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Reverse Mathematics in Lattice Theory

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2018

Acknowledgement

I thank Professor Wu Guohua, who has supervised me in those four years. He offers me a free environment to learn different topics in mathematics.

I also want to thank my office colleagues and friends Hong Nankun, Yuan Bowen, Wu Huishan, Huzhang Guangda, Zhang Zhibin and Quek Jiahao.

I sincerely thank my girl friend, Pan Hui, for her sustained encouragement, patient waiting and pure love.

Contents

Abstract

Reverse mathematics is a program of determining which axioms are required to prove theorems of mathematics. This thesis is devoted to the study of reverse mathematics in lattice theory, where several theorems and existence of objects of countable lattices are established from a reverse mathematics point of view. We will mainly focus on the following three subsystems: RCA_0, WKL_0 and ACA_0.

In Chapter 2, we first introduce three kinds of ideals in lattices: prime ideals, maximal ideals, relatively maximal ideals, respectively, and then we will investigate the logical strength of the existence of these ideals.

Theorem 2.2.5 (RCA_0) Every countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$ contains a maximal ideal.

Theorem 2.3.4 (RCA_0) Every countable distributive lattice $(L, \preceq, \vee, \wedge, 0, 1)$ contains a prime ideal.

We also study the relations among these ideals over RCA_0.

Theorem 2.2.6 (RCA_0) For a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$,

(1) Every maximal ideal is relatively maximal, but not conversely.

(2) Every prime ideal, if it exists, is relatively maximal, but not conversely.

Theorem 2.3.2 (RCA_0) For a countable distributive lattice $(L, \preceq, \vee, \wedge, 0, 1)$, an ideal of L is prime if and only if it is relatively maximal.

It is well-known that a lattice L is distributive if and only if neither M_3 nor N_5 can be embedded into L. It is also known that L is distributive if and only if every relatively maximal ideal in L is prime. We will prove in Chapter 2 that both

statements can be proved in RCA_0.

In mathematics, a representation theorem states that every abstract structure with certain properties is isomorphic to another structure, which is easy to understand. Because distributivity always holds for the set operations $\cup$ and $\cap$, it gave a motivation to construct isomorphisms between distributive lattices and certain spaces of sets. The first such representation theorem is Stone's Representation Theorem, which states that every Boolean algebra is isomorphic to a certain field of sets. For distributive lattices, we have Birkhoff's Representation Theorem (for finite distributive lattices) and Priestley's Representation Theorem (for all bounded distributive lattices).

In Chapter 2, we will show that Birkhoff's Representation Theorem can be proved over RCA_0.

In the proof of Stone's Representation Theorem (also in Priestley's Representation Theorem), the following separation property plays a central role.

Lemma 1.5.12 Let L be a distributive lattice, and $a, b \in L$ with $a \npreceq b$. Then there exists a prime ideal I such that $a \notin I$ and $b \in I$.

In Chapter 3, we will show that the separation property above (for countable distributive lattices) can be proved within WKL_0. Actually, we will prove a strong version.

Theorem 3.2.1(WKL_0) For a countable distributive lattice $(L, \preceq, \vee, \wedge, 0, 1)$, I an ideal, F a filter of L with $I \cap F = \emptyset$, there exists a prime ideal P containing I but disjoint from F.

This strong version is denoted by **DPI**. For general lattices, **DPI** is not guaranteed, and we can have a similar separation property, denoted by **RMI**, by using relatively maximal ideals. We will show in Chapter 4 that **RMI** can be proved over ACA_0.

Theorem 4.2.2 (ACA_0) In a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$, for an ideal I and a filter F of L, if $I \cap F = \emptyset$, there exists an ideal M which is maximal among those ideals containing I but disjoint from F.

In the proof of Birkhoff's Representation Theorem, the most important component is $\mathcal{J}(L)$, the set of join-irreducible elements. In Chapter 4, we will prove that for a countable lattice L, the existence of $\mathcal{J}(L)$ is equivalent to ACA_0 over RCA_0 (Theorem 4.4.1).

As a branch of order theory, domain theory studies directed complete partially ordered sets and domains, which formalize the intuitive ideas of approximation and convergence in a general way. In this area, Rudin's Lemma, a pure result of partially ordered sets, provides many applications in the study of quasi-continuous domains. In Chapter 5, we study Rudin's Lemma from a reverse mathematics point of view.

Theorem 5.3.1 The following are equivalent over RCA_0:

(1) ACA_0;

(2) Rudin's Lemma: Given a countable poset $(\mathrm{X},\preceq)$, and $\mathcal{F} = \langle F_i : i \in \mathbb{N}\rangle$ a $\sqsubseteq$-directed family of nonempty finite subsets of X, there exists a $\preceq$-directed subset $D \subseteq \bigcup\limits_{i\in\mathbb{N}} F_i$ that meets all F_i.

Chapter 1

Introduction

1.1 Background

The reverse mathematics in partially ordered sets has been studied in the last few years, in [14], [22], etc. Frittaion and Marcone studied scattered partial orders and partial orders satisfying finite antichain condition from a reverse mathematics point of view, and Lempp and Mummert explored the problem of constructing maximal filters on countable partially ordered sets. Brodhead, Khan, Kjos-Hanssen, Lampe, Nguyen and Shore [2] proved that the completeness of computable lattices are Π^1_1-complete. This thesis is a follow-up of this existing work, and we will focus on the reverse mathematics of theorems in lattice theory, including the existence of various ideals, separation properties and some basics of domain theory.

1.2 Reverse Mathematics

Second-order arithmetic, introduced by Hilbert and Bernays in [18] in the 1920's, is a collection of axiomatic systems that formalize the natural numbers and their subsets, allowing quantification over natural numbers and sets of natural numbers as well. Their purpose was to find a logic for analysis. In his PhD thesis [11], Friedman started to consider subsystems of second-order arithmetic, aiming to prove the necessary use of strong set-theoretic assumptions for ordinary mathematics.

Friedman presented his program of reverse mathematics in the ICM talk in 1974 (published in [12]), and this program has been further developed by Friedman, Simpson and many other mathematicians. Nowadays, reverse mathematics is an active field of research in logic. Two good references for reverse mathematics are Simpson's book [29] and Hirschfeldt's monograph [19].

The main question of reverse mathematics is: *Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?*

Here, ordinary mathematics means that it is independent of the introduction of abstract set-theoretic concepts, such as calculus, differential equations, real analysis, countable algebra, the topology of complete separable metric spaces, mathematical logic, and so on. The point is that in ordinary mathematics, sets are restricted to be countable-based (for example, the real line is uncountable, but it is a separable metric space), and the reason for this restriction is that set existence axioms needed for uncountable, set-theoretic mathematics are likely to be much stronger than those which are needed for ordinary mathematics.

In reverse mathematics, we work with subsystems of second-order arithmetic. Given a theorem γ, we want to find the weakest system S that suffices to prove γ, which means that S can prove γ and γ can also prove all axioms of S over a weaker system S_0. We will then say that γ is equivalent to S over S_0, and S is exactly the weakest system to prove theorem γ.

Over the past decades, theorems from different topics have been studied in reverse mathematics, and many of them happen to be equivalent to one of the five main subsystems, i.e., RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0, over RCA_0, where each subsystem contains basic axioms, induction axioms and some specific comprehension scheme. The following definitions are from Simpson's book [29].

Definition 1.2.1. *The axioms of second-order arithmetic consist of the universal closures of the following L_2-formulas (L_2 is the language of second-order arithmetic):*

(1) Basic axioms:

$$\begin{aligned}
&n+1 \neq 0,\\
&m+1 = n+1 \to m = n,\\
&m+0 = m,\\
&m+(n+1) = (m+n)+1,\\
&m \cdot 0 = 0,\\
&m \cdot (n+1) = (m \cdot n) + m,\\
&\neg m < 0,\\
&m < n+1 \longleftrightarrow (m < n \ \vee \ m = n).
\end{aligned}$$

(2) Induction axiom:

$$(0 \in X \wedge \forall n \ (n \in X \to n+1 \in X)) \to \forall n \ (n \in X).$$

(3) Comprehension scheme:

$$\exists X \ \forall n \ (n \in X \leftrightarrow \ \varphi(n)),$$

where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.
(4) By second-order arithmetic, denoted as Z_2, we mean the formal system in the language L_2 which are deducible from those axioms by means of the usual logical axioms and rules of inference.

By *a subsystem* of Z_2, we mean a formal system in the language L_2, each of whose axioms is a theorem of Z_2. In this thesis, we will be mainly focused on three subsystems: RCA_0, WKL_0 and ACA_0.

Definition 1.2.2. *(1) The system RCA_0 consists of the basic axioms, together with schemes of Σ^0_1 induction and Δ^0_1 comprehension, i.e.,*

$$\forall n(\phi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \phi(n)),$$

where $\phi(n)$ is any Σ^0_1 formula, $\psi(n)$ is any Π^0_1 formula.

(2) The system WKL_0 consists of RCA_0 and weak König's lemma: Every infinite subtree of $2^{<\omega}$ has an infinite path.

(3) The system ACA_0 consists of the basic axioms and the induction axiom, together with arithmetic comprehension, i.e., $\varphi(n)$ is an arithmetic formula in the comprehension scheme.

Proposition 1.2.3. *RCA_0 proves bounded Σ^0_1 comprehension:*

$$\forall n \exists X \forall i (i \in X \iff i < n \ \& \ \phi(i)),$$

where $\phi(i)$ is any Σ^0_1 formula in which X does not occur freely.

The following two theorems provide useful equivalences of ACA_0 and WKL_0, respectively. When we try to show the reversal parts, we often show that one of the equivalences can be proved.

Proposition 1.2.4. *The following are equivalent over RCA_0 :*

(1) WKL_0 : Every infinite subtree of $2^{<\omega}$ has an infinite path;

(2) If $f, g : \mathbb{N} \to \mathbb{N}$ are one-to-one functions with $\forall m \forall n f(m) \neq g(n)$, then there exists a set $X \subseteq \mathbb{N}$ such that

$$\forall n (f(n) \in X \ \& \ g(n) \notin X).$$

Proposition 1.2.5. *The following are equivalent over RCA_0 :*

(1) ACA_0;

(2) For every one-to-one function $f : \mathbb{N} \to \mathbb{N}$, there exists a set $X \subseteq \mathbb{N}$ such that

$$\forall n (n \in X \iff \exists m (f(m) = n));$$

(3) König lemma: Every infinite, finitely branching tree $T \subseteq \omega^{<\omega}$ has an infinite path.

1.3 Reverse mathematics in partially ordered sets

In this section, we give a brief review of results of existing work of reverse mathematics related to this thesis.

Definition 1.3.1. *A partially ordered set (poset for short) consists of a set* P *with a binary relation* $\preceq$ *such that the system* $(P, \preceq)$ *satisfies the following axioms:*

(1) reflexivity: $(\forall x \in P)(x \preceq x)$,

(2) anti-symmetricity: $(\forall x, y \in P)((x \preceq y) \wedge (y \preceq x) \rightarrow x = y)$,

(3) transitivity: $(\forall x, y, z \in P)((x \preceq y) \wedge (y \preceq z) \rightarrow x \preceq z)$.

We denote $x \prec y$ *if* $x \preceq y$ *but* $x \neq y$.

For a poset $(P, \preceq)$, let $\uparrow A$ denote the set $\{x \in P : \exists a \in A\ (x \succeq a)\}$ and call it the *upward closure* of A. A is called an *upper set* if $A = \uparrow A$. Note that $\uparrow A$ is the smallest upper set containing A. *Lower sets* are defined dually. In his thesis, Mummert proved that the existence of upward closure requires ACA_0.

Theorem 1.3.2. *(Mummert [25]) The following are equivalent over* RCA_0*:*

(1) ACA_0;

(2) Every subset of a countable poset has an upward closure.

