Convergence Behavior Analysis of FxLMS Algorithms with Different Leaky Term

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ABSTRACT

In order to improve the robustness of conventional filtered-X LMS (FXLMS) algorithm, different leaky terms are introduced into the cost function. With different leaky term, the convergence behavior differs much. The detailed derivation of LASSO based FXLMS (L1-FXLMS) and elastic-net based FXLMS (L1/2-FXLMS) algorithms are presented. In addition, sufficient conditions for guaranteed convergence are derived for FxLMS algorithms with different leaky term. Furthermore, the convergence behavior of L1-FXLMS and L1/2-FXLMS algorithms are compared with conventional FXLMS and leaky FXLMS (L2-FXLMS) algorithms. It is indicated that L1-FXLMS could have faster convergence speed than L2-FXLMS with the premise of guaranteed convergence. Simulation and experiments are conducted to test the sufficient boundaries for guaranteed convergence of different FXLMS algorithms.

1. INTRODUCTION

Filtered-X LMS (FXLMS) is the most popular algorithm in active noise control (ANC) problem, since it could make a good balance between guaranteed convergence and convergence speed, noise reduction, and computational complexity [1]. FXLMS algorithm is a typical ordinary least square (OLS) estimate which aims to minimize the residual squared error. OLS has two main drawbacks [2]. Firstly, OLS estimate often has a problem with prediction accuracy with low bias but large variance. Secondly, OLS estimate could not determine a smaller subset in large numbers of predictors that exhibit the strongest effects. These two drawbacks are inherited by FXLMS as well. Specifically, FXLMS is very sensitive to outlying data points which corrupt regular data points and model errors [3], which results FXLMS suffers from high possibility of divergence and instability during practical applications. In order to keep the system stable during the whole experiment, we usually choose an extremely small step size to do the adaptive control during real experiments. It will take a rather long time to achieve the convergence. In order to improve the robustness of conventional FXLMS, leaky FXLMS algorithms is proposed to balance robustness and loss of performance by adding a leaky term to the cost function [4,5]. A series of leaky FXLMS algorithms with different form of leaky term are designed: conventional leaky [6], effort weighting [7], uncertainty weighting [8], output weighting [9], and general robust. All these leaky FXLMS algorithms are derived from ridge regression or Tikhonov regularization with L2-norm leakage. We note these leaky algorithms as L2-FXLMS in this paper. The aim of adding this L2-norm leakage is to ensure the strictly positive and real (SPR) constraint [10-12], which assures the inflation and general instability are under control. Meanwhile,

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ridge regression is not able to do subset selection. Therefore, all the regular and outlying data will be regarded equally, which makes the algorithm could diverge with high possibilities.

Least absolute shrinkage and selection operator (LASSO) is proposed to solve these general problems exist in ridge regression estimates [13] by replacing the L2-norm leakage with L1-norm leakage. The constraint region is changed from elliptical to rotated square, which enables the estimator to set coefficients correspond to improper outlying data to zero. Thus, stability of the system is enhanced. L1-norm leakage is introduced to FXLMS in [14] to benefit from the advantages of LASSO, which is defined as L1-FXLMS algorithm here. However, in the process of deriving the gradient in the updating equation, approximation is applied to simplify the derivation. Actually, it is possible to derive the exact expression of the gradient without unnecessary approximation.

In this paper, we give an exact expression of gradient in updating equation of L1-FXLMS algorithm. The idea of elastic net is applied in ANC system, and L1/2-FXLMS is formed in this paper. Comparisons of the convergence behavior of four kinds of FXLMS algorithms are compared in detail.

This paper is organized as follows. Model and derivation of algorithm are formulated in section 2. Derivations of FXLMS algorithms with different cost function is presented in in section 3. Comparisons of the convergence behavior of the four FXLMS algorithms are conducted in section 4. Simulation results and analysis are presented in section 5. Conclusions are drawn in section 6.

2. MODEL FORMULATION

The block diagram of a typical feedforward single-reference multi-channel ANC (SR-MCANC) system is illustrated in Figure 1. This SR-MCANC system is composed of one reference microphones, K secondary sources, and M error microphones. Assuming that we will actively control any harmonic of the period sound, the analysis in the following will be considered in the frequency-domain. The following denotations obey the rules in [15].

If we only consider the nth harmonic, the reference input signal $X(\omega_n)$ will be a constant. Based on the system structure presented in Figure1, the error signal $\boldsymbol{E}(\omega_n)$ is formulated by the desired signal $\boldsymbol{D}(\omega_n)$, the reference signal $X(\omega_n)$, the complex response from the secondary sources to error microphones $\boldsymbol{S}(\omega_n)$, and the filter control weight in frequency domain $\boldsymbol{W}(\omega_n)$. Thus, the error signal is

$$\boldsymbol{E}(\omega_n) = \boldsymbol{D}(\omega_n) - \boldsymbol{S}(\omega_n)X(\omega_n)\boldsymbol{W}(\omega_n) \tag{1}$$

where

$$\boldsymbol{E}(\omega_n) = [E_1(\omega_n), E_2(\omega_n), \cdots, E_M(\omega_n)]^T$$

$$\boldsymbol{D}(\omega_n) = [D_1(\omega_n), D_2(\omega_n), \cdots, D_M(\omega_n)]^T$$

$$\boldsymbol{W}(\omega_n) = [W_1(\omega_n), W_2(\omega_n), \cdots, W_K(\omega_n)]^T$$

and

$$\boldsymbol{S}(\omega_n) = \begin{bmatrix} S_{11}(\omega_n) & \cdots & S_{1k}(\omega_n) \\ \vdots & \ddots & \vdots \\ S_{m1}(\omega_n) & \cdots & S_{mk}(\omega_n) \end{bmatrix}$$

More precisely,$\boldsymbol{D}(\omega_n) = \boldsymbol{P}(\omega_n)X(\omega_n)$, and $\boldsymbol{P}(\omega_n)$ is the complex response from the reference microphone to error microphones.

