+1}) \geq \ldots \geq G(\mathbf{Y}^{[fav],final}).$$

It is worth to mention some of the key differences of the PC methodology presented here and the original PC approach [1-3, 11, 12]. In the present approach, fewer numbers of samples were drawn from the uniform distribution of the individual agent's sampling interval. On the contrary, the original PC approach [1-3, 11, 12] used a Monte Carlo sampling method which was computationally expensive and slow as the number of samples needed was in the thousands or even millions. Most significantly, the sampling in further stages of the PC algorithm presented here was narrowed down in every iteration by selecting the sampling interval in the neighborhood of the most favorable value in the particular iteration. This ensures faster convergence and an improvement in efficiency over the original PC approach in which regression was necessary to sample the strategy values in the close neighborhood of the favorable value.

3.6 Validation of the Unconstrained PC

There are a number of benchmark test functions for contemporary optimization algorithms like GAs and evolutionary computation. The Rosenbrock function is an example of a nonlinear function having strongly coupled variables and is a real challenge for any optimization algorithm because of its slow convergence for most optimization methods [107, 108]. The Rosenbrock function with N number of variables is given by

$$G = \sum_{i=1}^{N-1} \left[100\left(x_i^2 - x_{i+1}\right)^2 + \left(1 - x_i\right)^2 \right] \tag{3.13}$$

$$\text{lower limit} \le x_i \le \text{upper limit}, \quad i = 1, 2, ..., N$$

and the global minimum is at, $x_i = 1, \quad \forall i \quad i = 1, 2, ..., N$, where $G^* = 0$. A difficulty arises from the fact that the optimum is located in a deep and narrow parabolic valley with a flat bottom. Also, gradient based methods may have to spend a large number of iterations to reach the global minimum.

In the context of PC, every function variable x_i, $i = 1, 2, ..., N$ was represented as an autonomous agent. These agents competed with one another to optimize their individual values and ultimately the entire function value (i.e. global objective G). The Rosenbrock function with 5 variables/agents ($N = 5$) was solved here in which each agent randomly selected the strategies from within a predefined sampling interval Ψ_i. Although the optimal value of each and every agent x_i is 1, the allowable sampling interval Ψ_i for each agent was intentionally assigned to be different (as shown in Table 3.1). The procedure explained in Section 3.4 was followed to reach convergence. For all the trials, the temperature step size α_T, the interval factor λ_{down} and sample size m_i for each agent chosen were 0.9, 0.1 and 42, respectively. It is worth to mention that the temperature T was reduced on completion of every 10 iterations.

3.6.1 Results and Discussion

The problem was coded in MATLAB 7.4.0 (R2007A) on Windows platform using a Pentium 4, 3GHz processor speed and 512MB RAM. The results of 5 trials are shown in Table 3.1. The convergence plot for Trial 5 is shown in Figure 3.3. For Trial 5, the value

of the function G and associated variable values at iteration 113 was accepted as the final solution as there was no change in the solution for a considerable number of iterations. The variations in the function value and the number of function evaluations among the trials are due to the randomness in selection of the strategies.

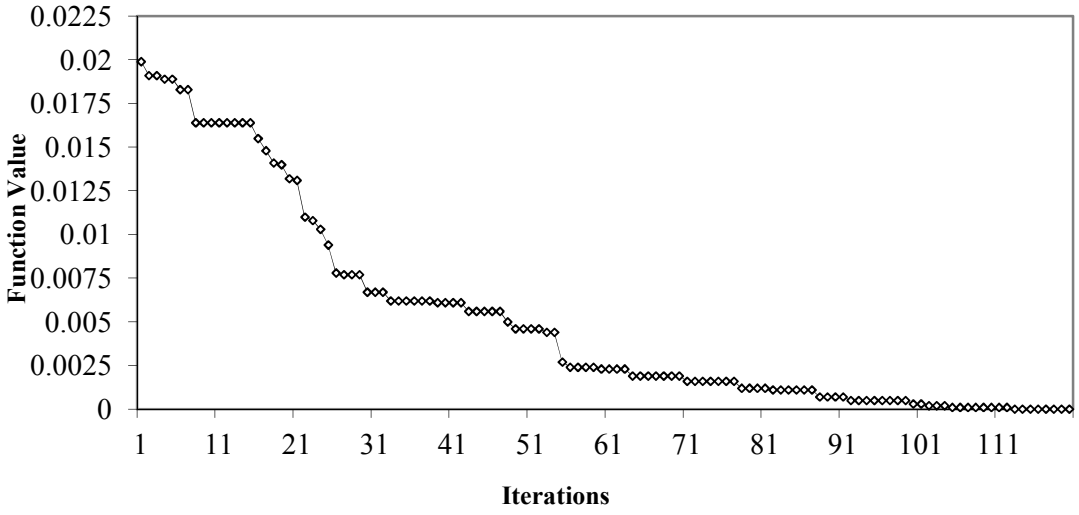


Figure 3.3: Convergence Plot for Trial 5

A number of researchers have solved the Rosenbrock function using various algorithms. Table 3.2 summarizes the results obtained by these various algorithms: Chaos Genetic Algorithm (CGA) [107], Punctuated Anytime Learning (PAL) system [108], Modified Differential Evolution (Modified DE) proposed in [109] and Loosely Coupled GA (LCGA) implemented in [110] are some of the algorithms demonstrated on the Rosenbrock function.

Table 3.1: Performance using PC Approach

Agents/(Variables)	Strategy Values Selected with Maximum Probability					
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Range of Values (Ψ_i)
Agent 1	1	0.9999	1.0002	1.0001	0.9997	-1.0 to 1.0
Agent 2	1	0.9998	1.0001	1.0001	0.9994	-5.0 to 5.0
Agent 3	1.0001	0.9998	1	0.9999	0.9986	-3.0 to 3.0
Agent 4	0.9998	0.9998	0.9998	0.9995	0.9967	-3.0 to 8.0
Agent 5	0.9998	0.9999	0.9998	0.9992	0.9937	1.0 to 10.0
Fun. Value G	2×10^{-5}	1×10^{-5}	2×10^{-5}	2×10^{-5}	5×10^{-5}	
Fun. Evaluations	288100	223600	359050	204750	249500	

Table 3.2: Performance Comparison of Various Algorithms solving Rosenbrock Function

Method	Number of Agents/Variables (N)	Function Value G	Function Evaluations.	Range of Values (Ψ_i)
CGA [107]	2	0.000145	250	-2.048 to 2.048
PAL [108]	2	≈ 0.01	5250	-2.048 to 2.048
	5	≈ 2.5	100000	-2.048 to 2.048
Modified DE 109]	2	1×10^{-6}	1089	-5 to 10
	5	1×10^{-6}	11413	-5 to 10
LCGA [110]	2	≈ 0.00003	--	-2.12 to 2.12

Within each of the results shown in Table 3.2, every variable was assigned an identical interval of allowable values, different from Table 3.1 where each variable was assigned a different allowable interval. Furthermore, even with the larger number of variables, the optimal function values in Table 3.1 are better than those in Table 3.2 except for the modified DE results. This makes it clear that the approach presented here produced fairly good comparable results to those produced by previous researchers.

The following few chapters discuss the efforts towards developing constrained handling techniques to incorporate into the PC in order to make it more generic and powerful optimization technique.

CHAPTER 4

CONSTRAINED PC (APPROACH 1)

4.1 Heuristic Approach

This chapter discusses an approach of handling the constraints in which the problem specific information is explicitly used. Basically, this approach involves development of problem specific heuristic techniques and further combines them with the unconstrained optimization technique to push the objective function into the feasible region.

The approach of PC was successfully combined with several problem specific heuristic techniques to handle the constraints. The approach was validated by solving two test cases of the Multiple Depot Multiple Traveling Salesmen Problem (MDMTSP) as well as several cases of the Single Depot MTSP (SDMTSP). In these cases the vehicles are considered as autonomous agents collectively searching the minimum cost path.

The following few sections describe the MTSP and Vehicle Routing Problem (VRP) in detail including the possible extension to Multiple Unmanned Aerial Vehicles (MUAVs). It is followed by the detailed explanation of the proposed MTSP specific heuristic techniques solving various cases of the MDMTSP and SDMTSP. The results are discussed at the end of the chapter.

4.2 The Traveling Salesman Problem (TSP)

The Traveling Salesman Problem (TSP) represents an important class of optimization problems also referred to as NP-hard. Although very easy to understand it is very hard to solve. This is because as the number of cities (n) increases, the number of possible routes

increases exponentially. According to [111] and [112], although these problems are inter-convertible, the computational complexity increases faster than any power of n. The inter-convertibility refers to an important characteristic that if a resource efficient solution to any one of them is discovered, it can be adapted to solve other NP-complete problems.

4.2.1 The Multiple Traveling Salesmen Problem (MTSP)

A generalization of the well known Traveling Salesman Problem (TSP) is the Multiple Traveling Salesman Problem (MTSP). As the name indicates, there is more than one traveling salesman in the MTSP. The different variants of the MTSP seem more appropriate for real-life applications [113]. The literature on TSP and MTSP suggests that as compared to the former, the latter received little attention from researchers. The generalized representation of MTSP is as follows.

Consider a complete directed graph $G = (\mathbf{N}, \mathbf{E})$, where $\mathbf{N} = \{1, 2, ..., n\}$ is the set of customer nodes or vertices or cities to be visited and $\mathbf{E}$ is the set of arcs connecting the customer nodes. The number of salesmen located at the depot is m. Generally, node 1 is considered as the depot which is the starting and end point of the salesmen's journey. All the salesmen start from the depot, visit the remaining nodes exactly once and return to the same depot. There is a non-negative cost c_{ij} associated with each arc connecting the nodes i and j which may represent the distance to travel or time to travel. Generally, the problems are symmetrical, hence $c_{ij} = c_{ji}$ for every set of arc (i, j). While following these routes, the total cost of visiting all the nodes is to be minimized. Generally, the cost metric is defined in terms of the total distance or the total time to travel the distance between the nodes, etc [114-117].

The applications of MTSP include the areas of logistics and manufacturing scheduling [23, 24], delivery and pick-up scheduling [25, 26], school bus delivery routing [27-29], satellite navigation systems [30], problems like overnight security problem [31], etc.

4.2.1.1 Variations of the MTSP

Depending on the applications and requirements, different constraints are added to the general MTSP resulting in different variations of the problem. These are discussed below.

The first variation to that is a single/multiple depot case in which all the salesmen return to their original depot/s at the end of their tours. This case is referred to as a fixed destination case [113].

The second variation is also a multiple depot case but it is not necessary that all the salesmen return to their original depots at the end of their tours but there is a constraint that all the depots should have the same number of salesmen at the end of all travels as at the beginning. This case is referred to as non-fixed destination case [113].

The third variation is that "the number of salesmen in the problem may be a bounded variable or fixed a priori" [113]. It is associated with assigning a fleet of vehicles to the predefined routes. Based on the demand, the fleet size may vary within the predefined limits. This has application to the transportation planning such as school bus routing, local truckload pick-up and delivery, as well as the satellite surveying system.

According to [113] the fourth variation is when the number of salesmen in the problem is not fixed. So whenever such salesmen are to be used, there is an associated cost with each salesman which is to be minimized by minimizing the number of salesmen in the solution.

The next variation is referred to as MTSP with Time Window (MTSPTW). This refers to the problem having the constraint that certain nodes need to be visited within specific period of time. The MTSP application to school-bus and airplane/ship fleet scheduling are characterized with such cases. MTSPTW has received attention recently for applications such as security crew scheduling. In MTSPTW, there is a Time Window ([earliest start time, latest end time]) applied to visit any customer node. The traveling salesman is allowed to visit the node within this Time Window only. If he arrives earlier than the 'earliest start time' he has to wait. If he arrives after 'latest end time' he will be penalized. The Time Window [earliest start time, latest end time] is also referred to as [lower bound, upper bound] [118]. The difference between the 'earliest start time' and the 'latest end time' is generally referred to as the service time. Sometimes there is only the upper bound associated with each customer node, then such MTSP can be referred to as MTSP with Time Deadlines (MTSPTD).

The Sixth variation is the restriction on the number of nodes each salesman visits. In addition to that, there can be an upper and lower limit on the distance traveled by a salesman.

4.2.2 The Vehicle Routing Problem (VRP)

The VRP is the extension to MTSP in which the cost of travel is minimized by considering traveling vehicles' capacity and the demand associated with each customer node.

Due to its economic importance in various distribution applications, VRP has received growing attention. For the general VRP, the routes are designed so that

1. Each demand location is served exactly once
2. The total demand on each route is less than or equal to the capacity of the vehicle assigned to that route
3. It has single depot (multiple depot in a few cases), i.e. all the vehicles start and complete the route at a single central depot.

In this sense, MTSP and VRP are closely related. In VRP there is capacity constraint which limits the amount of service demanded on a route. On the other hand, relaxing this constraint by giving unlimited capacity to the vehicles will turn the VRP into a MTSP. The generalized representation of VRP is as follows:

Let $G = (\mathbf{N}, \mathbf{E})$ be a graph, where $\mathbf{N} = \{1, 2, ..., n\}$ is the customer nodes set and $\mathbf{E}$ is the connecting arc set. Node 1 is the depot which is the starting and end point of the fleet of vehicles. Q is the capacity of an individual vehicle and in general, it is the same for all the vehicles. q_i is the demand at each customer node $i \in \mathbf{N} - \{1\}$. Similar to the MTSP, there is a non-negative cost c_{ij} associated with each arc connecting the nodes i and j which may represent the distance to travel or time to travel. Generally the problems are symmetrical, hence $c_{ij} = c_{ji}$ for every set of arc (i, j).

VRP has real-world applications in the manufacturing sector, including supply chains involving the delivery of goods from suppliers to factories and from factories to customers [118]. Some of the application areas according to [119] are the delivery of meals [120], packaged foods [121], chemical products [122], delivery of gas [123], etc. In such applications, the minimization of the number of vehicles is also important. The garbage

collection, mail delivery and public transportation are also some of the application areas in the service sector [118].

4.2.2.1 Variations of the VRP

Similar to the MTSP, there are some variations to the general VRP definition. As mentioned previously, these variants depend on the applications but more on the types of constraints to be satisfied. A detailed discussion on VRP variations can be found in [118].

Similar to the MTSP, there are variations to the VRP known as VRP with Time Window (VRPTW) and VRP with Time Deadlines (VRPTD).

There is a real world variation to VRP known as the Time-Dependent Vehicle Routing Problem (TDVRP) [118]. In this case, the traveling distance is fixed but the time is not fixed. This is because of peak (may take longer to complete a route) and non-peak (may be quicker as compared to peak hours) traffic hours.

Pick-up and Delivery Problem (PDP) is also a variation to the general VRP such that the corresponding pair of nodes should be on the same route. In this pair of nodes, one is a pick-up node (i-) and another is a delivery node (i+). So now this VRP is associated with two constraints. The first one is that the pick-up and delivery node should be on the same route. The second one is that delivery node should follow pick-up node. The former constraint is referred to as a 'pairing constraint' while the latter is referred to as a 'precedence constraint'. When Time Window is considered, the PDP is referred to as PDPTW [118].

Another real world variation is Vehicle Routing Problem with Backhauls (VRPB). In this variation, the customer nodes are partitioned into linehaul and backhaul nodes. The

linehaul nodes are the ones which are to be visited during the forward journey i.e. demand is to be delivered to these nodes, while the backhaul nodes are the ones from which the demand is brought back during the return journey to the depot. The precedence constraint has to be considered in this case. When Time Window is considered, the VRPB can be referred to as VRPBTW. According to the analysis of [124], the VRPB can be solved exactly for smaller instances and approximately for the larger sized problems.

In some of the cases where the fleet of vehicles is very limited or the length of the day is long as compared to the average duration of the routes, the VRP with Multiple Depots (MDVRP) is used. It is also referred to as Vehicle Routing Problem with Multiple Use of Vehicles (VRPM) [125].

A variation to the MDVRP is where the depots are considered as intermediate replenishment stations along the routes of the vehicles. Such MDVRP is referred to as Multi-Depot Vehicle Routing Problem with Inter-Depot Routes (MDVRPI) [126]. Applications such as the truck and trailer (leaving a trailer at a depot and carrying one for the next journey) can be solved using MDVRPI.

Some short details about other VRP variations such as multi-depot, mixed fleet and size, periodic vehicle routing, location-routing, inventory routing, stochastic vehicle routing, and dynamic vehicle routing problems can also be found in [118].

4.2.3 Algorithms and Local Improvement Techniques for solving the MTSP

There are various bio-inspired optimization algorithms such as GA, Ant Colony Optimization (ACO), Artificial Neural Networks (ANN), Particle Swarm Optimization (PSO) used to solve the variants of the TSP/MTSP/VRP. These algorithms are used in

conjunction with local improvement techniques to jump out of local minima and also to reduce the computational load and time. These algorithms are discussed, highlighting the various weaknesses when solving the MTSP/VRP. The respective importance of the local improvement techniques is also discussed. In the PC work presented here, efforts were taken to best exploit some of these local improvement techniques and also to overcome some of the weaknesses of the other algorithms.

4.2.3.1 Local Improvement Techniques

The commonly used local improvement techniques are insertion, savings principle, swapping of nodes, Simulated Annealing (SA), r-opt technique, etc. The insertion and the savings principle are generally applied in forming the routes while the swapping and r-opt techniques are applied for further improvement on forming of the feasible routes. In the insertion technique, the nodes are iteratively assigned to the vehicles to form the feasible routes. Once the feasible routes are formed, the above mentioned optimization techniques are applied to further optimize the routes. The savings principle is based on the traveling cost between the depot and two nodes d_{0i} and d_{0j} (where 0 is the depot, i and j are the nodes) and the traveling cost between the two nodes d_{ij}. The saving is computed as $S_{ij} = d_{0i} + d_{0j} - d_{ij}$. The savings principle ranks these savings in descending order, helping to cluster the closer nodes. The insertion and savings principle help to reduce the computational overload when forming the routes. These replace the possible exhaustive approach of using permutations to form the routes and selecting the best from them.

The nodes belonging to two routes are exchanged in the swapping technique which is a strong local improvement technique. In [127], the SA does the swapping by considering

the nodes from the neighborhood vehicle routes. In the r-opt technique, the total of r number of arcs is exchanged between the depots. The larger the value of r, the better the solution becomes. But the complexity increases as the operations required to test all the possible exchanges of r-arcs is proportional to n^r, where n is the number of nodes to be visited. This limits r to 2 or 3 which is commonly used.

There are two fundamental ways to deal with the variants of the TSP/MTSP/VRP. The first is the expansion of the MTSP into the standard TSP and the second is the simplification of the problem by using the approach of the cluster-first-route-second. Both the approaches are discussed below.

The work in [128, 129] using ANN is one of the earlier ones to deal with the MTSP by converting it into the standard TSP through the introduction of $m-1$ new imaginary/dummy bases/depots. This makes the total number of nodes to be $n+m-1$. The analysis of this approach by [25, 29, 128, 130, 131] indicates that this approach worsens the situation by increasing the size of the problem, to almost double. According to [132, 133], the extended cost matrix contains at least $m-1$ identical columns and rows. The problem becomes computationally tedious as compared to the standard TSP with the same number of nodes, i.e. ($n+m-1$). As indicated in [128, 129], when used with ANN this approach becomes computationally very expensive and could converge to only a valid and not an optimal solution.

Other similarly transformed TSP has been attempted by a few researchers using the ACO [131] and the GA [24]. The results demonstrated in both are encouraging and comparable but the performance declines as the number of nodes increase. This is because the problem size and the associated computational load increase with the addition of nodes.

An attempt is made to solve such transformed standard TSP using the PSO in [134] to solve the Single depot VRPTW. In this approach, the transformed standard TSP which is a route (sequence of nodes) of the vehicle is represented as a particle of $n+m-1$ dimensional vector. The formation of the particles is followed by the normalization in order to convert the particle sequences into integers or index sequences to compute the route cost.

An approach that reduces the computational load is the decomposition approach referred to as cluster-first-route-second [25, 31, 135-138]. The formation of the clusters of the nodes is based on the distances between them, the problem specific constraints such as turning radius limitations of the UAVs, [136] etc. It is followed by the formation of routes in the individual clusters and further modified for local improvement [135]. The alternative heuristic can be route-first-cluster-second where the longer routes can be constructed first followed by the shorter routes forming the clusters [118].

This approach to reduce the computational load is exploited in [139] using two Ant Colony Systems (ACSs) solving the VRPTW. The first ACS is for the minimization of the number of vehicles covering the maximum numbers of nodes per vehicle, i.e. one cluster is formed per vehicle. Once a feasible solution with this ACS is formed, a second ACS is used to minimize the traveling time of the vehicles covering all the points in the respective clusters, keeping the number of vehicles fixed. Both the colonies use the standard ACS framework.

Similarly, the decomposition approach is used in [137], in which the savings principle is used to form a complete solution with clusters of nodes. It is followed by the Sweep algorithm which exploits the information about the spatial coordinates and the centre of

gravity of the predefined clusters. The swapping of nodes, 2-opt techniques are also used for the local improvement. The results of this decomposition approach reveal that it can accommodate larger size problems as the clustering and local improvement technique reduce the computational efforts.

Such simplification techniques become important when computationally expensive and time consuming algorithms such as GA is used for solving combinatorial optimization problems such as the MTSP/VRP. The key part in the GA is the formation of the chromosomes population which plays an important role in the efficiency of the algorithm. Hence the search does not start from a single solution but from a population of solutions. Furthermore, with n number of nodes, the possible solutions or routes joining the nodes are as high as $n!$. Generating such possible solutions is quite expensive if done using the permutations, making the approach quite tedious at the first instance. This makes the GA limited to a small number of nodes and to a limited number of applications, such as the minimization of the newspaper printing time [23], where the number of nodes is quite few. On the other hand, if these n numbers of nodes are divided into k clusters, then the possible number of permutations required to form the chromosomes reduces to $(n/k)!k$ which is comparatively fewer than the approach without clusters. The approach of clustering the nodes around the depots based on the Euclidean distance is followed in [140, 141].

The type of ANN referred to as the Self-Organizing Map (SOM) is used to cluster/categorize/group the unorganized data. This makes the SOM useful for both the MTSP [130] and the VRP. This work is associated with the local optimization technique such as the 2-opt updating the routes by only considering the nodes in the neighborhood

zones/clusters. These clusters need to be very compact in order to reduce the computational overhead. This is a disadvantage of this approach restricting it to a fewer number of cities and also reducing the convergence speed. It outperforms the elastic net approach presented in [142] in terms of computational time because the elastic net approach considers all the nodes while updating the routes representing heavy interconnection.

This section described the various local improvement techniques necessary to be used along with the optimization algorithms. The next section summarizes these algorithms used to solve the MTSP/VRP.

4.2.3.2 Various Algorithms for Solving the MTSP

In the general ACO technique [143], the complexity in coding the ant trails and the associated local and the global updating makes the convergence quite slow and uncertain. This becomes worse when used NP-hard problems like the MTSP/VRP and needs to be used along with the heuristic techniques such as swapping of nodes, r-opt technique, node insertion, candidate list, etc. For example, the application of ACO for solving the VRPBTW in [137] uses the specially developed insertion technique for accommodating the linehaul and backhaul constraint. It is followed by local swapping. For a short discussion on some of the insertion and local swapping approaches as well as the one used in the current work, refer to Section 4.3.3 and 4.3.5.

As the PSO is weak in local search, it may get trapped in local minima. Special hybridization techniques are needed to jump out of the local optima [127]. Generally, the parameter of the PSO referred to as ‘inertia weight’ is used to jump out of the local optima. But this becomes quite ineffective when used for solving combinatorial optimization

problems such as MTSP/VRP [134]. One alternative suggested in [134] is to increase the number of particles to at least 10 times more than the number of nodes, resulting in computational overload. In addition, increasing the number of iterations does not improve the solution when the number of particles reaches above a certain level. Furthermore, in the applications of the VRP, the handling of the capacity constraints increases the computational overload further and also needs swapping, insertion, etc techniques as repair work to restore feasibility [127].

In the recent work on PSO in [134], the penalty coefficient is directly incorporated into the objective function to move the solution towards feasibility. The hybridization is presented in [127] combining the Discrete PSO (DPSO) with a local improvement/search technique such as SA. The job of the SA is to swap the nodes from the neighborhood vehicle routes. Similar to the mutation operation in GA, SA exchanges pairs in a vehicle route. The results indicate that the hybrid DPSO outperforms the GA used with the 2-opt technique.

According to [24], the GA also needs special attention in using the crossover operators solving the MTSP/VRP. This is because the direct exchanging of the parts of the parent solutions may produce infeasible solutions (a node may appear more than once or may not appear at all). This necessitates repair work to be done to make the offspring solutions feasible. The worse part is that the repair work may destroy the characteristics of the parent solutions as well as the offspring solutions. Eventually the algorithm becomes very slow due to the delayed convergence. In order to balance between the rapid convergence and avoidance of entrapment in local optima due to premature convergence, the general GA is modified and referred to as the Modified GA (MGA). In the MGA, the best

chromosome or solution found so far is accepted as one of the parent solutions and another parent is selected randomly. The crossover between these two parents is done using a 'cycle crossover' (CX) method with a high possibility of retaining the characteristics of the best parent [24].

The performance of the steady state GA, in which the best found offspring replaces the worst chromosome in the population, may become worse when used with combinatorial optimization problems like the MTSP/VRP. In order to make this approach faster at the first instance, the chromosomes are formed by inserting the nodes to form the vehicle routes [144]. The crossover and mutation may produce infeasible solutions. Similar to [134], the repair work is totally avoided in [144] by incorporating the penalty coefficient into the objective function. The results using this modified steady state GA indicates that a large population and high computational efforts are needed to reach convergence.

The computationally exhaustive approach such as ANN is first time used by Hopfield et al. [145] solving the TSP. According to the analysis of its re-implementation in [129, 146], the ANN approach is quite complex and unable to produce guaranteed MTSP solution. It is also not good at handling the capacity constraints and time window constraints and is essentially limited to spatial applications [118].

A unique type of SOM referred to as the elastic net approach originally proposed in [111] solving the standard TSP is also used to solve the MTSP [142]. The approach works on the basis of the pulling and attracting forces acting on the rubber band. The approach is essentially limited to the spatial applications minimizing the traveling distance between various points on the plane.

The tabu search is essentially a local search technique, improving the available feasible solution using the memory of the points visited previously. This avoids the repeated visiting of points for the predefined upcoming steps. Local search techniques such as diversification and intensification are used which are basically the combinations of the exchange of arcs in a depot and within the depots [119]. According to [125, 147-149], the tabu search finds the optimum when used with the limited class of problems such as MDVRP because of its tendency of getting trapped in local minima. This is because, although the memory of the visited points makes the algorithm efficient, some points which would have been useful at some moment may not be available. The next section discusses the application of the MTSP/VRP to the path planning of the MUAVs.

4.2.3.3 Solving Multiple Unmanned Aerial Vehicles (MUAVs) Path Planning Problem as a MTSP/VRP

The path planning/routing of MUAVs is an important potential application but there has been very few published work in the area of applying the MTSP/VRP to the MUAVs. The possible application areas are planning the path of MUAVs to visit the target points the locations of which are known in advance, planning the path for surveillance [138], etc. Applications in military intelligence, reconnaissance and surveillance with the objective of the total travel time and distance minimization of the multiple UAVs are some of the areas highlighted in [150]. This underscores the suitability of the MTSP approach for the above objective. In the practical sense, the implementations by [135, 136, 151] considers the famous Dubin's car model [152] in order to take into account the dynamic constraints such as the turning radius of the vehicles.

The fundamental approaches discussed previously in Sections 4.2.3.1 and 4.2.3.2 are also found in the published work on MUAVs. The UAV path planning to visit the predefined target locations [136] is based on the spanning tree formation using the Prim's algorithm, which is basically an insertion algorithm iteratively adding the nodes. The cluster-first-route-second and insertion technique is combined in [135]. The clusters belonging to the vehicles are formed by iteratively adding unselected nodes and avoiding the further repair work as in [24, 127]. The best possible routes in the individual clusters optimized using the Concorde TSP Solver [153]. The computationally exhaustive tool such as Branch and Bound (B & B) technique with best-first-search approach is used. Also the memory requirement for the B & B technique is very high and may worsen if number of nodes increase.

Depending on the fuel capacity, every Unmanned Combat Aerial Vehicle (UCAV) in a fleet is assigned a cluster of target nodes using the tabu search technique [138]. Once the clusters are finalized, the corresponding routes are formed using the branch and bound approach combined with an associated insertion heuristic technique proposed in [132]. The approach was quicker to converge but produced sub-optimal solution.

