SIAM J. OPTIM.
Vol. 14, No. 2, pp. 500–506


# RELATING HOMOGENEOUS CONES AND POSITIVE DEFINITE CONES VIA $T$-ALGEBRAS*

CHEK BENG CHUA†

**Abstract.** $T$-algebras are nonassociative algebras defined by Vinberg in the early 1960s for the purpose of studying homogeneous cones. Vinberg defined a cone $K(\mathcal{A})$ for each $T$-algebra $\mathcal{A}$ and proved that every homogeneous cone is isomorphic to one such $K(\mathcal{A})$. We relate each $T$-algebra $\mathcal{A}$ with a space of linear operators in such a way that $K(\mathcal{A})$ is isomorphic to the cone of positive definite self-adjoint operators. Together with Vinberg's result, we conclude that every homogeneous cone is isomorphic to a "slice" of a cone of positive definite matrices.

**Key words.** homogeneous cones, $T$-algebras, positive definite cones

**AMS subject classification.** 90C25

**DOI.** S1052623402406765

**1. Introduction.** Due to the generality of interior-point methods, they have been successfully applied to a wide class of conic programming problems; one of the more prominent of these classes is semidefinite programming (SDP), whose underlying cone is the set of positive semidefinite symmetric matrices.

Positive semidefinite cones are examples of homogeneous cones. A full-dimensional cone $K$ in $\mathbb{R}^n$ is homogeneous if the group of automorphisms of the cone acts transitively on it (i.e., for every $x, y \in K$, there exists a linear automorphism $A$ of $K$ such that $Ax = y$). Homogeneous cones were studied by Vinberg [4], who associated homogeneous cones with certain nonassociative algebras called $T$-algebras. Through $T$-algebras, Vinberg classified all homogeneous self-dual cones.

From the association of homogeneous cones with $T$-algebras, we show that homogeneous cones are "slices" of positive definite cones. More precisely, we show that for some $m \le n$, there exists an injective linear map $M : \mathbb{R}^n \to \mathbb{S}^{m\times m}$ such that $M(K) = \mathbb{S}^{m\times m}_{++} \cap M(\mathbb{R}^n)$, where $\mathbb{S}^{m\times m}$ is the space of $m$-by-$m$ symmetric matrices and $\mathbb{S}^{m\times m}_{++}$ is the cone of positive definite symmetric $m$-by-$m$ matrices.[1]

(After the first version of this paper appeared, Faybusovich pointed out that the same conclusion follows from his work [2]. Indeed, by recognizing the cone $K(\mathcal{A})$ as a cone of "squares" in the context of [2], it follows from the construction in [2] that $K(\mathcal{A})$ is a "slice" of a positive definite cone. However, this construction requires

*Received by the editors April 30, 2002; accepted for publication (in revised form) May 12, 2003; published electronically November 6, 2003. This research was performed as part of the author's Ph.D. study at Cornell University.

http://www.siam.org/journals/siopt/14-2/40676.html

†Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, Canada N2L 3G1 (cbchua@math.uwaterloo.ca).

[1]The converse is not true. For example, consider the cone $K = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_1 > 0, x_3 > \sqrt{x_1^2 + x_2^2}\}$. This is a "slice" of the positive definite cone $\mathbb{S}^{3\times 3}_{++}$, as can be seen by taking $M : \mathbb{R}^3 \to \mathbb{S}^{3\times 3}$ to be the injective linear map

$$M : (x_1, x_2, x_3)^T \mapsto \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_3 - x_1 & x_2 \\ 0 & x_2 & x_3 + x_1 \end{bmatrix}.$$

From Vinberg's classification of homogeneous cones (see [4]), a three-dimensional homogeneous cone is linearly isomorphic to either the positive orthant or the second-order cone. Therefore, $K$ is not homogeneous.

an order $n$ positive definite cone, i.e., a cone of symmetric positive definite $n$-by-$n$ matrices. In this paper, our construction may produce a cone of a lower order with the proper choice of some index set $I$.)