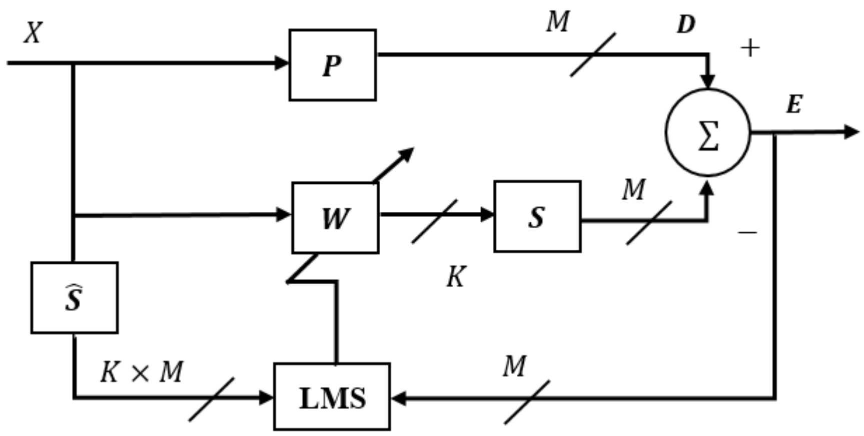


Figure 1: Block diagram of a SR-MCANC system

Under the assumption that the input signal is sine tonal, $X(\omega_n)$ is a constant. Thus, SR-MCANC system could be simplified as Figure 2, defining $\boldsymbol{X}_s(\omega_n) = \boldsymbol{S}(\omega_n)X(\omega_n)$ with dimension of $M \times K$. Thus, the error signal could be rewritten as

$$\boldsymbol{E}(\omega_n) = \boldsymbol{D}(\omega_n) - \boldsymbol{X}_s(\omega_n)\boldsymbol{W}(\omega_n) \tag{2}$$

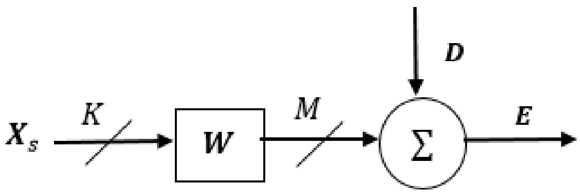


Figure 2: Block diagram of simplified SR-MCANC system

The basic idea of ANC system is adaptively controlling the reference signal $X(\omega_n)$ with control filter $\boldsymbol{W}(\omega_n)$, then producing the anti-signal $\boldsymbol{Y}(\omega_n) = X(\omega_n)\boldsymbol{W}(\omega_n)$ pass through the secondary path to attenuate the primary noise $\boldsymbol{D}(\omega_n)$ locally or globally.

3. DERIVATIONS OF ALGORITHMS WITH DIFFERENT COST FUNCTION

Based on the simplified structure of SR-MCANC system, we revisit the conventional FXLMS and L2-FXLMS algorithm. Furthermore, L1-FXLMS algorithm based on previous research work [14] is revised. In addition, L1/2-FXLMS algorithm is proposed. Three important parts of these FXLMS algorithms: cost function, weight updating equation, and optimal weight are compared and discussed.

3.1. Conventional FXLMS algorithm

Based on theory of linear regression, considering the propagation delay exist in ANC system, the cost function of FXLMS algorithm is

$$J_c = \boldsymbol{E}^H \boldsymbol{E} \tag{3}$$

where the superscript H denotes the Hermitian transpose (conjugate transpose) of a vector or a matrix. It is a typical regression problem. The method of steepest descent is used to minimize this cost function. [1] has provided the weight update gradient $\boldsymbol{g}_c$ and the optimal weight $\boldsymbol{W}_{optc}$

$$\boldsymbol{g}_c = -2\boldsymbol{X}_s^H \boldsymbol{E}, \tag{4}$$

$$\boldsymbol{W}_{optc} = [\boldsymbol{X}_s^H \boldsymbol{X}_s]^{-1} \boldsymbol{X}_s^H \boldsymbol{D}. \tag{5}$$

The weight update equation is

$$\boldsymbol{W}(n+1) = \boldsymbol{W}(n) - \frac{\mu}{2}\boldsymbol{g}, \tag{6}$$

where n is the iteration number, μ is the step size. Combining Equation 4 and 5, we can rewrite Equation 6 as

$$\left(\boldsymbol{W}(n+1) - \boldsymbol{W}_{optc}\right) = [I - \mu \boldsymbol{X_s^H X}_s]\left(\boldsymbol{W}(n) - \boldsymbol{W}_{optc}\right). \tag{7}$$

Assuming $\boldsymbol{W}(0) = \boldsymbol{0}$, $\left(\boldsymbol{W}(n) - \boldsymbol{W}_{optc}\right)$ can be repeatedly derived as

$$\boldsymbol{W}(n) - \boldsymbol{W}_{optc} = -[I - \mu \boldsymbol{X_s^H X}_s]^n \boldsymbol{W}_{optc}, \tag{8}$$

Thus, the convergence behavior of FXLMS algorithm is only related to the input reference signal and the complex response from the secondary sources to error microphones.