We call a poset $(P, \preceq)$ *scattered* if $(\mathbb{Q}, \leq)$, i.e., the set of rational numbers with the standard order, doesn't embed into $(P, \preceq)$. The following connection is due to Bonnet.

Theorem 1.3.3. *(Bonnet [1]) If* P *is an infinite, scattered poset with no infinite antichains, then the set of lower sets of* P *has the same cardinality as* P.

The converse of above result may not hold in general. Frittaion and Marcone [14] proved that the converse holds when P is countable.

Theorem 1.3.4. *A countable poset is scattered and has no infinite antichain if and only if it has countably many lower sets.*

Frittaion and Marcone considered individual implications in this theorem and show that:

Theorem 1.3.5. *(Frittaion and Marcone [14])*

(1) (RCA_0) Every countable poset with countably many lower sets is scattered.

(2) (WKL_0) Every countable poset with countably many lower sets has no infinite antichains.

(3) The statement in (2) cannot be proved within RCA_0.

A question left open from [14] is whether the statement in (2) is equivalent to WKL_0 over RCA_0.

Theorem 1.3.6. *(Frittaion and Marcone [14]) The following are equivalent over ACA_0:*

(1) ATR_0;

(2) Every countable scattered poset with no infinite antichain has countably many lower sets.

Compared with upper sets, filters of posets have more structure. F, a nontrivial subset of a poset $(P, \preceq)$, is a *filter* if F is an upper set, and for every two elements a, b in F, there exists an element $c \in F$ such that $c \preceq a, b$. *Ideals* are defined dually. We call a filter maximal if it is not contained in a strictly larger filter. The reverse mathematics of the existence of maximal filters in posets was studied by Lempp and Mummert:

Theorem 1.3.7. *(Lempp and Mummert [22]) The following are equivalent over RCA_0:*

(1) ACA_0;

(2) Every countable poset has a maximal filter.

The extension of filters is more complicated than the existence. Mummert shows that "any filter in a poset can be extended to a maximal filter" is equivalent to Π^1_1-CA$_0$ over RCA$_0$:

Theorem 1.3.8. *(Mummert [25]) The following are equivalent over RCA$_0$:*

(1) Π^1_1*-CA$_0$;*

(2) Every filter of a countable poset extends to a maximal filter.

A subset $A \subseteq P$ is an *antichain* if elements in A are pairwise incomparable. A subset $S \subseteq P$ is a *strong antichain* if elements in S are pairwise incompatible in P, i.e., any two elements of S have no common upper bound in P. Antichains and strong antichains of posets have a close relationship to ideals.

Theorem 1.3.9. *(1) (Bonnet [1]) A poset has no infinite antichains if and only if every lower set is a finite union of ideals.*

(2) (Erdös and Tarski [9]) A poset has no infinite strong antichains if and only if it is a finite union of ideals.

Frittaion considered the logical strength of them correspondingly in [14].

Theorem 1.3.10. *(RCA$_0$) If a countable poset is a finite union of ideals, then it has no infinite strong antichains.*

Theorem 1.3.11. *The following are equivalent over RCA$_0$:*

(1) ACA$_0$;

(2) If a countable poset has no infinite antichain, then every lower set is a finite union of ideals;

(3) If a countable poset has no infinite strong antichain, then it is a finite union of ideals.

Frittaion also proved that over WKL$_0$, if every lower set of a countable poset is a finite union of ideals, then it has no infinite antichains. A question left open is whether it is equivalent to WKL$_0$ over RCA$_0$.

For posets, Frittaion and Marcone [14] considered the reverse mathematics of the following separation property, an analogue of a property in distributive lattices in our consideration.

Theorem 1.3.12. *The following are equivalent over RCA_0:*

(1) WKL_0;

(2) Let A, B be subsets of a countable poset P such that for any $x \in A$, $y \in B$, $y \not\leq x$. Then there exists a lower set I of P such that $A \subseteq I$ and $B \cap I = \emptyset$.

1.4 Ideals in commutative rings

In this section, we will review some well-known results of the existence of various ideals in countable commutative rings from a reverse mathematics point of view. In this thesis,we will study the reverse mathematics of the existence of various ideals in countable lattices. While statements in these two areas are similar to each other, the reverse mathematics considered is quite different.

Friedman, Simpson and Smith considered in [13] the reverse mathematics of the existence of prime ideals and of maximal ideals.

Theorem 1.4.1. *The following are equivalent over RCA_0:*

(1) WKL_0;

(2) Every countable commutative ring with identity contains a prime ideal.

Theorem 1.4.2. *The following are equivalent over RCA_0:*

(1) ACA_0;

(2) Every countable commutative ring with identity contains a maximal ideal.

We know that a field is a commutative ring with only two ideals, the zero ideal and the field itself. Conversely, if a commutative ring R is not a field, then R has a nontrivial ideal. Downey, Lempp and Mileti proved in [8] that the existence of nontrivial proper ideals is equivalent to WKL_0 over RCA_0.

Theorem 1.4.3. *(Downey, Lempp and Mileti [8]) The following are equivalent over RCA_0:*

(1) WKL_0;

(2) Every countable commutative ring which isn't a field contains a nontrivial proper ideal.

This theorem says that for some computable commutative rings, it is impossible to construct a nontrival proper ideal effectively. Now, consider the following example. Let a be a nonzero and nonunit element, then $I = (a)$ is a principal ideal which is nontrival and proper, and $b \in I \iff \exists r \in R\ (r \cdot a = b)$. Though this ideal is very simple, and is generated by one element, we still need Σ^0_1-comprehension to obtain the existence.

Theorem 1.4.4. *(Downey, Lempp and Mileti [8]) The following are equivalent over RCA_0:*

(1) ACA_0;

(2) Every countable commutative ring which isn't a field contains a nontrivial proper principal ideal;

(3) Every countable commutative ring which isn't a field contains a nontrivial proper finitely generated ideal;

In a commutative ring, there are several ways to define maximal ideals. When we define a maximal ideal M of R over RCA_0, we will use the following definition:

$$M \text{ is an ideal } \& \ \forall r((r \in R \ \& \ r \notin M) \to \exists s(s \in M \ \& \ r \cdot s - 1 \in M)).$$

RCA_0 can prove the equivalence between this definition and R/M is a field.

1.5 Lattice Theory

Lattice theory has been studied in many mathematical subdisciplines. It is an outgrowth of the study of Boolean algebras, and provides a framework for unifying the study of classes of ordered sets in mathematics. It can be characterized as partially ordered structure or algebraic structure theory.

From an order theory point of view, a *lattice* is a partially ordered set $(L, \preceq)$ such that any two elements of L have a least upper bound and a greatest lower bound with respect to $\preceq$. It means that for every two elements a, b in L, there exist two elements c and d in L such that: (1) $c \preceq a, b$; (2) $a, b \preceq d$; (3) $\forall x \in L((a, b \preceq x) \Rightarrow d \preceq x)$; (4) $\forall x \in L((a, b \succeq x) \Rightarrow c \succeq x)$. We use $a \vee b$ and $a \wedge b$ to denote the least upper bound and the greatest lower bound of a and b, respectively, and we call them the join and the meet of a, b correspondingly. A *bounded lattice* is a lattice with a greatest element 1 and a least element 0.

From an abstract algebra point of view, a *lattice* is an algebraic structure $(L, \vee, \wedge)$ where $\vee$ and $\wedge$ are two binary operations satisfying commutative laws, associative laws and absorption laws, i.e., for all a, b in L, $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$. A bounded lattice is a lattice with 0, the identity element for $\vee$, and 1, the identity element for $\wedge$.

The two definitions above are equivalent. The order-theoretic definition of a lattice $(L, \preceq)$ gives rise to two binary operations $\vee$ and $\wedge$ satisfying the three laws above, which makes $(L, \vee, \wedge)$ into a lattice in the algebraic sense. Conversely, given an algebraic structure $(L, \vee, \wedge)$, we can define a partial order $\preceq$ on L by setting that for all $a, b \in L$, $a \preceq b \iff a \vee b = b \iff a \wedge b = a$, and the relation $\preceq$ introduced defines a partial order such that the original operations $\vee$ and $\wedge$ are exactly the meet and join w.r.t. $\preceq$.

In this thesis, we use the order-theoretic definition of lattice $(L, \preceq)$ and we assume that the join and meet operations are given. Also when we say a countable lattice, we always mean a countable bounded lattice $(L, \preceq, \vee, \wedge, 0, 1)$. This approach is used in Brodhead, Khan, Kjos-Hanssen, Lampe, Nguyen and Shore's paper [2].

1.5.1 Complete Lattices

A lattice $(L, \preceq, \vee, \wedge, 0, 1)$ is *complete* if for each subset $S \subseteq L$, S has a least upper bound and a greatest lower bound, i.e., there exist two elements $c, d \in L$ such that: (1) $\forall x \in S\ (c \preceq x \preceq d)$; (2) $(\forall a \in L)(\forall x \in S(x \preceq a) \Rightarrow d \preceq a)$; (3) $(\forall a \in L)(\forall x \in S(x \succeq a) \Rightarrow c \succeq a)$. We use $\bigwedge S$ and $\bigvee S$ to denote the greatest lower bound and the least upper bound of S, respectively. Complete lattices must be bounded since $\bigvee L$ and $\bigwedge L$ are the greatest element and least element, respectively.

Brodhead, Khan, Kjos-Hanssen, Lampe, Nguyen and Shore proved in [2] that it is Π^1_1-hard to verify whether a computable lattice is complete or not.

Theorem 1.5.1. *(Brodhead, Khan, Kjos-Hanssen, Lampe, Nguyen and Shore [2]) The set of indices of computable lattices that are complete is* Π^1_1*-complete.*

Sato and Yamazaki [27] gives the strength of the statement "every complete semilattice is a complete lattice".

Theorem 1.5.2. *(Sato and Yamazaki [27]) The following are equivalent over* RCA_0*:*

(1) ACA_0*;*

(2) For a countable lattice L, if it is a complete join semilattice, i.e., for any subset S of L, $\bigvee S$ *exists, then L is a complete lattice;*

(3) For a countable lattice L, if it is a complete meet semilattice i.e., for any subset S of L, $\bigwedge S$ *exists, then L is a complete lattice;*

In the proof of the direction from (1) to (2), we really need that the lattice L is bounded. For a given subset $S \subseteq L$, we will construct $S' = \{x \in L : \forall y \in S\ (x \preceq y)\}$ by arithmetical comprehension, and argue that $\bigvee S' = \bigwedge S$. But here we need L to contain the least element 0 to ensure that S' is nonempty.

In their proof, Sato and Yamazaki introduced the technical notion of α-true stage, which helps to handle the reversal part of a given statement about lattices to ACA_0. Sato and Yamazaki also obtain the strength of the Knaster-Tarski fixed point theorem and its converse:

Theorem 1.5.3. *(Sato and Yamazaki [27]) (RCA_0) Let L be a countable complete lattice. If $F : L \to L$ is order-preserving, then F has a fixed point. Furthermore, all fixed points of F form a countable complete lattice.*

Theorem 1.5.4. *(Sato and Yamazaki [27]) The following are equivalent over RCA_0:*

(1) ACA_0;

(2) If L is a countable lattice such that every order-preserving mapping $F : L \to L$ has a fixed point, then L is complete.

In Chapter 4, we will consider the connection between complete lattices and the ascending chain condition (ACC for short).

Theorem 4.3.4 (ACA_0) If a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$ satisfies ACC, then for every subset S of L, $\bigvee S$ exists. Furthermore, L is a complete lattice.