One consequence of this result is that we can model conic programming problems over homogeneous cones as SDP problems, which are studied much more thoroughly than homogeneous cone programming (see, e.g., [1]). However, from a practical point of view, modeling a conic programming problem over a homogeneous cone as an SDP may not be the best thing to do. For example, to optimize over an $n$-dimensional second-order cone (i.e., Lorentz cone), we can use the standard logarithmic barrier, which has a complexity value of 2. Modeling it as an SDP would embed the second-order cone into the cone of positive definite $(n-1)$-by-$(n-1)$ matrices. Thus we would be using a barrier of complexity value $n-1$ instead of 2 if we solve a second-order programming as an SDP. In fact, Güler and Tuncel [3] showed that the best barrier parameter for a homogeneous cone is the same as the rank $r$ of the cone, which is an algebraic property of the cone. In the same paper, a barrier of complexity value $r$ is given. However, the applicability of this barrier in implementations of interior-point methods for optimization over homogeneous cones depends on the efficient computability of its gradient and Hessian, which is still not addressed.

This paper is organized as follows. We begin by describing $T$-algebras as defined in [4]. We then state the main result in [4] that associates homogeneous cones with $T$-algebras. In section 3, we associate $T$-algebras with spaces of linear operators; in particular, we define, for each $T$-algebra, an injective linear map $L$ that maps elements in the $T$-algebra to linear operators. The special structure of $T$-algebras allows us to derive important properties of $L$, which is used in the proof of our main theorem. In the last section, we prove the main theorem: every homogeneous cone is a "slice" of some cone of positive definite linear operators.

**2. *T*-algebras and homogeneous cones.** This section is devoted to the description of $T$-algebras and the association of homogeneous cones with $T$-algebras.

A *homogeneous cone* $K$ is a full-dimensional convex pointed cone in a finite-dimensional space such that the group of linear automorphisms of $K$ acts transitively on it (i.e., for every $x, y \in K$, there exists a linear map $A$ such that $Ax = y$ and $AK = K$).

A *matrix algebra of rank* $r$ is an algebra $\mathcal{A} = \bigoplus_{i,j=1}^{r} \mathcal{A}_{ij}$ such that

$$\mathcal{A}_{ij}\mathcal{A}_{\ell k} \subset \begin{cases} \mathcal{A}_{ik} & \text{if } j = \ell, \\ \{0\} & \text{if } j \neq \ell. \end{cases}$$

Denote the dimension of $\mathcal{A}_{ij}$ by $n_{ij}$.

If we represent each $a \in \mathcal{A}$ by the generalized matrix $(a_{ij})_{i,j=1}^{r}$, where $a_{ij}$ denotes the projection of $a$ onto $\mathcal{A}_{ij}$, then the representation of $ab$ is given by the matrix product $(a_{ij})(b_{ij})$. For example, suppose $\mathcal{A}$ is a matrix algebra of rank 2 and $a = a_{11} + a_{12} + a_{21} + a_{22}$. It is easy to see that $(ab)_{ij} = \sum_{k=1}^{2} a_{ik}b_{kj}$, which corresponds to the usual matrix multiplication.

An *involution of a matrix algebra* $\mathcal{A}$ is an linear map $^*$ of $\mathcal{A}$ onto itself such that

1. $a^{**} = a$,
2. $(ab)^* = b^*a^*$, and
3. $\mathcal{A}_{ij}^* \subset \mathcal{A}_{ji}$.

In its matrix representation, an involution corresponds to taking the transpose, i.e., $(a^*)_{ij} = a_{ji}^*$. A consequence of the existence of an involution is that $n_{ij} = n_{ji}$.

EXAMPLE 2.1 (real matrices). *The algebra $\mathbb{R}^{r\times r}$ of real $r$-by-$r$ matrices is a matrix algebra of rank $r$. In this matrix algebra, $\mathcal{A}_{ij}$ is the subspace of matrices that are zero outside the $(i,j)$th entry, and $n_{ij} = 1$. The transposition of matrices is an involution for the matrix algebra.*

EXAMPLE 2.2 (real vectors). *When $n_{ij} = 0$ for all $i \neq j$, we get the algebra of real $r$-vectors, where the multiplication of two vectors is given by their componentwise product. The involution is the identity map.*

Henceforth, $\mathcal{A}$ will be a matrix algebra with involution.