3.2. Leaky FXLMS algorithm with L2-norm leakage (L2-FXLMS)

L2-FXLMS based on ridge regression is proposed to solve improve the robustness [4]. The cost function of L2-FXLMS is

$$J_2 = \boldsymbol{E}^H\boldsymbol{E} + \beta||\boldsymbol{W}||_2 \tag{9}$$

where $||\cdot||_2$ denotes the L2-norm of a matrix or a vector, and β is the positive effort coefficient. The weight update gradient $\boldsymbol{g}_2$ and the optimal weight $\boldsymbol{W}_{opt2}$ are derived in [15]

$$\boldsymbol{g}_2 = -2[\boldsymbol{X_s^H E} - \beta\boldsymbol{W}] \tag{10}$$

$$\boldsymbol{W}_{opt2} = [\boldsymbol{X_s^H X}_s + \beta\boldsymbol{I}]^{-1}\boldsymbol{X_s^H D} \tag{11}$$

Following the same derivation procedure as FLXMS, assuming $\boldsymbol{W}(0) = \boldsymbol{0}$, the weight update equation of L2-FXLMS is

$$\boldsymbol{W}(n) - \boldsymbol{W}_{optc} = -[I - \mu(\boldsymbol{X_s^H X}_s + \beta\boldsymbol{I})]^n \boldsymbol{W}_{optc} \tag{12}$$

3.3. Leaky FXLMS algorithm with L1-norm leakage (L1-FXLMS)

The leaky FXLMS with L1-norm leakage is first proposed by [14], to benefit from LASSO, which could improve the robustness against outliers. However, in the process of deriving the gradient, unnecessary approximation is introduced to simplify the derivation. The cost function of L1-FXLMS is

$$J_1 = \boldsymbol{E}^H\boldsymbol{E} + \beta||\boldsymbol{W}||_1 \tag{13}$$

where $||\cdot||_1$ denotes the L1-norm of a matrix or a vector, and β is the positive effort coefficient as well. Since $\boldsymbol{W}$ is a complex matrix, we assume that $\boldsymbol{W} = \boldsymbol{W}_R + j\boldsymbol{W}_I$, where $\boldsymbol{W}_R$ and $\boldsymbol{W}_I$ are entirely real. Based on the relationship

$$||\boldsymbol{W}||_1 = \sqrt{(\boldsymbol{W}_R)^2 + (\boldsymbol{W}_I)^2} \le (||\boldsymbol{W}_R||_1 + ||\boldsymbol{W}_I||_1) \tag{14}$$

Thus, the expression of gradient in (11) is approximated as

$$\boldsymbol{g}_1 = -2[\boldsymbol{X_s^H E} - \beta(\frac{\partial\left||\boldsymbol{W}_R|\right|_1}{\partial\boldsymbol{W}} + \frac{\partial\left||\boldsymbol{W}_I|\right|_1}{\partial\boldsymbol{W}})] \tag{15}$$

In this paper, this approximation is unnecessary. The gradient of L1-FXLMS could be directly derived as

$$\boldsymbol{g}_1 = \frac{\partial J_1}{\partial\boldsymbol{W}_R} + j\frac{\partial J_1}{\partial\boldsymbol{W}_I} = -2\left[\boldsymbol{X_s^H E} - \frac{\beta\boldsymbol{W}}{\left||\boldsymbol{W}|\right|_1}\right] \tag{16}$$

Thus, the weight updating equation of L-FXLMS algorithm is

$$\boldsymbol{W}(n+1)=\boldsymbol{W}(n)-\mu\left[\boldsymbol{X}_{\boldsymbol{s}}^{H}\boldsymbol{E}-\frac{\beta\boldsymbol{W}(n)}{||\boldsymbol{W}(n)||_{1}}\right] \tag{17}$$

The optimal weight of L1-FXLMS is derived based on LASSO related research (9)

$$\boldsymbol{W}_{opt1}=[\boldsymbol{X}_{\boldsymbol{s}}^{H}\boldsymbol{X}_{s}+\beta\boldsymbol{W}^{-}]^{-1}\boldsymbol{X}_{\boldsymbol{s}}^{H}\boldsymbol{D} \tag{18}$$

3.4. Leaky FXLMS algorithm with L1/2-norm leakage (L1/2-FXLMS)

LASSO has the advantage of conducting continuous shrinkage and automatic variable selection simultaneously. However, for some cases, it has empirically been observed that the prediction performance of LASSO is not as good as ridge regression. Elastic net is proposed to take advantage of both ridge regression and LASSO. Based on the idea of elastic net, L1/2-FXLMS is proposed in this paper. The cost function of L1/2 FXLMS is

$$J_{1/2}=\boldsymbol{E}^{H}\boldsymbol{E}+\beta_{1}||\boldsymbol{W}||_{1}+\beta_{2}||\boldsymbol{W}||_{2} \tag{19}$$

where β_1 and β_2 are the positive effort coefficient of L1-norm and L2-norm leakage. The complex gradient vector $\boldsymbol{g}_{1/2}$ for L1/2-FXLMS could be derived as

$$\boldsymbol{g}_{1/2}=2\left[\boldsymbol{X}_{\boldsymbol{s}}^{H}\boldsymbol{E}-\frac{\beta_{1}\boldsymbol{W}}{||\boldsymbol{W}||_{1}}-\beta_{2}\boldsymbol{W}\right] \tag{20}$$

As far as we know, current research works on elastic net haven't give the exact expression for its optimal weight as the exact expression is quite complicated with L1-norm and L2-norm leakage in the same time. Thus, we don't give the exact optimal weight for L1/2-FXLMS. The weight updating equation is enough for implementing the algorithm.