The above discussion on the various techniques and algorithms used so far in solving the MTSP/VRP indicates that they become slow as the problem size increases and limits the problem size. On the other hand, the distributed, decentralized approach can divide the computational load which can be handled more effectively and moreover may accommodate larger size problems. As a part of this research effort, PC is applied to the Multi Depot MTSP (MDMTSP) [84] in order to test its effectiveness in this area. The next

few sections discuss the implementation of constrained PC solving the two test cases of the MDMTSP.

4.3 Solution to the Multiple Depot MTSP (MDMTSP) using PC

In the proposed Multi Depot MTSP (MDMTSP), the traveling vehicles were considered as autonomous agents. These vehicles were assigned the routes, where every route was considered as a strategy. These routes were formed from randomly sampled nodes. This is in contrast to the Rosenbrock function problem in Section 3.6, where every agent randomly selected strategies within the range of values specified to each agent.

The sampling of routes was an important step in applying PC to the MTSP. The sampling procedure is explained in the following section by an illustrated example with 3 vehicles and 15 nodes.

4.3.1 Sampling

In problems like the MTSP, there is a strong possibility that some nodes may get sampled by more than one vehicle and also some may not get selected. This was avoided in the first sampling phase forming the individual agent routes. As there were 3 vehicles and 15 nodes, a grid of 3 × 7 was formed making sure that every vehicle is assigned at least one node. This is shown in Table 4.1 and the procedure is explained below.

Table 4.1: Initial Sampling using a 3 × 7 Grid

Vehicle 1		3	5		15		1
Vehicle 2	4	8	11	10		13	
Vehicle 3	6	9		2	12	14	7

The first node (node 1) randomly selects a cell from the grid, thus assigning itself to the corresponding vehicle. The cell once selected becomes unavailable for further selection by the remaining nodes. This process continues until all the nodes are assigned to the vehicles. The nodes assigned to a particular vehicle form an individual vehicle route. These routes are the strategies randomly assigned to the vehicles. According to the grid shown in Table 4.1, 4 nodes are assigned to vehicle 1 forming a route/strategy represented as D1-3-5-15-1-D1. Similarly, route/strategy formed by vehicle 2 is D2-4-8-11-10-13-D2 and by vehicle 3 is D3-6-9-2-12-14-7-D3, where D1, D2 and D3 represent the depots corresponding to the vehicles.

The above process continued for 45 times forming 45 possible combinations covering all the nodes using three vehicles. Out of these 45 combinations, the first 15 were assigned to vehicle 1 forming 15 combined strategy sets. Similarly, the next 15 combined strategy sets were assigned to vehicle 2 and the remaining 15 to vehicle 3. An example of a combined strategy set for a vehicle is represented as

$$\mathbf{Y} = \{[3 \quad 5 \quad 15 \quad 1],[4 \quad 8 \quad 11 \quad 10 \quad 13],[6 \quad 9 \quad 2 \quad 12 \quad 14 \quad 7]\} \qquad (4.1)$$

According to the PC framework, in the combined strategy set shown in equation (4.1), the first route D1-3-5-15-1-D1 was assumed to be selected by vehicle 1 while the remaining two routes were guessed by it as the strategy sets selected by the remaining two vehicles. In order to reduce the computational load and reach convergence faster, the number of combined strategies/routes per vehicle can be limited to a small number (15 in the current example). This makes it clear that there were 15 strategies/routes considered per vehicle and they were assigned uniform starting probability values (1/15 to each route in the current example) by the corresponding vehicle.

4.3.2 Formation of Intermediate Combined Route Set

Solving this problem according to the algorithmic procedure explained in Chapter 3, every vehicle converged to a probability distribution clearly distinguishing the most favorable route i.e. the one with the highest probability value. As discussed in the preceding Section 4.3.1, every vehicle has a strategy set with randomly sampled routes as its strategies. These strategies are represented on the X-axis of the probability distribution plots shown in Figure 4.1. The routes are denoted in the form R##. For example, R11 represents route 1 of vehicle 1, while R115 represents route 15 of vehicle 1. Similarly, R21 represents route 1 of vehicle 2 and so on.

As shown by an example illustration in Figure 4.1, vehicle 1 converges to R16, vehicle 2 converges to R29 and vehicle 3 converges to R314. These most favorable routes are joined to form the 'intermediate combined route set' represented as $\{[R16],[R29],[R314]\}$.

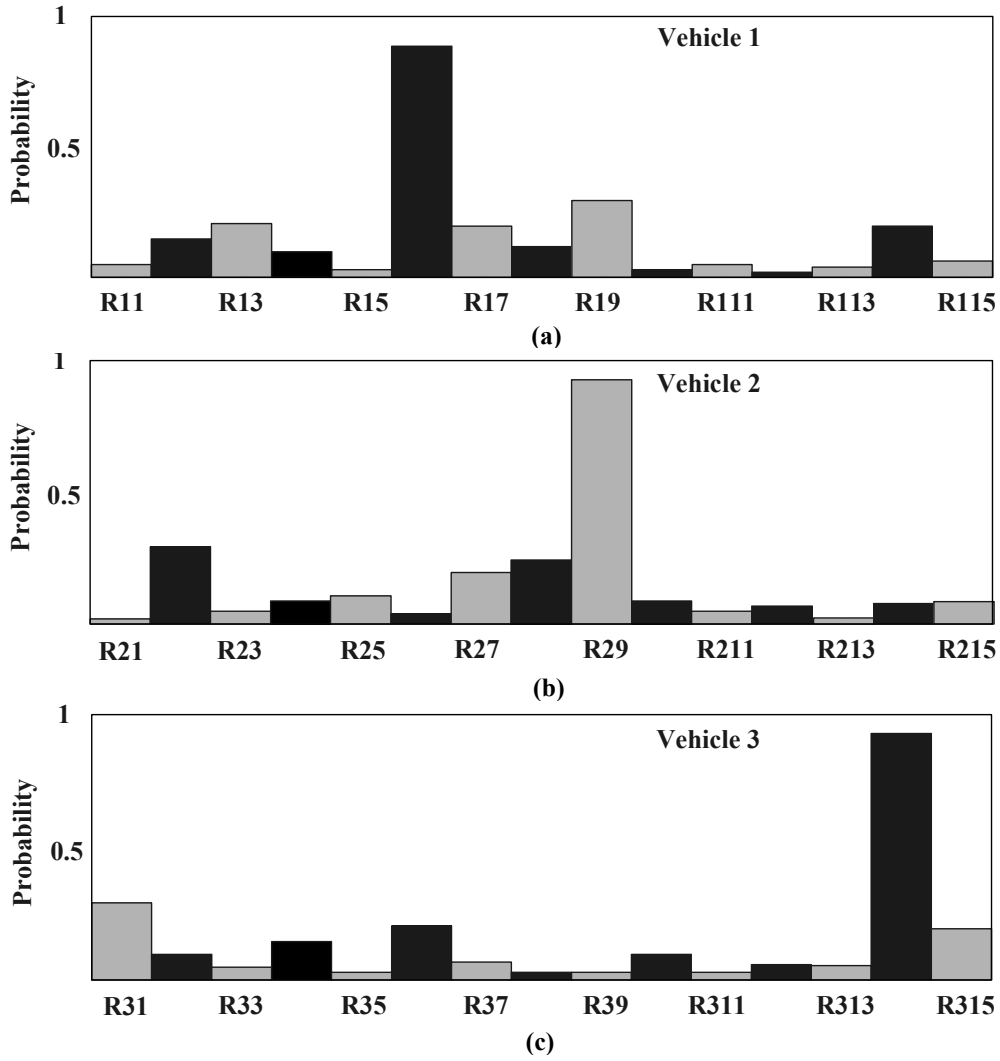


Figure 4.1: Illustration of Converged Probability Plots

This intermediate combined route set (also referred to as the 'intermediate solution') has a high probability of being infeasible. Infeasibility can be caused by one or more of the following four reasons:

1. The total number of nodes in the intermediate solution is less than the total number of available nodes.
2. A node has been selected by more than one vehicle, causing repetition error.
3. Some nodes are left unselected because of reason 2.

4. The total number of nodes in the combined strategy/route set formed from the most favorable strategies/routes is more than the total number of available nodes.

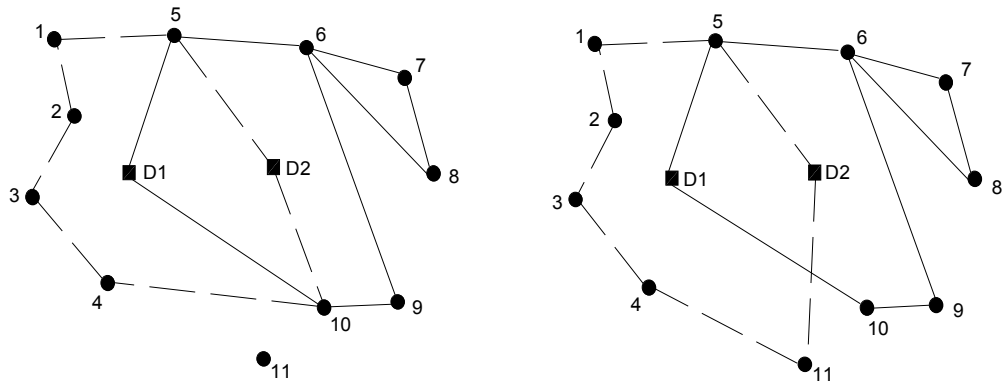
The above reasons arise because of the nature of the PC algorithm. The most favorable routes selected by each vehicle are generally from different combined strategy/route sets belonging to the individual vehicles. Reason 1 was avoided by repeating the sampling process until the valid intermediate solution is achieved. The next couple of sections discuss the implementation of the heuristic techniques employed to deal with reasons 2 to 4. Although applied to the entire intermediate solution, for simplicity and better clarity the heuristic techniques are described only for two vehicles' most favorable routes.

4.3.3 Node Insertion and Elimination Heuristic

The insertion heuristic was carried out in order to deal with reasons 2 and 3. Generally in the insertion technique a node is iteratively added to the available route and checked for feasibility and improvement. The insertion technique was also carried out in [135] where the Branch and Bound technique was implemented to perform Best-First Search for Unmanned Aerial Vehicle (UAV) path planning. The insertion heuristic steps 1 to 4 followed in this report are explained below.

1. The number of nodes of the combined strategy/route set formed from the most favorable strategy/route set was counted.
2. The missing nodes were identified.
3. The repeated nodes were replaced by the missing ones based on the total cost improvement. It is illustrated in Figure 4.2 with an example of 2 vehicles and 11 nodes. Figure 4.2(a) indicates that nodes 5 and 10 are visited by both vehicles and node 11 is left unselected. Accordingly, there are four ways to insert node 11 into a route. These

are inserting it in place of the repeated node 10 or 5 along the route belonging to either vehicle. Figure 4.2(b) shows the lowest cost combined route/strategy set among these four different ways.



Vehicle 1: D1-5-6-7-8-6-9-10-D1
Vehicle 2: D2-5-1-2-3-4-10-D2

Vehicle 1: D1-5-6-7-8-6-9-10-D1
Vehicle 2: D2-5-1-2-3-4-11-D2

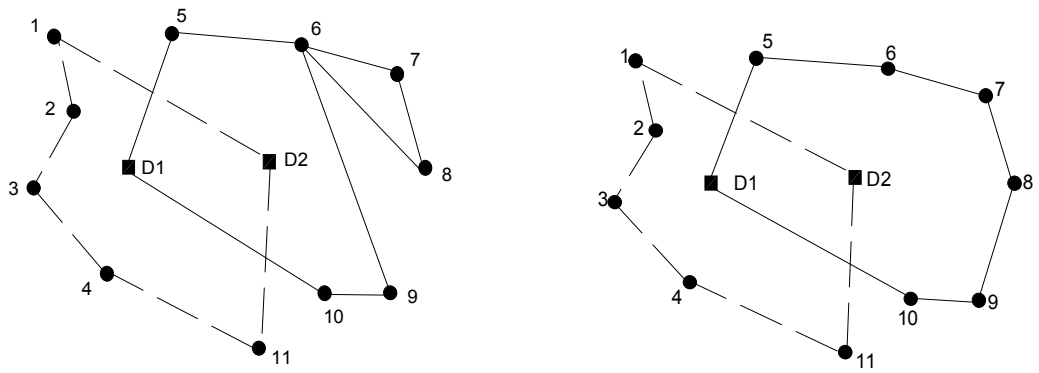
(a) **(b)**

Figure 4.2: Insertion Heuristic

4. The steps from 1 to 3 are repeated until all the nodes are covered and appear in the intermediate solution. If the number of nodes in the intermediate solution is equal to the total number of nodes, it indicates that the solution is feasible and therefore accept the current solution and skip to step 6; otherwise continue with step 5.
5. As mentioned previously in reason 4, in some cases it is found that although all the nodes are covered, the number of nodes in the intermediate solution is more than the total number of nodes. It is an indication that some nodes are visited more than once. The elimination heuristic steps are as follows:

   5.1 The repeated nodes are identified.

   5.2 For the inter-vehicle repetition, the elimination of the repeated node from the different vehicle routes is accepted if the new solution leads to improvement. It is illustrated in Figure 4.3(a).



Vehicle 1: D1-5-6-7-8-6-9-10-D1
Vehicle 2: D2-1-2-3-4-11-D2

Vehicle 1: D1-5-6-7-8-9-10-D1
Vehicle 2: D2-1-2-3-4-11-D2

(a) **(b)**

Figure 4.3: Elimination Heuristic

Continuing from Figure 4.2(b), there are two possible remedies to inter-vehicle repetition involving node 5. The first one is assigning node 5 to the route belonging to vehicle 1 and second is assigning it to the route belonging to vehicle 2. Figure 4.3(a) shows the better of the two options.

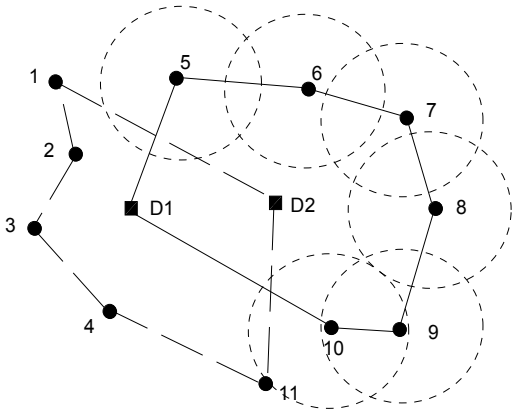
5.3 The intra-vehicle repetition was treated based on the elimination of the repeated node and the associated improvement in the feasible solution. Continuing from Figure 4.3(a), there are two possible remedies to intra-vehicle repetition to eliminate the repeated node 6. The first one is forming the route of vehicle 1 as D1-5-7-8-6-9-10-D1 and the second option is D1-5-6-7-8-9-10-D1. The second option shown in Figure 4.3(b) is the better one and valid as well. If the number of nodes covered in the intermediate solution is equal to the total number of nodes, it indicates that the solution is feasible. This intermediate solution is accepted as the current solution.

In this way, the most favorable routes were modified by repairing the infeasible intermediate solution to convert it into the feasible intermediate solution.

4.3.4 Neighboring Approach for Updating the Sample Space

The standard procedure of updating the sample space and further sample in the neighborhood of the most favorable strategy is explained in Step 7 of Section 3.4. It was applied in the MDMTSP solution procedure only when the feasible solution was achieved. The feasibility here refers to the solution having all the nodes selected with no repetition and at least one node is selected by each vehicle. The corresponding heuristic techniques are discussed in Section 4.3.3.

Instead of using the interval factor explained in Step 7 of Section 3.4, neighboring radii were used to update the sample space in the solution procedure of MDMTSP. For the first few iterations, sampling was carried out in the neighborhood of the nodes of a route by considering very large neighboring radii covering all the nodes. This facilitates every vehicle to select any available node and helps the algorithm to have an expanded search. The neighboring radii were reduced progressively only when there was no change in the solution for a predefined number of iterations. It is important to mention here that an identical neighboring radius was applied to every node of a route corresponding to the particular vehicle. Moreover, the nodes covered in that neighboring zone were eligible to be selected by the vehicle for further iterations. An example with arbitrary neighboring radius at some arbitrary iteration is shown in Figure 4.4. The circles around the nodes belonging to vehicle 1 represent the neighboring zones. The nodes (5, 6, 7, 8, 9, 10 and 11) in these zones are available to be sampled randomly for vehicle 1 for the next iteration.



Vehicle 1: D1-5-6-7-8-9-10-D1
Vehicle 2: D2-1-2-3-4-11-D2

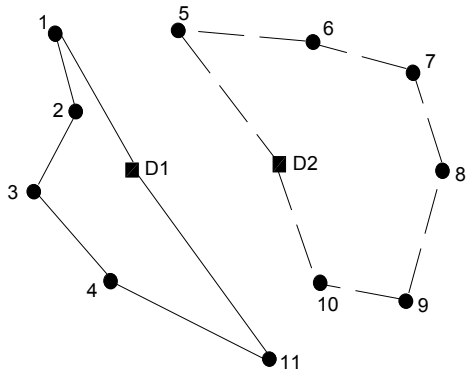
Figure 4.4: Neighboring Approach

The procedure from Section 4.3.1 to 4.3.4 was repeated until there is no change in the final solution or for a predefined number of iterations i.e. until convergence. On convergence, the node swapping heuristic technique was applied. This is discussed in the next section.

4.3.5 Node Swapping Heuristic

The heuristic of swapping the set of arcs is one of the techniques used in the multi-depot case in [154] and represented as inter-depot mutation by exchanging a set of arcs. In the current work presented here, an approach similar to that in [154] but swapping the entire set of nodes was implemented when there was no change in the current solution for a predefined number of iterations. The entire sets of nodes belonging to two depots were exchanged and the improved solution was accepted. This heuristic proved to be very useful for the PC algorithm solving the two cases of the MDMTSP presented in the following sections. This heuristic also helped in jumping out of local optima. Continuing

from the feasible solution shown in Figure 4.3(b), the node swapping is demonstrated in Figure 4.5.



Vehicle 1: D1-1-2-3-4-11-D1
Vehicle 2: D2-5-6-7-8-9-10-D2

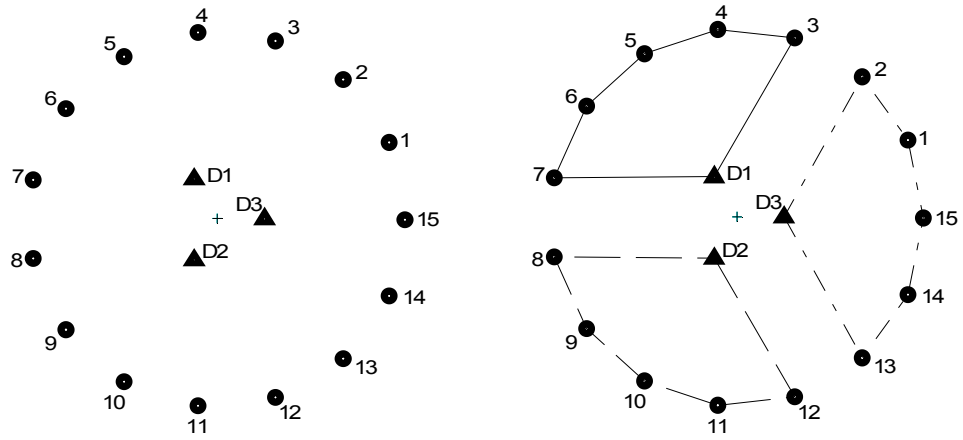
Figure 4.5: Swapping Heuristic

4.4 Test Cases of the Multiple Depot MTSP (MDMTSP)

PC accompanied with the above described heuristic techniques was applied to two test cases of the MDMTSP. Similar to the Rosenbrock function, the test cases were coded in MATLAB 7.4.0 (R2007A) on Windows platform using Pentium 4, 3GHz processor speed and 512MB RAM. Furthermore, for both the cases the set of parameters chosen was as follows: (a) individual vehicle sample size $m_i = 15$, (b) interval factor associated with the neighboring radius $\lambda_{down} = 0.05$.

Case 1 is presented in Figure 4.6(a) where three depots are placed 120º apart from one another on the periphery of an inner circle. Fifteen equidistant nodes are placed on the outer circle. The angle subtended between adjacent nodes is 24º. The diameters of the inner circle and the outer circle are 10 and 40 units, respectively. One traveling vehicle is assigned per depot. The vehicles start their journeys from their assigned depots and return

to their corresponding depots. According to the geometry of the problem and also assuming that the problem is symmetric (i.e. the cost of traveling between any two nodes is the same in both directions), the true optimum traveling cost is attained when vehicle 1 at depot 1 traveled the route D1-3-4-5-6-7-D1, vehicle 2 at depot 2 traveled the route D2-8-9-10-11-12-D2, and vehicle 3 at depot 3 traveled the route D3-13-14-15-1-2-D3. The true optimum solution was achieved by the PC approach and is plotted in Figure 4.6(b).



(a) Case 1 **(b) Solution to Case 1**

Figure 4.6: Test Case 1

According to the coordinates of the nodes and the depots, the optimum cost of traveling each of these routes is 134.7870 units. Hence, the optimum total traveling cost is 404.3610 units. A total of 50 runs were conducted for Case 1 and the true optimum was reached in every run. The time to reach the optimum varied between 0.35 minutes and 3.83 minutes with an average time of 2.09 minutes. The convergence plot of one of the runs is shown in Figure 4.7 in which the solution at iteration 21 was accepted as the final solution as there was no change in the solution for a considerable number of iterations.

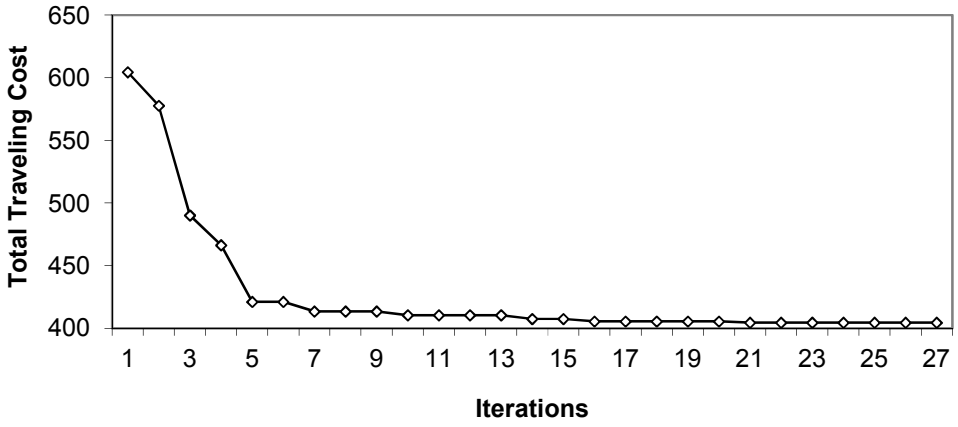
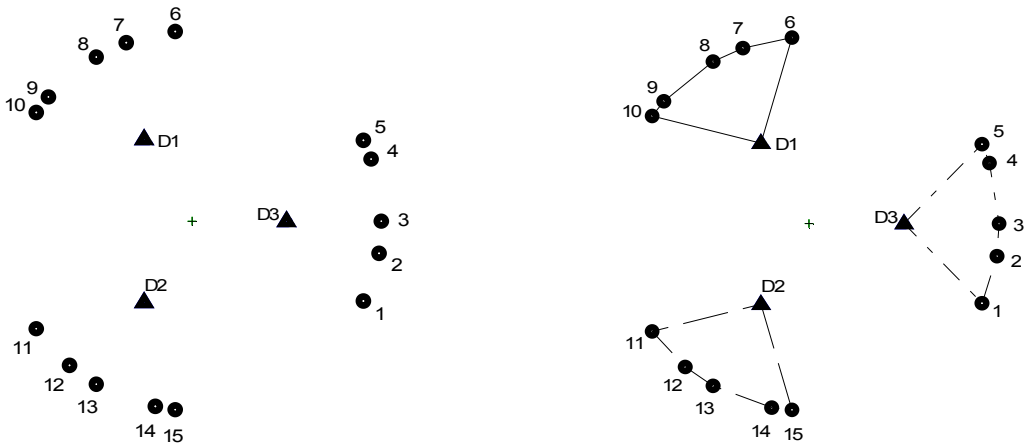


Figure 4.7: Convergence Plot for Test Case 1

Case 2 is presented in Figure 4.8(a) where three depots are placed 120º apart from one another on the periphery of an inner circle. Fifteen nodes are placed on the outer circle. These nodes are arranged in three clusters. In every cluster, the nodes are intentionally placed at uneven angles apart. The diameters of the inner circle and the outer circle are 5 and 10 units, respectively. Similar to Case 1, one vehicle is assigned per depot. The vehicles start their journeys from their assigned depots and return to their corresponding depots. According to the geometry of the problem and also assuming that the problem is symmetric, the optimum traveling cost is attained when vehicle 1 at depot 1 traveled the route D1-6-7-8-9-10-D1, vehicle 2 at depot 2 traveled the route D2-11-12-13-14-15-D2, and vehicle 3 at depot 3 traveled the route D3-1-2-3-4-5-D3. The true optimum solution was achieved by the PC approach and is plotted in Figure 4.8(b).



(a) Case 2 **(b) Solution to Case 2**

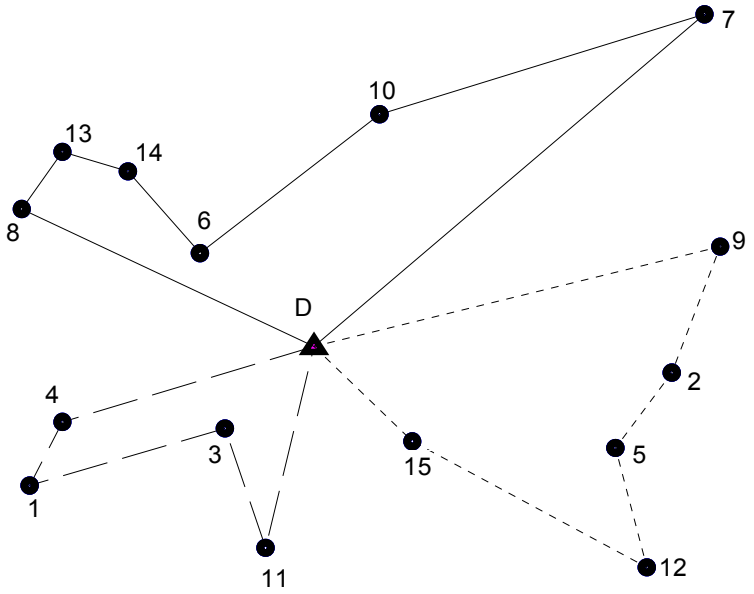
Figure 4.8: Test Case 2

According to the coordinates of the nodes and the depots, the optimum cost of traveling each of these routes is 20.4264 units. Hence, the optimum total traveling cost is 61.2792 units. For case 2, more than 50 runs were conducted and the true optimum was achieved in every run. The time to reach the optimum varied between 0.76 minutes and 3.10 minutes with an average time of 1.27 minutes. The convergence plot for one of the runs is shown in Figure 4.9 in which the solution at iteration 33 was accepted as the final solution as there was no change in the solution for a considerable number of iterations.

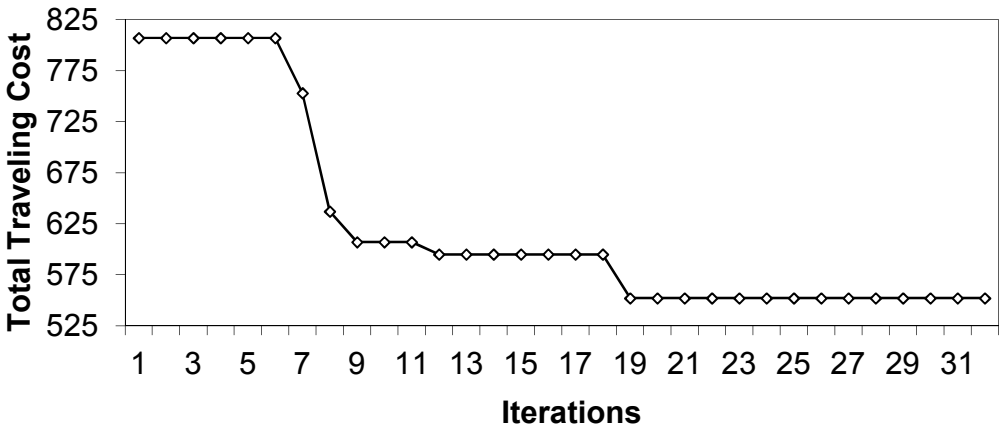
Figure 4.9: Convergence Plot for Test Case 2

4.5 Test Cases of the Single Depot MTSP (SDMTSP) with Randomly Located Nodes

Using the same procedure as the above test cases of the MDMTSP, over 30 cases of the Single Depot MTSP (SDMTSP) with 3 vehicles and 15 nodes were solved. In all the cases, the single depot and the 15 nodes were randomly located in an area of 100 × 100 units. Two sample cases are shown in Figure 4.10(a) and 4.11(a) and the corresponding convergence plots in 4.10(b) and 4.11(b). In the first sample, the solution at iteration 19 and for the second sample case, the solution at iteration 33 was accepted as the final ones as there was no change in the respective solutions for a considerable number of iterations. In all the cases with randomly located nodes, the approximate convergence time was found to vary between 1 minute and 6 minutes with an average time of 3.34 minutes.