1.5.2 Notions in Lattices

Recall that a nontrivial subset F of a poset $(P, \preceq)$ is a filter if F is an upper set, and for every two elements a, b in F, there exists a common lower bound of a, b in F. If we apply this definition to a lattice $(L, \preceq, \vee, \wedge, 0, 1)$, then the existence of a common lower bound of a, b in F is equivalent to $a \wedge b \in F$ because F is an upper set. Hence in a lattice $(L, \preceq, \vee, \wedge, 0, 1)$, a nontrivial subset F is a filter if F is an upper set and it is closed under meet, i.e., for all a, b in F, $a \wedge b$ is also in F. Similarly, in a lattice $(L, \preceq, \vee, \wedge, 0, 1)$, a nontrivial subset I is an ideal if I is a lower set and it is closed under join, i.e., for all a, b in I, $a \vee b$ is also in I.

Now, we recall some basic notions of lattices, which will be used in the remainder of the introduction.

Definition 1.5.5. *Let $(L, \preceq, \vee, \wedge, 0, 1)$ be a countable lattice and S be a nonempty subset of L. S is a sublattice of L, if for all a, b in S, $a \vee b$ and $a \wedge b$ are also in S.*

Definition 1.5.6. *Let $(L, \preceq, \vee, \wedge, 0, 1)$ be a countable lattice and I be a ideal of L.*

(1) I is a prime ideal in L, if for all a, b in L, $a \wedge b \in I \Rightarrow a \in I$ or $b \in I$;

(2) I is a maximal ideal in L, if for all a in L, $a \notin I \Rightarrow \exists b \in I\ (a \vee b = 1)$.

Definition 1.5.7. *Let $(L, \preceq, \vee, \wedge, 0, 1)$ be a countable lattice and S be a sublattice of L. Call an ideal I relatively maximal with respect to S, if (i) $I \cap S = \emptyset$ and (ii) $\forall a \in L\ (a \notin I \Rightarrow \exists b \in I\ \exists c \in S\ (a \vee b \succeq c))$.*

Note that if I is a relatively maximal ideal w.r.t. S, then there is no ideal properly containing I and disjoint from S. We call an ideal I *relatively maximal* if there is some sublattice S such that I is relatively maximal w.r.t. S.

Definition 1.5.8. *Let $(P, \preceq_P)$ and $(Q, \preceq_Q)$ be countable posets. A mapping $\varphi : P \longrightarrow Q$ is an order-isomorphism, if*

(1) φ is an order-embedding: for all x, y in P, $x \preceq_P y \iff \varphi(x) \preceq_Q \varphi(y)$;

(2) φ maps P onto Q.

Obviously, an order-isomorphism is one-to-one.

Definition 1.5.9. *Let $(L, \preceq_L, \vee_L, \wedge_L, 0_L, 1_L)$ and $(K, \preceq_K, \vee_K, \wedge_K, 0_K, 1_K)$ be countable lattices. A mapping $\eta : L \longrightarrow K$ is a lattice embedding if*

(1) η is a lattice homomorphism: for all x, y in L, $\eta(x \vee_L y) = \eta(x) \vee_K \eta(y)$ and $\eta(x \wedge_L y) = \eta(x) \wedge_K \eta(y)$;

(2) η is one-to-one: for all x, y in L, $x \neq y \Rightarrow \eta(x) \neq \eta(y)$.

It is easy to see that if $\eta : L \longrightarrow K$ is a lattice embedding, then $\eta(L)$ is a sublattice of K.

Definition 1.5.10. *Let $(L, \preceq_L, \vee_L, \wedge_L, 0_L, 1_L)$ and $(K, \preceq_K, \vee_K, \wedge_K, 0_K, 1_K)$ be countable lattices. A mapping $\eta : L \longrightarrow K$ is a lattice isomorphism, if (i) η is a lattice homomorphism; (ii) η is a bijection.*

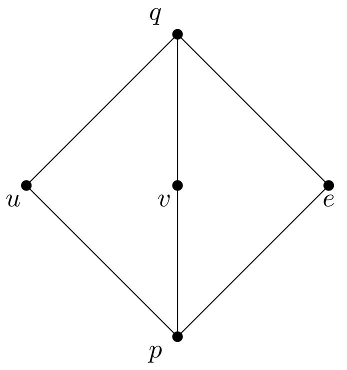


Figure 1.1: M_3

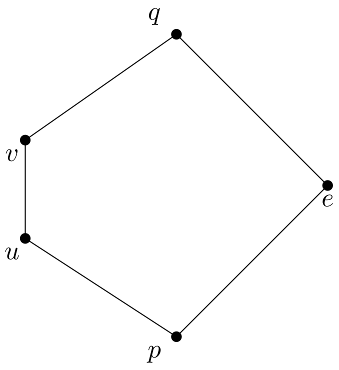


Figure 1.2: N_5

The notions of order-isomorphisms and lattice isomorphisms will be used in the representation theorems.

1.5.3 Distributive Lattices

A lattice $(L, \preceq, \vee, \wedge, 0, 1)$ is *distributive* if for all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. The distributive law above can be replaced by the dual version: for all $a, b, c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. Dedekind [7] and Birkhoff [3] gave a sufficient and necessary condition for a lattice to be distributive.

Dedekind-Birkhoff Theorem : A lattice L is distributive if and only if neither M_3 (see Figure 1.1) nor N_5 (see Figure 1.2) can be embedded into L.

In Chapter 2, we will prove the Dedekind-Birkhoff theorem for countable lattices within RCA_0.

For a lattice L, $a \in L$ is *join-irreducible* if $a \neq 0$ and $a = b \vee c$ implies $a = b$ or $a = c$. We use $\mathcal{J}(L)$ to denote the set of all join-irreducible elements of L, and $\mathcal{O}(\mathcal{J}(L))$ to denote the set of all lower sets of $\mathcal{J}(L)$ with respect to the order $\subseteq$.

Since the intersection and the union of sets satisfy the distributive law, with regard to the partial order $\subseteq$, we can try to obtain isomorphic copies of distributive lattices by using sets, which is the main theme of representation theorems.

Birkhoff's representation theorem. Let L be a finite distributive lattice. Then the map

$$\eta : L \longrightarrow \mathcal{O}(\mathcal{J}(L))$$

$$a \longmapsto \{x \in \mathcal{J}(L) : x \preceq a\}$$

is a lattice isomorphism.

The theorem says that finite distributive lattices mimic finite lower set lattices, where each lower set consists of join-irreducible elements. So for finite distributive lattices, $\mathcal{J}(L)$ is "big" enough to reflect the structure of L.

In the proof of Birkhoff's Representation Theorem, the following lemma about separation properties plays a critical role.

Lemma 1.5.11. *Let L be a finite lattice. Suppose that $a, b \in L$ with $a \npreceq b$, then there exists an $x \in \mathcal{J}(L)$ such that $x \preceq a$ and $x \npreceq b$.*

The lemma says that for any two distinct elements a and b, we can use a join-irreducible element to separate them to ensure that $\eta(a) \neq \eta(b)$. We call such a property "separation". In fact, this separation property is exactly what we need to guarantee that there are enough join-irreducible elements to make η an embedding.

Join-irreducible elements served well as building blocks for finite distributive lattices. But for an infinite distributive lattice L, $\mathcal{J}(L)$ may be empty (see the following example), and we need an alternative if we are to try the same thing for infinite distributive lattices.

Example: Let $L = 2^{<\omega} \cup \{\varepsilon\}$, i.e., the binary strings with a least element ε. Define the order $\preceq$ such that $\alpha \preceq \beta$ if $\beta \subset \alpha$, then the least element ε is the meet of any two incompatible strings, which guarantees $(L, \preceq)$ to be a lattice. Since each element (except ε) is a join of its successors, i.e., $\alpha = \alpha^\frown 0 \vee \alpha^\frown 1$, it is not join-irreducible.

For representations for infinite distributive lattices, two natural questions arise:

Question 1: Which alternatives shall we take to replace join-irreducible elements?

Question 2: Is there a similar property of "separation" for these alternatives?

Note that if x is a join-irreducible element, then $I = L \setminus {\uparrow} x = \{y \in L : x \not\preceq y\}$ is a prime ideal of L. Thus for $a, b \in L$ with $a \not\preceq b$, we can separate a from b by a prime ideal. It gives the motivation to replace the join-irreducible elements by prime ideals, leading to Priestley's Representation Theorem.

For a countable lattice L, let $\mathcal{I}_p(L)$ be the set of all prime ideals of L. For an ordered topological space $(X; \preceq, \tau)$, we use $\mathcal{O}^\tau(X)$ to denote the set of all clopen lower subsets of $(X; \preceq, \tau)$.

Priestley's Representation Theorem. Let L be a bounded distributive lattice. Then the map

$$\eta : L \longrightarrow \mathcal{O}^\tau(\mathcal{I}_p(L))$$

$$a \longmapsto \{I \in \mathcal{I}_p(L) : a \notin I\}$$

is a lattice isomorphism. Here, the order $\preceq$ on $\mathcal{I}_p(L)$ is set-inclusion, i.e., $I \preceq J \iff I \subseteq J$ for $I, J \in \mathcal{I}_p(L)$, and the topology τ on $\mathcal{I}_p(L)$ is generated by the topological subbasis $\{\eta(b) : b \in L\} \cup \{\mathcal{I}_p(L) \setminus \eta(c) : c \in L\}$.

So Question 1 is solved by using prime ideals to replace join-irreducible elements. Question 2 is solved by the following lemma.

Lemma 1.5.12. *For a distributive lattice L, if $a, b \in L$ with $a \not\preceq b$, then there exists a prime ideal I such that $a \notin I$ and $b \in I$.*

Again, this separation property is what we need to ensure that L has enough prime ideals for η to be an embedding.

In Chapter 2, we will show that RCA_0 is strong enough to prove Lemma 1.5.11 and Birkhoff's Representation Theorem. In Chapter 3, we will show that WKL_0 can prove Lemma 1.5.12 for countable lattices.

1.5.4 Boolean Algebras

For representations, we mention here a classic representation theorem, due to Stone, for Boolean algebras. In a bounded lattice, two elements x and y are complements of each other if $x \vee y = 1$ and $x \wedge y = 0$. A *boolean algebra* is a bounded distributive complemented lattice. Note that distributivity ensures that the complement of each element is unique.

In 1936, Stone [30] gave a representation theorem for Boolean algebras, providing a connection between Boolean algebras and topological spaces, now called Stone spaces. Stone's Representation Theorem is the first representation theorem which gave a duality between topological spaces and partially ordered sets.

Let $\mathcal{I}_p(B)$ be the space of all prime ideals of B. For a topological space $(X;\tau)$, we denote $\mathcal{P}^\tau(X)$ as the set of all clopen subsets of $(X;\tau)$.

Stone's Representation Theorem. Let B be a Boolean algebra. Then the map

$$\eta : B \longrightarrow \mathcal{P}^\tau(\mathcal{I}_p(B))$$

$$a \longmapsto \{I \in \mathcal{I}_p(B) : a \notin I\}$$

is a Boolean algebra isomorphism. Here, the topology τ on $\mathcal{I}_p(B)$ is generated by the topological basis $\{\eta(a) : a \in B\}$.

Again, Lemma 1.5.12 plays a central role in the proof of Stone's Representation Theorem. In Chapter 3, we will study a strong version of Lemma 1.5.12 and show that WKL_0 can prove it.

Theorem 3.2.1(WKL_0) For a countable distributive lattice $(L, \preceq, \vee, \wedge, 0, 1)$, I an ideal, F a filter of L with $I \cap F = \emptyset$, there exists a prime ideal P containing I but disjoint from F.

This strong version is denoted as **DPI**, which says that in distributive lattices, we can always separate an ideal and a filter, if they are disjoint, by a prime ideal. Such a prime ideal separation property is only true for distributive lattices, since in general, lattices may not have prime ideals. Instead, we will use relatively maximal ideals for such a separation, denoted as **RMI**, and in Chapter 4, we will show that **RMI** can be proved over ACA_0.