4. COMPARISON OF CONVERGENCE BEHAVIOR

The most common method adopted to analyze convergence behavior is bridging $(\boldsymbol{W}(n+1)-\boldsymbol{W}_{opt})$ and $(\boldsymbol{W}(n)-\boldsymbol{W}_{opt})$. For conventional FXLMS and L2-FXLMS algorithm, the relationship between $(\boldsymbol{W}(n+1)-\boldsymbol{W}_{opt})$ and $(\boldsymbol{W}(n)-\boldsymbol{W}_{opt})$ has already been established in previous research works. Under the assumption that $\boldsymbol{W}(n)=\boldsymbol{0}$, $\boldsymbol{W}(n)$ at nth iteration could be represented as Equation 8 and 12. Conducting eigendecomposition on both $\boldsymbol{X}_{\boldsymbol{s}}^{H}\boldsymbol{X}_{s}$ and $\boldsymbol{X}_{\boldsymbol{s}}^{H}\boldsymbol{X}_{s}+\beta\boldsymbol{I}$, they can be composed of normalized eigenvectors, $\boldsymbol{Q}_c$ and $\boldsymbol{Q}_2$, and diagonal matrix, $\boldsymbol{\Lambda}_c=diag(\lambda_{c,1},\lambda_{c,2},\cdots,\lambda_{c,K})$ and $\boldsymbol{\Lambda}_2=diag(\lambda_{2-1},\lambda_{2-2},\cdots,\lambda_{2-K})$

$$\boldsymbol{X}_{\boldsymbol{s}}^{H}\boldsymbol{X}_{s}=\boldsymbol{Q}_{c}\boldsymbol{\Lambda}_{c}\boldsymbol{Q}_{c}^{H} \tag{21}$$

$$\boldsymbol{X}_{\boldsymbol{s}}^{H}\boldsymbol{X}_{s}+\beta\boldsymbol{I}=\boldsymbol{Q}_{2}\boldsymbol{\Lambda}_{2}\boldsymbol{Q}_{2}^{H} \tag{22}$$

The subscript c means conventional FXLMS algorithm, and 2, 1, 1/2 correspond to L2-FXLMS, L1-FXLMS, and L1/2-FXLMS algorithm, respectively. Following the analysis method in [12], we define the principal coordinates of the control system to be

$$\boldsymbol{V}_{c}(n)=\boldsymbol{Q}_{c}^{H}\left(\boldsymbol{W}_{c}(n)-\boldsymbol{W}_{optc}\right) \tag{23}$$

$$\boldsymbol{V}_{2}(n)=\boldsymbol{Q}_{2}^{H}\left(\boldsymbol{W}_{2}(n)-\boldsymbol{W}_{opt2}\right) \tag{24}$$

Thus, Equation 8 and 12 could be rewritten as

$$\boldsymbol{V}_{c}(n)=[\boldsymbol{I}-\mu\boldsymbol{\Lambda}_{c}]^{n}\,\boldsymbol{V}_{c}(0) \tag{25}$$

$$\boldsymbol{V}_{2}(n)=[\boldsymbol{I}-\mu\boldsymbol{\Lambda}_{2}]^{n}\,\boldsymbol{V}_{2}(0) \tag{26}$$

Then, the kth component of $\boldsymbol{V}_c(n)$ and $\boldsymbol{V}_2(n)$ are respectively

$$V_{c-k}(n) = (1-\mu\lambda_{c-k})^n V_{c-k}(0) \tag{27}$$

$$V_{2-k}(n) = (1-\mu\lambda_{2-k})^n V_{2-k}(0) \tag{28}$$

Thus, the stability conditions of step size are

$$0 < \mu_c < \frac{2}{\lambda_{c,k}}, for\ all\ k \tag{29}$$

$$0 < \mu_2 < \frac{2}{\lambda_{2,k}}, for\ all\ k \tag{30}$$

The relationship between $\lambda_{c,k}$ and $\lambda_{2,k}$ is

$$\lambda_{2,k} = \lambda_{c,k} + \beta \tag{31}$$

Since the effort coefficient β in L2-FXLMS is positive, $\lambda_{2,k} > \lambda_{c,k}$. Thus, the convergence boundary for L2-FXLMS is smaller than conventional FXLMS.

For L1-FXLMS, as the existence of L1-norm, we could not directly establish the relationship between $(\boldsymbol{W}(n+1) - \boldsymbol{W}_{opt})$ and $(\boldsymbol{W}(n) - \boldsymbol{W}_{opt})$. In order to analysis the convergence behavior of L1-FXLMS, we modify the $\boldsymbol{W}_{opt1}$ defined in Equation 18

$$\boldsymbol{W}'_{opt1} = -\left[\boldsymbol{X}_s^H \boldsymbol{X}_s + \beta \boldsymbol{\gamma}_{opt} \boldsymbol{I}\right]^{-1} \boldsymbol{X}_s^H \boldsymbol{D} \tag{32}$$

where $\boldsymbol{\gamma}_{opt} = diag(\frac{1}{|W'_{opt1,1}|}, \frac{1}{|W'_{opt1,2}|}, \cdots, \frac{1}{|W'_{opt1,K}|})$. Therefore, we can rewrite Equation 17 as