Let

$$\mathcal{T} := \sum_{i\leq j} \mathcal{A}_{ij}$$

be the subspace of $\mathcal{A}$ whose elements are represented by upper-triangular matrices, and let

$$\mathcal{H} := \{a \in \mathcal{A} : a = a^*\}$$

be the subspace of $\mathcal{A}$ whose elements are represented by "symmetric" matrices.

Suppose $\mathcal{A}_{ii}$ is isomorphic to the field $\mathbb{R}$ of real numbers for each $i$. We let $\rho_i : \mathcal{A}_{ii} \to \mathbb{R}$ denote the isomorphism and $e_i$ denote the unit element of $\mathcal{A}_{ii}$. Since the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto \rho_i(\rho_i^{-1}(x)^*)$ is a linear automorphism on $\mathbb{R}$, it is the identity map. Hence $a_{ii}^* = a_{ii}$ for all $a_{ii} \in \mathcal{A}$. The *trace* of an element $a \in \mathcal{A}$ is defined as

$$\operatorname{tr} a := \sum_{i=1}^{r} \rho_i(a_{ii}).$$

A *$T$-algebra* is a matrix algebra $\mathcal{A}$ of rank $r$ with involution $^*$ that satisfies the following axioms:

(I) $\mathcal{A}_{ii}$ is isomorphic to $\mathbb{R}$.
(II) $e_i a_{ij} = a_{ij} e_j = a_{ij}$ for all $a_{ij} \in \mathcal{A}_{ij}$.
(III) $\operatorname{tr} ab = \operatorname{tr} ba$.
(IV) $\operatorname{tr} a(bc) = \operatorname{tr}(ab)c$.
(V) $\operatorname{tr} a^*a > 0$ unless $a = 0$.
(VI) $t(uw) = (tu)w$ for all $t, u, w \in \mathcal{T}$.
(VII) $t(uu^*) = (tu)u^*$ for all $t, u \in \mathcal{T}$.

In a $T$-algebra $\mathcal{A}$, the element with $a_{ii} = e_i$ and $a_{ij} = 0$, $i \neq j$, is the unit element $e$ of $\mathcal{A}$.

From axiom (V), we see that $\langle a, b\rangle := \operatorname{tr} a^*b$ is an inner product on $\mathcal{A}$. Under this inner product, $\mathcal{A}_{ij}$ is orthogonal to $\mathcal{A}_{k\ell}$ unless $(i,j) = (k,\ell)$.

Let

$$\mathcal{I} := \{t \in \mathcal{T} : \rho_i(t_{ii}) > 0 \text{ for } 1 \leq i \leq r\}$$

be the subgroup of upper-triangular matrices whose diagonal elements are positive, and let

$$K(\mathcal{A}) := \{tt^* : t \in \mathcal{I}\} \subset \mathcal{H}.$$

Vinberg [4] proved the following important result that relates homogeneous cones with the cones $K(\mathcal{A})$.

THEOREM 2.3. *A cone $K$ is homogeneous if and only if there exists a $T$-algebra $\mathcal{A}$ such that $K$ is isomorphic to $K(\mathcal{A})$.*

**3. *T*-algebras and linear operators.** Let

$$\hat{\mathcal{V}} := \sum_{i=1}^{r} \mathcal{A}_{i1}$$

be the subspace of "vectors." Each $a \in \mathcal{A}$ defines a linear operator $\hat{L}_a : \hat{\mathcal{V}} \to \hat{\mathcal{V}}$ by $v \mapsto av$. Since $\mathcal{A}$ is nonassociative in general, we cannot expect $\hat{L}_a\hat{L}_b = \hat{L}_{ab}$ to hold in general, where $\hat{L}_a\hat{L}_b$ is the composition of $\hat{L}_a$ and $\hat{L}_b$. Still, $T$-algebras have enough structure to allow us to prove the following useful proposition.

PROPOSITION 3.1. *Let $\hat{L} : \mathcal{A} \to L[\hat{\mathcal{V}}, \hat{\mathcal{V}}]$ be as defined above. For every $a \in \mathcal{A}$ and $t, u \in \mathcal{T}$,*

(i) $\hat{L}_{a^*} = \hat{L}_a^*$, *where $\hat{L}_a^*$ denotes the adjoint of $\hat{L}_a$ under $\langle \cdot, \cdot \rangle$;*
(ii) $\hat{L}_t\hat{L}_u = \hat{L}_{tu}$; *and*
(iii) $\hat{L}_t\hat{L}_{t^*} = \hat{L}_{tt^*}$.