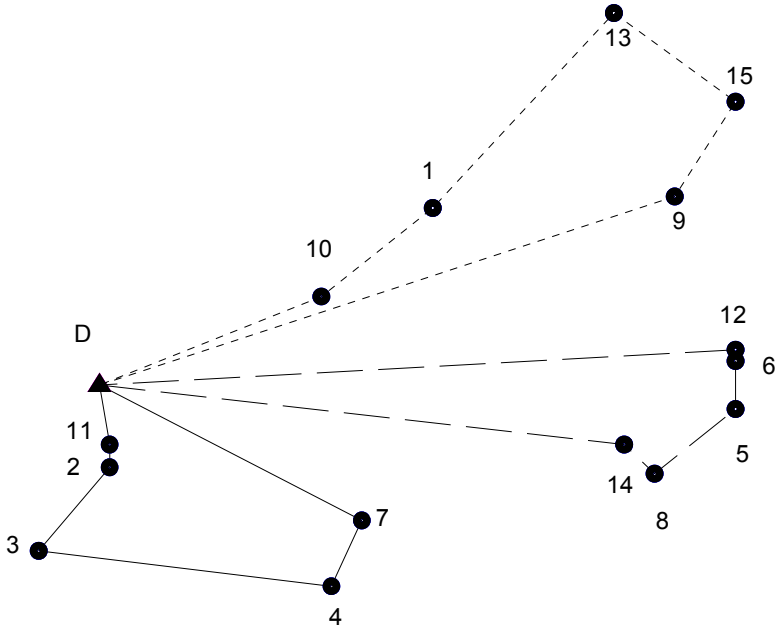


(a)



(b)

Figure 4.10: Randomly Located Nodes (Sample Case 1)



(a)

(b)

Figure 4.11: Randomly Located Nodes (Sample Case 2)

4.6 Comparison and Discussion

The above solution to the MDMTSP and SDMTSP using PC indicates that it can successfully be used to solve the combinatorial optimization problems. As shown in Table 4.2, the results are compared with those of some other methods solving the problem with

sizes close to the specially developed test cases solved here. It is worth mentioning that the technique of converting the MTSP into the standard TSP using Miliotis reverse algorithm [154], cutting plane algorithm [133], elastic net approach [142] as well as Branch and Bound techniques such as 'branch on an arc' and 'branch on a route' with different lower bounds (LB0, LB2, etc.) in [132] did not produce the optimum solution in every run. It therefore requires the algorithms to be run for a number of trials and then accepting the average from among the trials as the solution. On the other hand, every run of the PC converged to the actual optimal solution for the MDMTSP in quite reasonable CPU time. In case of the SDMTSP, the percentage deviation of total traveling cost from the average solution over 30 runs of PC was recorded to be 2.94%. Furthermore, the CPU time varied within a very small range. The time variation exists because of the probabilistic nature of PC algorithm and random initial sampling of the routes affecting every subsequent step. As per the literature review in Section 4.2.3, the popular optimization techniques are computationally time consuming and expensive. It is also worth to mention that the above insertion, elimination and swapping heuristics could be easily accommodated into PC.

Table 4.2: Performance Comparison of Various Algorithms solving the MTSP

Method	Nodes	Vehicles	Avg. CPU Time (Minutes)	% Deviation
MTSP to std. TSP [154]	20	2	2.05	--
	20	3	2.47	
	20	4	2.95	
Cutting Plane [133]	20	2	1.71	--
	20	3	1.50	--
	20	4	1.44	
Elastic Net^{+} [142]	22	2	12.03	28.71
	22	3	13.10	74.18
	22	4	12.73	33.33
Branch on an Arc with LB0 [132]	15	3	0.56	--
Branch on an Arc with LB2 [132]	15	3	1.01	--
	20	4	2.32	
Branch on a Route with LB0 [132]	15	3	0.44	--
PC (MDMTSP)	**15 (Case 1)**	**3**	**2.09**	**0.00**
	15 (Case 2)	**3**	**1.27**	**0.00**
PC (SDMTSP)	**15**	**3**	**3.34**	**2.94**

+: unable to reach the optimum

Besides the advantages of PC, some weaknesses associated with the above solution procedure solving the MTSP were also identified. As an effort to deal with constraints, various heuristic techniques were incorporated which served as repair techniques to push the solution into feasibility. This may not be an elegant option since they cannot be used for handling generic constraints. If complexity of the problem and related constraints increase, the repair work may become more tedious and may add further computational load. In addition, the number of function evaluations can become very large because of its dependence on the number of strategies in every agent's strategy sets. Both of these disadvantages may limit its use to smaller size problems with fewer constraints. In order to avoid this limitation and develop a more generic constraint handling technique, a penalty

function approach is incorporated in to PC algorithm. This is described in the following chapter.

CHAPTER 5

CONSTRAINED PC (APPROACH 2)

There are a number of traditional constraint handling methods available, such as gradient projection method, reduced gradient method, Lagrange multiplier method, aggregate constraint method, feasible direction based method, penalty based method, etc [96, 97]. The penalty based methods can be referred to as generalized constraint handling methods because of their simplicity, ability to handle nonlinear constraints and can be used with most of the unconstrained optimization methods [96, 97]. They basically transform a constrained optimization problem into an unconstrained optimization problem.

A constrained PC algorithm is proposed in this chapter by incorporating the penalty function approach and associated modifications to the unconstrained PC approach described in Chapter 3. The constrained PC approach was validated by successfully solving three constrained test problems [6]. The resulting constrained PC algorithm flowchart is presented in Figure 5.1.

5.1 Penalty Function Approach

Consider a general constrained problem (in the minimization sense) as follows:

$$\text{Minimize} \quad G \tag{5.1}$$

$$\text{Subject to} \quad g_j \le 0, \quad j = 1, 2, ..., s$$
$$h_j = 0, \quad j = 1, 2, ..., t$$

For each of its combined strategy set $\mathbf{Y}_i^{[r]}$, each agent i computes the corresponding system objective $G\left(\mathbf{Y}_i^{[r]}\right)$ [1, 2], i.e. each agent i computes m_i values of the system

objective $G\left(\mathbf{Y}_i^{[r]}\right)$. In addition, each agent i computes m_i vectors of the corresponding constraints $\mathbf{C}\left(\mathbf{Y}_i^{[r]}\right)$ as follows:

$$\begin{aligned}
\mathbf{C}\left(\mathbf{Y}_i^{[1]}\right) &= \left[g_1\left(\mathbf{Y}_i^{[1]}\right) \quad g_2\left(\mathbf{Y}_i^{[1]}\right) \quad \ldots \quad g_s\left(\mathbf{Y}_i^{[1]}\right) \quad h_1\left(\mathbf{Y}_i^{[1]}\right) \quad h_2\left(\mathbf{Y}_i^{[1]}\right) \quad \ldots \quad h_t\left(\mathbf{Y}_i^{[1]}\right) \right] \\
&\vdots \\
\mathbf{C}\left(\mathbf{Y}_i^{[r]}\right) &= \left[g_1\left(\mathbf{Y}_i^{[r]}\right) \quad g_2\left(\mathbf{Y}_i^{[r]}\right) \quad \ldots \quad g_s\left(\mathbf{Y}_i^{[r]}\right) \quad h_1\left(\mathbf{Y}_i^{[r]}\right) \quad h_2\left(\mathbf{Y}_i^{[r]}\right) \quad \ldots \quad h_t\left(\mathbf{Y}_i^{[r]}\right) \right] \\
&\vdots \\
\mathbf{C}\left(\mathbf{Y}_i^{[m_i]}\right) &= \left[g_1\left(\mathbf{Y}_i^{[m_i]}\right) \quad g_2\left(\mathbf{Y}_i^{[m_i]}\right) \quad \ldots \quad g_s\left(\mathbf{Y}_i^{[m_i]}\right) \quad h_1\left(\mathbf{Y}_i^{[m_i]}\right) \quad h_2\left(\mathbf{Y}_i^{[m_i]}\right) \quad \ldots \quad h_t\left(\mathbf{Y}_i^{[m_i]}\right) \right]
\end{aligned} \tag{5.2}$$

where s is the number of inequality constraints and t is the number of equality constraints.

In order to incorporate the constraints into the problem, each agent i forms the pseudo-objective function as follows [96, 97]:

$$\phi\left(\mathbf{Y}_i^{[r]}\right) = G\left(\mathbf{Y}_i^{[r]}\right) + \theta \left\{ \sum_{j=1}^{s} \left[g_j^{+}\left(\mathbf{Y}_i^{[r]}\right) \right]^2 + \sum_{j=1}^{t} \left[h_j\left(\mathbf{Y}_i^{[r]}\right) \right]^2 \right\} \tag{5.3}$$

where $g_j^{+}\left(\mathbf{Y}_i^{[r]}\right) = \max\left(0, g_j\left(\mathbf{Y}_i^{[r]}\right)\right)$ and θ is the scalar penalty parameter.

The ultimate goal of every agent i is to identify its own strategy value which contributes the most towards the minimization of the sum of these system objective values, i.e. $\sum_{r=1}^{m_i} \phi(\mathbf{Y}_i^{[r]})$, referred to as the collection of pseudo-system objectives.

In addition to the parameters listed in the unconstrained PC algorithm in Section 3.4, four more parameters are required in the constrained algorithm, namely the number of test iterations n_{test} which decides the iteration number from which the convergence condition needs to be checked, the scalar penalty parameter θ, the associated update factor α_θ for

the parameter θ and a constraint violation tolerance μ. Similar to the other parameters, the values of these parameters are chosen based on the preliminary trials of the algorithm. Furthermore, at the beginning of the algorithm, the tolerance μ is initialized to a very large prescribed value. The value of μ is tightened iteratively as the algorithm progresses.

The system objective $G\left(\mathbf{Y}^{[fav]}\right)$ and corresponding $\mathbf{Y}^{[fav]}$ are accepted if and only if the largest absolute component from the constraint vector $\mathbf{C}\left(\mathbf{Y}^{[fav]}\right)$ referred to as C_{max} is not worse than the constraint violation tolerance μ, i.e., $C_{max} \le \mu$, and the value of μ is updated to C_{max}, i.e., $\mu = C_{\max}$. In this way the algorithm progresses by iteratively tightening the constraint violation.

The detailed illustration of the above mentioned modifications to the unconstrained PC algorithm (discussed in Chapter 3) corresponding to the decision of acceptance/rejection of the solution, the updating of the parameters are represented in windows A and B respectively, of the constrained PC algorithm flowchart shown in Figure 5.1.

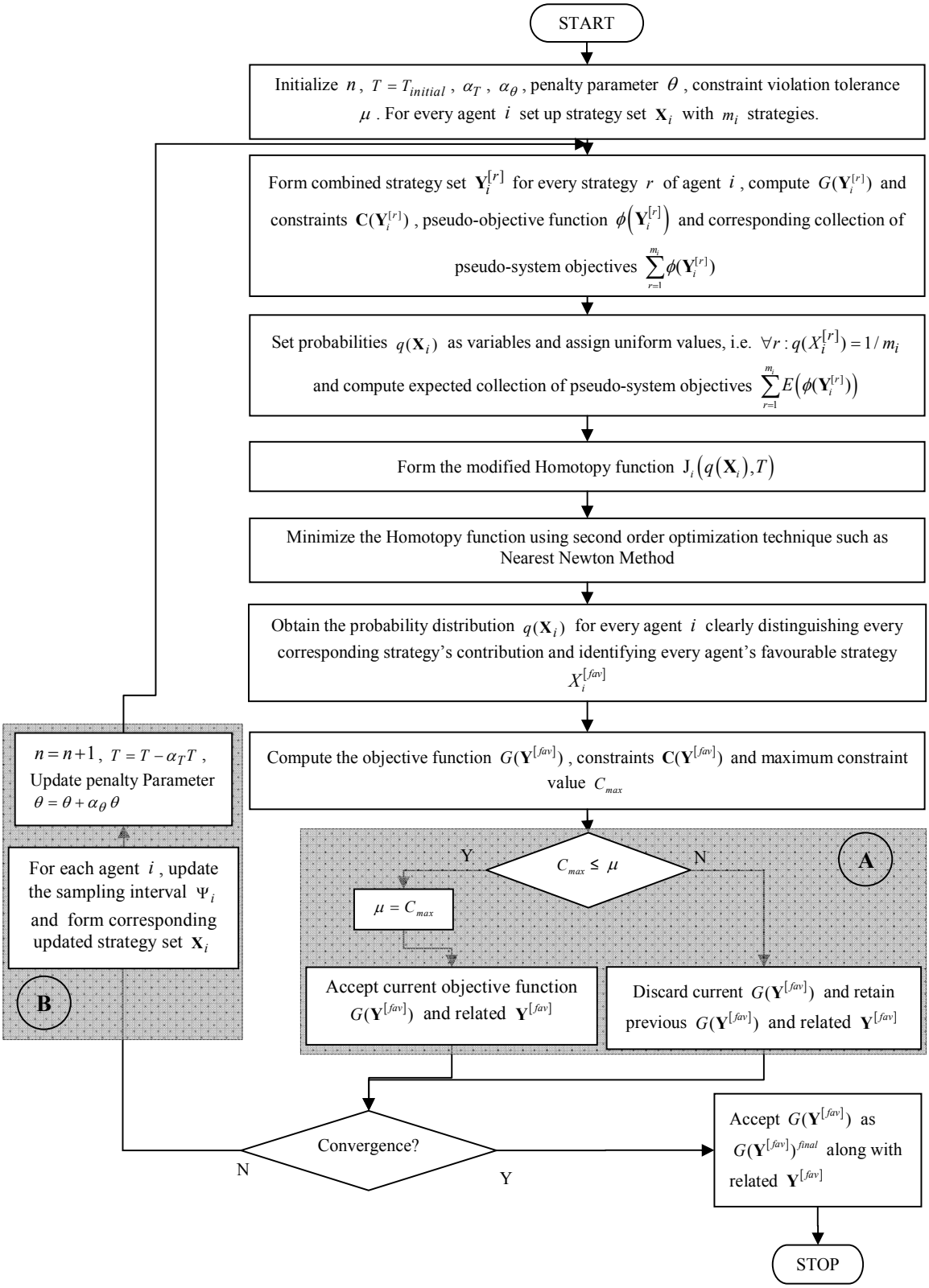


Figure 5.1: Constrained PC Algorithm Flowchart (Penalty Function Approach)

5.2 Solutions to Constrained Test Problems

The constrained PC approach was successfully tested for solving three problems with equality and inequality constraints. These problems have already been well studied in the literature and used to compare the performance of various optimization algorithms. The various associated constraint handling techniques are discussed below in brief.

Some of the approaches [155-159] focused on overcoming the limitations of the classic penalty approach. A self adaptive penalty approach [155] and dynamic penalty scheme [157] produced premature convergence and similar to stratum approach [158] were quite sensitive to the additional set of parameters. A stochastic bubble sort algorithm in [156] attempted the direct and explicit balance between the objective function and the penalty function. Although a variety of constrained problems could be solved, it was sensitive to the associated probability parameter and needed several preliminary trials. The approach of penalty parameter was avoided using a feasibility-based rule in [159]. It required an additional fitness function and failed to produce optimum solution in every run. In addition, the quality of the solution was governed by the population in hand. The need of additional fitness function was avoided in Hybrid PSO [160] by incorporating the feasibility-based rules [159] into PSO.

The GEnetic algorithm for Numerical Optimization of COnstrained Problems (GENOCOP) specifically for linear constraints [161, 162], the PSO [163] and the Homomorphous Mapping [164] needed initial feasible solutions along with the problem dependent parameters to be set. Furthermore, for some of the problems, it was almost impossible to generate initial feasible solutions, which might need some additional techniques. The lack of diversity of the population was noticed in the cultural algorithm

[165] and Cultural Differential Evolution (CDE) [166] which divide the search space into feasible, semi-feasible and infeasible regions.

An interactive approach of constraint correction algorithm [96] needed an additional constraint and several preliminary trials. The gradient repair method [167] was embedded into PSO [168] and its performance was governed by the number of solutions undergoing repair [169].

The multi-objective approach in Filter SA (FSA) [170] required numerous problem dependent parameters to be tuned along with the associated preliminary trials. Inspired from [170], GA was applied to find the non-dominated solution vector in [171]. The multi-criteria approach problem in [172-174] could not solve the fundamental problem of balancing the objective function and the constraint violations. The results of the two multi-objective approaches in [175] highlighted the necessity of an additional search bias technique to locate the feasible solutions.

In this current work, the constrained PC algorithm was coded in MATLAB 7.4.0 (R2007A) and the simulations were run on a Windows platform using a Pentium 4, 3GHz processor speed and 512MB memory capacity. The results and the comparison with other algorithms are discussed below.

5.2.1 Test Problem 1

This is a tension/compression spring design problem described in [96, 155, 160, 165, 171]. It has the aim of minimizing the weight $f(\mathbf{X})$ of a tension/compression spring (as shown in Figure 5.2) subject to constraints on minimum deflection, shear stress, surge frequency, limits on outside diameter and on design variables. The design variables are the

mean coil diameter, the wire diameter x_1 and the number of active coils x_3. This problem is categorized as a quadratic problem with three variables, six boundary conditions, one linear inequality and three nonlinear inequality constraints.

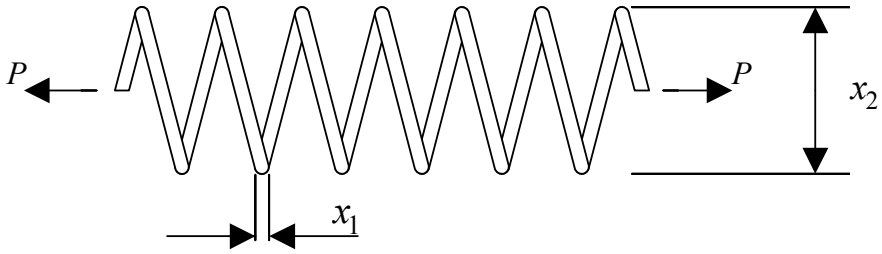


Figure 5.2: Tension/Compression Spring

The mathematical formulation of this problem can be described as follows:

$$\text{Minimize} \quad f(\mathbf{X}) = (x_3 + 2)\, x_2 x_1^2$$

$$\text{Subject to} \quad g_1(\mathbf{X}) = 1 - \frac{x_2^3 x_3}{71785 x_1^4} \le 0$$

$$g_2(\mathbf{X}) = \frac{4x_2^2 - x_1 x_2}{12566\left(x_2 x_1^3 - x_1^4\right)} + \frac{1}{5108 x_1^2} - 1 \le 0$$

$$g_3(\mathbf{X}) = 1 - \frac{140.45 x_1}{x_2^2 x_3} \le 0$$

$$g_4(\mathbf{X}) = \frac{x_1 + x_2}{1.5} - 1 \le 0$$

where $0.05 \le x_1 \le 2,\ 0.25 \le x_2 \le 1.3,\ 2 \le x_3 \le 15$

This problem was solved in [96, 155, 160, 165, 171]. The best solutions obtained by these approaches along with those by the PC approach are listed in Table 5.1. The PC solution to the problem produced competent results at low computational cost (computational costs of the other approaches were not available). The best, mean and worst function values found from ten runs were 0.01350, 0.02607, 0.05270, respectively. Furthermore, the average CPU time in seconds and average number of function

evaluations were 24.5 and 5214, respectively. Moreover, in every iteration, the parameters such as the number of strategies m_i and interval factor λ_{down} were 6 and 0.1, respectively. The best PC solution was within about 6.63% of the best-reported solution [160]. The convergence plot for the best PC solution is shown in Figure 5.3.

Table 5.1: Performance comparison of various Algorithms solving Problem 1

Design Variables	Best Solutions Found					
	Coello et al. (2004) [165]	Arora (2004) [96]	Coello (2000) [155]	Coello et al. (2002) [171]	He (2007) [160]	**PC**
x_1	0.05000	0.05339	0.05148	0.05198	0.05170	**0.05060**
x_2	0.31739	0.39918	0.35166	0.36396	0.35712	**0.32781**
x_3	14.03179	9.18540	11.63220	10.89052	11.26508	**14.05670**
$g_1(\mathbf{X})$	0.00000	0.00001	-0.00330	-0.00190	-0.00000	**-0.05290**
$g_2(\mathbf{X})$	-0.00007	-0.00001	-0.00010	0.00040	0.00000	**-0.00740**
$g_3(\mathbf{X})$	-3.96796	-4.12383	-4.02630	-4.06060	-4.05460	**-3.70440**
$g_4(\mathbf{X})$	-0.75507	-0.69828	-0.73120	-0.72270	-0.72740	**-0.74769**
$f(\mathbf{X})$	0.01272	0.01273	0.01270	0.01268	0.01266	**0.01350**

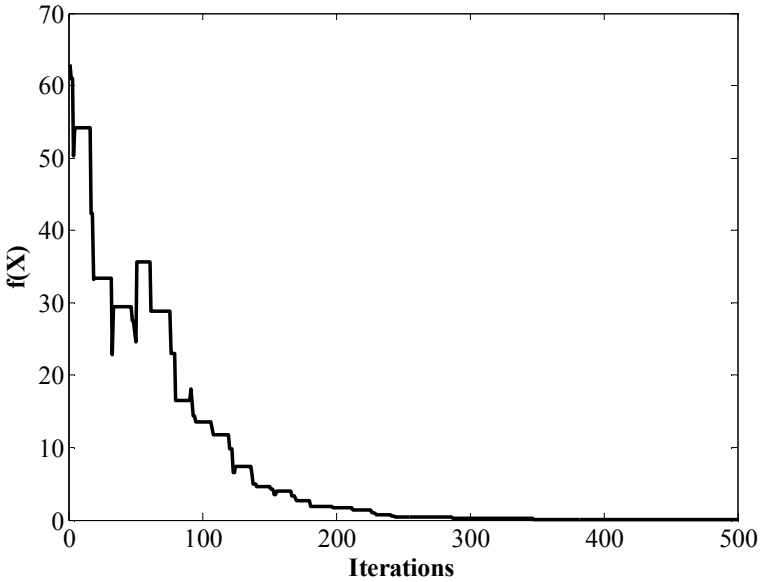


Figure 5.3: Convergence Plot for Problem 1

5.2.2 Test Problem 2

This is Himmelblau's nonlinear optimization problem [176]. It has five design variables $(x_1, x_2, x_3, x_4, x_5)$ and ten boundary conditions. According to [53], the ratio of the size of the feasible search space to the size of the whole search space (which represent the degree of difficulty) is 52.123%. There are six nonlinear inequality constraints out of which two are active at the optimum. The problem can be stated as follows:

$$\text{Minimize} \quad f(\mathbf{X}) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141$$

$$\text{Subject to} \quad g_1(\mathbf{X}) = 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 - 92 \le 0$$

$$g_2(\mathbf{X}) = -85.334407 - 0.0056858x_2x_5 - 0.0006262x_1x_4 + 0.0022053x_3x_5 \le 0$$

$$g_3(\mathbf{X}) = 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3^2 - 110 \le 0$$

$$g_4(\mathbf{X}) = -80.51249 - 0.0071317x_2x_5 - 0.0029955x_1x_2 - 0.0021813x_3^2 + 90 \le 0$$

$$g_5(\mathbf{X}) = 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 - 25 \le 0$$

$$g_6(\mathbf{X}) = -9.300961 - 0.0047026x_3x_5 - 0.0012547x_1x_3 - 0.0019085x_3x_4 + 20 \le 0$$

where $78 \le x_1 \le 102,\ 33 \le x_2 \le 45,\ 27 \le x_i \le 45,\ (i = 3,4,5)$

This problem was solved in [155-160, 163-170, 173, 175]. The best, mean, and worst solutions obtained by these approaches along with those from ten runs of the PC approach are listed in Table 5.2. The PC solution to this problem produced competent results at reasonable computational cost. In addition, the best PC solution was within about 0.078% of the best-reported solution [156]. Furthermore, the average CPU time was 11 minutes. Moreover, in every iteration, the parameters such as the number of strategies m_i and interval factor λ_{down} were 100 and 0.01, respectively. The convergence plot for the best PC solution is shown in Figure 5.4.

Table 5.2: Performance comparison of various Algorithms solving Problem 2

Algorithm	Best	Mean	Worst	Std Dev	Average Function Evaluations
Coello et al. (2004) [165]	-30665.5000	-30662.5000	-30636.2000	9.3	
Becerra et. al. (2006) [166]	-30665.5386	-30665.5386	-30665.5386	0.00000	
Koziel et al. (1999) [164]	-30664.0000	-30655.0000	-30645.0000	--	1400000
Hedar et al. (2006) [170]	-30665.5380	-30665.4665	-30664.6880	0.173218	86154
Coello (2000) [155]	-31020.8590	-30984.2400	-30792.4070	73.633	--
Chootinan et al. (2006) [167]	-30665.5386	-30665.3538	-30665.5386	0.000000	26981
Runarsson et al. (2005) [175]	-30665.5386	-30665.3539	-30665.5386	0.000000	66400
Runarsson et al. (2000) [156]	-30665.5390	-30665.5390	-30665.5390	1.1E-11	350000
Zahara et al. (2008) [168]	-30665.5386	-30665.35386	-30665.5386	0.000000	19658
Deb (2000) [159]	-30665.5370	--	29846.6540	--	
Farmani et al. (2003) [157]	-30665.5000	-30665.2000	-30663.3000	4.85E-01	--
Hu et al. (2002) [163]	30665.5000	30665.5000	30665.5000	--	--
Dong et al. (2005) [169]	-30664.7000	-30662.8000	-30656.1000	--	--
Ray et al. (2001) [173]	-30651.6620	-30647.1050	-30619.0470	--	35408
He et al. (2007) [160]	-30665.5390	-30665.5390	-30665.5390	1.7E-06	
Homaifar et al. (1994) [158]	-30373.9500	--	-30175.8040	--	--
PC	**-30641.5702**	**-30635.4157**	**-30626.7492**	**7.5455**	**2780449**

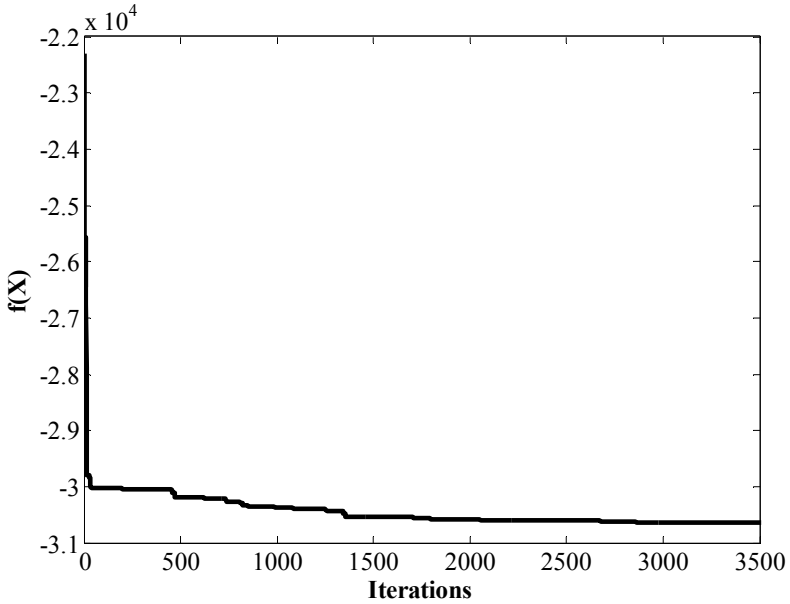


Figure 5.4: Convergence Plot for Problem 2

5.2.3 Test Problem 3

This problem is categorized as logarithmic nonlinear with ten variables, ten boundary conditions and three linear equality constraints. The problem can be stated as follows:

$$\text{Minimize } f(\mathbf{X}) = \sum_{j=1}^{10} x_j \left(c_j + \ln \frac{x_j}{x_1 + x_2 + \ldots + x_{10}} \right)$$

$$\text{Subject to } h_1(\mathbf{X}) = x_1 + 2x_2 + 2x_3 + x_6 + x_{10} - 2 = 0$$

$$h_2(\mathbf{X}) = x_4 + 2x_5 + x_6 + x_7 - 1 = 0$$

$$h_3(\mathbf{X}) = x_3 + x_7 + x_8 + 2x_9 + x_{10} - 1 = 0$$

$$x_i \geq 0.000001, \quad i = 1, 2, \ldots, 10$$

where

$$c_1 = -6.089 \quad c_2 = -17.164 \quad c_3 = -34.054 \quad c_4 = -5.914 \quad c_5 = -24.721$$
$$c_6 = -14.986 \quad c_7 = -24.100 \quad c_8 = -10.708 \quad c_9 = -26.662 \quad c_{10} = -22.179$$

This problem is a modified version of the chemical equilibrium problem in complex mixtures originally presented in [177]. The modified problem was solved in [162, 178]. The best solutions obtained by these approaches along with those by the PC approach are listed in Table 5.3. The PC solution to this problem produced competent results but at high computational cost (computational costs of the other approaches were not available). The best, mean and worst function values found from ten runs were -46.7080572120, -45.6522267370, -44.4459333503, respectively, with standard deviation 0.7893. Furthermore, the average CPU time for PC solution was 21.60 min. Moreover, in every iteration, the parameters such as the number of strategies m_i and interval factor λ_{down} were 6 and 0.1, respectively. The best PC solution was within about 2.20% of the best-reported solution [162]. The convergence plot for the best PC solution is shown in Figure 5.5.