Theorem 4.2.2 (ACA_0) In a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$, for an ideal I and a filter F of L, if $I \cap F = \emptyset$, then there exists an ideal M which is maximal among those ideals containing I but disjoint from F.

1.5.5 Domain Theory

In this subsection, we introduce some basic notions of *dcpo* and *domains*, which will be the main topic in Chapter 5.

Definition 1.5.13. *Let* $(P, \preceq)$ *be a countable poset. A subset* D *of* P *is directed if for any* x, y *in* D*, there exists* $z \in D$ *such that* $x, y \preceq z$*.*

Note that a poset is a directed complete partial order, *dcpo* for short, if the join of each directed subset exists.

Sato and Yamazaki [27] studied some fixed point theorems for countable dcpos from a reverse mathematics point of view. The following theorems about fixed points can be proved over RCA_0:

(1) (Tarski-Kantorovitch fixed point theorem) Let P be a countable dcpo and $F : P \to P$ be a continuous mapping on P. Then the set of all fixed points of F is a dcpo.

(2) (Bourbaki-Witt fixed point theorem) Let P be a countable dcpo and $F : P \to P$ be an inflationary mapping on P. Then F has a maximal fixed point.

(3) (Abian-Brown fixed point theorem) Let P be a countable dcpo and $F : P \to P$ be an order-preserving mapping on P. Then F has a maximal fixed point.

Theorem 1.5.14. *(Sato and Yamazaki [27]) The following are equivalent over* RCA_0*:*

(1) ACA_0*;*

(2) If P *is a countable poset such that every order-preserving mapping* $F : P \to P$ *has a least fixed point, then* P *is a dcpo.*

In $\mathbb{R}$, the real space, with the standard Euclidean order $\leq$, we can approximate each real by some elements from $\mathbb{Q}$, which is a countable subset of $\mathbb{R}$. It gives us an idea of seeking approximations of elements in a poset by using “much simpler” elements. This leads to the class of continuous posets.

Definition 1.5.15. *Let $(P, \preceq)$ be a countable poset. Say that x is way-below y, $x \ll y$, if for all directed subsets $D \subseteq P$ for which $\bigvee D$ exists and $y \preceq \bigvee D$, there is an element $d \in D$ with $x \preceq d$.*

A poset L is called *continuous* if every element can be obtained as the join of a directed set of elements that are way-below this element.

A dcpo which is continuous as a poset will be called a *domain.* [15] provides a good reference for domain theory. The basic idea of domains was proposed independently by Scott (domains) in paper [28] and Ershov (f_0-spaces) in paper [10] around the 1970's. Scott's domains are used to investigate denotational semantics of lambda calculus, while the basic idea of Ershov's f_0-spaces is to generalize the notion of approximation by finite objects. In the development of domain theory, various topologies on partially ordered sets are studied , such as the Alexandrov topology, the Scott topology, and the Lawson topology, etc. In search of a suitable complete lattice on which the Lawson topology is exactly Hausdorff, Gierz, Lawson and Stralka introduced quasicontinuous domains in [16].

As pointed out in Mummert's thesis [25], there has been little work to determine which set existence axioms are required to prove theorems of domain theory. In this thesis, in Chapter 5, we study Rudin's Lemma, a crucial lemma in the development of quasicontinuous domains, from a reverse mathematics point of view. Here, quasicontinuous domains generalized domains with many good properties of domains remaining valid, such like the interpolation property, the soberness of its Scott topology. We will not introduce these properties here, as our focus will be on the Rudin's Lemma.

Definition 1.5.16. *Let $(X, \preceq)$ be a countable poset. For $A, B \subseteq X$, define $A \sqsubseteq B$ if $\uparrow B \subseteq \uparrow A$. The relation $\sqsubseteq$ on the power set of X is called the Smyth preorder.*

Theorem 5.3.1 The following are equivalent over RCA_0:

(1) ACA_0;

(2) Rudin's Lemma: Given a countable poset (X,$\preceq$), and $\mathcal{F} = \langle F_i : i \in \mathbb{N}\rangle$ a $\sqsubseteq$-directed family of nonempty finite subsets of X, there exists a $\preceq$-directed subset $D \subseteq \bigcup\limits_{i\in\mathbb{N}} F_i$ that meets all F_i.

Rudin's Lemma is a fundamental tool to prove that the way-below relation is interpolative, which makes the Scott topology and Lawson topology well defined on the quasicontinuous domains. We also consider a variant of Rudin's Lemma, and prove that it is equivalent to ACA_0.

Theorem 5.4.1 The following are equivalent over RCA_0:

(1) ACA_0;

(2) Given a countable poset $(X, \preceq)$, and $\mathcal{F} = \langle F_i : i \in \mathbb{N}\rangle$ a $\sqsubseteq$-directed family of disjoint nonempty finite subsets of X, then there exists a $\preceq$-directed subset $E \subseteq \bigcup\limits_{i\in\mathbb{N}} F_i$ such that each $E \cap F_i$ is a singleton.

Chapter 2

Basic properties provable within $\mathbf{RCA}_0$

2.1 Introduction

In this chapter, we will consider properties of lattices provable within RCA_0. For posets, Lempp and Mummert proved in [22] that the "existence of maximal filters" is equivalent to ACA_0 over RCA_0, and Mummert proved in his PhD thesis [25] that "every filter can be extended to a maximal filter" is equivalent to Π^1_1-CA_0 over RCA_0. For lattices, we have two additional operations, meet and join, which help a lot in the constructions of objects wanted. In Section 2.2, we first show that for a countable lattice, RCA_0 is strong enough to prove the existence of maximal ideals. We will also consider the relations among prime ideals, maximal ideals and relatively maximal ideals in lattices, and show that RCA_0 can prove that prime ideals and maximal ideals are both relatively maximal, but the converse is not necessarily true.

In Section 2.3, we will show that RCA_0 can prove that prime ideals and relatively maximal ideals coincide for distributive lattices. In fact, "relatively maximal ideals are prime" is a necessary and sufficient condition for lattices to be distributive, and we will prove that RCA_0 can prove this.

In Section 2.3, we also prove that over RCA$_0$, every countable distributive lattice contains a prime ideal.

In Section 2.4, we prove Birkhoff's Representation Theorem over RCA$_0$. Birkhoff's Representation Theorem gives a one-to-one correspondence between the class of finite distributive lattices and the class of finite posets, providing a powerful tool for the study of finite distributive lattices.

Birkhoff's Representation Theorem: (1) For a finite distributive lattice $(L, \preceq, \vee, \wedge, 0, 1)$, the mapping

$$\eta : L \longrightarrow \mathcal{O}(\mathcal{J}(L))$$

defined by

$$\eta(a) = \{x \in \mathcal{J}(L) : x \preceq a\}$$

is a lattice isomorphism.

(2) For a finite poset $(P, \preceq)$, the mapping

$$\gamma : P \longrightarrow \mathcal{J}(\mathcal{O}(P))$$

defined by

$$\gamma(a) = \{x \in P : x \preceq a\}$$

is an order isomorphism.

Recall that $\mathcal{O}(\mathcal{J}(L))$ denotes the set of all lower sets of $\mathcal{J}(L)$, $(\mathcal{O}(\mathcal{J}(L)), \subseteq, \emptyset, \mathcal{J}(L), \bigcup, \bigcap)$ is a finite distributive lattice.

The two theorems above show that if we start from a finite distributive lattice L, then $\mathcal{O}(\mathcal{J}(L))$ is isomorphic to L, and if we start from a finite poset P, then $\mathcal{J}(\mathcal{O}(P))$ is isomorphic to P.

2.2 Ideals in lattices

We first list some basic properties of lattices which can be proved over RCA$_0$ directly:

Proposition 2.2.1. *(RCA_0) Let* $(L, \preceq, \vee, \wedge, 0, 1)$ *be a countable lattice, then for all* $a, b, c \in L$ *we have:*

(1) $a \preceq b \iff a \vee b = b \iff a \wedge b = a$;

(2) $a \vee b = b \vee a$ *and* $a \wedge b = b \wedge a$;

(3) $(a \vee b) \vee c = a \vee (b \vee c)$;

(4) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$;

(5) $a \wedge (b \vee c) \succeq (a \wedge b) \vee (a \wedge c)$;

(6) $a \vee (b \wedge c) \preceq (a \vee b) \wedge (a \vee c)$;

(7) $(\forall a, b, c \in L)[a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)]$
$\iff (\forall a, b, c \in L)[a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)]$.

Remark 2.2.2. *Combining the proposition (5) and (7), we have that L is nondistributive if and only if there exist* $a, b, c \in L$ *such that* $a \wedge (b \vee c) \succ (a \wedge b) \vee (a \wedge c)$.

The following lemma is about the ideal closure of sets, which will be used in the remainder of this section. Here, in a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$, for a nonempty subset A of L, $\{x \in L : \exists a_0, a_1, \cdots, a_n \in A \ (x \preceq a_0 \vee a_1 \vee \cdots \vee a_n)\}$, is called the ideal closure of A, denoted as $Cl(A)$.

Lemma 2.2.3. *(RCA_0) Let A be a nonempty subset of a countable lattice* $(L, \preceq, \vee, \wedge, 0, 1)$ *and suppose that $Cl(A)$ exists. Then $Cl(A)$ is the smallest ideal that contains A.*

Proof. First we show that $Cl(A)$ is an ideal of L. (1) Suppose $x \in Cl(A)$ and $y \preceq x$. Then $x \preceq a_0 \vee a_1 \vee \cdots \vee a_n$, for some $a_0, a_1, \cdots, a_n \in A$, and hence $y \preceq x \preceq a_0 \vee a_1 \vee \cdots \vee a_n$, $y \in Cl(A)$. (2) Suppose $x, y \in Cl(A)$, then $x \preceq a_0 \vee a_1 \vee \cdots \vee a_n$, for some $a_0, a_1, \cdots, a_n \in A$, and $y \preceq b_0 \vee b_1 \vee \cdots \vee b_m$, for some $b_0, b_1, \cdots, b_m \in A$. Then $x \vee y \preceq a_0 \vee a_1 \vee \cdots \vee a_n \vee b_0 \vee b_1 \vee \cdots \vee b_m$, and $x \vee y \in Cl(A)$.

Now we show that $Cl(A)$ is the smallest ideal containing A. For $x \in Cl(A)$, $x \preceq a_0 \vee a_1 \vee \cdots \vee a_n$, for some $a_0, a_1, \cdots, a_n \in A$. So, for any ideal $I \supseteq A$, as $a_0, a_1, \cdots, a_n$ are also in I, $a_0 \vee a_1 \vee \cdots \vee a_n \in I$, $x \in I$. Hence $Cl(A) \subseteq I$. □

Remark 2.2.4. *If $A = \{a_0, a_1, \cdots, a_n\}$ is a finite subset of L, then*

$$Cl(A) = \{x \in L : x \preceq a_0 \vee a_1 \vee \cdots \vee a_n\}.$$

So $Cl(A)$ exists by Σ^0_0-comprehension and hence the existence of $Cl(A)$ can be proved in RCA$_0$. In this case, $Cl(A)$ is a principal ideal, i.e., $Cl(A) = \downarrow(\bigvee A)$.

Theorem 1.3.7 implies that for a countable poset, the existence of maximal ideals is equivalent to ACA$_0$ over RCA$_0$. We will prove that for a countable lattice, the existence of maximal ideals only requires RCA$_0$.