*Furthermore, $\hat{L}_a$ is the zero map if and only if $a_{ji} = a_{ij} = 0$ for all $i$ with $n_{i1} \neq 0$ and all $j \geq i$.*

*Proof.* (i) For any $u, v \in \hat{\mathcal{V}}$, $\langle \hat{L}_a^* u, v\rangle = \langle u, \hat{L}_a v\rangle = \operatorname{tr} u^*(av) = \operatorname{tr}(u^*a)v = \operatorname{tr}(a^*u)^*v = \langle a^*u, v\rangle = \langle \hat{L}_{a^*}u, v\rangle$ by axiom (IV). It follows that $\hat{L}_a^* = \hat{L}_{a^*}$.

(ii) By axiom (VI), $\hat{L}_{u^*}\hat{L}_{t^*}v = \hat{L}_{u^*}(t^*v) = u^*(t^*v) = (u^*t^*)v = \hat{L}_{u^*t^*}v$. This implies that $\hat{L}_{u^*}\hat{L}_{t^*} = \hat{L}_{u^*t^*}$. Taking $^*$ on both sides, we get $\hat{L}_t\hat{L}_u = \hat{L}_{tu}$.

(iii) By axiom (VII), $\hat{L}_t\hat{L}_{t^*}v = \hat{L}_t(t^*v) = t(t^*v) = (tt^*)v = \hat{L}_{tt^*}v$. So, $\hat{L}_t\hat{L}_{t^*} = \hat{L}_{tt^*}$.

Suppose that $\hat{L}_a$ is the zero map and $n_{i1} \neq 0$. Then, for any $v_{i1} \in \mathcal{A}_{i1}$ with $v_{i1} \neq 0$, $a_{ji}v_{i1} = (\hat{L}_a v_{i1})_{j1} = 0$. So, for any $j \geq i$,

$$\begin{aligned}
0 &= \operatorname{tr}(a_{ji}v_{i1})(a_{ji}v_{i1})^* \\
&= \operatorname{tr} a_{ji}(v_{i1}(v_{i1}^* a_{ji}^*)) && \text{by axiom (IV)} \\
&= \operatorname{tr}((v_{i1}v_{i1}^*)a_{ji}^*)a_{ji} && \text{by axioms (III) and (VII)} \\
&= \operatorname{tr}(v_{i1}v_{i1}^*)(a_{ji}^* a_{ji}) && \text{by axiom (IV)} \\
&= \rho_i(v_{i1}v_{i1}^*)\rho_i(a_{ji}^* a_{ji}),
\end{aligned}$$

implying that $\rho_i(a_{ji}^* a_{ji}) = 0$ since $\rho_i(v_{i1}v_{i1}^*) \neq 0$ when $v_{i1} \neq 0$. Therefore, we conclude that $a_{ji} = 0$. Since $\hat{L}_{a^*} = \hat{L}_a^*$ is also the zero map, the same argument shows that $(a^*)_{ji} = 0$, from which we conclude that $a_{ij} = (a_{ij}^*)^* = ((a^*)_{ji})^* = 0^* = 0$.

Conversely, suppose that $a \in \mathcal{A}$ is such that $a_{ij} = a_{ji} = 0$ for all $i$ with $n_{i1} \neq 0$ and all $j \geq i$. Let $v \in \hat{\mathcal{V}}$ be arbitrary. Consider $L_a v_{i1}$ for each $1 \leq i \leq r$. Clearly, $L_a v_{i1} = 0$ if $n_{i1} = 0$. If $n_{i1} \neq 0$, consider $(L_a v_{i1})_{j1}$ for each $1 \leq j \leq r$. If $n_{j1} = 0$, then $(L_a v_{i1})_{j1} \in \mathcal{A}_{j1} \implies (L_a v_{i1})_{j1} = 0$. Otherwise, we have either $i \leq j$ or $j \leq i$ (or $i = j$). In either case, $a_{ij} = a_{ji} = 0$ by assumption. Hence, $(L_a v_{i1})_{j1} = a_{ji}v_{i1} = 0$. Consequently, $L_a v_{i1} = 0$ when $n_{i1} \neq 0$. Thus, $L_a v = \sum_{i=1}^{r} L_a v_{i1} = 0$ for any $v \in \hat{\mathcal{V}}$. □