Table 5.3: Performance Comparison of Various Algorithms Solving Problem 3

Design Variables	Best Solutions Found		
	Hock et al. (1981) [178]	Michalewicz (1997) [162]	**PC**
x_1	0.01773548	0.04034785	**0.0308207485**
x_2	0.08200180	0.15386976	**0.2084261218**
x_3	0.88256460	0.77497089	**0.6708869580**
x_4	0.0007233256	0.00167479	**0.0371668767**
x_5	0.4907851	0.48468539	**0.3510055351**
x_6	0.0004335469	0.00068965	**0.1302810195**
x_7	0.01727298	0.02826479	**0.1214712339**
x_8	0.007765639	0.01849179	**0.0343070642**
x_9	0.01984929	0.03849563	**0.0486302636**
x_{10}	0.05269826	0.10128126	**0.0486302636**
$h_1(\mathbf{X})$	8.6900E-08	6.0000E-08	**-0.0089160590**
$h_2(\mathbf{X})$	0.0141	1.0000E-08	**-0.0090697995**
$h_3(\mathbf{X})$	5.9000E-08	-1.0000E-08	**-0.0047181958**
$f(\mathbf{X})$	-47.707579	-47.760765	**-46.7080572120**
Average Function Evaluations	--	--	**389546**

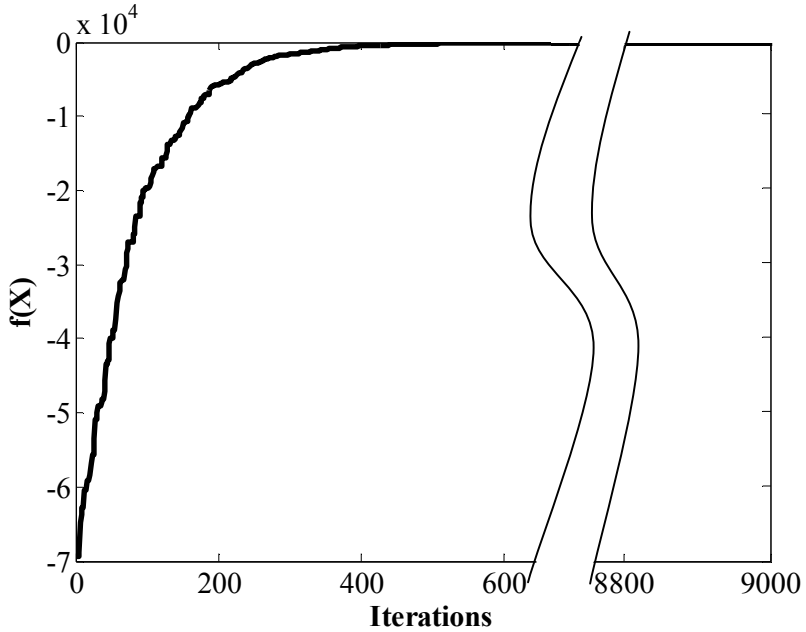


Figure 5.5: Convergence Plot for Problem 3

5.3 Discussion

The above solutions using PC indicated that it can successfully be used to solve a variety of constrained optimization problems. The results also demonstrated its competitiveness with those of some other methods. Furthermore, it was evident from the results that the approach is sufficiently robust and produced consistent results in every run. It implies that the rational behavior of the agents could be successfully formulated and demonstrated.

The worthy features of the above constrained PC approach are discussed here by comparing with some other methods. Approaches such as PSO [163], Homomorphous Mapping [164], GENOCOP [162] require initial feasible solution, whereas the constrained PC approach proposed here can solve the problems starting from randomly generated and completely infeasible solutions. The approach presented here can handle both equality as well as inequality constraints, whereas the approach of concurrent co-evolution in [155] is applicable only to inequality constraints with further additional computations involved. The method of GENOCOP [161, 162] is suitable only for the linear constraints while the proposed approach can handle both linear and nonlinear constraints.

Besides the advantages of the present approach, some limitations were also identified. The rate of convergence and the quality of the solution was dependent on the parameters such as the number of strategies m_i in every agent's strategy set $\mathbf{X}_i$ and the interval factor λ_{down}. Some preliminary trials were also necessary for fine-tuning these parameters. Based on the results and the trials performed, it was also noticed that the PC method is quite efficient in searching the feasible region and the local minima (in the vicinity of the true

optimum) but demonstrates weak local search ability which may be necessary to reach the true optimum.

The next chapter discusses two variations of the Feasibility-based Rule as an effort to make PC a more versatile optimization approach.

CHAPTER 6

CONSTRAINED PC (APPROACH 3)

This chapter demonstrates further efforts to develop a generic constraint handling technique for PC in order to make it a more versatile optimization algorithm. Two variations of the Feasibility-based Rule referred to as Feasibility-based Rule I and II originally proposed in [159] and further implemented in [160, 179-183] were incorporated into the unconstrained PC approach from Chapter 3.

Consider a general constrained problem (in the minimization sense) as follows:

$$\text{Minimize} \quad G \tag{6.1}$$

$$\text{Subject to} \quad g_j \le 0, \quad j = 1, 2, ..., s$$
$$h_j = 0, \quad j = 1, 2, ..., w$$

According to [172-174], the equality constraint $h_j = 0$ can be transformed into a pair of inequality constraints using a tolerance value δ as follows:

$$h_j = 0 \quad \Rightarrow \begin{cases} g_{s+j} = h_j - \delta \le 0 & j = 1, 2, ..., w \\ g_{s+w+j} = -\delta - h_j \le 0 \end{cases} \tag{6.2}$$

Thus, w equality constraints are replaced by $2w$ inequality constraints with the total number of constraints given by $t = s + 2w$. Then a generalized representation of the problem in equation (6.1) can be stated as follows:

$$\text{Minimize} \quad G \tag{6.3}$$

$$\text{Subject to} \quad g_j \le 0, \quad j = 1, 2, ..., t$$

The Feasibility-based Rule I assisted with the perturbation approach, updating of the sampling space in the neighborhood of the favorable strategy and the modified convergence criterion are presented in Section 6.1 and the validation of the approach by solving two cases of the Circle Packing Problem (CPP) is presented in Section 6.2. In addition, attaining the true optimum solution to the CPP using PC clearly demonstrated its inherent ability to avoid the tragedy of commons.

In addition to the two cases of the CPP, as a distributed optimization approach PC was also tested for its capability to deal with the agent failure scenario. The solution highlights its strong potential to deal with the agent failure which may arise in real world complex problems including urban traffic control, formation of airplanes fleet and mid-air collision avoidance, etc.

The constraint handling approach using Feasibility-based Rule II is discussed in Section 6.4 followed by the validation of its performance by solving three variations of the Senor Network Coverage Problem (SNCP).

6.1 Feasibility-based Rule I

This rule allows the objective function and the constraint information to be considered separately. Similar to the penalty function approach presented in Chapter 5, the constraint violation tolerance is iteratively tightened in order to obtain the fitter solution and further drive the convergence towards feasibility. More specifically, at the beginning of the PC algorithm, the constraint violation tolerance μ is initialized to the number of constraints $|\mathbf{C}|$, i.e. $\mu = |\mathbf{C}|$ where $|\mathbf{C}|$ refers to the cardinality of the constraint vector $\mathbf{C} = [g_1 \quad g_2 \quad ... \quad g_t]$. The value of μ is tightened iteratively as the algorithm progresses.

As mentioned previously, the rule assisted with the perturbation approach, updating of the sampling space in the neighborhood of the favorable strategy and the modified convergence criterion are incorporated into the PC framework. The rule and these assisting techniques are discussed below as modifications to the corresponding steps of the unconstrained PC approach presented in Section 3.4. The following Sections 6.11 to 6.13 should therefore be read in conjunction with the unconstrained PC algorithm steps described in Section 3.4.

6.1.1 Modifications to Step 5 of the Unconstrained PC Approach

The *step 5* of the unconstrained PC procedure discussed in Section 3.4 is modified by using the following rule:

(a) Any feasible solution is preferred over any infeasible solution

(b) Between two feasible solutions, the one with better objective is preferred

(c) Between two infeasible solutions, the one with fewer violated constraints is preferred.

The detailed formulation of the above rule is explained below and further presented in window A of the constrained PC algorithm flowchart in Figure 6.1.

(a) If the current system objective $G(\mathbf{Y}^{[fav]})$ as well as the previous solution are infeasible, accept the current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$ as the current solution if the number of constraints violated $C_{violated}$ is less than or equal to μ, i.e. $C_{violated} \leq \mu$, and then the value of μ is updated to $C_{violated}$, i.e. $\mu = C_{violated}$.

(b) If the current system objective $G(\mathbf{Y}^{[fav]})$ is feasible, and the previous solution is infeasible, accept the current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$ as the current solution, and then the value of μ is updated to 0, i.e. $\mu = C_{violated} = 0$.

(c) If the current system objective $G(\mathbf{Y}^{[fav]})$ is feasible, i.e. $C_{violated} = 0$ and is not worse than the previous feasible solution, accept the current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$ as the current solution.

(d) If all the above conditions (a) to (c) are not met, then discard current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$, and retain the previous iteration solution.

6.1.2 Modifications to Step 7 of the Unconstrained PC Approach

As mentioned previously, similar to [159, 160, 179-183] where additional techniques were implemented to avoid premature convergence, a perturbation approach was also incorporated. It perturbs the individual agent's favorable strategy set based on its reciprocal and associated predefined interval. The solution is accepted if the feasibility is maintained. In this way, the algorithm continues until convergence by selecting the samples from the neighborhood of the recent favorable strategies. Unlike the unconstrained PC approach described in Chapter 3 and the Penalty Function Approach discussed in Chapter 5 where the sampling space was reduced to the neighborhood of the favorable strategy, in this case the sampling space in the neighborhood of the favorable strategy is reduced or expanded according to the improvement in the system objective for a predefined number of iterations. The detailed formulation of the updating of the

sampling space and the perturbation approach is explained below and further presented in window B of the constrained PC algorithm flowchart in Figure 6.1.

On completion of the pre-specified n_{test} iterations of the PC algorithm, the following conditions are checked in every further iteration.

(a) If $G(\mathbf{Y}^{[fav],n}) \le G(\mathbf{Y}^{[fav],n-n_{test}})$, then every agent shrinks its sampling interval as follows:

$$\Psi_i \in \left[\left(X_i^{[fav]} - \lambda_{down}\left\|\Psi_i^{upper} - \Psi_i^{lower}\right\|\right), \left(X_i^{[fav]} + \lambda_{down}\left\|\Psi_i^{upper} - \Psi_i^{lower}\right\|\right)\right],\ 0 < \lambda_{down} \le 1 \tag{6.4}$$

where λ_{down} is referred to as the interval factor corresponding to the shrinking of sample space.

(b) If $G(\mathbf{Y}^{[fav],n})$ and $G(\mathbf{Y}^{[fav],n-n_{test}})$ are feasible and $\left\|G(\mathbf{Y}^{[fav],n}) - G(\mathbf{Y}^{[fav],n-n_{test}})\right\| \le \varepsilon$, the system objective $G(\mathbf{Y}^{[fav],n})$ can be referred to as a stable solution $G(\mathbf{Y}^{[fav],s})$ or possible local minimum.

In order to jump out of this possible local minimum, every agent *i* perturbs its current favorable strategy $X_i^{[fav]}$ by a perturbation factor $fact_i$ corresponding to the reciprocal of its favorable strategy $X_i^{[fav]}$ as follows:

$$X_i^{[fav]} = X_i^{[fav]} \pm \left(X_i^{[fav]} \times fact_i\right) \tag{6.5}$$

where $fact_i = \begin{cases} random\,value \in \left(\sigma_1^{lower}, \sigma_1^{upper}\right) & if\ \dfrac{1}{X_i^{[fav]}} \le \gamma \\ random\,value \in \left(\sigma_2^{lower}, \sigma_2^{upper}\right) & if\ \dfrac{1}{X_i^{[fav]}} > \gamma \end{cases}$

and $\sigma_1^{lower}, \sigma_1^{upper}, \sigma_2^{lower}, \sigma_2^{upper}$ are randomly generated values between 0 and 1 i.e., $0 < \sigma_1^{lower} < \sigma_1^{upper} \leq \sigma_2^{lower} < \sigma_2^{upper} < 1$. The value of γ as well as '+' or '-' sign in equation (6.5) are chosen based on the preliminary trials of the algorithm.

It gives a chance to every agent *i* to jump out of the local minima and may further help to search for a better solution. The perturbed solution is accepted if and only if the feasibility is maintained. Furthermore, every agent expands its sampling interval as follows:

$$\Psi_i \in \left[\left(\Psi_i^{lower} - \lambda_{up}\left\|\Psi_i^{upper} - \Psi_i^{lower}\right\|\right), \left(\Psi_i^{upper} + \lambda_{up}\left\|\Psi_i^{upper} - \Psi_i^{lower}\right\|\right)\right], \quad 0 < \lambda_{up} \leq 1 \tag{6.6}$$

where λ_{up} is referred to as the interval factor corresponding to the expansion of sample space.

6.1.3 Modifications to Step 6 of the Unconstrained PC approach

The detailed formulation of the modified convergence criterion is explained below and further presented in window C of the constrained PC algorithm flowchart in Figure 6.1.

The current stable system objective $G(\mathbf{Y}^{[fav],s})$ and corresponding $\mathbf{Y}^{[fav],s}$ are accepted as the final solution referred to as $G(\mathbf{Y}^{[fav],final})$ and $\mathbf{Y}^{[fav],final} = \left\{X_1^{[fav],final}, X_2^{[fav],final}, ..., X_{N-1}^{[fav],final}, X_N^{[fav],final}\right\}$, if and only if either of the following conditions is satisfied.

(a) If temperature $T = T_{final}$ or $T \to 0$.

(b) If there is no significant change in the successive stable system objectives (i.e. $\left\| G(\mathbf{Y}^{[fav],s}) - G(\mathbf{Y}^{[fav],s-1}) \right\| \leq \varepsilon$) for two successive implementations of the perturbation approach.

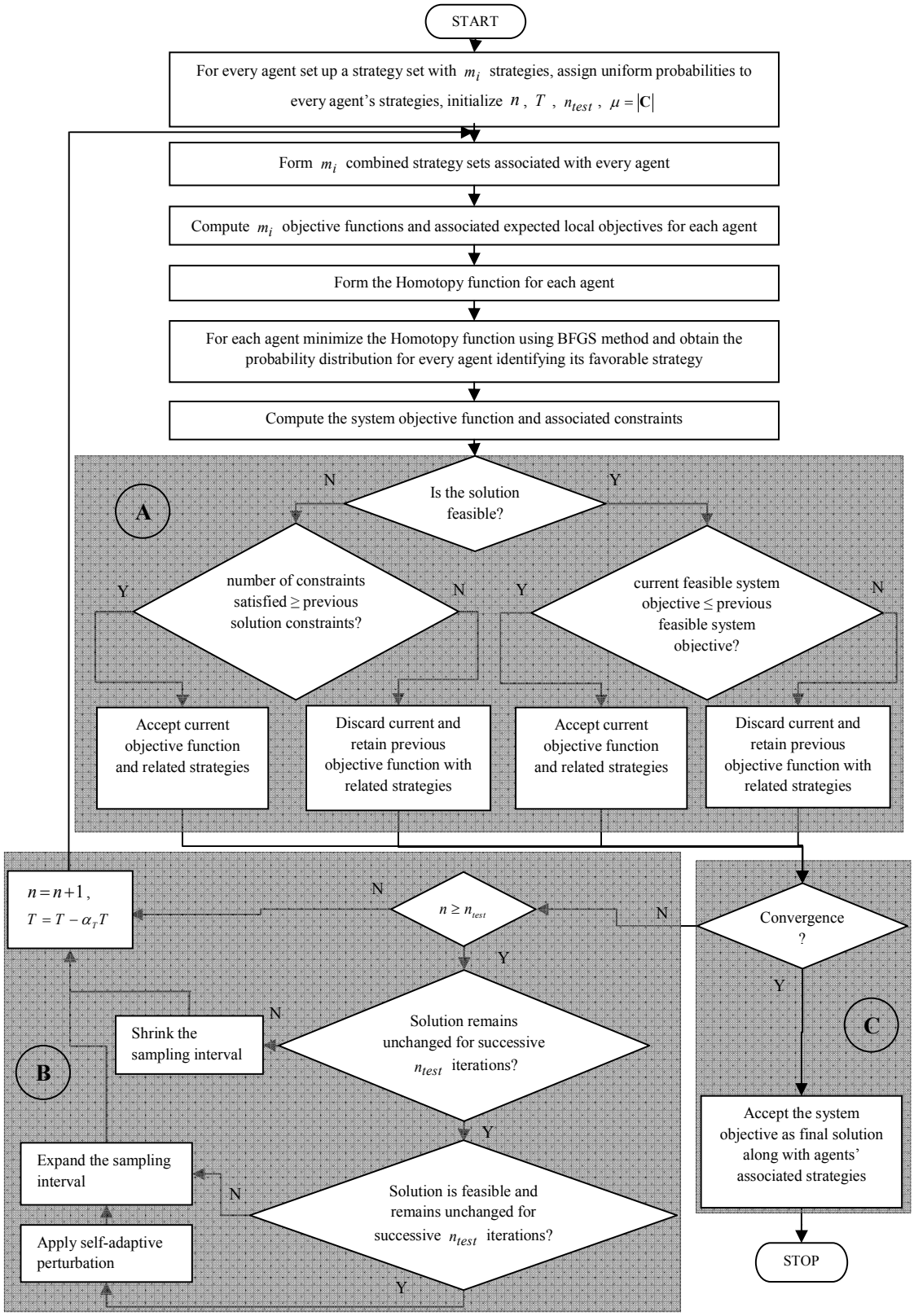


Figure 6.1: Constrained PC Algorithm Flowchart (Feasibility-based Rule I)

6.2 The Circle Packing Problem (CPP)

A generalized packing problem consists of determining how best to pack z objects into a predefined bounded space that yields best utilization of space with no overlap of object boundaries [184, 185]. The bounded space can also be referred to as a container. The packing objects and container can be circular, rectangular or irregular. Although the problem appears rather simple and in spite of its practical applications in production and packing for the textile, apparel, naval, automobile, aerospace, food industries, etc. [32] the CPP received considerable attention in the 'pure' mathematics literature but only limited attention in the operations research literature [33]. As it is proven to be a NP-hard problem [185-188] and cannot be effectively solved by purely analytical approaches [189-199], a number of heuristic techniques were proposed solving the CPP [184, 185, 200-212]. Most of these approaches address the CPP in limited ways, such as close packing of fixed and uniform sized circles inside a square or circle container [185, 189-198], close packing of fixed and different sized circles inside a square or circle container [184, 32, 205-212], simultaneous increase in the size of the circles covering the maximum possible area inside a square [201-204], etc.

As per knowledge of the author of this report; the CPP was never solved in a distributed way. In this report, as a distributed MAS, every individual circle changes its size and position autonomously. This allows for addressing the important issue of the avoidance of the tragedy of commons which was also never addressed before in the context of the CPP. The next few sections describe the mathematical formulation and the solution to two cases of the CPP.

6.2.1 Formulation of the CPP

The objective of the CPP solved here was to cover the maximum possible area within a square by z number of circles without overlapping one another or exceeding the boundaries of the square. In order to achieve this objective, all the circles were allowed to increase their sizes as well as change their locations. The problem is formulated as follows:

$$\text{Minimize} \quad f = L^2 - \sum_{i=1}^{z} \pi r_i^2 \tag{6.7}$$

Subject to

$$\sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2} \geq r_i + r_j \tag{6.8}$$

$$x_i - r_i \geq x_l \tag{6.9}$$

$$x_i + r_i \leq x_u \tag{6.10}$$

$$y_i - r_i \geq y_l \tag{6.11}$$

$$y_i + r_i \leq y_u \tag{6.12}$$

$$0.001 \leq r_i \leq L/2 \tag{6.13}$$

$$i, j = 1, 2, ..., z \quad i \neq j \tag{6.14}$$

where

L = length of the side of the square

r_i = radius of circle i

x_i, y_i = x and y coordinates of the center of circle i

x_l, y_l = x and y coordinates of the lower left corner of the square

x_u, y_u = x and y coordinates of the upper right corner of the square

In solving the proposed CPP using constrained PC approach presented in Section 6.1, the circles were considered as autonomous agents. These circles were assigned the strategy sets of coordinates and y coordinates of the center and the radius. Two cases of the CPP were solved. These cases differ from each other based on the initial configuration (location) of the circles as well as the way constraints are handled solving each case.

In Case 1, the circles were randomly initialized inside the square and were not allowed to cross the square boundaries. The constraints in equation (6.8) were satisfied using the Feasibility-based Rule I described in Section 6.1. And the constraints in equations (6.9) to (6.12) were satisfied in every iteration of the algorithm using a repair approach. The repair approach refers to pushing the circles inside the square if they crossed the boundaries of it. It is similar to the one proposed in Chapter 4 solving the MTSP. In Case 2, the circles were randomly located in-and-around the square and all the constraints from (6.8) to (6.12) were satisfied using the Feasibility-based Rule I described in Section 6.1. The initial configuration of Case 1 and Case 2 is shown in Figure 6.2(a) and 6.5(a), respectively.

The constrained PC algorithm solving both the cases was coded in MATLAB 7.8.0 (R2009A) and the simulations were run on a Windows platform using an Intel Core 2 Duo, 3GHz processor speed and 3.25GB memory capacity. Furthermore, for both the cases the set of parameters chosen was as follows: (a) individual agent sample size $m_i = 5$, (b) number of test iterations $n_{test} = 20$, (c) the shrinking interval factor $\lambda_{down} = 0.05$, (d) the expansion interval factor $\lambda_{up} = 0.1$, (e) perturbation parameters $\sigma_1^{lower} = 0.001$, $\sigma_1^{upper} = 0.01$, $\sigma_2^{lower} = 0.5$, $\sigma_2^{upper} = 0.7$, $\gamma = 0.99$ and the sign in equation (6.5) was chosen to be '-'. In addition to it, a voting heuristic was also incorporated in the constrained PC algorithm. It is described in the Section 6.2.4.

6.2.2 Case 1: CPP with Circles Randomly Initialized Inside the Square

In this case of the CPP, five circles ($z = 5$) were initialized randomly inside the square without exceeding the boundary edges of the square. The length of the side of the square was five units (i.e. $L = 5$). More than 30 runs of the constrained PC algorithm described in

Section 6.1 were conducted solving the Case 1 of the CPP with different initial configurations of the circles inside the square. The true optimum solution was achieved in every run with the average CPU time of 14.05 minutes and average number of function evaluations is 17515.

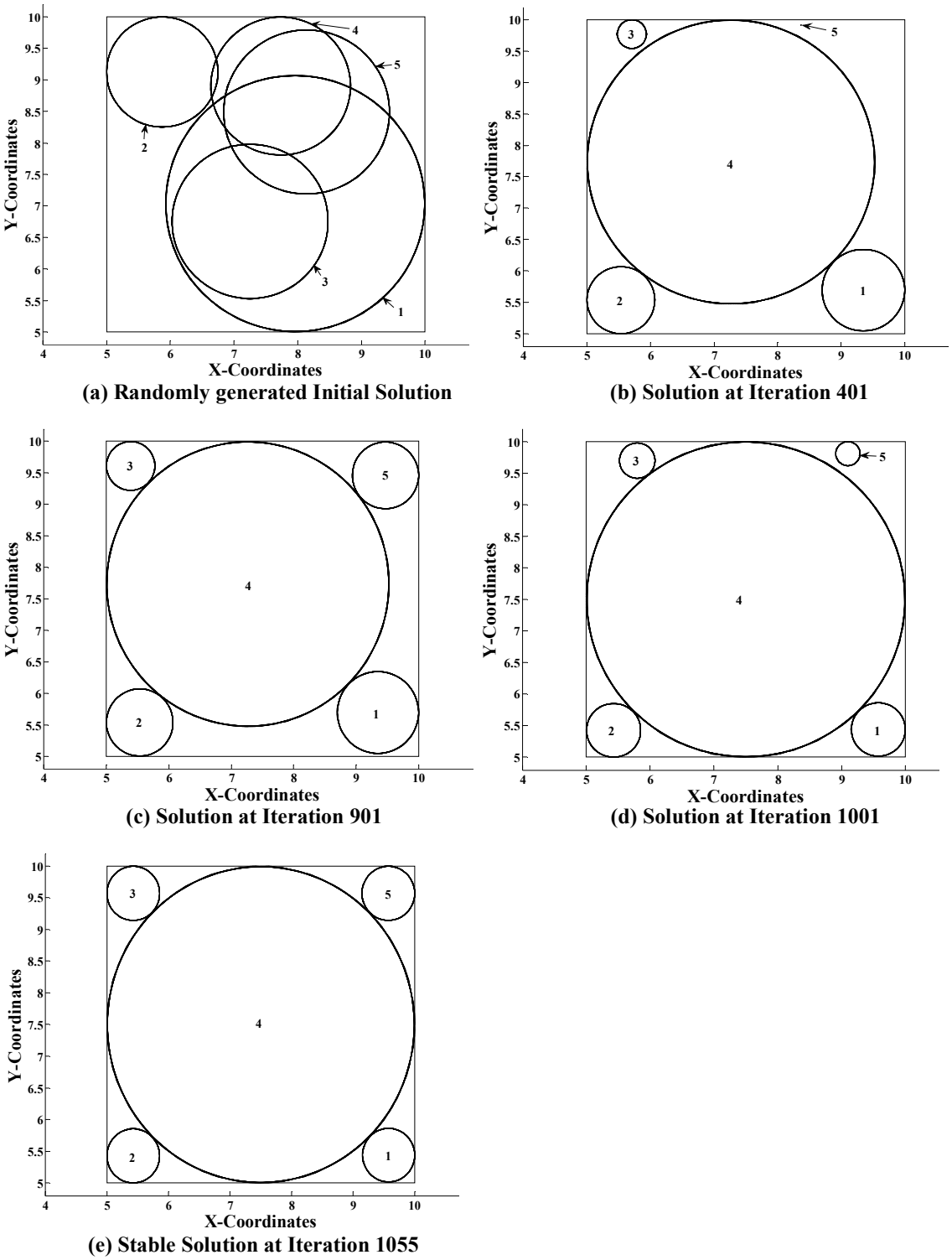


Figure 6.2: Solution History for Case 1

The randomly generated initial solution, the intermediate iteration solutions, and the

converged true optimum solution from one of the instances are presented in Figure 6.2. The corresponding convergence plot of the system objective is presented in Figure 6.3. The convergence of the associated variables such as radius of the circles, x coordinates and y coordinates of the center of the circles is presented in Figure 6.4(a), 6.4(b) and 6.4(c), respectively. The solution was converged at iteration 1035 with 26910 function evaluations. The true optimum value of the objective function (f) achieved was 3.0807 units.