$$\left(\boldsymbol{W}(n+1) - \boldsymbol{W}'_{opt1}\right) = \left(\boldsymbol{W}(n) - \boldsymbol{W}'_{opt1}\right)[\boldsymbol{I} - \mu(\boldsymbol{X}_s^H \boldsymbol{X}_s + \beta \boldsymbol{\gamma}(n) \boldsymbol{I})] \tag{33}$$

where $\boldsymbol{\gamma}(n) = diag(\frac{1}{|W_1(n)|}, \frac{1}{|W_2(n)|}, \cdots, \frac{1}{|W_K(n)|})$. Different from L2-FXLMS and conventional FXLMS algorithm, the essential matrix $\boldsymbol{X}_s^H \boldsymbol{X}_s + \beta \boldsymbol{\gamma}(n) \boldsymbol{I}$ is closely related to the module of $\boldsymbol{W}(n)$ at every updating step. Hence, fluctuation is allowed for the whole eigenstructure of $\boldsymbol{X}_s^H \boldsymbol{X}_s + \beta \boldsymbol{\gamma}(n) \boldsymbol{I}$ to achieve convergence. A rather restricted convergence condition is casted

$$\left|1 - \mu_1 \lambda_{1,k}(n)\right| < 1, for\ all\ k\ and\ n \tag{34}$$

where $\lambda_{1,k}(n)$ is the mth eigenvalue of $\boldsymbol{X}_s^H \boldsymbol{X}_s + \beta \boldsymbol{\gamma}(n) \boldsymbol{I}$ at every updating step. Thus, a sufficient condition for convergence of L1-FXLMS algorithm is

$$0 < \mu_1 < \frac{2}{\lambda_{1,k}}, for\ all\ k\ and\ n \tag{35}$$

where $\lambda_{1,k} = \lambda_{c,k} + \beta\gamma_k(n)$. Since β and $\gamma_k(n)$ are both positive, the convergence boundary for L1-FXLMS is smaller than conventional FXLMS as well. If the effort coefficient β in L1-FXLMS and L2-FXLMS are the same, $\gamma_k(n)$ determines the relationship between convergence boundary of L1-FXLMS and L2-FXLMS. Therefore, the primary path, the secondary path and spectrum of input reference signal will decide this relationship together.

In addition, as the vacancy of exact expression of the optimal weight of L1/2-FXLMS algorithm, it is difficult to bridge $(\boldsymbol{W}(n+1) - \boldsymbol{W}_{opt})$ and $(\boldsymbol{W}(n) - \boldsymbol{W}_{opt})$. Therefore, the corresponding convergence behavior analysis is not included here for now.

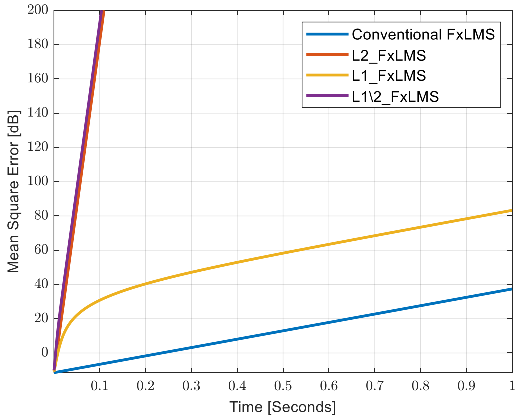
As presented in Equation 29, 30, 35, it is clear that for persistently excited single mode the boundary for conventional FxLMS is the widest among all four FxLMS algorithms. The stable range of step size for three leaky FxLMS algorithms will varies with the effort coefficients for different leaky term. All the above derivation about the stable range of step size are established on the assumption that the input noise is sine tonal noise, and persistently excites the corresponding mode. For broadband noise, multiple modes of the system will be excited by the broadband input noise, the convergence behavior will be quite complex. In previous research, only [16] mentioned about rough boundaries for the step size with broadband noise input. This rough boundary is only related to the signal power, filter length and secondary path delay, and unrelated to the algorithm itself. However, based on the research works on OLS, with complex broadband input noise, the stability of leaky FxLMS algorithm will be better than the conventional FxLMS algorithm. Thus, simulation will be conducted to validate the convergence behavior of these four algorithms in the next section.

5. SIMULATION AND ANALYSIS

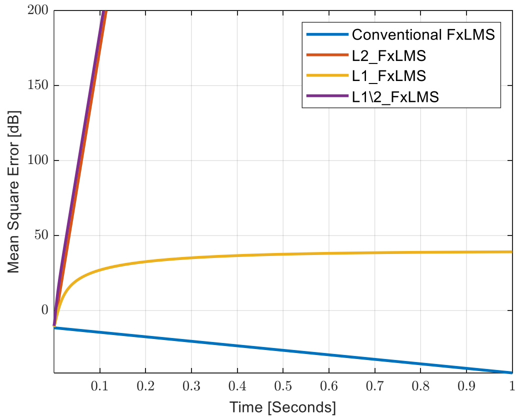
For boundary validation, s simple four channel decentralized ANC system is established to compare the best noise reduction performance and convergence behavior of four kind of FXLMS algorithms. There are one reference microphone, four secondary loudspeakers, and four error microphones, which means $K = M = 4$.
System parameters are defined as follows. The primary path, which is the acoustic delay from reference microphone to secondary path, is defined as $\boldsymbol{\Delta_P}$ with dimension of 4×1. The secondary path, which is the acoustic delay from error microphone to secondary loudspeaker, is defined as $\boldsymbol{\Delta_S}$. The sampling rate is 10 kHz. The primary path and secondary path are set as $\boldsymbol{\Delta_P} = [15\ 15\ 15\ 15]$, $\boldsymbol{\Delta_S} = \begin{bmatrix} 2 & 4 & 8 & 4 \\ 4 & 2 & 4 & 8 \\ 8 & 4 & 2 & 4 \\ 4 & 8 & 4 & 2 \end{bmatrix}$. The complex response from the secondary sources to error microphones $\boldsymbol{S}(\omega_n)$ and the complex response from the reference microphone to error microphones $\boldsymbol{P}(\omega_n)$ could be calculated out based on the acoustic wave propagation equation