For each $i$, $\mathcal{A}^{(i)} := \sum_{k,l=i}^{r} \mathcal{A}_{kl}$ is clearly a subalgebra of $\mathcal{A}$. In fact, it is a $T$-algebra with involution $^*$. Thus, we can define the subspace of "vectors" $\mathcal{V}^{(i)}$ in $\mathcal{A}^{(i)}$, and the linear operator $L_a^{(i)} : \mathcal{V}^{(i)} \to \mathcal{V}^{(i)}$ by $v \mapsto av$ for each $a \in \mathcal{A}^{(i)}$. Note that $\mathcal{A}^{(1)} = \mathcal{A}$, $\mathcal{V}^{(1)} = \hat{\mathcal{V}}$, and $L^{(1)} = \hat{L}$. For each subset $I \subset \{1, \ldots, r\}$, let $\mathcal{V}^I$ denote the subspace $\sum_{i \in I} \mathcal{V}^{(i)} \subset \mathcal{T}^*$. Define the map $L_a^I : \mathcal{V}^I \to \mathcal{V}^I$ by $L_a^I v = \sum_{i \in I} L_{a^{(i)}}^{(i)} v^{(i)}$, where $a^{(i)}$ and $v^{(i)}$ denote projections of $a$ and $v$ onto $\mathcal{A}^{(i)}$ and $\mathcal{V}^{(i)}$, respectively. By observing that $(L_a v)^{(i)} = L^{(i)}_{a^{(i)}} v^{(i)}$, we can easily see that the first three statements in the above proposition hold for the map $L^I : \mathcal{A} \to L[\mathcal{V}^I, \mathcal{V}^I]$.

Suppose that $I$ is chosen to satisfy the following condition:

(∗) For all $1 \le j \le r$, there exists $i \in I$ such that $i \le j$ and $n_{ji} \ne 0$.

Clearly, the choice $I = \{1, \ldots, r\}$ satisfies this condition. Whenever $I$ satisfies (∗), we call the map $L^I$ the *real matrix representation of $\mathcal{A}$ with respect to $I$*.

EXAMPLE 3.2 (real matrices (cont'd)). *When $\mathcal{A}$ is the algebra of real $r$-by-$r$ matrices, the choice $I = \{1\}$ satisfies (∗) since $n_{ji} = 1 \ne 0$ for all $1 \le i, j \le r$. With this choice, $\mathcal{V}^I$ can be regarded as the space of real $r$-vectors, and $L^I_a$ is the map represented by the matrix $a$.*

EXAMPLE 3.3 (real vectors (cont'd)). *When $\mathcal{A}$ is the algebra of real $r$-vectors, the only $I$ satisfying (∗) is $I = \{1, \ldots, r\}$ since $n_{ji} = 0$ for all $j \ne i$.*

Suppose that $L^I_a$ is the zero map. Then, for each $i \in I$, $L^{(i)}_{a^{(i)}}$ is the zero map. Now, fix an arbitrary $1 \le j \le r$ and choose an $i \in I$, $i \le j$, for which $n_{ji} \ne 0$. By applying the above proposition to $L^{(i)}$, we conclude that $a_{kj} = a_{jk} = 0$ for all $k \ge j$. Since $j$ is arbitrary, we have $a = 0$. Thus, $L^I$ is injective when $I$ satisfies (∗).

Conversely, suppose that for some $1 \le j \le r$, $n_{ji} = 0$ for all $i \in I$ such that $i \le j$. It follows that $L^I_{e_j} v = \sum_{i \in I} L^{(i)}_{e_j^{(i)}} v^{(i)} = \sum_{i \in I,\, i \le j} L^{(i)}_{e_j} v^{(i)} = \sum_{i \in I,\, i \le j} e_j v_{ji} = 0$ for any $v \in \mathcal{V}^I$. Hence, $L^I$ is not injective when $I$ violates (∗).

Thus, we have proven the following proposition.