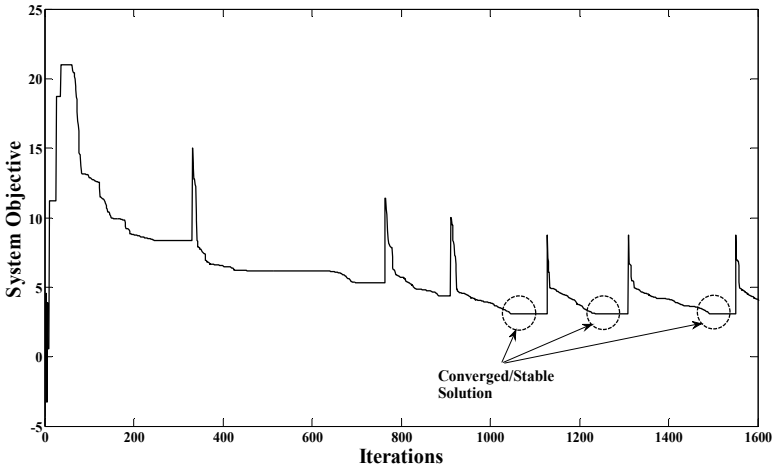
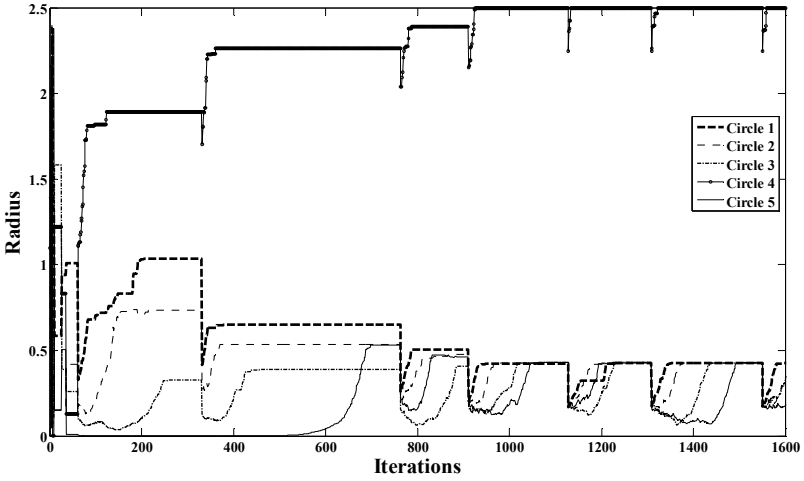
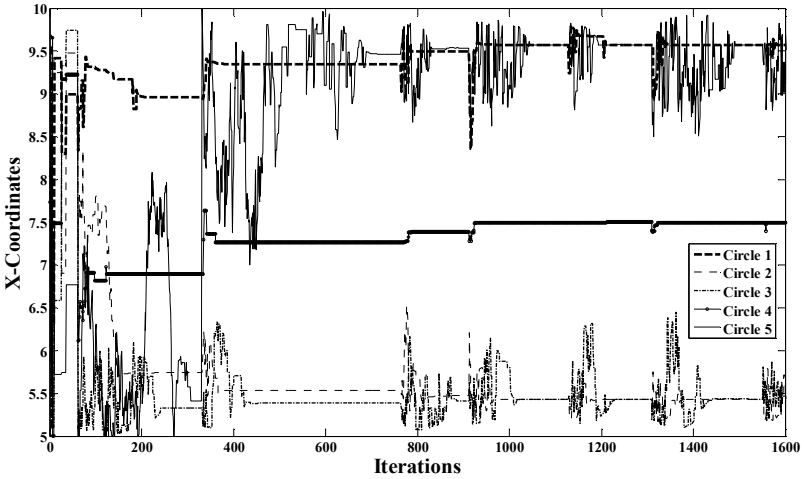


Figure 6.3: Convergence of the Objective Function for Case 1

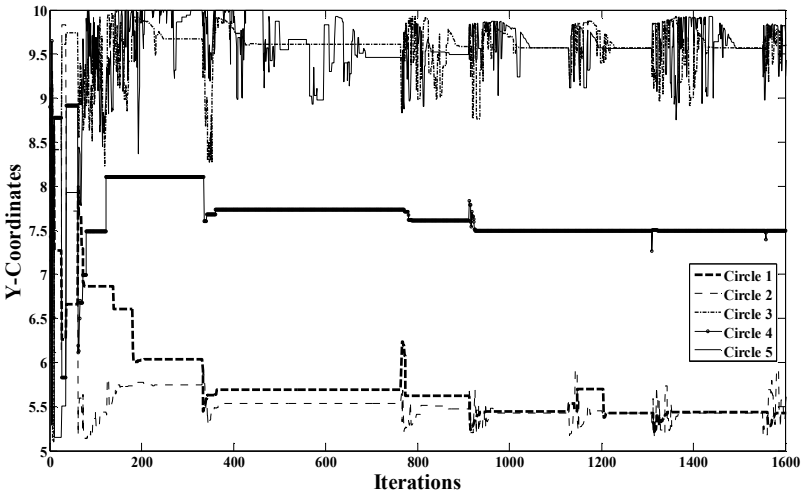
As mentioned before, the algorithm was assumed to have converged when successive implementations of the perturbation approach stabilize to equal objective function value. It is evident from Figure 6.2, 6.3 and 6.4 that the solution was converged to true optimum at iteration 1035 as the successive implementations of the perturbation approach produced stable and equal objective function values. Furthermore, it is also evident from Figure 6.3 and 6.4 that the solution was perturbed at iteration 778, 901, 1136 and 1300. It is clear that the implementation of the perturbation approach at iteration 901 helped the solution to jump out of the local minima and further achieve the true optimum solution at iteration 1035.



(a) Convergence of the Radius



(b) Convergence of the X-Coordinates of the Center



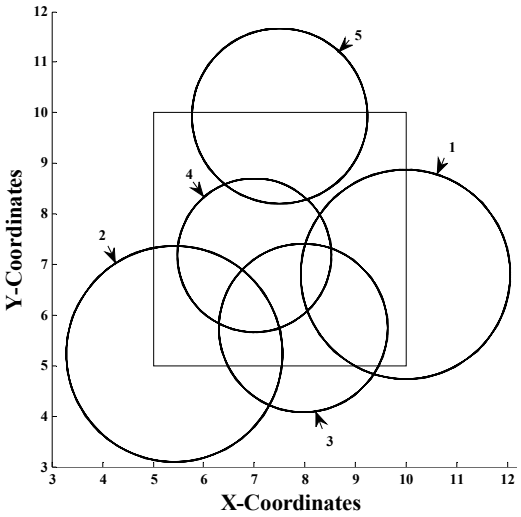
(c) Convergence of the Y-Coordinates of the Center

Figure 6.4: Convergence of the Strategies for Case 1

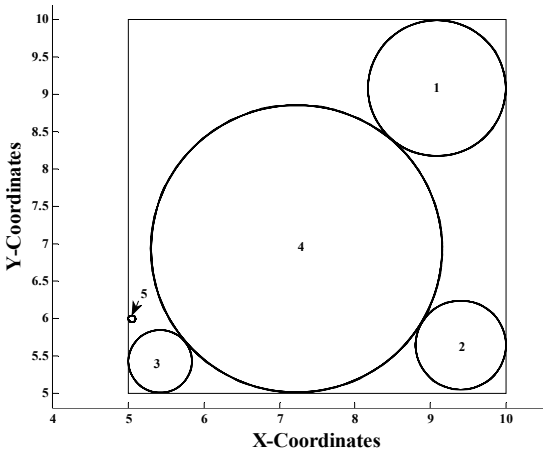
6.2.3 Case 2: CPP with Circles Randomly Initialized

In this case of the CPP, five circles ($z = 5$) were initialized randomly in the space with no restriction as in Case 1 where circles were randomly placed inside the square. The length of the side of the square was five units (i.e. $L = 5$). Similar to Case 1, more than 30 runs of the constrained PC algorithm described in Section 6.1 with different initial configuration of the circles were conducted solving the Case 2. The true optimum solution was achieved in every run with the average CPU time of 14.05 minutes and average number of function evaluations is 68406.

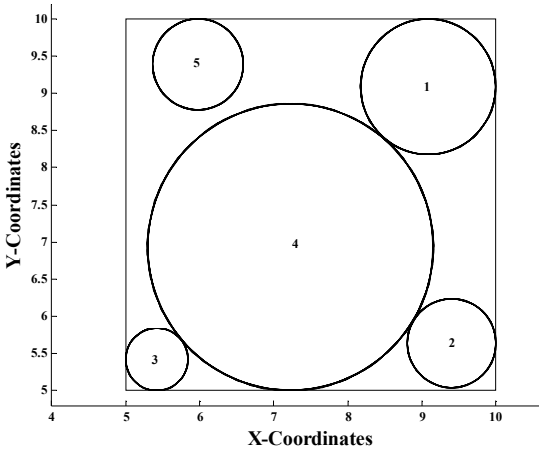
The randomly generated initial solution, the intermediate iteration solutions, and the converged true optimum solution from one of the instances of Case 2 are presented in Figure 6.5. The corresponding convergence plot of the system objective is presented in Figure 6.6. The convergence of the associated variables such as radius of the circles, x coordinates and y coordinates of the center of the circles is presented in Figure 6.7(a), 6.7(b) and 6.7(c), respectively. The solution was converged at iteration 955 with 24830 function evaluations. The true optimum value of the objective function (f) achieved was 3.0807 units.



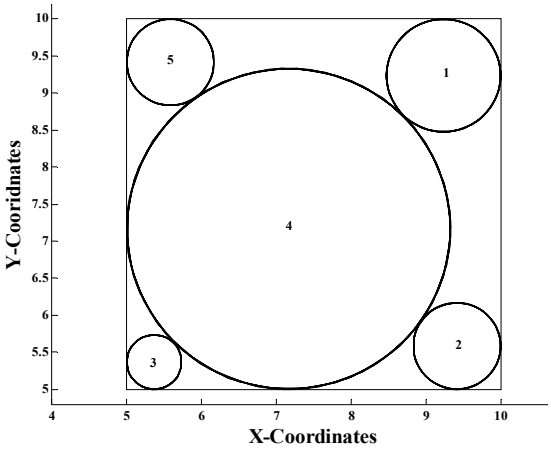
(a) Randomly generated Initial Solution



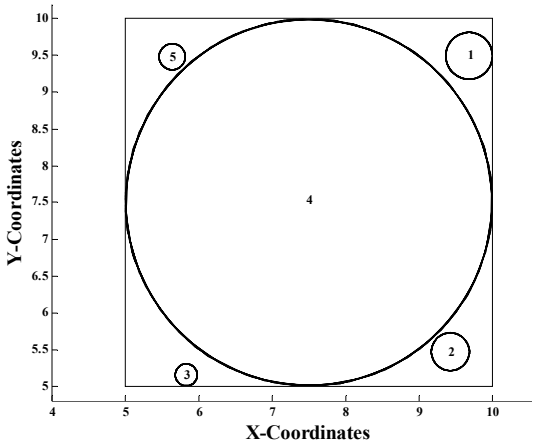
(b) Solution at Iteration 301



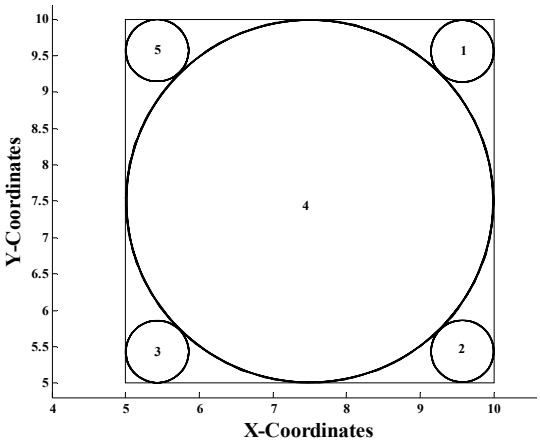
(c) Solution at Iteration 401



(d) Solution at Iteration 601



(e) Solution at Iteration 801



(f) Stable Solution at Iteration 955

Figure 6.5: Solution History for Case 2

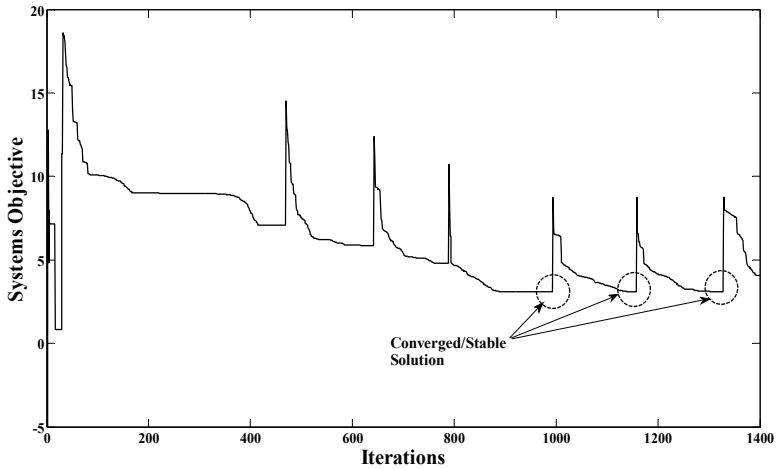
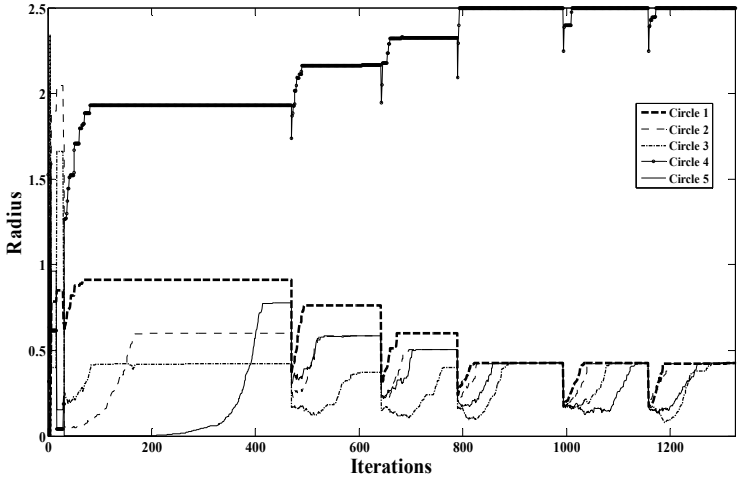
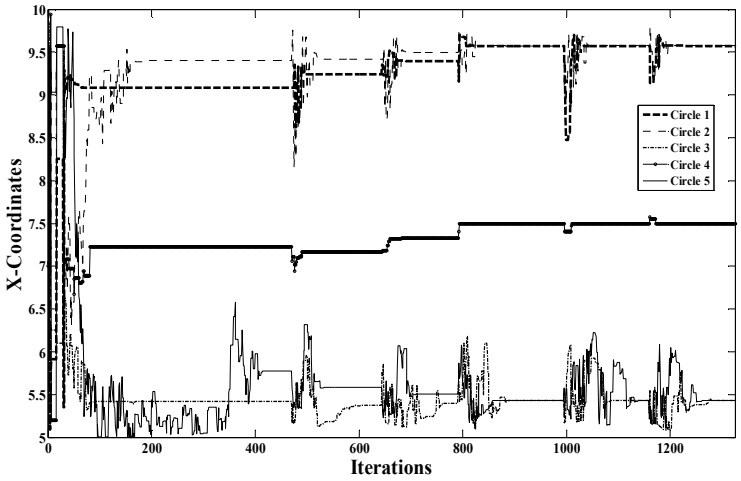


Figure 6.6: Solution Convergence Plot for Case 2

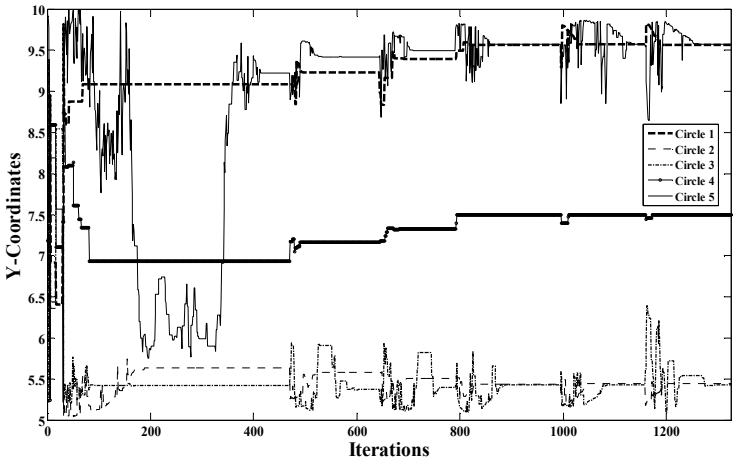
In the instance of the Case 2 presented here, the solution was perturbed at iteration 788, 988, 1170 and 1355. It is clear that the implementation of the perturbation approach at iteration 788 helped the solution to jump out of the local minima and further achieve the true optimum solution at iteration 955. It is important to mention that the instance illustrated here did not require the voting heuristic to be applied.



(a) Convergence of the Radius



(b) Convergence of the X-Coordinates of the Center

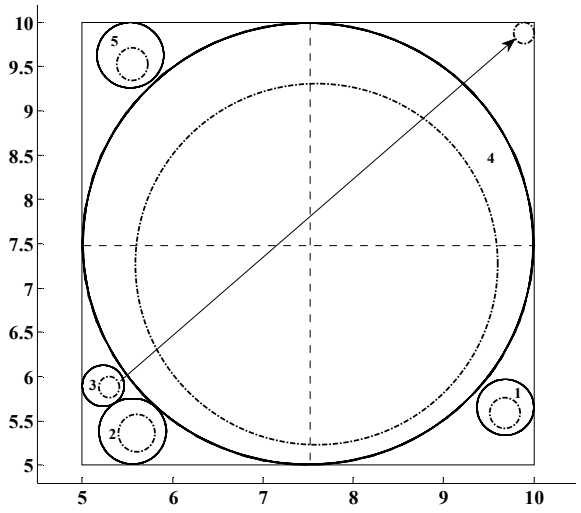


(c) Convergence of the Y-Coordinates of the Center

Figure 6.7: Convergence of the Strategies for Case 2

6.2.4 Voting Heuristic

In a few instances of the CPP cases solved here, in order to jump out of the local minimum, a voting heuristic was required. It was implemented in conjunction with the perturbation approach. Once the solution was perturbed, every circle voted 1 for each quadrant which it does not belong to at all, and voted 0 otherwise. The circle with the smallest size shifted itself to the extreme corner of the quadrant with the highest number of votes, i.e. the winner quadrant. The new position of the smallest size circle was confirmed only when the solution remained feasible and the algorithm continues. If all the quadrants acquire equal number of votes, no circle moves its position and the algorithm continues. The voting heuristic is demonstrated in Figure 6.8.



(a) A Case for Voting Heuristic

Q 2

1	2	3	4	5
1	1	1	0	0

Total votes = 3

Q 1

1	2	3	4	5
1	1	1	0	1

Total votes = 4

Q 3

1	2	3	4	5
1	0	0	0	1

Total votes = 2

Q 4

1	2	3	4	5
0	1	1	0	1

Total votes = 3

(b) Voting Grids

Figure 6.8: Voting Heuristic

A voting grid corresponding to every quadrant of the square in Figure 6.8(a) is represented in Figure 6.8(b). The solid circles represent the solution before perturbation while corresponding perturbed ones are represented in dotted lines. The votes given by the perturbed circles (dotted circles) to the quadrants are presented in the grid. As the

maximum number of votes are given to quadrant 1 (i.e. Q 1), the circle with smallest size (circle 3) shifts to the extreme corner of the quadrant Q 1 and confirms the new position as the solution remains feasible. Based on the trials conducted so far, it was noticed that the voting heuristic was not necessary to be implemented in every run of the constrained PC algorithm solving the CPP. Moreover, in those of the few runs in which the voting heuristic was required, it was required to be implemented only once in the entire execution of the algorithm. A variant of the voting heuristic was also implemented in conjunction with energy landscape paving algorithm [32, 206, 207] in which, the smallest circle was picked and placed randomly at the vacant place to produce a new configuration. It was claimed that this heuristic helped the algorithm jump out of the local minima. Furthermore, this heuristic was required to be implemented in every iteration of the algorithm.

6.2.5 Agent Failure Case

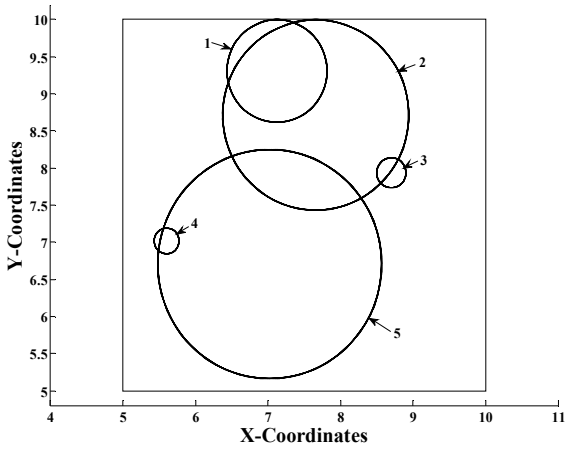
As discussed before, a centralized system is vulnerable to single point failures and its performance may be severely affected if an agent in the system fails. On the other hand, in case of a distributed and decentralized algorithm, the system is more robust as the system is controlled by its autonomous subsystems.

As mentioned in section 3.2, the failed agent in PC can be considered as the one that does not communicate with other agents in the system and does not update its probability distribution. This does not prevent other agents from continuing further, by simply considering the failed agent's latest communicated strategies as the current strategies.

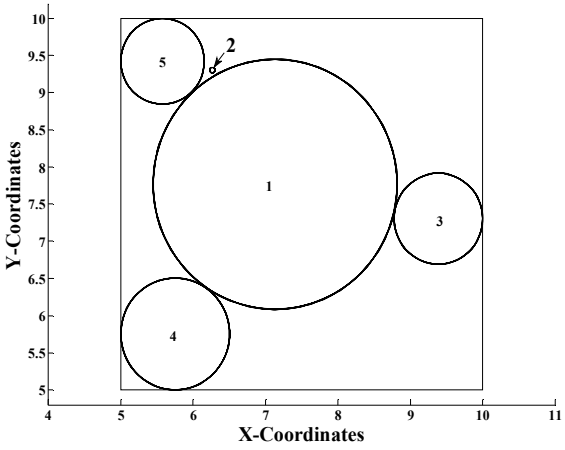
The immunity to agent failure is essential in the field of UAV formation and collaborative path planning because failure of the centralized controller would be

devastating and may result in collision. This is addressed in the related work on UAVs [2, 56-62].

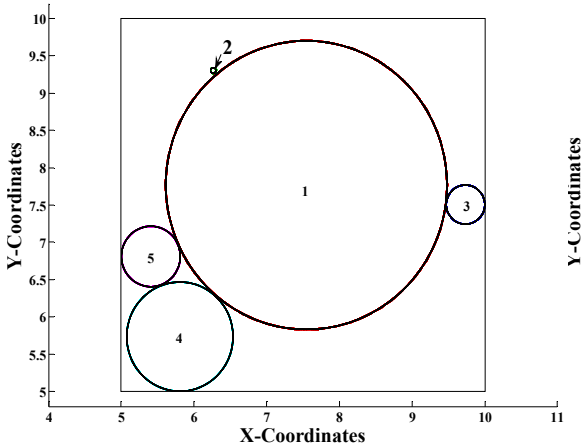
In the context of the CPP, as a variation of Case 1, circle 2 was failed at a randomly chosen iteration 30, i.e. circle 2 does not update its x and y coordinates as well as its radius after iteration 30. More than 30 cases of the constrained PC algorithm with Feasibility-based Rule I were conducted solving the CPP with agent failure case and the average function evaluations were 17365. For the case presented here, the number of function evaluations was 19066 and the randomly generated initial solution, the intermediate iteration solutions, and the converged true optimum solution are shown in Figure 6.9. The corresponding convergence plot of the system objective is presented in Figure 6.10. The convergence of the associated variables such as radius of the circles, x coordinates and y coordinates of the center of the circles is presented in Figure 6.11(a), 6.11(b) and 6.11(c), respectively.



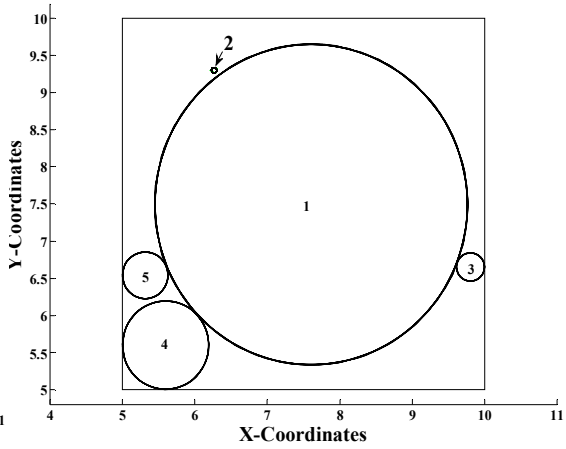
(a) Randomly generated Iteration Solution



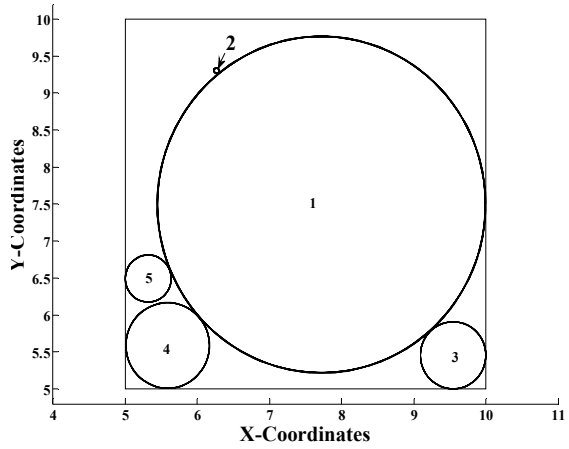
(b) Solution at Iteration 124



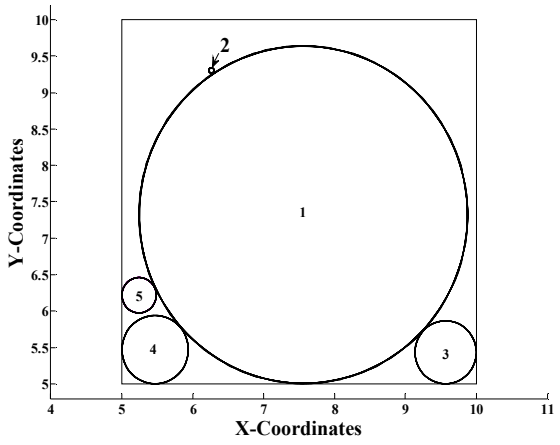
(c)Solution at Iteration 231



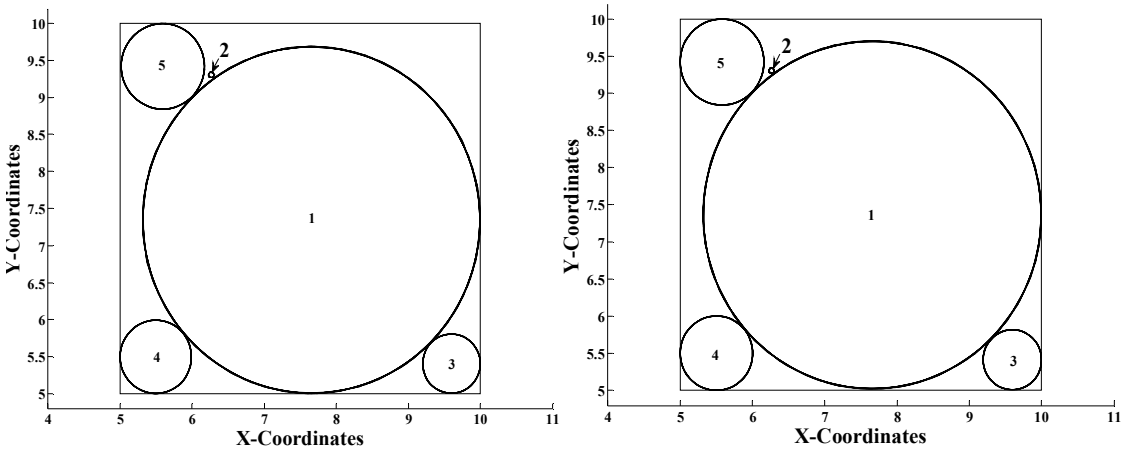
(d) Solution at Iteration 377



(e) Solution at Iteration 561



(f) Solution at Iteration 723



(g) Stable Solution at Iteration 901 (h) Stable Solution at Iteration 1051

Figure 6.9: Solution History for Agent Failure Case

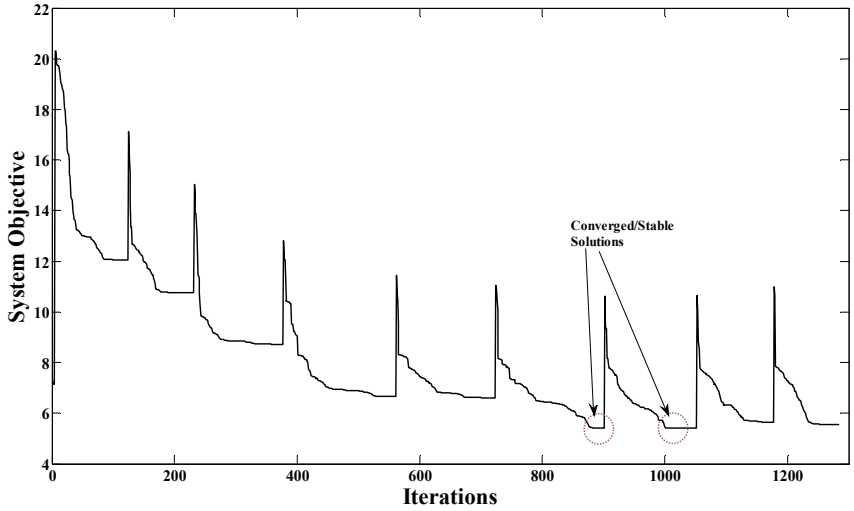
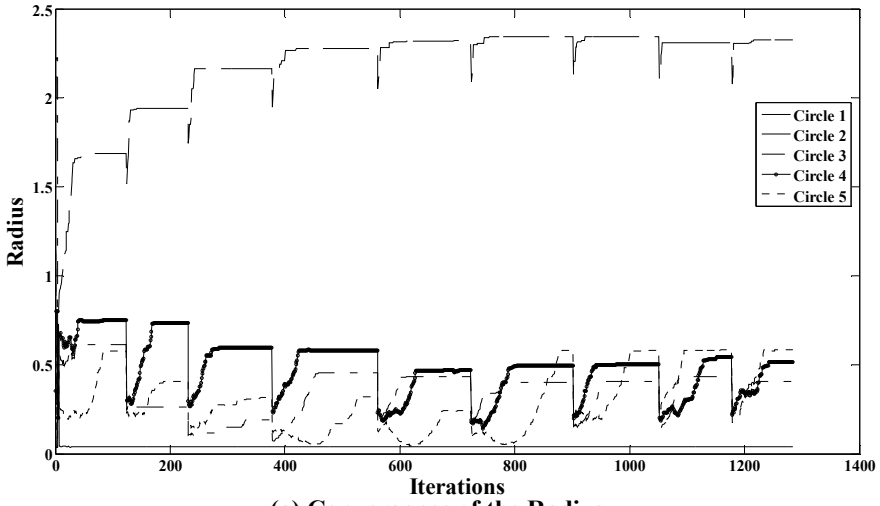
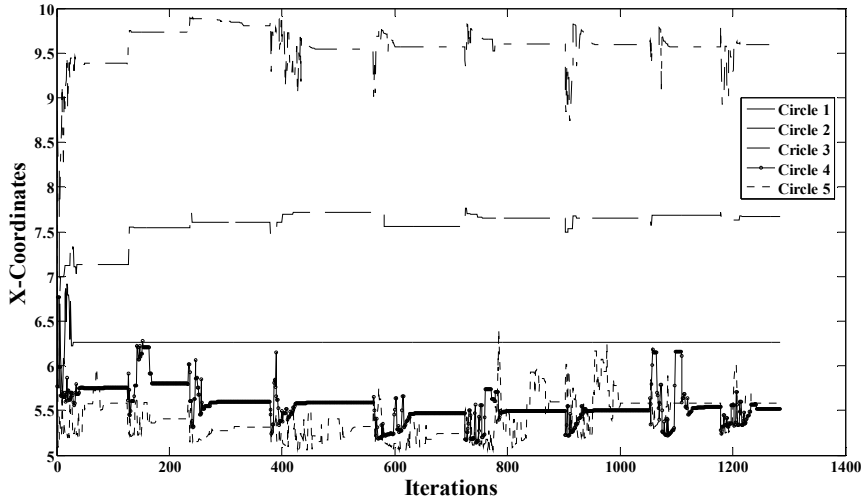