Theorem 2.2.5. *(RCA$_0$) Every countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$ contains a maximal ideal.*

Proof. Say that a finite subset $X \subseteq L$ is "good" if $1 \neq \bigvee X$. Since X is finite, say $X = \{x_1, x_2, \cdots, x_p\}$, $\bigvee X = x_1 \vee x_2 \vee \cdots \vee x_p$, which can be computed over RCA$_0$. Define $f : \mathbb{N} \to \mathbb{N}$ as follows:

$$f(n) = \begin{cases} 1 & \text{if } \{m : m < n \text{ and } f(m) = 1\} \cup \{n\} \text{ is good,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $M = \{n \in \mathbb{N} : f(n) = 1\}$ is a maximal ideal of L. □

Now, we consider relations among maximal ideals, relatively maximal ideals and prime ideals in lattices, and also the logical strength of implications involved.

Theorem 2.2.6. *(RCA$_0$) For a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$,*

(1) Every maximal ideal is relatively maximal, but not conversely.

(2) Every prime ideal, if it exists, is relatively maximal, but not conversely.

Proof. If I is a maximal ideal of L, then I is a relatively maximal ideal w.r.t. to $\{1\}$. If I is a prime ideal of L, then $F = L \setminus I$ is a filter of L (and hence a sublattice) because

(1) For $a \in F$ and $a \preceq b$, $b \in F$, as otherwise, $b \in L \setminus F = I$, and hence $a \in I$, a contradiction.

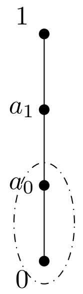


Figure 2.1: $\{0, a_0\}$ is relatively maximal ideal w.r.t. $\{a_1, 1\}$, but not maximal

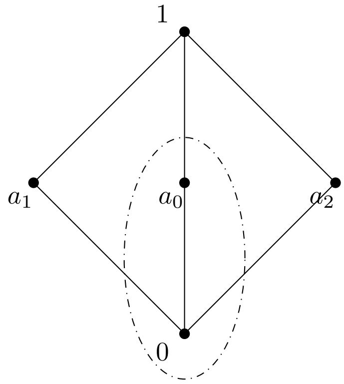


Figure 2.2: $\{0, a_0\}$ is relatively maximal ideal w.r.t. $\{a_1, 1\}$, but not prime

(2) For $a, b \in F$, $a \wedge b \notin F$, as otherwise, $a \wedge b \in L \setminus F = I$. Since I is prime, either $a \in I$ of $b \in I$, a contradiction.

Thus, I is a relatively maximal ideal w.r.t. F.

In Figure 2.1, the ideal $\{0, a_0\}$ is relatively maximal (w.r.t. $\{a_1, 1\}$), but $\{0, a_0\}$ is not maximal. Note that the lattice in Figure 2.1 is distributive. In Figure 2.2, the ideal $\{0, a_0\}$ is relatively maximal (w.r.t. $\{a_1, 1\}$), but $\{0, a_0\}$ is not prime.

□

2.3 Distributive lattices

For distributivity, a classic theorem is the Dedekind-Birkhoff Theorem, which says that a lattice L is distributive if and only if L contains no copy of M_3 or N_5. A close check of the proof of this theorem shows that it can be proved in RCA_0.

Theorem 2.3.1. *(RCA_0) A countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$ is distributive if and only if neither M_3 nor N_5 can be embedded into L.*

Proof. The following standard proof can be conducted within RCA_0. Suppose that L contains a copy of M_3, like the one in Figure 2.3, then

$$v \wedge (u \vee e) = v \wedge q = v \neq p = p \vee p = (v \wedge u) \vee (v \wedge e).$$

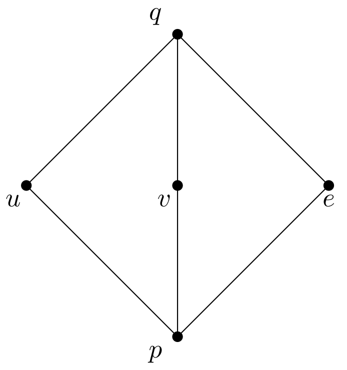


Figure 2.3: M_3

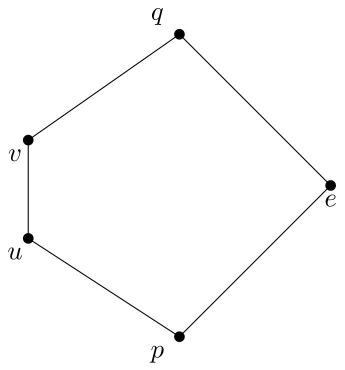


Figure 2.4: N_5

Suppose that L contains a copy of N_5, like the one in Figure 2.4, we have

$$v \wedge (u \vee e) = v \wedge q = v \neq u = u \vee p = (v \wedge u) \vee (v \wedge e).$$

Thus, L isn't distributive in both cases.

For the other direction, suppose that L is nondistributive, we have the following two cases:

Case 1: L is modular. That is, for all $a, b, c \in L$, if $a \succeq c$, then

$$a \wedge (b \vee c) = (a \wedge b) \vee c.$$

Since L is nondistributive, by Remark 2.2.2, there are $a, b, c \in L$ such that

$$(a \wedge b) \vee (a \wedge c) \prec a \wedge (b \vee c).$$

We construct $\{p, q, u, v, e\}$ from a, b, c by letting

$$\begin{aligned} p &= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c), \\ q &= (a \vee b) \wedge (a \vee c) \wedge (b \vee c), \\ u &= (a \wedge q) \vee p, \\ v &= (b \wedge q) \vee p, \\ e &= (c \wedge q) \vee p. \end{aligned}$$

Claim: $\{p, q, u, v, e\}$ forms a copy of M_3 in L as shown in Figure 2.3.

In order to prove the claim, we only need to check that this subset has the correct joins and meets, and $p \neq q$ ($p \preceq q$ is trivial). We will repeatedly apply the modular law, and for each application, we underline the two elements involved.

$$\begin{aligned}
u \wedge v &= ((a \wedge q) \vee \underline{p}) \wedge (\underline{(b \wedge q) \vee p}) \\
&= ((a \wedge q) \wedge ((b \wedge \underline{q}) \vee \underline{p})) \vee p \\
&= ((a \wedge q) \wedge (b \vee p) \wedge q) \vee p \\
&= ((a \wedge q) \wedge (b \vee p)) \vee p \\
&= ((a \wedge (b \vee c)) \wedge (b \vee (a \wedge c))) \vee p \\
&= (a \wedge ((\underline{b \vee c}) \wedge (\underline{b} \vee (a \wedge c)))) \vee p \\
&= (a \wedge (((b \vee c) \wedge (a \wedge c)) \vee b)) \vee p \\
&= (\underline{a} \wedge ((\underline{a \wedge c}) \vee b)) \vee p \\
&= ((a \wedge b) \vee (a \wedge c)) \vee p \\
&= p.
\end{aligned}$$

In exactly the same way, we can show that $u \wedge e = p$ and $v \wedge e = p$. Similar calculations give us $u \vee v = u \vee e = v \vee e = q$.

Again since

$$\begin{aligned}
a \wedge q &= a \wedge (b \vee c), \\
a \wedge p &= \underline{a} \wedge (\underline{(a \wedge b) \vee (a \wedge c)} \vee (b \wedge c)) \\
&= ((a \wedge b) \vee (a \wedge c)) \vee (a \wedge (b \wedge c)) \\
&= (a \wedge b) \vee (a \wedge c),
\end{aligned}$$

and by the assumption $a \wedge (b \vee c) \succ (a \wedge b) \vee (a \wedge c)$, we have $a \wedge q \succ a \wedge p$, and so $p \neq q$.

Case 2: L is nonmodular. Then there exist $a, b, c \in L$ such that $a \succ c$ and $a \wedge (b \vee c) \neq (a \wedge b) \vee c$, and hence $a \wedge (b \vee c) \succ (a \wedge b) \vee c$.

We construct a copy of N_5, $\{p, q, u, v, e\}$ from a, b, c, as follows:

$$p = a \wedge b, \qquad q = b \vee c, \qquad e = b,$$

$$u = (a \wedge b) \vee c, \qquad v = a \wedge (b \vee c).$$

By the assumption that $u \prec v$, we only need to check $v \vee e = u \vee e = q$ and $v \wedge e = u \wedge e = p$. First, $v \vee e = u \vee e = q$ because

$$q = b \vee c = u \vee e \preceq v \vee e \preceq b \vee c \vee b = b \vee c = q.$$

Similarly, $v \wedge e = u \wedge e = p$ since

$$p = a \wedge b = a \wedge e = (a \wedge e) \wedge e \preceq u \wedge e \preceq v \wedge e = a \wedge e = a \wedge b = p.$$

□

We have seen in Theorem 2.2.6 that maximal ideals are all relatively maximal, and the converse is not true, even for distributive lattices (the lattice in Figure 2.1 is distributive). The next theorem says that, for distributive lattices, prime ideals and relatively maximal ideals coincide.

Theorem 2.3.2. *(RCA$_0$) For a countable distributive lattice $(L, \preceq, \vee, \wedge, 0, 1)$, an ideal of L is prime if and only if it is relatively maximal.*

Proof. Theorem 2.2.6 already shows that prime ideals are all relatively maximal. So we only need to show that for distributive lattices, every relatively maximal ideal is prime.

Let L be a countable distributive lattice, and for a contradiction, suppose that $R \subseteq L$ is a relatively maximal ideal w.r.t. a sublattice S of L and that R is not prime. Then there exist two elements a, b in L such that $a \wedge b \in R$ but $a, b \notin R$. Since R is relatively maximal w.r.t. S, there exist $r_1, r_2 \in R$, $s_1, s_2 \in S$ such that

$$s_1 \preceq r_1 \vee a \preceq r_1 \vee r_2 \vee a \quad \text{and} \quad s_2 \preceq r_2 \vee b \preceq r_1 \vee r_2 \vee b.$$

Thus,

$$s_1 \wedge s_2 \preceq (r_1 \vee r_2 \vee a) \wedge (r_1 \vee r_2 \vee b) = r_1 \vee r_2 \vee (a \wedge b).$$

We have the last equality by the distributivity of L. This shows that $s_1 \wedge s_2 \in R$ because R is an ideal and both $r_1 \vee r_2$ and $a \wedge b$ are in R. Note that $s_1 \wedge s_2$ is also in S because S is a sublattice. This contradicts the assumption that $R \cap S = \emptyset$. Hence R is prime. □

Thus, the following well-known relation between maximal ideals and prime ideals for distributive lattices can be proved in RCA_0.

Corollary 2.3.3. *(RCA_0) In countable distributive lattices, maximal ideals are prime, and there are prime ideals which aren't maximal.*

Proof. From Theorem 2.3.2, the prime ideals and relatively maximal ideals in distributive lattices are exactly the same. The lattice in the counterexample in Theorem 2.2.6(1) (Figure 2.1) is also distributive, hence the prime ideal is not maximal. □

Now, we can get the existence of prime ideals for distributive lattices over RCA_0.

Theorem 2.3.4. *(RCA_0) Every countable distributive lattice $(L, \preceq, \vee, \wedge, 0, 1)$ contains a prime ideal.*

Proof. By Theorem 2.2.5, there exists a maximal ideal M of L. By Corollary 2.3.3, M is a prime ideal. □

Theorem 2.3.2 says that if L is a distributive lattice, then every relatively maximal ideal is prime. In fact, the converse is also true.

Theorem 2.3.5. *(RCA_0) For a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$, if every relatively maximal ideal is prime, then L is distributive.*

Proof. Assume that L is non-distributive, then by Theorem 2.3.1, we have that L contains a copy of M_3 or N_5.

We show how to obtain a contradiction from the assumption that L contains a copy of N_5 (the one in Figure 2.4). The argument for the case that L contains a copy of M_3 is the same as the one for N_5.

Let $\{x_0, x_1, x_2, \cdots\}$ be an enumeration of the members of L with $x_0 = u$. We will construct a relatively maximal ideal containing u but not v.