PROPOSITION 3.4. *Let $L^I : \mathcal{A} \to L[\mathcal{V}^I, \mathcal{V}^I]$ be as defined above. For every $a \in \mathcal{A}$ and $t, u \in \mathcal{T}$,*

(i) $L^I_{a^*} = (L^I_a)^*$;
(ii) $L^I_t L^I_u = L^I_{tu}$ *(equivalently, $L^I|_{\mathcal{T}^*}$ is an isomorphism of algebras); and*
(iii) $L^I_t L^I_{t^*} = L^I_{tt^*}$.

*Furthermore, $L^I$ is injective if and only if $I$ satisfies (∗).*

Henceforth, we will fix an $I$ that satisfies (∗). To simplify notation, we shall drop the superscript $I$ from $L^I$ and $\mathcal{V}^I$.

We end this section with two remarks on the map $L$.

REMARK 3.5. *By observing that each $t \in \mathcal{I}$ has a right inverse $u \in \mathcal{I}$ such that $tu = e$, we see that $L_t$ is invertible for any $t \in \mathcal{I}$. Since $L|_{\mathcal{T}^*}$ is an isomorphism of algebras, $t$ is also invertible with inverse $t^{-1}$ satisfying $L_{t^{-1}} = L_t^{-1}$. It follows from $L_{(t^*)^{-1}} = L_{t^*}^{-1} = (L_t^*)^{-1} = (L_t^{-1})^* = (L_{t^{-1}})^* = L_{(t^{-1})^*}$ that $(t^*)^{-1} = (t^{-1})^*$.*

REMARK 3.6. *It is easy to see that $t = e$ is the only $t \in \mathcal{I}$ that satisfies $tt^* = e$. Suppose $tt^* = uu^*$ for some $t, u \in \mathcal{I}$. Then $L_{(t^{-1}u)(t^{-1}u)^*} = L_{t^{-1}u} L^*_{t^{-1}u} = L_t^{-1} L_u L_u^* (L_t^{-1})^* = L_t^{-1} L_{uu^*} (L_t^{-1})^* = L_t^{-1} L_{tt^*} (L_t^{-1})^* = L_t^{-1} L_t L_t^* (L_t^{-1})^* = L_{t^{-1}t} L^*_{t^{-1}t}$ is the identity map, implying that $t = u$. Hence, the relation $a = tt^*$ sets up a one-to-one correspondence between each $a \in K(\mathcal{A})$ and $t \in \mathcal{I}$.*

**4. Homogeneous cones and cones of positive definite operators.** Before we proceed to the main theorem, let us apply the result of the previous section to produce an easy proof of the fact that $K(\mathcal{A})$ is homogeneous.

For each $t \in \mathcal{I}$, define the map $\tau(t) : uu^* \mapsto (tu)(tu)^*$. By Remark 3.6, $\tau(t)$ is well defined. $\tau(t)$ is clearly a map of $K(\mathcal{A})$ into itself. In fact, by observing that every $u \in \mathcal{I}$ has an inverse in $\mathcal{I}$, we see that $\tau(t)$ maps $K(\mathcal{A})$ onto itself and $\{\tau(t) : t \in \mathcal{I}\}$ acts transitively on $K(\mathcal{A})$. From Proposition 3.4, $L_{(tu)(tu)^*} = L_t L_u L_u^* L_t^* = L_t L_{uu^*} L_t^*$, which implies that $\tau(t)$ acts linearly on $K(\mathcal{A})$. By extending $\tau(t)$ to a linear automorphism of the subspace $\mathcal{H}$, we can prove the "if" part of Theorem 2.3.

THEOREM 4.1. *For each $t \in \mathcal{I}$, let $\bar{\tau}(t)$ be the extension of $\tau(t)$ to the subspace $\mathcal{H}$. The subgroup of automorphisms $\{\bar{\tau}(t) : t \in \mathcal{I}\}$ of $\mathcal{H}$ is an invariant and transitive subgroup for the cone $K(\mathcal{A})$. Consequently, $K(\mathcal{A})$ is homogeneous.*

Finally, we give the main theorem.