$$p(r, k) = \frac{A}{r} e^{\pm ikr} \tag{36}$$

where A is the amplitude of input signal, r is the acoustic delay, and k is the wavenumber of input signal. In the following, the single frequency signal $f = 1000Hz$, which is the edge frequency of active noise control frequency area, is adopted as the reference input signal. The amplitude of the input signal is set as 4. Uniformly, we set the effort coefficient for L1-norm and L2-norm leakage as 0.1 in the three leaky FXLMS algorithms. Thus, we substitute all the variables in Equation 29 and 30, the convergence boundary of step size for conventional FXLMS and L2-FXLMS algorithms are respectively: $\mu_c = 0.2199$, $\mu_2 = 0.2175$, Several simulations are conducted to validate these boundaries. We gradually change the value of step size to manually get the convergence boundary for L1-FXLMS and L1/2-FXLMS: $\mu_1 = 0.2181$, $\mu_2 = 0.2158$. All the simulation results are presented in Figure 3. Conclusions for sine tonal noise can be drawn as: the robustness of these four FXLMS algorithms ranks: L1/2-FXLMS<L2-FXLMS<L1-FXLMS<Conventional FXLMS; the noise reduction performance of these four FXLMS algorithms ranks: Conventional FXLMS>L1-FXLMS>L2-FXLMS>L1/2-FXLMS.



(a)



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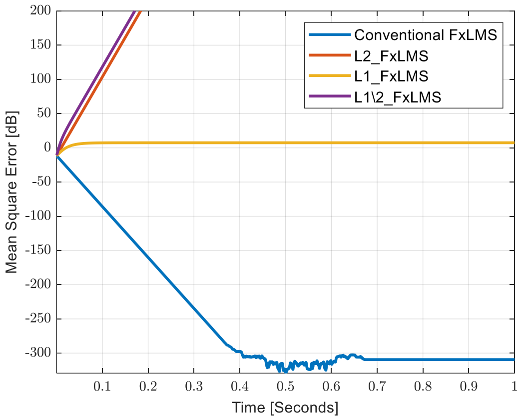
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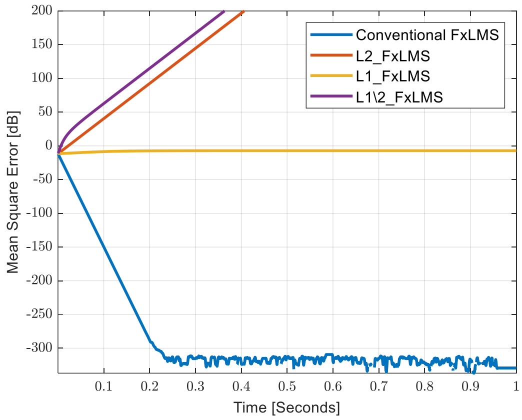
L2_FxLMS
L1_FxLMS
L1\2_FxLMS
Mean Square Error [dB]
Time [Seconds]

(c)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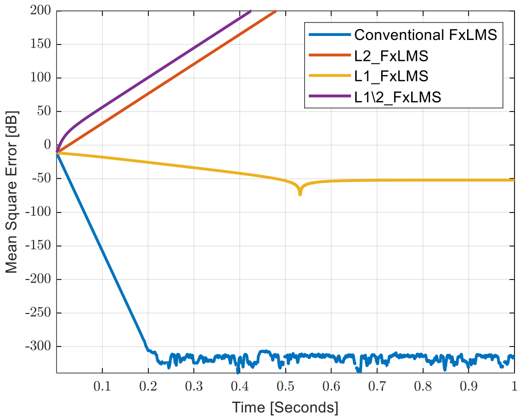
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Time [Seconds]

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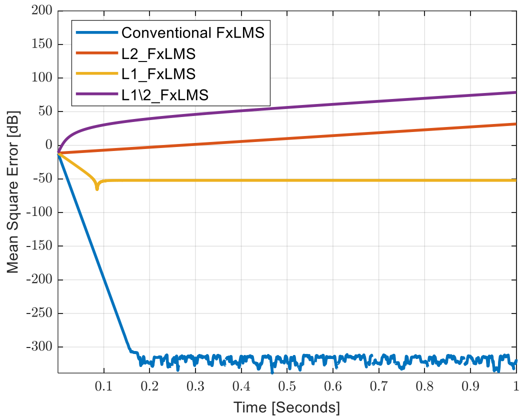
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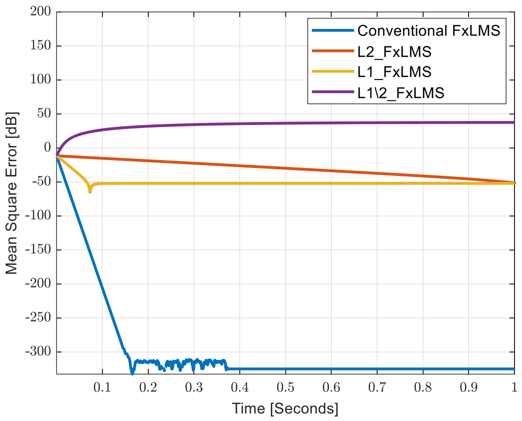
Mean Square Error [dB]
Time [Seconds]