Let $X_0 = \{x_0\}$, i.e., $X_0 = \{u\}$,

$$X_{n+1} = \begin{cases} X_n \cup \{x_{n+1}\} & \text{if } v \not\preceq \bigvee(X_n \cup \{x_{n+1}\}), \\ X_n & \text{otherwise.} \end{cases}$$

Let $I = \bigcup_n X_n$, and I exists over RCA$_0$ because $x_n \in I \iff x_n \in X_n$. From the construction, we see that $\bigvee X_n \not\succeq v$ for all n.

Claim: I is a relatively maximal ideal with respect to the sublattice $\{v\}$.

- I is an ideal of L.

Suppose that $x_n \in I$, i.e., $x_n \in X_n$, and $x_m \preceq x_n$, we will show that $x_m \in I$.

(1) If $n < m$, then as $x_m \preceq x_n$, $x_m \preceq \bigvee X_n \preceq \bigvee X_{m-1}$, $\bigvee(X_{m-1} \cup \{x_m\}) = \bigvee X_{m-1}$. By $v \not\preceq \bigvee X_{m-1}$, $v \not\preceq \bigvee(X_{m-1} \cup \{x_m\})$. So $X_m = X_{m-1} \cup \{x_m\}$, $x_m \in I$.

(2) If $n > m$, then by $x_m \preceq x_n$ and $X_{m-1} \subseteq X_n$, $\bigvee(X_{m-1} \cup \{x_m\}) \preceq \bigvee X_n$. By $v \not\preceq \bigvee X_n$, $v \not\preceq \bigvee(X_{m-1} \cup \{x_m\})$, and hence $X_m = X_{m-1} \cup \{x_m\}$, $x_m \in I$.

Suppose that $x_n, x_m \in I$ with $n < m$, i.e., $x_n, x_m \in X_m$. Let $x_k = x_n \vee x_m$, we will show that $x_k \in I$.

(1) If $k > m$, then $x_k = x_m \vee x_n \preceq \bigvee X_m \preceq \bigvee X_{k-1}$, and $\bigvee(X_{k-1} \cup \{x_k\}) = \bigvee X_{k-1}$. By $v \not\preceq \bigvee X_{k-1}$, and hence $v \not\preceq \bigvee(X_{k-1} \cup \{x_k\})$, $x_k \in X_k$, $x_k \in I$.

(2) If $k < m$, then by $x_n, x_m \in X_m$ and $x_k = x_m \vee x_n$, $\bigvee(X_{k-1} \cup \{x_k\}) \preceq \bigvee X_m$. Again since $v \not\preceq \bigvee X_m$, $v \not\preceq \bigvee(X_{k-1} \cup \{x_k\})$, and hence $x_k \in X_k$, $x_k \in I$.

- It is obvious that $u \in I$ but $v \notin I$.

- I is relatively maximal respect to $\{v\}$.

Otherwise, there exists $x_k \notin I$, such that for all $x_m \in I$, $x_k \vee x_m \not\succeq v$. Since I is an ideal and $X_{k-1} \subseteq I$, $\bigvee X_{k-1} \in I$ and hence $(\bigvee X_{k-1}) \vee x_k \not\succeq v$, which implies that $x_k \in X_k \subseteq I$, a contradiction.

Thus, by our assumption, I is prime. By $v \wedge e = p \in I$, $v \in I$ or $e \in I$. As $v \notin I$, $e \in I$. Therefore, $q = u \vee e \in I$. This implies that $v \in I$ as $v \preceq q$, a contradiction. □

Theorem 2.3.6. *(RCA$_0$) A countable lattice L is distributive if and only if every relatively maximal ideal in L is prime.*

Proof. Directly from Theorem 2.3.5 and Theorem 2.3.2. □

2.4 Birkhoff's Representation Theorem

In this section, we will show that Birkhoff's Representation Theorem can be proved over RCA$_0$. Say that a poset $(P, \preceq)$ satisfies the *ascending chain condition*, *ACC* for short, if for any ascending chain of elements $x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots$ in P, it eventually stabilizes. In other words, there exists an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $x_n = x_{N_0}$. The *descending chain condition*, *DCC* for short, is defined dually. For *ACC*, RCA$_0$ can prove the following equivalent statement.

Lemma 2.4.1. *(RCA$_0$) A countable poset $(P, \preceq)$ satisfies ACC if and only if every non-empty subset A of P has a maximal element.*

Proof. We first assume that P satisfies *ACC*. Given a nonempty subset A of P, let $A = \{a_0, a_1, \cdots, a_n, \cdots\}$. Here , for $i \neq j$, a_i and a_j can be equal to each other, so the case "A is finite" is included. Define $f : \mathbb{N} \to \mathbb{N}$ as follows:

$$\begin{aligned} f(0) &= 0, \\ f(n+1) &= \begin{cases} n+1, & \text{if } a_{f(n)} \preceq a_{n+1}; \\ f(n), & \text{otherwise.} \end{cases} \end{aligned}$$

Let $X = \{a_{f(i)} : i \in \mathbb{N}\}$. X exists, by Σ^0_0-comprehension, because

$$a_i \in X \iff \exists n \leq i \; (a_n = a_i \;\&\; (n = f(n))).$$

Since P satisfies *ACC*, for the ascending chain $a_{f(0)} \preceq a_{f(1)} \preceq \cdots \preceq a_{f(n)} \preceq \cdots$, there exists an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $a_{f(n)} = a_{f(N_0)}$. Thus $a_{f(N_0)}$ is a maximal element of A.

Now for the other direction, we assume that every non-empty subset of P has a maximal element. Given any ascending sequence of elements $x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$ in P, by assumption, it contains a maximal element x_{N_0}. This gives that for all $n \geq N_0$, $x_n = x_{N_0}$.

□

Dually, we can prove that within RCA$_0$, a countable poset $(P, \preceq)$ satisfies *DCC* if and only if every nonempty subset A of P has a minimal element.

Before giving a proof of Birkhoff's Representation Theorem, we show that RCA$_0$ can prove some properties about join-irreducible elements and join-dense sets.

Lemma 2.4.2. *(RCA$_0$) If a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$ satisfies DCC, then for any $a, b \in L$ with $a \npreceq b$, there exists $x \in \mathcal{J}(L)$ such that $x \preceq a$ and $x \npreceq b$.*

Proof. Let $a \npreceq b$, and let $S = \{x \in L : x \preceq a \text{ and } x \npreceq b\}$. S exists by Σ^0_0-comprehension, and S is nonempty because $a \in S$. Since L satisfies *DCC*, by the dual version of Lemma 2.4.1, there exists a minimal element x of S.

We claim that x is join-irreducible. Suppose that $x = c \vee d$ with $c \prec x$ and $d \prec x$. By the minimality of x, neither c nor d is in S. Hence $c \preceq b$ and $d \preceq b$, $x = c \vee d \preceq b$, contradicting the assumption that $x \in S$. Thus x is join-irreducible. □

For a countable lattice L, a subset S of L is *join-dense* if for every element $x \in L$, there there exists a subset A of S such that $x = \bigvee A$. We will show that RCA$_0$ can prove that $\mathcal{J}(L)$ is join-dense for any finite lattice L. Actually, RCA$_0$ can prove that for a countable lattice L satisfying *DCC* , if $\mathcal{J}(L)$ exists, then it is join-dense.

Theorem 2.4.3. *(RCA$_0$) If a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$ satisfies DCC and $\mathcal{J}(L)$ exists, then $\mathcal{J}(L)$ is join-dense in L.*

Proof. It's sufficient to prove $a = \bigvee\{x \in \mathcal{J}(L) : x \preceq a\}$ for all $a \in L$. Fix $a \in L$, let $A = \{x \in \mathcal{J}(L) : x \preceq a\}$. A exists by Σ^0_0-comprehension, because we have assumed that $\mathcal{J}(L)$ exists. Clearly, a is an upper bound of A.

Let b be another upper bound of A. We claim that $a \preceq b$. Otherwise, $a \npreceq b$, then, by Lemma 2.4.2, there exists an $x \in \mathcal{J}(L)$ with $x \preceq a$ and $x \npreceq b$. Hence $x \in A$. But b is an upper bound of A, so $x \preceq b$, a contradiction. Thus we have $a \preceq b$. This shows that a is the least upper bound of A.

Thus, RCA_0 proves that $\mathcal{J}(L)$ is join-dense in L. □

If L is finite, then $\mathcal{J}(L)$ exists over RCA_0. So when we prove Birkhoff's Representation Theorem, finite distributive lattices are considered, and hence over RCA_0 we can show that $\mathcal{J}(L)$ exists and $\mathcal{J}(L)$ is join-dense in L.

In Chapter 4, we will prove that if L is a countable lattice, then the existence of $\mathcal{J}(L)$ is equivalent to ACA_0 over RCA_0.

For distributive lattices, there is an equivalent statement to verify whether an element is join-irreducible or not.

Lemma 2.4.4. *(RCA_0) Let $(L, \preceq, \vee, \wedge, 0, 1)$ be a countable distributive lattice. For all $x \in L$ with $x \neq 0$, the following are equivalent:*

(1) x is join-irreducible;

(2) if $a, b \in L$ and $x \preceq a \vee b$, then $x \preceq a$ or $x \preceq b$.

Proof. We will reason within RCA_0. First we show (1) $\Rightarrow$ (2). Let x be join-irreducible and $x \preceq a \vee b$, then $x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b)$. Thus $x = x \wedge a$ or $x \wedge b$, and hence $x \preceq a$ or $x \preceq b$.

For (2) $\Rightarrow$ (1), suppose that $x = a \vee b$, then by $x \preceq a \vee b$, $x \preceq a$ or $x \preceq b$. Since $x = a \vee b \succeq a, b$, $x = a$ or $x = b$. □

Now we prove Birkhoff's Representation Theorem in RCA_0.

Theorem 2.4.5. *(RCA_0) For a finite distributive lattice $(L, \preceq, \vee, \wedge, 0, 1)$, the mapping*

$$\eta : L \longrightarrow \mathcal{O}(\mathcal{J}(L))$$

defined by

$$\eta(a) = \{x \in \mathcal{J}(L) : x \preceq a\}$$

is a lattice isomorphism.

Proof. As L is finite, RCA$_0$ can prove the existence of $\mathcal{J}(L)$, $\mathcal{O}(\mathcal{J}(L))$ and η. Since finite unions and finite intersections of lower sets are still lower sets, $(\mathcal{O}(\mathcal{J}(L)), \subseteq, \bigcup, \bigcap, \emptyset, \mathcal{J}(L))$ is a finite distributive lattice. It is clear that $\eta(a) \in \mathcal{O}(\mathcal{J}(L))$. We will show that η is a lattice isomorphism.

(1) η is lattice homomorphism, i.e., for all $a, b \in L$, $\eta(a \vee b) = \eta(a) \bigcup \eta(b)$ and $\eta(a \wedge b) = \eta(a) \bigcap \eta(b)$. Suppose that $x \in \eta(a \vee b)$, then x is join-irreducible and $x \preceq a \vee b$. So by Lemma 2.4.4, $x \preceq a$ or $x \preceq b$, and $x \in \eta(a) \bigcup \eta(b)$. Thus, $\eta(a \vee b) \subseteq \eta(a) \bigcup \eta(b)$. Conversely, suppose that $x \in \eta(a) \bigcup \eta(b)$. Then x is join-irreducible and $x \preceq a$ or $x \preceq b$, and hence $x \preceq a \vee b$. Thus, $x \in \eta(a \vee b)$. Therefore, $\eta(a) \bigcup \eta(b) \subseteq \eta(a \vee b)$. This shows that $\eta(a \vee b) = \eta(a) \bigcup \eta(b)$. $\eta(a \wedge b) = \eta(a) \bigcap \eta(b)$ can be proved in the same way.