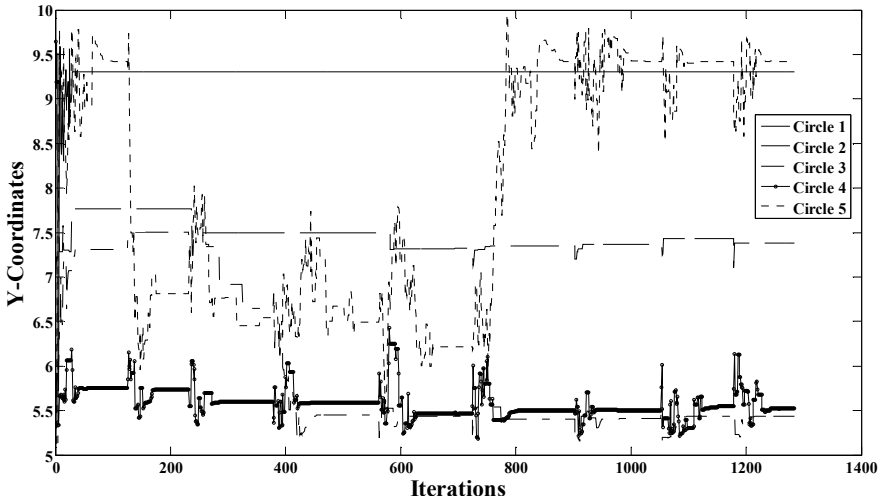
Figure 6.10: Convergence of the Objective Function for Agent Failure Case



(a) Convergence of the Radius



(b) Convergence of the X-Coordinates of the Center



(c) Convergence of the Y-Coordinates of the Center

Figure 6.11: Convergence of the Strategies for Agent Failure Case

It is worth to mention that the voting heuristic was not required in the case presented in Figure 6.9. This is because according to the voting heuristic the smallest circle which moves itself to the new position is itself the failed agent. In addition, the same set of parameter as values listed in Section 6.2.1 was used for the agent failure case.

6.3 Discussion

The above sections described the successful implementation of a generalized constrained PC approach using a variation of the feasibility-based rule originally proposed in [159]. The results indicated that it could successfully be used to solve constrained optimization problems such as the CPP. It is evident from the results that the approach was sufficiently robust and produced true optimum results in every run of Case 1 and 2. It implies that the rational behavior of the agents could be successfully formulated and demonstrated. Moreover, it is also evident that because of the inherent distributed nature of the PC algorithm, it can easily accommodate the agent failure cases. The solution highlights its strong potential to deal with the agent failure which may arise in real world complex problems including urban traffic control, formation of airplanes fleet and mid-air collision avoidance, etc.

In addition, the feasibility-based rule in [159, 160, 179-183] suffered from maintaining the diversity and further required additional techniques such as niching [159], SA [160], modified mutation approach [181, 182], and several associated trials in [179-182], etc. It may require further computations and memory usage. On the other hand, in order to jump out of the possible local minima, a simple perturbation approach was successfully

incorporated into the constrained PC algorithm. It is worth to mention that the perturbation approach was computationally cheaper and required no additional memory usage.

It is important to mention that the concept of the avoidance of tragedy of commons was also successfully demonstrated solving the two cases (Case 1 and Case 2) of the CPP. More specifically, as all the agents were supposed to collectively cover the maximum possible area within the square, every circle/agent could have in a greedy way, increased its own radius (or size) [7, 9]. This would have further resulted into the suboptimal solution. However, in the PC solution to the two cases of the CPP presented here, every circle/agent selected its individual radius (or size) as well as its x and y coordinates in order to collectively achieve the true optimum solution (i.e. collectively cover the maximum possible area within the square) by avoiding the tragedy of commons. Although only inequality constraints were handled in both the cases of the CPP solved here, the approach of transformation of equality constraints into inequality constraints [172-174] can be easily implemented for problems with equality constraints.

In agreement with the no-free-lunch theorems [213], some limitations were also identified. The rate of convergence and the quality of the solution was dependent on the parameters such as the number of strategies m_i in every agent's strategy set $\mathbf{X}_i$, the interval factor λ and also the perturbation parameters. It is also necessary that some preliminary trials are performed for fine-tuning these parameters. Additionally, in order to confirm the convergence, the algorithm was required to be run beyond the convergence for a considerable number of iterations.

The following sections discuss the Feasibility-based Rule II, associated formulation, implementation and validation of results solving three cases of the Sensor Network Coverage Problem (SNCP).

6.4 Feasibility Based Rule II

This rule is a variation of the Feasibility-based Rule I discussed in Section 6.1 and also allows the objective function and the constraint information to be considered separately. In addition to the iterative tightening of the constraint violation tolerance in order to obtain fitter solution and further drive the convergence towards feasibility, the Feasibility based Rule II helps the solution jump out of possible local minima.

At the beginning of the PC algorithm, the number of constraints improved μ is initialized to zero, i.e. $\mu = 0$. The value of μ is updated iteratively as the algorithm progresses. Similar to the Feasibility-based Rule I, Feasibility-based Rule II is also assisted with the modified approach of updating of the sampling space as well as the convergence criterion. The rule and these assisting techniques are discussed below as the modifications to the unconstrained PC approach presented in Section 3.4.

6.4.1 Modifications to Step 5 of the Unconstrained PC Approach

The *step 5* of the unconstrained PC procedure discussed in Section 3.4 is modified by using the following rule:

(a) Any feasible solution is preferred over any infeasible solution

(b) Between two feasible solutions, the one with better objective is preferred

(c) Between two infeasible solutions, the one with more number of improved constraint violations is preferred.

(d) If the solution remains feasible and unchanged for successive predefined number of iterations, and current feasible system objective is worse than the previous iteration feasible solution, accept the current solution. Similar to SA [95-97, 185], this may help jump out of the local minima.

The detailed formulation of the above rule is explained below and further presented in window A of the constrained PC algorithm flowchart in Figure 6.12.

(a) If the current system objective $G\left(\mathbf{Y}^{[fav]}\right)$ as well as the previous solution are infeasible, accept the current system objective $G\left(\mathbf{Y}^{[fav]}\right)$ and corresponding $\mathbf{Y}^{[fav]}$ as current solution if the number of improved constraints is greater than or equal to μ, i.e. $C_{improved} \geq \mu$, and then the value of μ is updated to $C_{improved}$, i.e. $\mu = C_{improved}$.

(b) If the current system objective $G(\mathbf{Y}^{[fav]})$ is feasible, and the previous solution is infeasible, accept the current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$ as the current solution, and then the value of μ is updated to 0, i.e. $\mu = C_{improved} = 0$.

(c) If the current system objective $G(\mathbf{Y}^{[fav]})$ is feasible and is not worse than the previous feasible solution, accept the current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$ as the current solution.

(d) If all the above conditions (a) to (c) are not met, then discard current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$, and retain the previous iteration solution.

(e) If the solution remains feasible and unchanged for successive pre-specified n_{test} iterations i.e. $G(\mathbf{Y}^{[fav],n})$ and $G(\mathbf{Y}^{[fav],n-n_{test}})$ are feasible and $G(\mathbf{Y}^{[fav],n}) = G(\mathbf{Y}^{[fav],n-n_{test}})$, and the current feasible system objective is worse than the previous iteration feasible solution, accept the current system objective $G(\mathbf{Y}^{[fav]})$ and corresponding $\mathbf{Y}^{[fav]}$ as the current solution.

In this way the algorithm progresses by iteratively tightening the constraint violation.

6.4.2 Modifications to Step 7 of the Unconstrained PC Approach

It is important to mention that once the solution becomes feasible, every agent i shrinks its sampling interval to the local neighborhood of its current favorable strategy $X_i^{[fav]}$. This is done as follows:

$$\Psi_i \in \left[\left(X_i^{[fav]} - \lambda_{down}\left\|\Psi_i^{upper} - \Psi_i^{lower}\right\|\right), \left(X_i^{[fav]} + \lambda_{down}\left\|\Psi_i^{upper} - \Psi_i^{lower}\right\|\right)\right], \quad 0 < \lambda_{down} \leq 1 \tag{6.15}$$

where λ_{down} is referred to as the interval factor corresponding to the shrinking of sample space. The modification of updating the sampling space is presented in window B of the constrained PC algorithm flowchart in Figure 6.12. The associated modified convergence criterion is discussed below.

6.4.3 Modifications to Step 6 of the Unconstrained PC Approach

The detailed formulation of the modified convergence criterion is explained below and further presented in window C of the constrained PC algorithm flowchart in Figure 6.12.

The current system objective $G(\mathbf{Y}^{[fav],n})$ and corresponding $\mathbf{Y}^{[fav],n}$ are accepted as the final solution referred to as $G(\mathbf{Y}^{[fav],final})$ and $\mathbf{Y}^{[fav],final} = \left\{ X_1^{[fav],final}, X_2^{[fav],final}, ..., X_{N-1}^{[fav],final}, X_N^{[fav],final} \right\}$, respectively, if and only if any of the following conditions are satisfied.

(a) If temperature $T = T_{final}$ or $T \to 0$

(b) If maximum number of iterations exceeded

(c) If there is no significant change in the system objective (i.e. $\left\| G(\mathbf{Y}^{[fav],n}) - G(\mathbf{Y}^{[fav],n-1}) \right\| \le \varepsilon$) for successive considerable number of iterations.

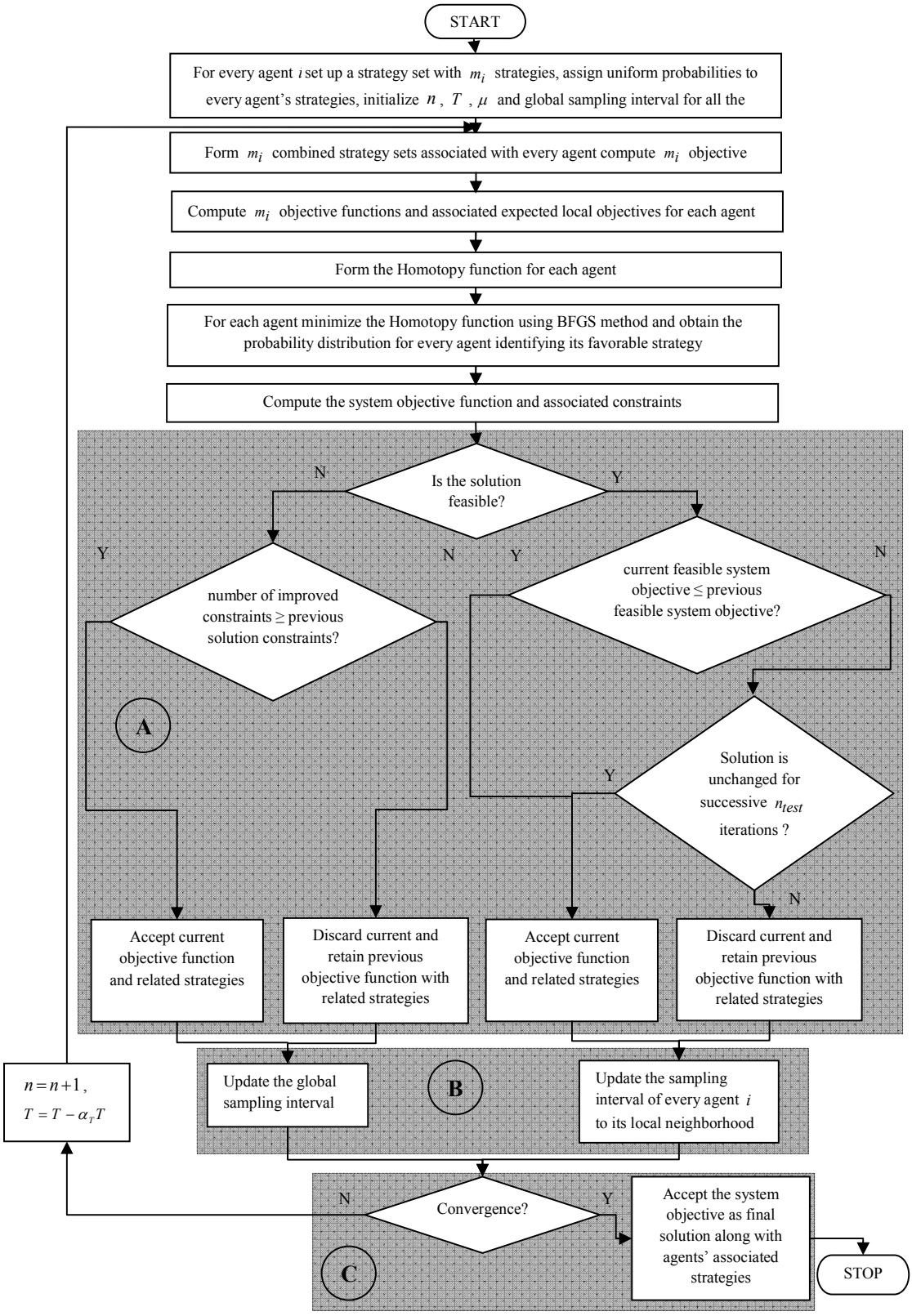


Figure 6.12: Constrained PC Algorithm Flowchart (Feasibility-based Rule II)

6.5 The Sensor Network Coverage Problem (SNCP)

The sensor network plays a significant role in various strategic applications such as hostile and hazardous environmental and habitat exploration and surveillance, critical infrastructure monitoring and protection, situational awareness of battlefield and target detection, natural disaster relief, industrial sensing and diagnosis, biomedical health monitoring, seismic sensing, etc. [35-43]. The deployment or positioning of the individual sensor directly affects the coverage, detection capability, connectivity and associated communication cost and resource management of the entire network [37, 38, 44, 214]. According to [39, 44, 215-217], coverage is the important performance metric that quantifies the quality and effectiveness of the surveillance/monitoring provided by the sensor network. This highlighted the requirement for an effective sensor deployment algorithm to optimize the coverage of the entire network [39, 44, 214]. Furthermore, according to [218, 219] the connected coverage is important by which all the sensors in the network can communicate with one another over multiple hops.

The sensor network coverage can be classified as deterministic or stochastic coverage [220]. The deterministic coverage refers to the static deployment of the sensors over a predefined Field of Interest (FoI) [221-225]. It includes the uniform coverage of the FoI as well as weighted coverage of certain section of the FoI [45, 216]. The stochastic coverage refers to the random deployment of the sensors over the FoI. The sensor positions are selected based on the uniform, Gaussian, Poisson or any other problem specific distribution [216]. The deterministic coverage may provide worst-case performance which may not be achievable by stochastic coverage [39, 217]. According to [226], the coverage can also be classified into three types. The first is blanket coverage, which represents the

static arrangement of the sensors to maximize the total detection area; the second is barrier coverage, which aims to arrange the sensors to form a barrier in order to minimize the probability of undetected penetration/intrusion through the barrier [217, 227]; and the third is sweep coverage which is a moving barrier of sensors that aims to maximize the number of detections per unit time and minimize the number of missed detections per unit area. The blanket coverage can be further classified into two types [35]. The first is point-set coverage, which aims to cover a set of discrete points scattered over a certain field and the second is FoI coverage, which aims to cover a certain region to the maximum possible extent.

This following section addresses the problem of deployment of a set of homogeneous (identical characteristics) sensors over a certain predefined FoI in order to achieve the maximum possible deterministic connected blanket coverage using the method of PC.

6.5.1 Formulation of the SNCP

The SNCP addressed here is as follows: given a set of z homogeneous sensors with equal and fixed sensing range r_s for every sensor $i \in \{1, 2, ..., z\}$, and the sensing area of every sensor i modeled as a circle with the sensor located at its centre; find the deployment of all the sensors to collectively cover the maximum possible area (hereafter referred to as collective coverage $A_{collective}$) of a square FoI without overlapping one another and without exceeding the boundaries of the square FoI, and importantly maintaining connectivity between them. The connectivity here refers to the constraint which limits the distance $d(i, j)$ between two adjacent sensors i and j to a maximum predefined value γ, i.e. $d(i, j) \le \gamma$, $i \sim j$, $i \ne j$, $i, j = 1, 2, ..., z$ [218, 219] where adjacency is represented by

the symbol ' ~ '. The network is considered connected if there is at least one connected path between each and every pair of sensors. As mentioned previously, it ensures that all the sensors in the network can communicate with one another over multiple hops. Variations to this connectivity constraint based on the number of sensors required [218], and the specific deployment pattern under consideration [218, 219], can be found in the literature. An example of a SNCP with three sensors ($z = 3$) is illustrated in Figure 6.13 in which the distance $d(1,2)$ between adjacent sensors 1 and 2 and $d(2,3)$ between adjacent sensors 2 and 3 is represented by dashed lines with arrows at both the ends.

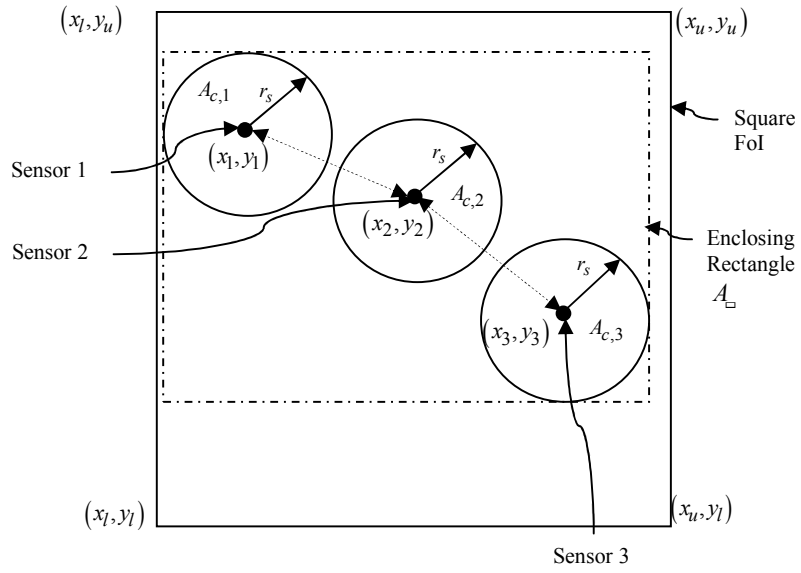


Figure 6.13: An Example of a Sensor Network Coverage Problem

In order to achieve the above mentioned objective of maximum possible collective coverage $A_{collective}$ of the square FoI and yet maintaining connectivity of the sensors the system objective is the area $A_{\square}$ of an enclosing rectangle for which the sensing areas of all the sensors can be fitted within. The sensors are considered as autonomous agents and

are assigned the individual x coordinates and y coordinates as their strategy sets. The SNCP is formulated as follows:

$$\text{Minimize} \quad A_{\square} = \left(\left(\max\left(x_1, x_2, ..., x_i, ..., x_z\right) + r_s\right) - \left(\min\left(x_1, x_2, ..., x_i, ..., x_z\right) - r_s\right)\right) \times \left(\left(\max\left(y_1, y_2, ..., y_i, ..., y_z\right) + r_s\right) - \left(\min\left(y_1, y_2, ..., y_i, ..., y_z\right) - r_s\right)\right) \tag{6.16}$$

Subject to

$$d\left(i, j\right) \geq 2r_s, \qquad i, j = 1, 2, ..., z, \qquad i \neq j \tag{6.17}$$

$$x_i - r_s \geq x_l \tag{6.18}$$

$$x_i + r_s \leq x_u \tag{6.19}$$

$$y_i - r_s \geq y_l \tag{6.20}$$

$$y_i + r_s \leq y_u \tag{6.21}$$

$$i = 1, 2, ..., z$$

$$d\left(i, j\right) \leq \gamma \qquad i \sim j, \quad i \neq j, \quad i, j = 1, 2, ..., z \tag{6.22}$$

where

$A_{\square}$ = area of the enclosing rectangle

z = number of sensors

r_s = sensing range of every sensor $i \in \{1, 2, ..., z\}$

x_i, y_i = x and y coordinates of sensor i (or x and y coordinates of the center of the circle i)

x_l, y_l = x and y coordinates of the lower left corner of the square FoI

x_u, y_u = x and y coordinates of the upper right corner of the square FoI

$d\left(i, j\right) = \sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2}$ = distance between two sensors i and j

The constraints in equation (6.17) are to prevent overlapping of the sensor circles and the constraints in equations (6.18) to (6.21) are to prevent the sensor circles from exceeding the FoI boundaries. The constraints in equation (6.22) are the connectivity

constraints and are checked in every iteration using the well known Depth First Search (DFS) algorithm [228] only when all the other constraints are satisfied.

The associated collective coverage $A_{collective}$ is calculated as follows:

$$A_{collective} = \sum_{i=1}^{z} A_{c,i} \tag{6.23}$$

where $A_{c,i} =$ FoI coverage achieved by sensor i

In the three sensor example illustrated in Figure 6.13, the individual sensor coverage area is $A_{c,i} = \pi r_s^2$ and the collective coverage is $A_{collective} = \sum_{i=1}^{3} A_{c,i} = 3\pi r_s^2$. However, there can be many possible instances where the sensing areas of the sensors may not be completely within the FoI, and the required computation for these different individual sensor coverage cases are discussed in Appendix D.

6.5.2 Variations of the SNCP Solved

In solving the SNCP using constrained PC approach, the sensors were considered as autonomous agents. These sensors were assigned the individual x coordinates and y coordinates as their strategy sets. Two variations of the SNCP were solved. In Variation 1, the number of sensors and their sensing ranges were chosen such that the area $A_{\square}$ of the converged enclosing rectangle should be significantly lesser than the area of the square FoI being covered. Unlike Variation 1, in Variation 2 the number of sensors and the associated sensing range were chosen such that the area $A_{\square}$ of the converged enclosing rectangle will be significantly larger than the area of the square FoI being covered. Three cases of Variation 2 were solved. These cases differ from one another based on the

sensing range of the sensors as well as the number of sensors being deployed to cover the square FoI.

In both the variations, sensors were randomly located in-and-around the square FoI and all the constraints presented in equations (6.17) to (6.21) were treated using the feasibility-based approach described in Section 6.4. It is important to mention here that on satisfying the inter-agent overlapping constraints presented in equation (6.17) the sampling space of every individual agent was shrunk to its local neighborhood. The approach of shrinking the sampling space is discussed in equation (6.15).

The constrained PC algorithm solving both the variations was coded in MATLAB 7.8.0 (R2009A) and the simulations were run on a Windows platform using an Intel Core 2 Duo, 3GHz processor speed and 3.25GB memory capacity. Furthermore, for both the variations the set of parameters chosen was as follows: (a) individual agent sample size $m_i = 5$, (b) the shrinking interval factor $\lambda_{down} = 0.05$, (c) number of test iterations $n_{test} = 20$. In addition, the maximum allowable distance between two adjacent sensors γ was chosen as $0.2 + 2r_s$ units.

6.5.2.1 Variation 1

In this variation of the SNCP, five sensors ($z = 5$) each with sensing radius 0.5 ($r_s = 0.5$) units were initialized randomly in-and-around a square FoI. The length of the side of the square FoI was five units. In total, 10 runs of the constrained PC algorithm described in Section 6.4 with different initial locations of the sensors were performed. The summary of results including the average CPU time, average number of function

evaluations, the minimum, maximum and average values of the collective coverage achieved and the area of the enclosing rectangle are listed in Table 6.1.

The iterative progress of the solution including randomly generated initial solution (the set of gray circles) and converged optimum solution (the set of black circles) from one of the instances solving Variation 1 is presented in Figure 6.14. The corresponding convergence plot of the area of the enclosing rectangle is presented in Figure 6.15(a) and the convergence plot of the associated collective coverage is presented in Figure 6.15(b). It is clear from these convergence plots that the solution was converged at iteration 2198.

6.5.2.2 Variation 2

Three distinct cases of Variation 2 were solved. Case 1 includes five sensors ($z = 5$) each with sensing radius 1.2 ($r_s = 1.2$), Case 2 includes ten sensors ($z = 10$) each with sensing radius 1 ($r_s = 1$) and Case 3 includes twenty sensors ($z = 20$) each with sensing radius 0.6 ($r_s = 0.6$) units. In each of the three cases, sensors were initialized randomly in-and-around the square FoI. The length of the side of the square FoI was five units. In total, 10 runs of the constrained PC algorithm described in Section 6.4 with different initial locations of the sensors were conducted for each of the three cases.

The iterative progress of the solution including randomly generated initial solution (the set of gray circles) and the converged optimum solution (the set of black circles) from one of the instances each of Case 1, 2 and 3 are illustrated in Figure 6.16, 6.18 and 6.20, respectively. In addition, the corresponding convergence plots of the area of the enclosing rectangle are presented in Figure 6.17(a), 6.19(a) and 6.21(a), and the convergence plots of the collective coverage are presented in Figure 6.17(b), 6.19(b) and 6.21(b), respectively.

It is clear from these convergence plots that the solution for Case 1, Case 2 and Case 3 was converged at iteration 6725, 17484 and 42218, respectively.

The summary of results corresponding to Case 1, 2 and 3 are listed in Table 6.1.