(f)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Mean Square Error [dB]
Time [Seconds]

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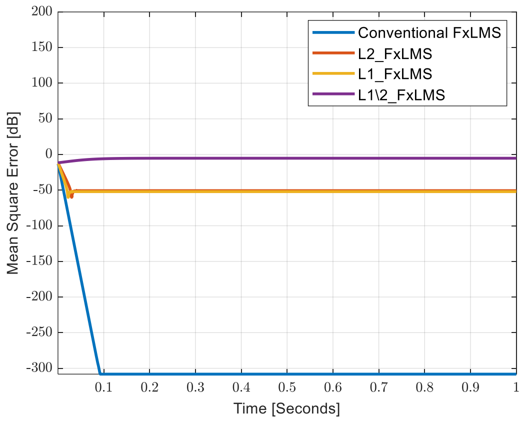
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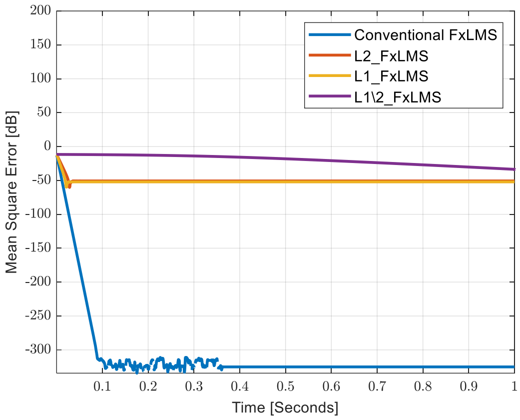
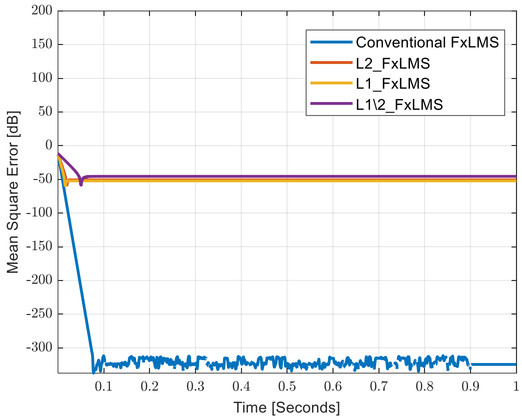
Mean Square Error [dB]
Time [Seconds]

(h)



(i)

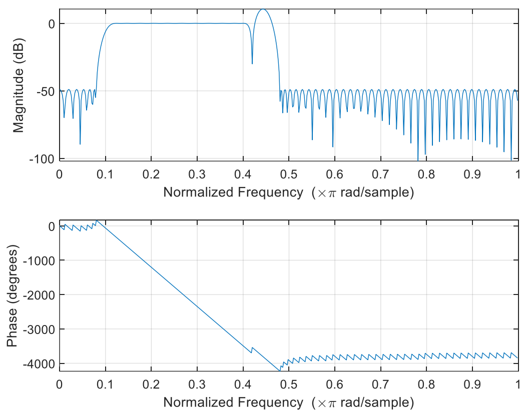


(j)

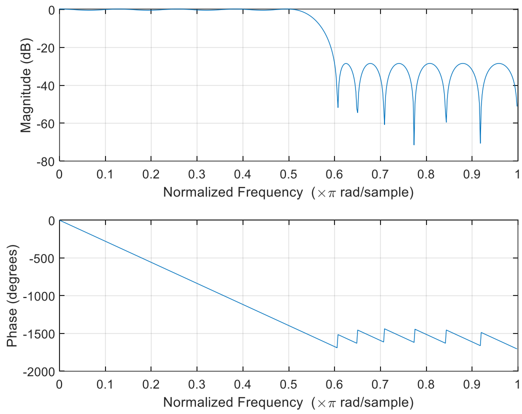
Figure 3: Noise reduction of four FXLMS algorithms

when (a) μ=0.22 ;(b) μ=0.2199; (c) μ=0.219 ; (d) μ=0.2182; (e) μ=0.2181 ; (f) μ=0.2176; (g) μ=0.2175 ;(h) μ=0.2159;(i) μ=0.2158 ;(j) μ=0.215.

For broadband noise, a single channel ANC system is established to compare the best noise reduction performance and convergence behavior of four kind of FXLMS algorithms. The impulse responses of the primary and secondary paths are plotted in Figure 4. Other general system parameters are set as: the filter length $L = 512$, and sampling frequency $f_s = 10\ kHz$. Uniformly, we set the effort coefficient for L1-norm and L2-norm leakage as 0.1 in the three leaky FXLMS algorithms.