THEOREM 4.2. *For each $a \in \mathcal{A}$, $a \in K(\mathcal{A})$ if and only if $L_a$ is positive definite and self-adjoint. Consequently, $L$ embeds $K(\mathcal{A})$ into some cone of positive definite self-adjoint linear operators.*

*Proof.* For the "only if" part, suppose that $a = tt^* \in K(\mathcal{A}) \subset \mathcal{H}$ for some $t \in \mathcal{I}$. Then, by Proposition 3.4, $L_a^* = L_{a^*} = L_a$ and $\langle v, L_a v\rangle = \langle v, L_{tt^*} v\rangle = \langle v, L_t L_t^* v\rangle = \langle L_t^* v, L_t^* v\rangle > 0$ for all $v \in \mathcal{V}$, $v \neq 0$, since $L_t$ is nonsingular, and so $L_t^* v \neq 0$.

For the "if" part, we shall proceed by induction on the rank of $\mathcal{A}$.[2] If $\mathcal{A}$ has rank 1, then $\mathcal{A}$ is isomorphic to the algebra of the reals, and every positive definite $a$ can be written as $(\sqrt{\rho_1(a_1)}e_1)(\sqrt{\rho_1(a_1)}e_1)^*$ with $\rho_1(a_1) > 0$. Suppose that $\mathcal{A}$ has rank $r > 1$, and that the "if" part is true for all $T$-algebras of rank less than $r$. Suppose $L_a$ is positive definite and self-adjoint. Let $\bar{\mathcal{A}} := \sum_{i,j=1}^{r-1} \mathcal{A}_{ij}$ be a rank $r-1$ $T$-algebra. Let $\bar{a} = \sum_{i,j=1}^{r-1} a_{ij} \in \bar{\mathcal{A}}$ and $a_r = \sum_{i=1}^{r-1} a_{ir}$. Let $\bar{\mathcal{V}} := \sum_{i\in I}\sum_{j=1}^{r-1} \mathcal{A}_{ji} \subset \mathcal{V}$. The orthogonal complement of $\bar{\mathcal{V}}$ in $\mathcal{V}$ is $\bar{\bar{\mathcal{V}}} := \sum_{i\in I} \mathcal{A}_{ri}$. For any $v \in \mathcal{V}$, there exist $\bar{v} \in \bar{\mathcal{V}}$ and $\bar{\bar{v}} \in \bar{\bar{\mathcal{V}}}$ such that $v = \bar{v} + \bar{\bar{v}}$; and

$$\begin{aligned} L_a v =& \sum_{i\in I} a^{(i)} v^{(i)} = \sum_{i\in I} (\bar{a}^{(i)} + a_r^{(i)}) v^{(i)} + \sum_{i\in I} ((a_r^{(i)})^* + a_{rr}) v^{(i)} \\ =& \sum_{i\in I} \bar{a}^{(i)} (\bar{v})^{(i)} + \sum_{i\in I} a_r^{(i)} (\bar{\bar{v}})^{(i)} + \sum_{i\in I} (a_r^{(i)})^* (\bar{v})^{(i)} + \sum_{i\in I} a_{rr} (\bar{\bar{v}})^{(i)} \\ =& L_{\bar{a}}\bar{v} + L_{a_r}\bar{\bar{v}} + L_{a_r}^*\bar{v} + \rho_r(a_{rr})\bar{\bar{v}}, \end{aligned}$$

where $L_{\bar{a}}\bar{v} + L_{a_r}\bar{\bar{v}} \in \bar{\mathcal{V}}$ and $L_{a_r}^*\bar{v} + \rho_r(a_{rr})\bar{\bar{v}} \in \bar{\bar{\mathcal{V}}}$. By $(*)$, both $\bar{\mathcal{V}}$ and $\bar{\bar{\mathcal{V}}}$ have positive dimensions. So, $\hat{L}_a$ is positive definite and self-adjoint only if $\rho_r(a_{rr}) > 0$ and $L_{\bar{a}} - \rho_r(a_{rr})^{-1} L_{a_r} L_{a_r}^*$ is positive definite over $\bar{\mathcal{V}}$. Therefore,

$$\begin{aligned} L_{\rho_r(a_{rr})\bar{a} - a_r a_r^*} &= \rho_r(a_{rr}) L_{\bar{a}} - L_{a_r a_r^*} \\ &= \rho_r(a_{rr}) L_{\bar{a}} - L_{a_r} L_{a_r}^* \qquad \text{(by Proposition 3.4(iii))} \\ &= \rho_r(a_{rr})(L_{\bar{a}} - \rho_r(a_{rr})^{-1} L_{a_r} L_{a_r}^*) \end{aligned}$$