Table 6.1: Summary of Results

Sr. No	Particulars	Variation 1	Variation 2		
1	Cases	--	Case 1	Case 2	Case 3
2	Number of Sensors (z)	5	5	10	20
3	The Sensing Range (r_s)	0.5	1.2	1	0.6
4	Average Collective Coverage	3.927	18.5237	19.4856	16.3631
5	Minimum and Maximum Collective Coverage	3.9270, 3.9270	18.0920, 18.7552	17.5427, 20.8797	15.5347, 17.3377
6	Standard Deviation associated with Collective Coverage	0.0000	0.1687	1.1837	1.2217
7	Average area of the Enclosing Rectangle	5.8311	34.3014	49.0938	39.3480
8	Minimum and Maximum area of the Enclosing Rectangle	5.7046, 5.9750	33.0448, 39.7099	44.7135, 52.6277	34.1334, 43.8683
9	Standard Deviation associated with the area of Enclosing Rectangle	0.1040	1.9899	2.6995	2.8829
10	Average CPU time (Approx.)	20 Mins	1 Hr	2Hrs	3.5 Hrs
11	Average number of Function Evaluations	90417	315063	1172759	3555493

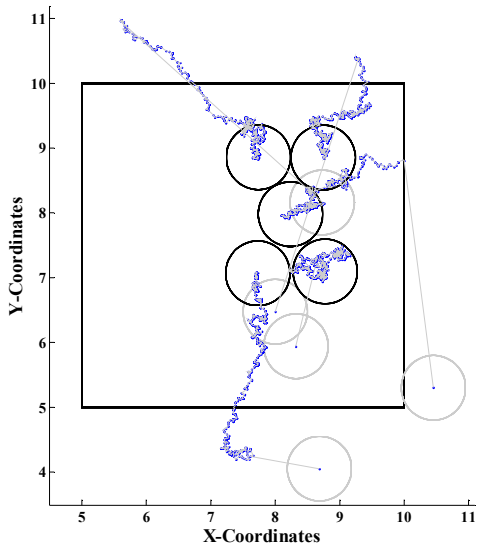
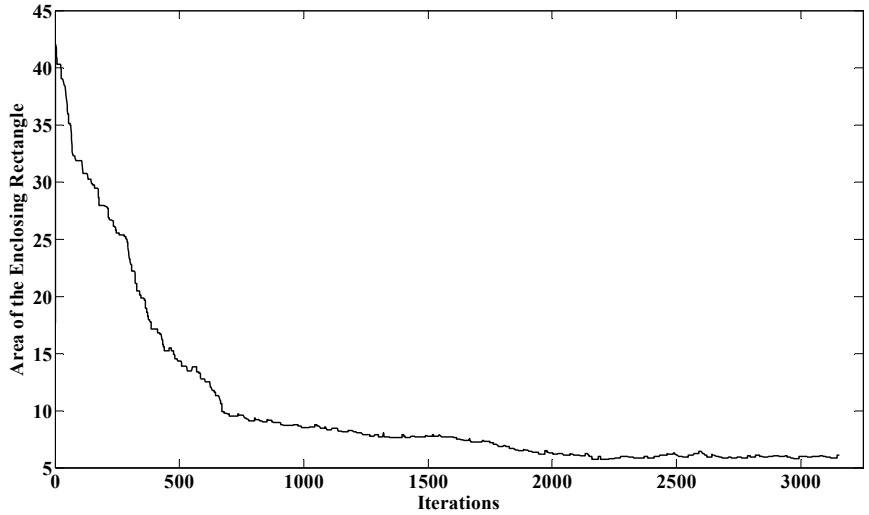
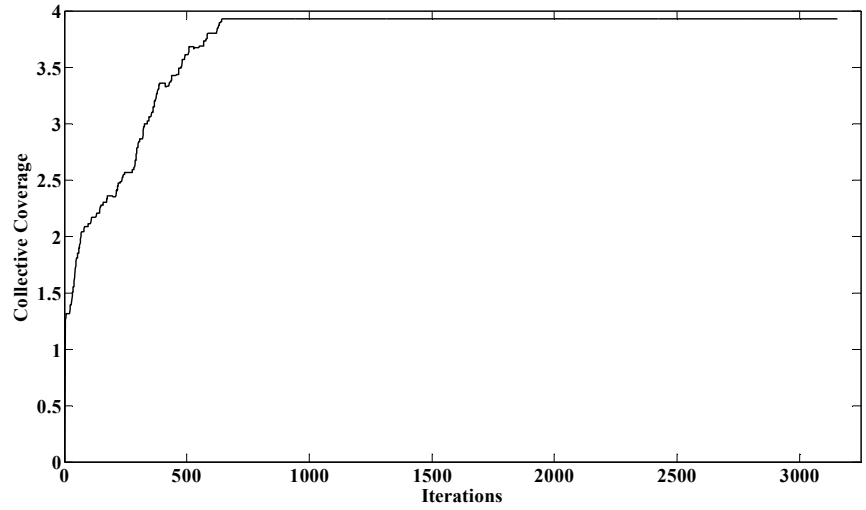


Figure 6.14: Solution History for Variation 1



(a) Convergence of the Area of the Enclosing Rectangle



(b) Convergence of the Collective Coverage

Figure 6.15: Convergence of the Area of the Enclosing Rectangle and Collective Coverage for Variation 1

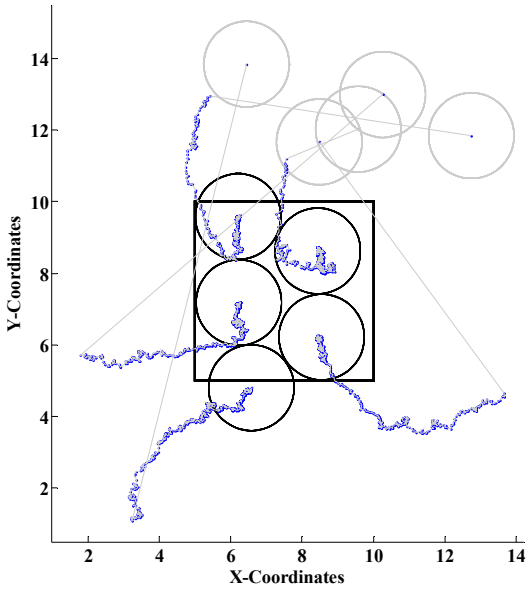
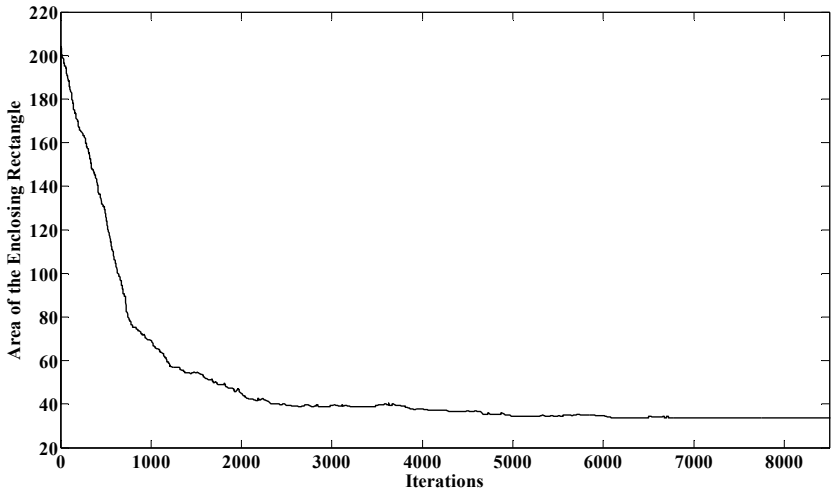
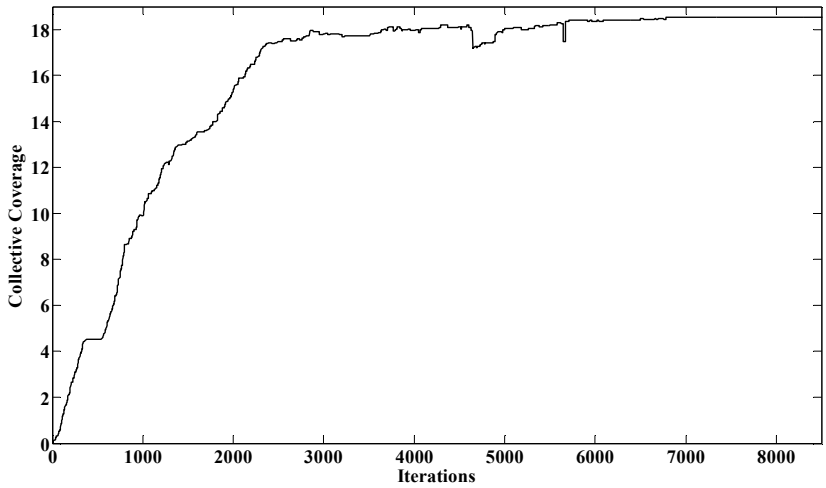


Figure 6.16: Solution History for Case 1 of Variation 2



(a) Convergence of Area of the Enclosing Rectangle



(b) Convergence of the Collective Coverage

Figure 6.17: Convergence of the Area of the Enclosing Rectangle and Collective Coverage for Case 1 of Variation 2

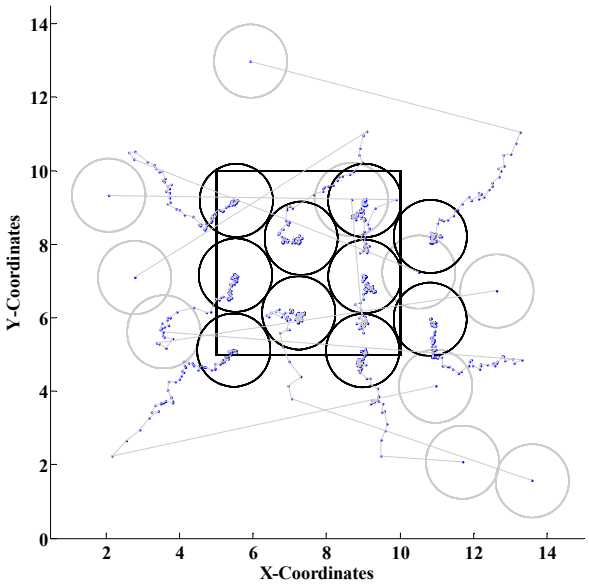
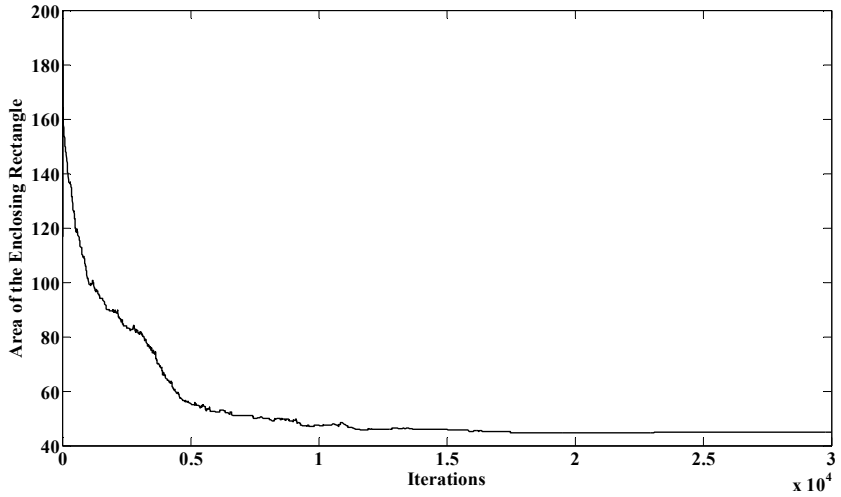
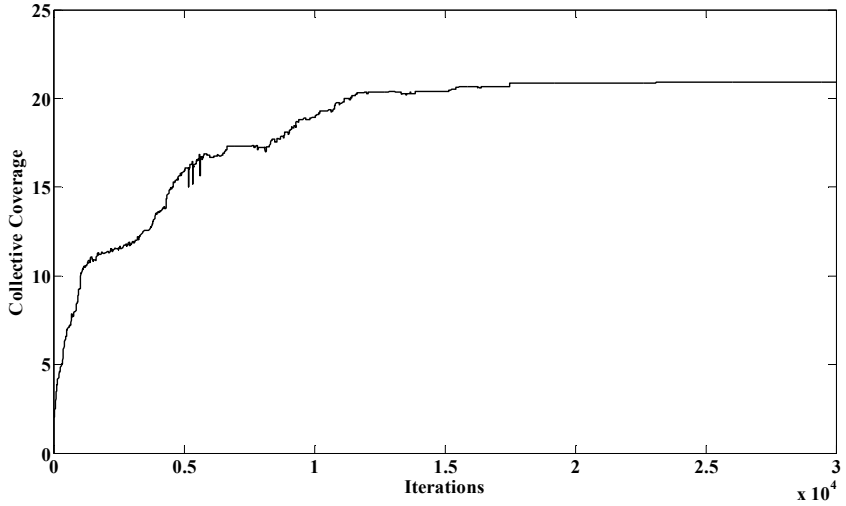


Figure 6.18: Solution History for Case 2 of Variation 2



(a) Convergence of the Area of the Enclosing Rectangle



(b) Convergence of the Collective Coverage

Figure 6.19: Convergence of the Area of the Enclosing Rectangle and Collective Coverage for Case 2 of Variation 2

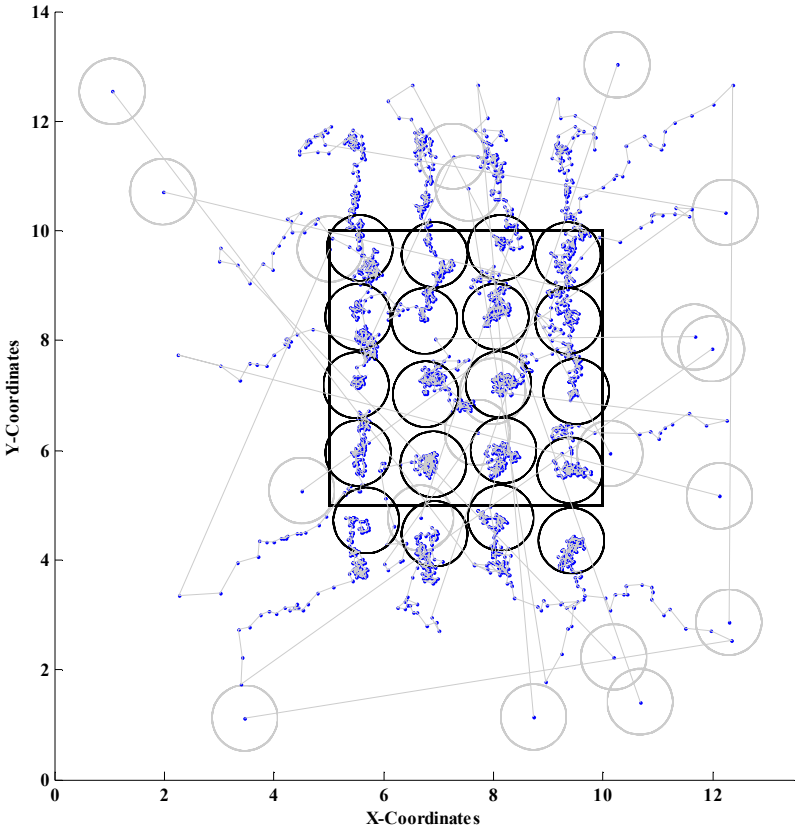
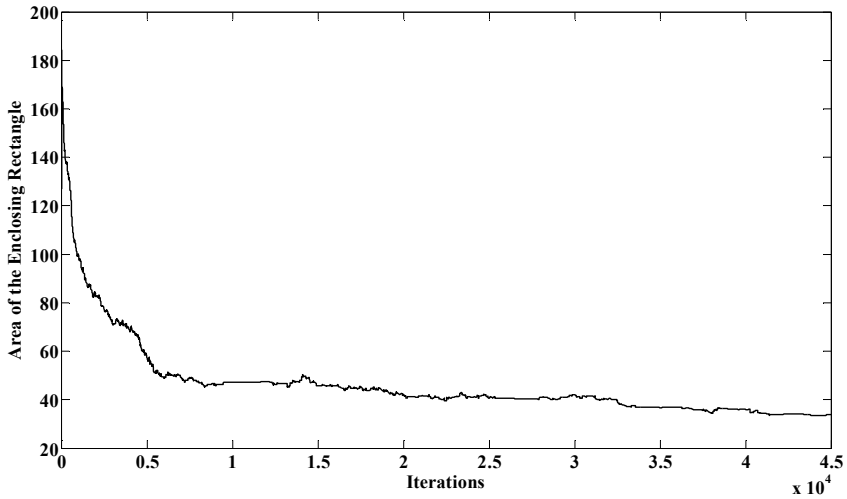
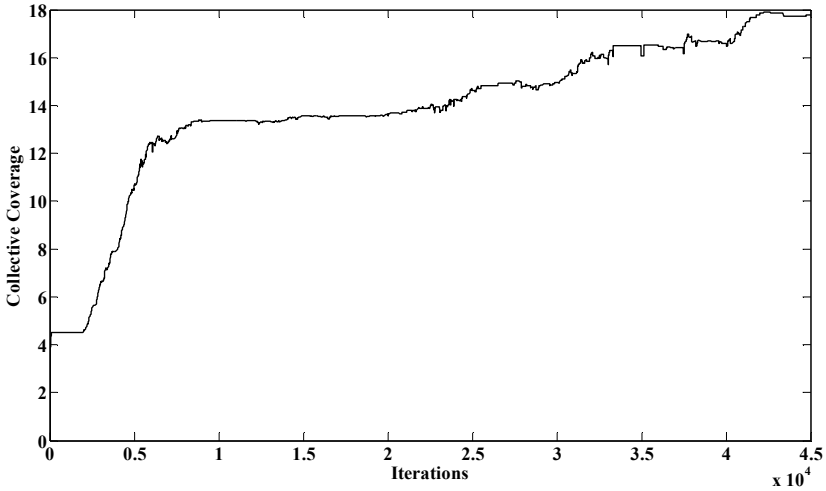


Figure 6.20: Solution History for Case 3 of Variation 2



(a) Convergence of the Area of the Enclosing Rectangle



(b) Convergence of the Collective Coverage

Figure 6.21: Convergence of the Area of the Enclosing Rectangle and Collective Coverage for Case 3 of Variation 2

6.6 Discussion

The above solutions using constrained PC indicated that it could successfully be used to solve constrained optimization problems such as the SNCP. Moreover, connectivity constraint was satisfied in every run of the constrained PC solving the SNCP. It is worth to mention that the DFS algorithm was successfully implemented for checking the connectivity constraint. It is evident from the results presented in Table 6.1, that the

approach produced sufficiently robust results solving all the variations of the SNCP. It implies that the rational behavior of the agents could be successfully formulated and demonstrated. It is important to highlight that the distributed nature of the PC approach will allow the total number of function evaluations to be equally divided among the agents of the system. This can be made practically evident by implementing the PC approach on a real distributed platform assigning separate workstations carrying out the computations independently. These advantages along with the directly incorporated uncertainty using the real valued probabilities treated as variables suggest that PC can potentially be applied to real world complex problems.

In addition, the feasibility-based rule in [159, 160, 179-183] suffered from maintaining the diversity and further required additional techniques such as niching [159], SA [160], modified mutation approach [181, 182], and several associated trials in [179-182], etc. It may require further computations and memory usage. Moreover, in order to jump out of local minima, feasibility-based Rule I was required to be accompanied with a perturbation approach which required several associated parameters to be tuned. In case of the Feasibility-based Rule II proposed in Section 6.4, the perturbation approach was completely replaced by the addition of a generic rule of accepting the worse feasible solution when the solution remains feasible and unchanged for a successive predefined number of iterations. From the convergence plot and the associated results it is evident that this additional generic rule helped the solution jump out of local minima.

In agreement with the no-free-lunch theorems [213], some limitations were also identified. The rate of convergence and the quality of solution was dependent on the parameters such as the number of strategies m_i in every agent's strategy set $\mathbf{X}_i$, the

interval factor λ_{down} and the number of test iterations n_{test} . It necessitated some preliminary trials for fine-tuning these parameters. Additionally, a high computational cost achieving the optimum solution was noticed. However, computational time can be reduced by implementing the PC approach on an actual distributed platform.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

The PC methodology was improved and demonstrated successfully within a COIN framework. The original PC approach was improved with a reduction in the computational complexity. A scheme for updating the solution space was developed which contributed to faster convergence and improved efficiency of the overall algorithm. In addition, PC was assumed to be converged when there was no further improvement in the final goal and/or exceeds a predefined number of iterations. The modified PC algorithm was validated successfully optimizing the Rosenbrock Function.

Moreover, various inherent and desirable characteristics of the PC methodology of being a distributed, decentralized and cooperative approach were investigated and exploited. This includes the successful demonstration of the important characteristic of avoidance of tragedy of commons as well as the capability of PC to deal with the failure of a subsystem or an agent. Furthermore, the concept of Nash equilibrium was successfully formalized and demonstrated.

In the attempts to exploit the key characteristics and benefits of the PC methodology of being distributed, decentralized and cooperative approach as well as make it a more generic and powerful optimization approach, a number of constraint handling techniques in the form of penalty function and feasibility-based rules were developed for incorporating into the overall algorithm. These techniques were tested and validated by successfully solving a number of challenging problems from variegated areas.

In the first attempt to develop a constraint handling technique, several problem specific heuristic techniques were proposed and incorporated into the PC algorithm solving the challenging NP-hard Multiple Depot Multiple Traveling Salesmen Problem (MDMTSP) and Single Depot Multiple Traveling Salesmen Problem (SDMTSP). The implementation of these techniques helped the algorithm to converge faster as well as to jump out of local minima. It is worth to mention that for the first time the MDMTSP and the SDMTSP were solved using a distributed, decentralized and cooperative approach such as PC. The true optimum solution was produced in every run of the PC solving two specially developed test cases of the MDMTSP. The MDMTSP and SDMTSP were solved in a reasonably short time.

In the second attempt, a generalized constrained PC approach was proposed using the simple penalty function method. The dependence on the penalty parameter was reduced through the approach of iteratively biasing the fitter solution, i.e. iteratively tightening the tolerance to constraint violation. The constrained PC approach was tested successfully on a variety of optimization problems with equality and inequality constraints. It could produce competitive, if not better, results and appeared to be a promising distributed approach capable of handling the constraints.

In the third attempt, two variations of the Feasibility-based Rule were proposed. The Feasibility-based Rule I was successfully formalized and demonstrated for solving two cases of the Circle Packing Problem (CPP). In both the cases, the approach could find the true optimal solution in every run of the constrained PC in reasonable computational efforts. Similar to the penalty function approach, the constraint violation tolerance was iteratively tightened in order to obtain the fitter solution. In addition to the two test cases,

the potential of PC was successfully tested for the case of agent failure which yielded promising results. It is worth to mention here that in order to make the solution jump out of possible local minima and further reach the true optimum, the rule was assisted with the perturbation approach as well as the problem specific voting heuristic. Most importantly, the PC solution to the CPP successfully demonstrated the important and desirable characteristic in the PC algorithm of avoiding the tragedy of commons.

The Feasibility-based Rule II was successfully applied for solving two variations of the SNCP. Furthermore, the constrained PC approach produced sufficiently robust results solving all the variations of the SNCP. It is important to mention that the Feasibility-based rule II did not require the perturbation approach to be implemented for the solution to jump out of local minima. In addition, in the solution procedure of the SNCP, the depth-first-search algorithm was successfully implemented. Furthermore, in solving the CPP and the SNCP, the BFGS method was successfully used as an alternative to the existing Nearest Newton Descent Scheme for minimizing the Homotopy function.

Besides the above discussed improvements in the PC algorithm, a few disadvantages were also identified. A high computational cost was required in achieving the optimum solution. However, real (wall clock) time can be reduced by implementing the PC approach on an actual distributed (parallel) platform. Furthermore, the rate of convergence and the quality of solution was dependent on a few parameters. This necessitated a few preliminary trials for the fine-tuning of these parameters.

According to the no-free-lunch theorem [175, 213], making a few assumptions about the set of problems being solved may make the search technique and associated constraint handling method more effective and efficient in solving those problems; however, the

possibility of performing worse or being trapped into local minima for other problems may be greater. It is important to mention here that in agreement with the no-free-lunch theorem, along with the preliminary trials for fine tuning of a few parameters, in order to make the PC algorithm jump out of possible local minima, various problem specific heuristic techniques were required to be developed and further incorporated into the PC algorithm.

7.2 Recommendations for Future Work

For future work, efforts will be taken to make the approach more generalized and increase the efficiency of the PC algorithm. This may include improving the diversification of sampling, the rate of convergence, computational cost, quality of results, etc. by developing a self adaptive scheme for the parameters such as the number of strategies m_i and interval factor λ and temperature step size α_T. In addition, a local search technique can be incorporated for assisting in a neighborhood search to reach the global optimum.

The author is also interested to extend and modify the PC methodology further to make it capable of solving a variety of discrete problems which may open up a wider range of applications including complex mechanical design and assembly problems [229-232] as well as disassembly problems [233]. In addition, the author also sees the potential of constrained PC approach for the design of composite structures with various manufacturing constraints [234].

In order to exploit its property of being distributed, the constrained PC approach can be used for solving more realistic and large size problems such as machine shop

scheduling, urban traffic control, etc. The author also sees some potential in the field of healthcare systems management [235]. In addition, more realistic path planning problems of the Multiple Unmanned Vehicles (MUVs) can then be solved with the MTSP and VRP approaches.

As a continuation and further development/improvement in the constraint handling technique, the author is currently working on the Multi-Objective Probability Collectives (MOPC) approach. The constraints in this approach are being considered as objectives to be optimized along with the system objective. The approach of considering constraints as objectives is inspired from similar work done in [172-174, 236] using GA. A short discussion on the MOPC approach as well as the potential of PC in urban traffic control is discussed in the following sections.

7.2.1 Multi-Objective Probability Collectives (MOPC)

As mentioned previously, the Evolutionary Algorithms (EAs) in general are unconstrained optimization techniques and their performance may be significantly affected when constraints are involved. Recently, some researchers [172-174, 237-245] have proposed the Multi-Objective optimization concept to handle the single objective constrained problems in which the constraints are treated as objectives to be optimized. This allows for the unconstrained optimization of every objective and further may provide the decision maker a set of candidate solutions. According to [235], this may allow the decision maker to deal with the soft constraints, i.e. a choice can be made from a set of candidate solutions with a slight constraint violation and substantial improvement in the objective function value. In addition, according to [238], the EAs when resort to this

Multi-Objective optimization approach, it is possible to quickly locate the feasible region but the extension to solve the single objective constrained problems is not straightforward.

In order to exploit above mentioned benefits the author of this thesis is currently working on applying the PC approach to address the Multi-Objective optimization approach solving the single objective constrained problems. Moreover, as per the knowledge of the author, this approach is never addressed in a distributed way. This may also help the solution to exploit the benefits associated with the distributed approach such as PC. In addition, the author intends to test the approach solving a variety of multi-objective test problems [246] as well as challenging machine shop scheduling problem [247].

7.2.2 PC for the Urban Traffic Control

In recent years, the ever growing traffic volume in urban areas motivated several researchers to address the traffic control problem as an optimization problem. Some of the researchers proposed traffic responsive controllers by focusing on a single intersection traffic light optimization, by isolating it from other neighboring intersections [248-255]. As the load on the traffic network increases, the performance of the traffic responsive controller decreases and tends to resemble that of a fixed-time controller [256, 257-259]. Moreover, such systems are difficult to set up, maintain and demand significant computing power [260]. The ever growing traffic in urban areas and above mentioned limitations triggered the need for a distributed control of the traffic at the corresponding intersections [261, 262].

In the effort of solving the traffic control problem using PC, a network of intersections can be considered. The PC approach can be applied to a predefined network of intersections. Each intersection can be represented as an agent deciding its own variable values such as green signal duration as well as the phase sequence by considering the traffic condition at the neighboring intersections in the network. The application of PC solving the traffic control problem may help to exploit the inherent characteristics of PC discussed in Section 3.2 and may further open the doors for applying the PC to real world complex problems.

APPENDIX A

Analogy of Homotopy Function to Helmholtz Free Energy

A.1 The Homotopy function $J_i\left(q\left(\mathbf{X}_i\right),T\right)$ is analogous to the Helmholtz free energy in statistical physics [100-106]. The Helmholtz free energy equation can be represented as follows [106]:

$$L = D - TS \tag{A.1}$$

where L is Helmholtz free energy available to do the work or to create a system when the environment is at temperature T. The term D is referred to as internal energy and S is the entropy of the physical system. The term TS is the spontaneous energy that can be transferred from the environment to the system.

As Helmholtz free energy L is actually the measure of the amount of internal energy D needed to create the system, achieving thermal equilibrium is nothing but minimizing the free energy L at temperature T, and hence directly minimizing the internal energy D. The procedure involves a series of iterations gradually reducing the entropy level of the system. This suggests the framework of an annealing schedule, i.e. start at a high temperature $T >> 0$ or $T = T_{initial}$ and successively reducing the temperature and the entropy level, achieving the thermal equilibrium at every successive temperature drop until $T \to 0$ or $T = T_{final}$. This process avoids the local minima of internal energy D and reaches the global minima at $T \to 0$ or $T = T_{final}$.