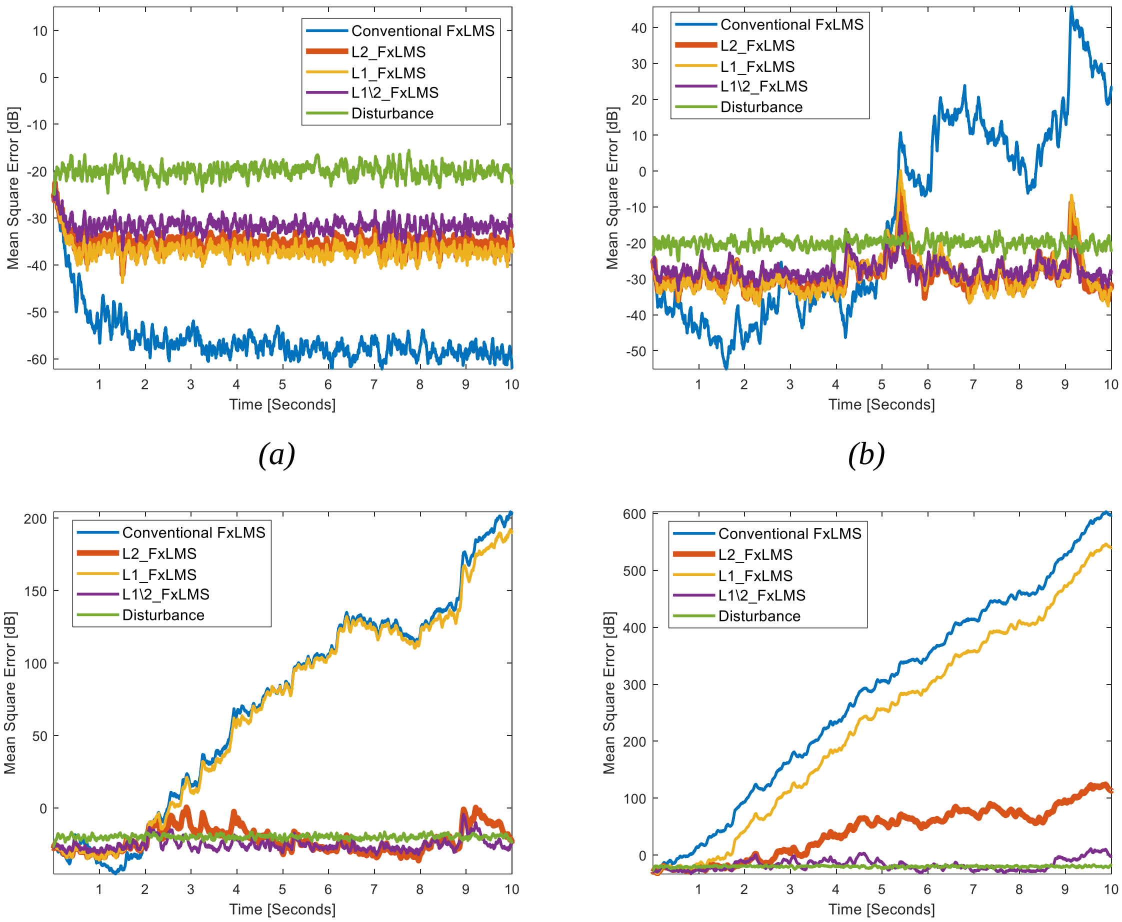
(a)



(b)

Figure 4: The impulsive response of (a) primary path and (b) secondary path

The above analysis about the convergence behavior is in frequency domain. And the detailed boundary for step size is only derived for single frequency component. $\lambda_{c,k}$ is the eigenvalue of $\boldsymbol{X}_s^H\boldsymbol{X}_s$ which is together determined by the input signal and the secondary path. However, the advantage of adding leaky term to the cost function of FxLMS algorithm is to speed up the very slow modes of convergence. Therefore, the precondition to have the advantage of leaky FxLMS algorithm is the input noise signal contains multiple modes. For the following simulations, we assume that the input noise is the WGN noise with frequency range from 200 to 800 Hz. For single frequency component, we could derive the exact boundary for the step size, but for broad band signal, it would be impossible to derive the exact boundary for the complex signal compounded of multiple modes. We gradually adjust the step size during simulation, four main results are extracted and presented in Figure 5.



(a) *(b)*

(c) *(d)*

Figure 5: Noise reduction of four FXLMS algorithms
when (a) μ=0.002, (b) μ=0.0052, (c) μ=0.0055 and (d) μ=0.0059

Simulation results in Figure 5 match with the theoretical boundary derivations. The robustness of the newly proposed L1/2-FXLMS is the highest. However, if convergence is guaranteed, the final noise reduction performance is not as good as the rest three FxLMS algorithms. There is a compromise between robustness and noise reduction performance.

6. CONCLUSIONS

The L1-FXLMS algorithm is revised to derive the precise version. The idea of elastic net is applied in ANC system, and L1/2-FXLMS is formed in this paper. The convergence behavior of four kinds of FXLMS algorithms are compared in detail. Simulation results proves that for sine tonal noise, the robustness of these four FXLMS algorithms ranks as follows: L1/2-FXLMS<L2-FXLMS<L1-FXLMS<Conventional FXLMS; the noise reduction performance of these four FXLMS algorithms ranks: Conventional FXLMS>L1-FXLMS>L2-FXLMS>L1/2-FXLMS. However, if the input noise is broadband noise, Conventional FXLMS <L1-FXLMS<L2-FXLMS< L1/2-FXLMS; the noise reduction performance of these four FXLMS algorithms ranks: Conventional FXLMS>L1-FXLMS>L2-FXLMS>L1/2-FXLMS. The convergence boundary of step size for L1/2-FXLMS could be larger, and the robustness could be improved, by adjusting the effort coefficient of L1 and L2 norm leakage.

7. ACKNOWLEDGEMENTS

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