(2) η is one-to-one. Let $a, b \in L$ with $\eta(a) = \eta(b)$. As L a is finite lattice, RCA$_0$ proves that L satisfies *DCC* and $\mathcal{J}(L)$ exists. By Theorem 2.4.3, $a = \bigvee \eta(a) = \bigvee \eta(b) = b$. So η is one-to-one.

(3) η is onto. First note that $\emptyset = \eta(0)$. Let $\emptyset \neq F \in \mathcal{O}(\mathcal{J}(L))$ and suppose that $F = \{a_0, a_1, \cdots, a_k\}$. Let $a = a_0 \vee a_1 \vee \cdots \vee a_k$. We claim that $F = \eta(a)$. As all elements in F are join-irreducible, $F \subseteq \eta(a)$, by the definition of $\eta(a)$. On the other hand, for $x \in \eta(a)$, as x is join-irreducible and $x \preceq a_0 \vee a_1 \vee \cdots \vee a_k$, by Lemma 2.4.4, $x \preceq a_i$ for some $0 \leq i \leq k$. Because F is downwards closed, $x \in F$. Thus $F = \eta(a)$ and η is onto.

Thus RCA$_0$ proves that η is an isomorphism.

□

Theorem 2.4.6. *(RCA$_0$) For a finite poset $(P, \preceq)$, the mapping*

$$\gamma : P \longrightarrow \mathcal{J}(\mathcal{O}(P))$$

defined by

$$\gamma(a) = \{x \in P : x \preceq a\}$$

is an order isomorphism. Here $\mathcal{J}(\mathcal{O}(P)) = (\mathcal{J}(\mathcal{O}(P)), \subseteq)$ *is a finite poset.*

Proof. We first show that $\gamma(a) \in \mathcal{J}(\mathcal{O}(P))$, i.e., $\gamma(a)$ is a join-irreducible element of $\mathcal{O}(P)$. Suppose that $\gamma(a) = U \cup V$ for $U, V \in \mathcal{O}(P)$. Since $a \in \gamma(a)$, $a \in U \cup V$. Without loss of generality, suppose that $a \in U$. As U is a lower set, $\gamma(a) \subseteq U$. By $\gamma(a) = U \cup V \supseteq U$, we have $\gamma(a) = U$. This shows that $\gamma(a) \in \mathcal{J}(\mathcal{O}(P))$.

For order-embedding, $a \preceq b \iff \gamma(a) \subseteq \gamma(b)$ is obvious. So it's sufficient to show that γ is onto. Let $U \in \mathcal{J}(\mathcal{O}(P))$, since P is finite, U is also finite. Let $U = \{a_0, a_1, \cdots, a_k\}$. Then $U \subseteq \gamma(a_0) \cup \gamma(a_1) \cup \cdots \cup \gamma(a_k)$. Since U is join-irreducible in $\mathcal{O}(P)$ and $\mathcal{O}(P) = (\mathcal{O}(P), \subseteq, \emptyset, P, \cap, \cup)$ is a finite distributive lattice, we have that $U \subseteq \gamma(a_i)$ for some $0 \leq i \leq k$, by Lemma 2.4.4. As $a_i \in U$ and U is a lower set, $\gamma(a_i) \subseteq U$. Therefore, $U = \gamma(a_i)$. This shows that γ is onto.

This shows that Theorem 2.4.6 can be proved in RCA_0. $\square$

Chapter 3

DPI and $\mathbf{WKL}_0$

3.1 Introduction

In the proofs of Stone's Representation Theorem and Priestley's Representation Theorem, the separation property (Theorem 3.2.3) plays a critical role. This theorem says that we can use a prime ideal to separate two distinct elements, making the mapping involved to be one-to-one. This theorem has a strong version (Theorem 3.2.1), saying that given an ideal I and a filter F, if they are disjoint, then they can be separated by a prime ideal. Davey and Priestley [6] denote it as **DPI**, we will prove it over WKL_0 in Section 3.2.

Theorem 2.3.4 says that RCA_0 is strong enough to prove the existence of prime ideals of distributive lattices. In this chapter, we will also prove that over WKL_0, every proper ideal can be extended to a prime ideal of distributive lattices.

3.2 DPI and $\mathbf{WKL}_0$

In this section, we show that WKL_0 can prove a separation property, **DPI**.

Theorem 3.2.1. *(WKL_0) For a countable distributive lattice* $(L, \preceq, \vee, \wedge, 0, 1)$*, I an ideal, F a filter of L with $I \cap F = \emptyset$, there exists a prime ideal P containing I but disjoint from F.*

Proof. Let $\{a_0, a_1, a_2, \cdots\}$ be an enumeration of L. We assume that $a_0 = 0, a_1 = 1$. Let $T \subseteq 2^{<\omega}$ be the set of all strings $\sigma \in 2^{<\omega}$ such that:

- $a_i \in I$ implies $\sigma(i) = 1$;
- $a_i \in F$ implies $\sigma(i) = 0$;
- For all $i, j, k < lh(\sigma)$,
  (1) if $\sigma(i) = \sigma(j) = 1$ and $a_i \vee a_j = a_k$, then $\sigma(k) = 1$;
  (2) if $\sigma(i) = 1$ and $a_j \preceq a_i$, then $\sigma(j) = 1$;
  (3) if $\sigma(i) = \sigma(j) = 0$ and $a_i \wedge a_j = a_k$, then $\sigma(k) = 0$.

Clearly, T is a tree and T exists by Σ^0_0-comprehension. In order to show that T is infinite, we need the following binary subtree $S \subseteq 2^{<\omega}$.

We define a sequence of finite sets $X_s \subseteq L$ for each $s \in 2^{<\omega}$, beginning with $X_{\langle\rangle} = \{0\}$. Suppose that X_s has been defined. Let

$$lh(s) = 4 \cdot \langle\langle i, j\rangle, m\rangle + k,\ 0 \leq k < 4,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing function.

Case 1: $k = 0$. If $a_i \wedge a_j \in X_s$, then put $X_{s^\frown 0} = X_s \cup \{a_i\}$, $X_{s^\frown 1} = X_s \cup \{a_j\}$; otherwise put $X_{s^\frown 0} = X_s$, $X_{s^\frown 1} = \emptyset$.

Case 2: $k = 1$. Put $X_{s^\frown 0} = \emptyset$. If $a_i \in X_s$, $a_j \in X_s$, then put $X_{s^\frown 1} = X_s \cup \{a_i \vee a_j\}$; otherwise, put $X_{s^\frown 1} = X_s$.

Case 3: $k = 2$. Put $X_{s^\frown 0} = \emptyset$. If $a_i \in X_s$, $a_j \preceq a_i$, then put $X_{s^\frown 1} = X_s \cup \{a_j\}$; otherwise, put $X_{s^\frown 1} = X_s$.

Case 4: $k = 3$. Put $X_{s^\frown 0} = \emptyset$. If $X_s \cap F \neq \emptyset$, i.e., $\exists a \in X_s\ (a \in F)$, which is a Σ^0_0-sentence (as X_s is finite), then put $X_{s^\frown 1} = \emptyset$; otherwise, put $X_{s^\frown 1} = X_s$.

Let $S = \{s \in 2^{<\omega} : X_s \neq \emptyset\}$. Clearly, S is a tree and S exists by Σ^0_0-comprehension. We claim that for each $n \in \mathbb{N}$, there exists an $s \in S$ with $lh(s) = n$

such that $Cl(X_s) \cap F = \emptyset$. Since X_s is finite and F is a filter, the claim can be written as: For $n \in \mathbb{N}, \varphi(n)$, where

$$\varphi(n) := \exists s \in S \ (lh(s) = n \text{ and } \bigvee X_s \notin F).$$

Here, we prove $\varphi(n)$ by Σ^0_0-induction on n. $\varphi(0)$ is trivial. Suppose that $\varphi(n)$ is true for n. If $n \equiv 1, 2,$ or $3 \bmod 4$, then $\varphi(n+1)$ follows immediately. If $n \equiv 0 \bmod 4$, then there exists $s \in S$ such that $lh(s) = n$ and $\bigvee X_s \notin F$. We only need to consider the case $a_i \wedge a_j \in X_s$, i.e., $X_{s^\frown 0} = X_s \cup \{a_i\}$, $X_{s^\frown 1} = X_s \cup \{a_j\}$.

Claim: $\bigvee X_{s^\frown 0} \notin F$ or $\bigvee X_{s^\frown 1} \notin F$.

Otherwise, we have $(\bigvee X_s) \vee a_i = \bigvee X_{s^\frown 0} \in F$ and $(\bigvee X_s) \vee a_j = \bigvee X_{s^\frown 1} \in F$. By $a_i \wedge a_j \in X_s$ and F is filter, $\bigvee X_s = (\bigvee X_s) \vee (a_i \wedge a_j) \in F$ (by distributivity), a contradiction.

Hence $\bigvee X_{s^\frown 0} \notin F$ or $\bigvee X_{s^\frown 1} \notin F$, and so $\varphi(n+1)$ is true.

"$\forall n \in \mathbb{N} \ \varphi(n)$" implies that S is infinite. By WKL$_0$, S has an infinite path, g say. Now, coming back to T, we will show that T is infinite.

For $m \in \mathbb{N}$, let $Y_m = \{i < m : \exists n \ (a_i \in X_{g \restriction n})\}$. Y_m exists by bounded Σ^0_1-comprehension. Define $\gamma \in 2^{<\omega}$ such that $lh(\gamma) = m$ and for all $i < m$,

$$\gamma(i) = \begin{cases} 1 & \text{if } i \in Y_m, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\gamma \in T$. This proves that T is infinite. Hence, by WKL$_0$ again, there exists an infinite path f of T. Let $P = \{a_i : f(i) = 1\}$. P is the desired prime ideal. □

In **DPI**, by letting $F = \{1\}$, the proper ideal I can be extended to the prime ideal P, which implies the prime ideal extension theorem immediately.

Corollary 3.2.2. *(WKL$_0$) Every proper ideal on a countable distributive lattice extends to a prime ideal.*

The next theorem, a direct corollary of Theorem 3.2.1, plays a central role in the proofs of Stone's Representation Theorem and Priestley's Representation Theorem.

Theorem 3.2.3. *(WKL_0) Let* $(L, \preceq, \vee, \wedge, 0, 1)$ *be a countable distributive lattice. Suppose* $a, b \in L$ *and* $a \npreceq b$*, then there exists a prime ideal* P *such that* $a \notin P$ *and* $b \in P$*.*

Proof. Let $I = \downarrow b$ and $F = \uparrow a$. I and F exist by Σ^0_0 comprehension. Since $a \npreceq b$, $I \cap F = \emptyset$. By Theorem 3.2.1, we can have a prime ideal P containing I avoiding F (hence $a \notin P$ and $b \in P$) and this can be done within WKL_0. □

The question of whether **DPI** implies WKL_0 over RCA_0 remains open.

Chapter 4

RMI and ACA_0

4.1 Introduction

In this chapter, we study the properties of lattices within ACA_0. From the last chapter, we know that for distributive lattices, we can always separate an ideal and a filter by a prime ideal, if they are disjoint. In general, prime ideals may not exist for some lattice, e.g., M_3. So, we cannot prove the same result for general lattices. Instead, we will use relatively maximal ideals, as a substitute for prime ideals, to establish the separation. This separation is denoted by **RMI**.

In Section 4.2, we will prove that **RMI** can be proved within ACA_0. Furthermore, we prove that over RCA_0, **RMI** is equivalent to the statement "every proper ideal on a countable lattice extends to a maximal ideal".

Recall that in commutative algebra, Noetherian rings have the following equivalent definitions (see Hungerford's book [20]): for a commutative ring R,

(1) R is Noetherian, i.e., R satisfies the ascending chain condition on ideals.