A.2 The analogy of the Homotopy function $J_i\left(q\left(\mathbf{X}_i\right),T\right)$ to Helmholtz free energy L in statistical physics is the motivation behind minimizing the collection of system

objectives $\sum_{r=1}^{m_i} G(\mathbf{Y}_i^{[r]})$ [100-104]. As thermal equilibrium needs to be achieved to minimize the free energy L of the physical system given in equation (A.1), similarly, the equilibrium needs to be achieved to minimize the Homotopy function $J_i\left(q\left(\mathbf{X}_i\right),T\right)$ and that is referred to as Nash equilibrium.

According to [102, 103], careful annealing of physical systems ensures the convergence to the global minimum of the energy D. One of the most suitable approaches to deal with the Homotopy function such as $J_i\left(q\left(\mathbf{X}_i\right),T\right)$ and eventually the collection of system objectives $\sum_{r=1}^{m_i} G(\mathbf{Y}_i^{[r]})$ having many possible local minima is referred to as Deterministic Annealing (DA).

APPENDIX B

Nearest Newton Descent Scheme

In this scheme, the quadratic approximation of the Hessian of Homotopy function $J_i\left(q\left(\mathbf{X}_i\right),T\right)$ in equation (3.8) is carried out by every agent *i* using the probability space formed from the coupling of the individual agent's probability spaces rather than using the individual agent's probability distribution $q(\mathbf{X}_i)$. It is worth to mention that the Hessian in this scheme is positive definite. The important benefit is that it converts the Homotopy function $J_i\left(q\left(\mathbf{X}_i\right),T\right)$ into a convex function. The gradient and Hessian of the Homotopy function in equation (3.8) is used to find the individual agent probability distribution [2].

The simplified resulting probability variable update rule (descent rule) for each strategy r of agent i, referred to as the 'k-update' rule [2] which minimizes the Homotopy function in equation (3.8), is represented below:

$$q(X_i^{[r]})^{k+1} \leftarrow q(X_i^{[r]})^k - \alpha_{step}\, q(X_i^{[r]})^k k_{r.update} \tag{B.1}$$

where $$k_{r.update} = \frac{(Contribution\ of\ X_i^{[r]})^k}{T} + S_i^k + \log_2(q(X_i^{[r]})^k)$$

and $$\left(Contribution\ of\ X_i^{[r]}\right)^k = \left(E\left(G(\mathbf{Y}_i^{[r]})\right)\right)^k - \left(\sum_{r=1}^{m_i} E\left(G(\mathbf{Y}_i^{[r]})\right)\right)^k$$

where k is the corresponding update number and S_i^k is the corresponding entropy. The updating procedure is applied as follows:

(a) Set iteration number $k=1$, maximum number of updates k_{final}, and step size α_{step} ($0 < \alpha_{step} \le 1$). The values of k_{final} and α_{step} are held constant throughout the optimization and chosen based on preliminary trials of the algorithm.

(b) Update every agent's probability distribution $q(\mathbf{X}_i)^{(k)}$ using the update rule in equation (B.1).

(c) If $k \geq k_{final}$, stop, else update $k = k + 1$ and return to (b).

For each agent i, the above optimization process converges to a probability variable vector $q(\mathbf{X}_i)$ which can be seen as the individual agent's probability distribution clearly distinguishing every strategy's contribution towards the minimization of the expected collection of system objectives $\sum_{r=1}^{m_i} E\left(G(\mathbf{Y}_i^{[r]})\right)$. In other words, for every agent i, if strategy r contributes the most towards the minimization of the objective than other strategies, its corresponding probability certainly increases by some amount more than those for the other strategies' probability values, and so strategy r is distinguished from the other strategies. Such a strategy is referred to as a favorable strategy $X_i^{[fav]}$.

The initial uniform probability distribution, the intermediate iteration distributions, and the converged probability distribution clearly distinguishing the contribution of every strategy from one of the runs solving the tension/compression spring design problem discussed in Chapter 5 are illustrated in Figure B.1.

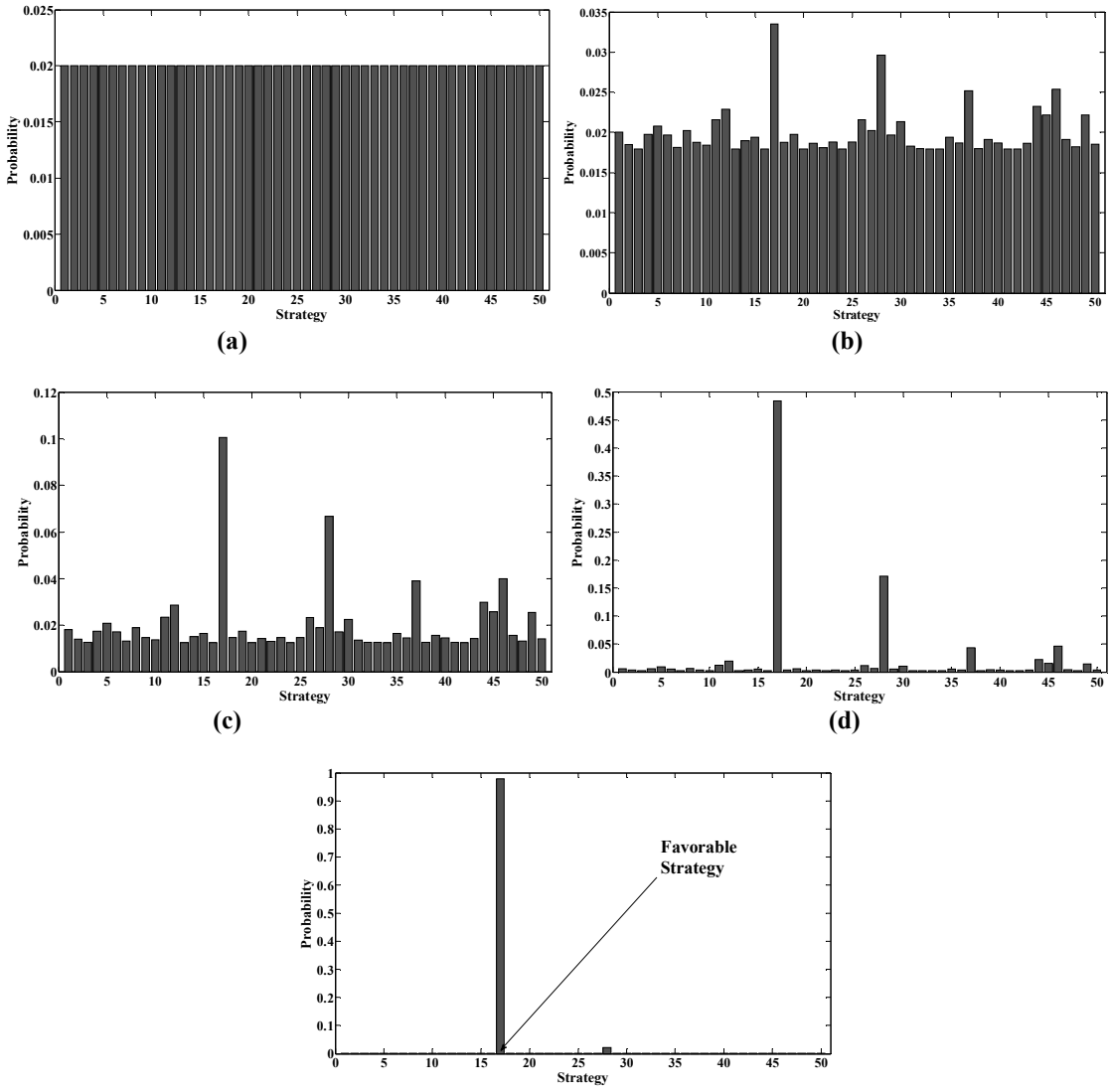


Figure B.1: Solution History of Probability Distribution using Nearest Newton Descent Scheme

APPENDIX C

Broyden-Fletcher-Goldfarb-Shanno (BFGS) Method for Minimizing the Homotopy Function

The minimization of the Homotopy function given in equation (3.8) was carried out using a suitable second order optimization technique such as Broyden-Fletcher-Goldfarb-Shanno (BFGS) method [6, 96, 97]. The approximated Hessian in this method is positive definite. Moreover, the updating of the Hessian also preserves the positive definiteness. The BFGS method minimizing the Homotopy function in equation (3.8) is discussed below

C.1 Set BFGS iteration counter $k = 1$, BFGS maximum number of iterations ν, and step size α_{step} ($0 < \alpha_{step} \leq 1$). The value of α_{step} is held constant throughout the optimization and chosen based on the preliminary trials of the algorithm.

C.1.1 Initialize the convergence criterion $q(\mathbf{X}_i)^k - q(\mathbf{X}_i)^{k-1} \leq \varepsilon_2$. The convergence parameter $\varepsilon_2 = 0.0001$ is equal for all the N agents.

C.1.2 Initialize the Hessian $\mathbf{H}_i^k$ to a positive definite matrix, preferably identity matrix $\mathbf{I}$ of size $m_i \times m_i$.

C.1.3 Initialize the probability variables as follows:

$$q(\mathbf{X}_i)^k = \left\{ \left(q(X_i^{[1]})^k = 1/_{m_i} \right), \left(q(X_i^{[2]})^k = 1/_{m_i} \right), ..., \left(q(X_i^{[m_i]})^k = 1/_{m_i} \right) \right\} \qquad \text{(C.1)}$$

i.e. assign uniform probabilities to the strategies of agent i . This is because, at the beginning, least information is available (largest uncertainty and highest entropy) about

which strategy is favorable for the minimization of the collection of system objectives $\sum_{r=1}^{m_i} G(\mathbf{Y}_i^{[r]})$.

C.1.4 Compute the gradient of the Homotopy function in equation (3.8) as follows:

$$\mathbf{C}^k = \left[\frac{\partial J_i\left(q(\mathbf{X}_i),T\right)^k}{\partial q\left(X_i^{[1]}\right)^k} \quad \frac{\partial J_i\left(q(\mathbf{X}_i),T\right)^k}{\partial q\left(X_i^{[2]}\right)^k} \quad \cdots \quad \frac{\partial J_i\left(q(\mathbf{X}_i),T\right)^k}{\partial q\left(X_i^{[m_i]}\right)^k} \right]$$
$$= \left[G\left(\mathbf{Y}_i^{[1]}\right).\prod q\left(X_{(i)}^{[?]}\right) + \frac{T}{\ln(2)}\left[1+\ln\left(q\left(X_i^{[1]}\right)^k\right)\right] \quad G\left(\mathbf{Y}_i^{[1]}\right).\prod q\left(X_{(i)}^{[?]}\right) + \frac{T}{\ln(2)}\left[1+\ln\left(q\left(X_i^{[2]}\right)^k\right)\right] \cdots \right.$$
$$\left. \cdots \; G\left(\mathbf{Y}_i^{[m_i]}\right).\prod q\left(X_{(i)}^{[?]}\right) + \frac{T}{\ln(2)}\left[1+\ln\left(q\left(X_i^{[m_i]}\right)^k\right)\right] \right] \tag{C.2}$$

C.2 Compute the search direction as $\mathbf{d}_i^k = -\mathbf{C}_i^k.\left(\mathbf{H}_i^k\right)^{-1}$.

C.3 Compute $\mathrm{J}_i\left(\left(q\left(\mathbf{X}_i\right)+\alpha_{step}.\mathbf{d}_i^k\right),T\right)$

C.4 Update the probability vector $q(\mathbf{X}_i)^{k+1} = q(\mathbf{X}_i)^k + \alpha_{step}.\mathbf{d}_i^k$

C.5 Update the Hessian $\mathbf{H}_i^{k+1} = \mathbf{H}_i^k + \mathbf{D}_i^k + \mathbf{E}_i^k$

where $\mathbf{D}_i^k = \dfrac{\mathbf{y}_i^k.\left(\mathbf{y}_i^k\right)^T}{\mathbf{y}_i^k.\mathbf{s}_i^k}$, $\mathbf{E}_i^k = \dfrac{\mathbf{C}_i^k.\left(\mathbf{C}_i^k\right)^T}{\mathbf{C}_i^k.\mathbf{d}_i^k}$, $\mathbf{s}_i^k = \alpha_{step}.\mathbf{d}_i^k$, $\mathbf{y}_i^k = \mathbf{C}_i^{k+1} - \mathbf{C}_i^k$ and

$$\mathbf{C}^{k+1} = \left[\frac{\partial J_i\left(q(\mathbf{X}_i),T\right)^{k+1}}{\partial q\left(X_i^{[1]}\right)^{k+1}} \quad \frac{\partial J_i\left(q(\mathbf{X}_i),T\right)^{k+1}}{\partial q\left(X_i^{[2]}\right)^{k+1}} \quad \cdots \quad \frac{\partial J_i\left(q(\mathbf{X}_i),T\right)^{k+1}}{\partial q\left(X_i^{[m_i]}\right)^{k+1}} \right]$$
$$= \left[G\left(\mathbf{Y}_i^{[1]}\right).\prod q\left(X_{(i)}^{[?]}\right) + \frac{T}{\ln(2)}\left[1+\ln\left(q\left(X_i^{[1]}\right)^{k+1}\right)\right] \cdots \right.$$
$$G\left(\mathbf{Y}_i^{[1]}\right).\prod q\left(X_{(i)}^{[?]}\right) + \frac{T}{\ln(2)}\left[1+\ln\left(q\left(X_i^{[2]}\right)^{k+1}\right)\right] \cdots$$
$$\left. \cdots \; G\left(\mathbf{Y}_i^{[m_i]}\right).\prod q\left(X_{(i)}^{[?]}\right) + \frac{T}{\ln(2)}\left[1+\ln\left(q\left(X_i^{[m_i]}\right)^{k+1}\right)\right] \right]$$

C.6. Accept the current probability distribution $q(\mathbf{X}_i)^k$, if $k \geq \nu$ or the condition $q(\mathbf{X}_i)^k - q(\mathbf{X}_i)^{k-1} \leq \varepsilon_2$ is true for successive considerable number of iterations, then stop, else update $k = k+1$ and go to C.2.

As an example, the Homotopy function minimization when solving the Sensor Network Coverage Problem (SNCP) is presented in Figure C.1. The initial uniform probability distribution, the intermediate iteration distributions, and the converged probability distribution clearly distinguishing the contribution of every strategy from one of the runs solving the SNCP discussed in Section 6.5.2 are illustrated in Figure C.2.

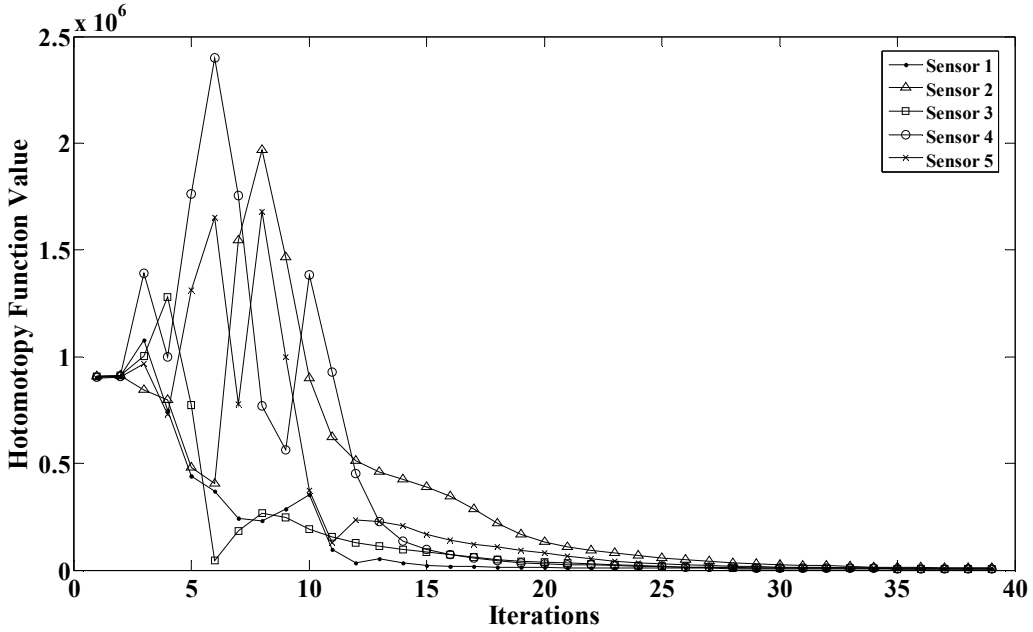


Figure C.1: Convergence of the Homotopy Functions for the SNCP using BFGS Method

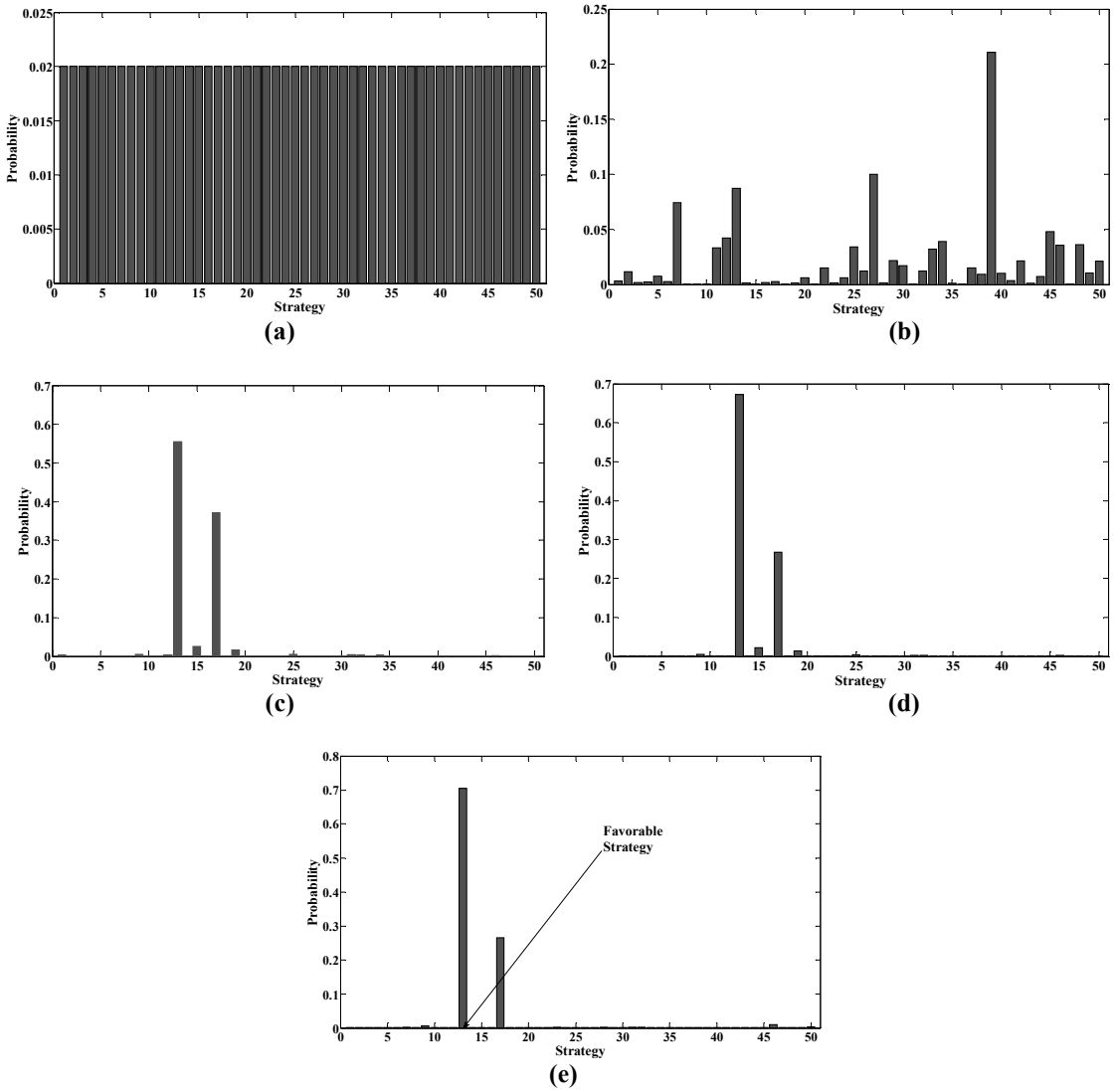


Figure C.2: Solution History of Probability Distribution using BFGS Method

APPENDIX D

Individual Sensor Coverage Calculation

There are four possible cases of partial coverage of square FoI in which the sensing area of the sensor *i* crosses one or more boundaries of the FoI. The four cases associated with it are discussed below with corresponding illustrations presented from Figure D.1 to D.4.

(a) Case 1: the sensor near a corner of the square FoI with the corner outside the sensing area

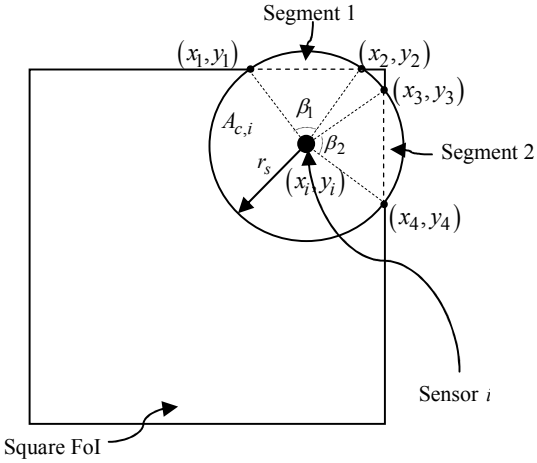


Figure D.1: Case 1

The area of Segment 1 $A_{Segment1}$ bounded by arc $\overset{\frown}{1\,2}$ associated with $\angle\beta_1$ and chord $\overline{1\,2}$, and area of Segment 2 $A_{Segment2}$ bounded by arc $\overset{\frown}{3\,4}$ associated with $\angle\beta_2$ and chord $\overline{3\,4}$ are calculated as follows:

$$A_{Segment1} = \left(r_s^2\left(\beta_1 - \sin\beta_1\right)\right)/2 \tag{D.1}$$

where

$$\beta_1 = \cos^{-1}\left[\left(d(i,1)^2 + d(i,2)^2 - d(1,2)^2\right)/\left(2\times d(i,1)^2\right)\right] \text{ and}$$

$$d(i,1) = r_s$$
$$d(i,2) = r_s$$
$$d(1,2) = x_2 - x_1 \quad ,$$

$$A_{Segment\,2} = \left(r_s^2 \left(\beta_2 - \sin\beta_2\right)\right)/2 \tag{D.2}$$

where

$$\beta_2 = \cos^{-1}\left[\left(d(i,3)^2 + d(i,4)^2 - d(3,4)^2\right)/\left(2 \times d(i,3)^2\right)\right]$$

$$d(i,3) = r_s$$
$$d(i,4) = r_s$$
$$d(3,4) = y_3 - y_4$$

The FoI coverage $A_{c,i}$ is calculated as follows:

$$A_{c,i} = \pi r_s^2 - \left(A_{Segment\,1} + A_{Segment\,2}\right) \tag{D.3}$$

(b) Case 2: the sensor near a corner of the square FoI with the corner inside the sensing area

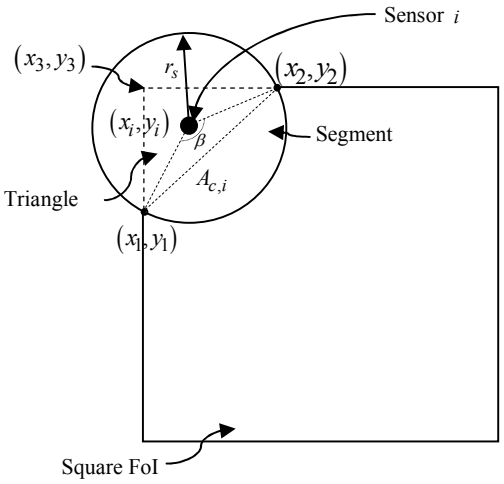


Figure D.2: Case 2

The area of Segment $A_{Segment}$ bounded by arc $\overset{\frown}{1\,2}$ associated with $\angle\beta$ and chord $\overline{1\,2}$ and the area of triangle $\Delta 123$, $A_{Triangle}$ is calculated as follows:

$$A_{Segment} = \left(r_s^2\left(\beta - \sin\beta\right)\right)/2 \tag{D.4}$$

where

$$\beta = \cos^{-1}\left[\left(d\left(i,1\right)^2 + d\left(i,2\right)^2 - d\left(1,2\right)^2\right)/\left(2\times d\left(i,1\right)^2\right)\right] \tag{D.5}$$

$$d\left(i,1\right) = r_s$$
$$d\left(i,2\right) = r_s$$
$$d\left(1,2\right) = \sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2}$$

$$A_{Triangle} = \left(d\left(1,3\right)\times d\left(2,3\right)\right)/2 \tag{D.6}$$

where

$$d\left(1,3\right) = y_3 - y_1$$
$$d\left(2,3\right) = x_2 - x_3$$

The FoI coverage $A_{c,i}$ is calculated as follows:

$$A_{c,i} = A_{Triangle} + A_{Segment} \tag{D.7}$$

(c) Case 3: sensing area crossing a boundary of the square FoI at a side with sensor outside the square FoI

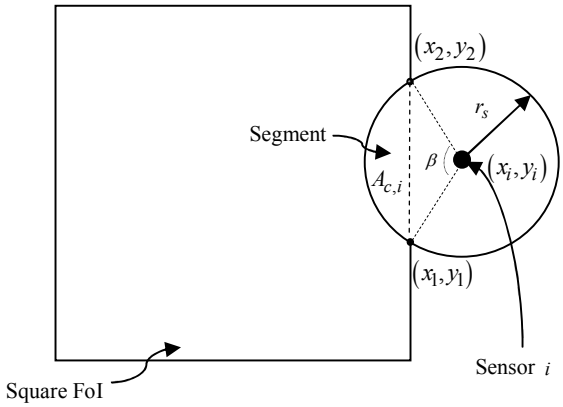


Figure D.3: Case 3

The area of Segment $A_{Segment}$ bounded by arc $\overset{\frown}{1\,2}$ associated with $\angle\beta$ and chord $\overline{1\,2}$ is calculated as follows:

$$A_{Segment} = \left(r_s^2\left(\beta - \sin\beta\right)\right)/2 \tag{D.8}$$

where

$$\beta = \cos^{-1}\left[\left(d\left(i,1\right)^2 + d\left(i,2\right)^2 - d\left(1,2\right)^2\right)/\left(2\times d\left(i,1\right)^2\right)\right] \tag{D.9}$$

$$\begin{aligned} d\left(i,1\right) &= r_s \\ d\left(i,2\right) &= r_s \\ d\left(1,2\right) &= y_2 - y_1 \end{aligned} \tag{D.10}$$

The FoI coverage $A_{c,i}$ is calculated as follows:

$$A_{c,i} = A_{Segment} \tag{D.11}$$

(d) Case 4: sensing area crossing the boundary of the square FoI at a side with sensor inside the square FoI

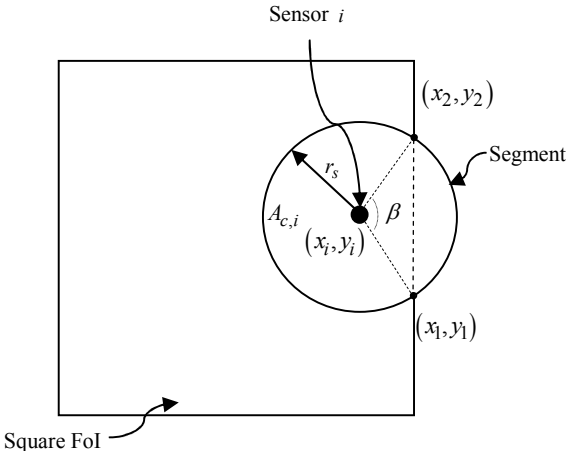


Figure D.4: Case 4

The area of Segment $A_{Segment}$ bounded by arc $\overset{\frown}{1\,2}$ associated with $\angle\beta$ and chord $\overline{1\,2}$ is calculated as follows:

$$A_{Segment} = \left(r_s^2\left(\beta - \sin\beta\right)\right)/2 \tag{D.12}$$

where

$$\beta = \cos^{-1}\left[\left(d(i,1)^2 + d(i,2)^2 - d(1,2)^2\right) \Big/ \left(2 \times d(i,1)^2\right)\right] \tag{D.13}$$

$$\begin{aligned} d(i,1) &= r_s \\ d(i,2) &= r_s \\ d(1,2) &= y_2 - y_1 \end{aligned} \tag{D.14}$$

The FoI coverage $A_{c,i}$ is calculated as follows:

$$A_{c,i} = \pi r_s^2 - A_{Segment} \tag{D.15}$$

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