is positive definite over $\bar{\mathcal{V}}$. It is clearly self-adjoint. Let $\bar{I} = I\backslash\{r\}$, which satisfies $(*)$ for $\bar{\mathcal{A}}$. Let $\bar{L}$ be the real matrix representation of $\bar{\mathcal{A}}$ with respect to $\bar{I}$. It is easy to check that $L_a|_{\bar{\mathcal{V}}} = \bar{L}_a$ for all $a \in \bar{\mathcal{A}}$. So, by the induction hypothesis, $\rho_r(a_{rr})\bar{a} - a_r a_r^* = tt^*$ for some $t \in \mathcal{I} \cap \bar{\mathcal{A}}$. Therefore,

$$(t + a_r + a_{rr})(t + a_r + a_{rr})^* = tt^* + a_r a_r^* + a_{rr}^2 + a_r a_{rr} + a_{rr} a_r^* = \rho_r(a_{rr}) a,$$

which implies that $a = uu^*$ with $u = (t + a_r + a_{rr})/\sqrt{\rho_r(a_{rr})}$.

Finally, since $L$ is injective, it is an embedding of $K(\mathcal{A})$ into the cone of positive definite self-adjoint linear operators over $\mathcal{T}^*$. ∎

COROLLARY 4.3. *If $K$ is a homogeneous cone in $\mathbb{R}^n$, then there exist an $m \leq n$ and an injective linear map $M : \mathbb{R}^n \to \mathbb{S}^{m\times m}$ such that $M(K) = \mathbb{S}_{++}^{m\times m} \cap M(\mathbb{R}^n)$,*

[2]The proof of this part resembles a proof of the Cholesky factorization of symmetric positive definite matrices. Indeed, in the case where our $T$-algebra is the algebra of real $r$-by-$r$ matrices and $L = L^I$ with $I = \{1\}$, the proof of this part would be a proof of Cholesky factorization.

*where* $\mathbb{S}^{m\times m}$ *is the space of m-by-m symmetric matrices and* $\mathbb{S}^{m\times m}_{++}$ *is the cone of positive definite symmetric m-by-m matrices.*

*Proof.* By Theorem 2.3, there exists a $T$-algebra $\mathcal{A}$ for which $K(\mathcal{A})$ is isomorphic to $K$. This isomorphism can be extended linearly to a linear bijection from $\mathbb{R}^n$ to $\mathcal{H}$, the space of "symmetric" matrices in $\mathcal{A}$. Pick an $I$ that satisfies $(*)$ for $\mathcal{A}$. Then the real matrix representation of $\mathcal{A}$ with respect to $I$ embeds $K(\mathcal{A})$ into the cone of positive definite self-adjoint linear operators on $\mathcal{V}^I$, which is of dimension $m := \sum_{i\in I}\sum_{j\geq i} n_{ji} \leq \sum_{i=1}^{r}\sum_{j\geq i} n_{ji} = n$. $M$ is then obtained by composing the bijection from $\mathbb{R}^n$ to $\mathcal{H}$ with the real matrix representation of $\mathcal{A}$ with respect to $I$. ◻

**Acknowledgments.** I would like to express my gratitude toward my thesis advisor Professor James Renegar for his motivation and encouragement. I would also like to thank Professor Adrian Lewis for his helpful comments which led to a shorter proof of the main proposition. I thank the referees for their very useful comments. I dedicate this paper to my wife.

## REFERENCES

[1] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, MPS/SIAM Ser. Optim., SIAM, Philadelphia, PA, 2001.

[2] L. Faybusovich, *On Nesterov's approach to semi-definite programming*, Acta Appl. Math., 74 (2002), pp. 195–215.

[3] O. Güler and L. Tuncel, *Characterization of the barrier parameter of homogeneous convex cones*, Math. Programming, 81 (1998), pp. 55–76.

[4] È. B. Vinberg, *The theory of convex homogeneous cones*, Trans. Moscow Math. Soc., 12 (1965), pp. 340–403.