(2) Every ideal of R is finitely generated.

(3) Every prime ideal of R is finitely generated.

For lattices, there is a similar characterization of the *ACC* property, showing that relatively maximal ideals in lattices behave quite similarly to the prime ideals in ring theory.

Theorem 4.1.1. *(Kinugawa and Hashimoto [21]) Given a lattice L, the following are equivalent:*

(1) L satisfies the ascending chain condition.

(2) Every ideal of L is finitely generated.

(3) Every relatively maximal ideal of L is finitely generated.

In Section 4.3, we will consider the strength of implications of these three statements above. In particular, we will show that RCA_0 is strong enough to prove (1) ⇒ (2) and (1) ⇒ (3). We will also show that ACA_0 can prove (2) ⇒ (1) and (3) ⇒ (1).

Furthermore, once a lattice satisfies the ascending chain condition, it must be a complete lattice, which can also be proved within ACA_0.

In Section 4.4, we will show that for a countable lattice L, the existence of $\mathcal{J}(L)$, i.e., the set of all join-irreducible elements, is equivalent to ACA_0 over RCA_0.

4.2 RMI and ACA_0

Lemma 2.2.3 says that $Cl(A)$, i.e., the ideal closure of $A \subseteq L$, is the smallest ideal containing A. If A is finite, then $Cl(A)$ exists over RCA_0. But if A is countably infinite, then the existence of $Cl(A)$ requires ACA_0.

Theorem 4.2.1. *(ACA_0) For every nonempty subset A of a countable lattice L, $Cl(A)$ exists.*

Proof. Let $(L, \preceq, \vee, \wedge, 0, 1)$ be a countable lattice, and A be a nonempty subset of L. Then $Cl(A) = \{x \in L : \exists a_0, a_1, \cdots, a_n \in A\ (x \preceq a_0 \vee a_1 \vee \cdots \vee a_n)\}$ exists by arithmetical comprehension. □

From the proof, we can compute $Cl(A)$ from A over ACA_0. Now we prove **RMI** within ACA_0, and the ideal closure will be used during the proof.

Theorem 4.2.2. *(ACA$_0$) In a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$, for an ideal I and a filter F of L, if $I \cap F = \emptyset$, there exists an ideal M which is maximal among those ideals containing I but disjoint from F.*

Proof. Say a subset $X \subseteq L$ is "good" if $Cl(X) \cap F = \emptyset$. Theorem 4.2.1 says that ACA$_0$ guarantees the existence of $Cl(A)$, and hence the concept of goodness can be defined over ACA$_0$. Define $f : \mathbb{N} \to \mathbb{N}$ as following:

$$f(n) = \begin{cases} 1 & \text{if } \{m < n : f(m) = 1\} \cup \{n\} \cup I \text{ is good,} \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $M = \{n \in \mathbb{N} : f(n) = 1\}$ is the desired ideal.

- M is an ideal of L.

Let $x \in M$, i.e., $f(x) = 1$. Suppose that $y \preceq x$.

If $x < y$, then $x \in Cl(\{m < y : f(m) = 1\})$. Since $y \preceq x$, $y \in Cl(\{m < y : f(m) = 1\})$, and furthermore

$$\text{Cl}(\{m < y : f(m) = 1\} \cup I) = \text{Cl}(\{m < y : f(m) = 1\} \cup \{y\} \cup I).$$

By the definition of f,

$$\text{Cl}(\{m < y : f(m) = 1\} \cup I) \cap F = \emptyset,$$

and hence $\{m < y : f(m) = 1\} \cup \{y\} \cup I$ is good. This implies that $f(y) = 1$ and $y \in M$.

If $x > y$, then $\{m < y : f(m) = 1\} \subseteq \{m < x : f(m) = 1\}$, and as $y \preceq x$,

$$\text{Cl}(\{m < y : f(m) = 1\} \cup \{y\} \cup I) \subseteq \text{Cl}(\{m < x : f(m) = 1\} \cup \{x\} \cup I).$$

By assumption, $f(x) = 1$, $\text{Cl}(\{m < x : f(m) = 1\} \cup \{x\} \cup I) \cap F = \emptyset$, and so $\text{Cl}(\{m < y : f(m) = 1\} \cup \{y\} \cup I) \cap F = \emptyset$. This implies $f(y) = 1$ and $y \in M$.

Now let $x, y \in M$ and we will show that $x \vee y$ is also in M. Let $z = x \vee y$ and assume that $x < y$ as numbers, then $x \in \{m < y : f(m) = 1\}$.

If $z \leq y$, then by $z = x \vee y$, $z \in Cl(\{m < y : f(m) = 1\} \cup \{y\} \cup I)$, and hence

$$\text{Cl}(\{m < z : f(m) = 1\} \cup \{z\} \cup I) \subseteq \text{Cl}(\{m < y : f(m) = 1\} \cup \{y\} \cup I).$$

Since $f(y) = 1$, $\mathrm{Cl}(\{m < y : f(m) = 1\} \cup \{y\} \cup I) \cap F = \emptyset$, $\mathrm{Cl}(\{m < z : f(m) = 1\} \cup \{z\} \cup I) \cap F = \emptyset$, implying that $f(z) = 1$ and hence $z \in M$.

If $z > y$, then by $z = x \vee y$, and both x, y are in $\{m < z : f(m) = 1\}$, $z \in Cl(\{m < z : f(m) = 1\})$. Thus we have

$$\mathrm{Cl}(\{m < z : f(m) = 1\} \cup I) {=} \mathrm{Cl}(\{m < z : f(m) = 1\} \cup \{z\} \cup I),$$

and hence by $\mathrm{Cl}(\{m < z : f(m) = 1\} \cup I) \cap F = \emptyset$, $\{m < z : f(m) = 1\} \cup \{z\} \cup I$ is good, implying that $f(z) = 1$ and $z \in M$.

- $I \subseteq M$ and $M \cap F = \emptyset$.

Since $I \cap F = \emptyset$, for all $x \in I$, $f(x) = 1$, and $I \subseteq M$. Also because for all n with $f(n) = 1$, $n \notin F$ and so $M \cap F = \emptyset$.

- M is relatively maximal with respect to F.

Otherwise, there exists an $x \notin M$ with $\mathrm{Cl}(M \cup \{x\}) \cap F = \emptyset$. Thus $\mathrm{Cl}(\{m : m < x \text{ and } f(m) = 1\} \cup \{x\} \cup I) \cap F = \emptyset$, and $f(x) = 1$, $x \in M$, a contradiction.

□

From Theorem 1.3.8, we know that over Π^1_1-CA$_0$, every ideal of a countable poset extends to a maximal ideal. Corollary 3.2.2 says that over WKL$_0$, every proper ideal of a countable distributive lattice extends to a prime ideal. Now we consider the maximal ideal extension for a given ideal in a countable lattice and prove that this extension property is equivalent to **RMI**.

Theorem 4.2.3. *The following are equivalent over RCA$_0$:*

1. ***RMI**: In a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$, for an ideal I, and a filter F of L, if $I \cap F = \emptyset$, then there exists an ideal M which is maximal among those ideals containing I but disjoint from F.*

2. *Every proper ideal on a countable lattice extends to a maximal ideal.*

Proof. ($1 \Rightarrow 2$): Let $(L, \preceq, \vee, \wedge, 0, 1)$ be a countable lattice, and I be a proper ideal. Let $F = \{1\}$. Obviously, $I \cap F = \emptyset$. By **RMI**, there exists an ideal M which is maximal among those ideals containing I and disjoint from F. This M is a maximal ideal containing I.

($2 \Rightarrow 1$): Let $(L, \preceq, \vee, \wedge, 0, 1)$ be a countable lattice, I be an ideal and F be a filter of L, with $I \cap F = \emptyset$. Define a new lattice $L' = \{1'\} \cup (L \setminus F)$ with the order $\preceq'$ defined as follows:

- for all a, b in $L \setminus F$, $a \preceq' b \iff a \preceq b$.
- for all a in $L \setminus F$, $a \preceq' 1'$.

The lattice $(L', \preceq', 0, 1', \vee', \wedge')$ can be defined over RCA$_0$, because for $a, b \in L \setminus F$,

$$a \vee' b = \begin{cases} a \vee b & \text{if } a \vee b \notin F, \\ 1' & \text{otherwise.} \end{cases} \qquad a \wedge' b = a \wedge b.$$

Note that I is also a proper ideal in L'. By the assumption, I can be extended to a maximal ideal M of L'. We claim that M is also an ideal of L, and is maximal among those ideals containing I and disjoint from F.

We already know that $I \subseteq M$. As $M \subseteq L' \setminus \{1'\} = L \setminus F$, $M \cap F = \emptyset$.

- M is an ideal of L.

(1) For $a \preceq b$ in L, if $b \in M$, then $a \in L \setminus F$ and hence $a \preceq' b$, implying that $a \in M$ because M is an ideal of L'.

(2) For $a, b \in M$, $a \vee' b \in M$, so $a \vee' b \neq 1'$. and hence $a \vee' b = a \vee b$, $a \vee b \in M$.

Now suppose for a contradiction that M is not maximal among ideals containing I but disjoint from F. Then there is some $c \notin M$ such that for any $m \in M$, $m \vee c \notin F$, implying that $m \vee' c \neq 1'$, which contradicts the assumption that M is maximal ideal in L'. This shows that M is maximal among those ideals containing I but disjoint from F. □

A question remaining open is whether **RMI** implies ACA$_0$ over RCA$_0$ or not.

4.3 Ascending chain condition

In classical lattice theory, we have the following equivalent statements [21]:

Theorem 4.3.1. *Given a lattice L, the following are equivalent:*

(1) L satisfies the ascending chain condition, ACC for short.

(2) Every ideal of L is finitely generated (we call an ideal I finitely generated if there exists a finite subset A of I such that $I = Cl(A)$).

(3) Every relatively maximal ideal of L is finitely generated.

We study the logical strength of implications involved as follows:

Theorem 4.3.2. *(RCA_0) If a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$ satisfies the ascending chain condition, then every ideal of it is finitely generated. Thus, (1)⇒ (2) and (1)⇒ (3) in Theorem 4.3.1 can be proved in RCA_0.*

Proof. Let $I = \{a_0, a_1, a_2, \cdots\}$ be an ideal of L. For each n, define

$$b_n = a_0 \vee a_1 \vee a_2 \vee \cdots \vee a_n.$$

Then each $b_n \in I$ and $b_0 \preceq b_1 \preceq \cdots \preceq b_n \preceq \cdots$ is an ascending chain. As L satisfies the ascending chain condition, there exists N_0 such that for all $n \geq N_0$, $b_n = b_{N_0}$ and hence $I = Cl(\{b_0, b_1, \cdots, b_{N_0}\}) = Cl(\{a_0, a_1, \cdots, a_{N_0}\})$, I is finitely generated. $\square$

Theorem 4.3.3. *(ACA_0) For a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$, if every relatively maximal ideal of L is finitely generated, then L satisfies the ascending chain condition.*

Proof. Suppose that $a_0 \preceq a_1 \preceq \cdots \preceq a_n \preceq \cdots$ is an ascending chain of L. Let $I = \{x \in L : \exists a_n \in L\ (x \preceq a_n)\}$ and $F = \{x \in L : \forall a_n \in L\ (x \succeq a_n)\}$, I and F exist by arithmetical comprehension, and they are nonempty because $0 \in I$ and $1 \in F$. It is easy to check that I is an ideal and F is a filter of L.

Claim: $I \cap F \neq \emptyset$. Otherwise, by **RMI**, there exists a relatively maximal ideal $M \supseteq I$ w.r.t. F. By assumption, M is generated by a finite subset $P \subseteq M$, i.e., $M = Cl(P) = \downarrow(\bigvee P)$ (by Remark 2.2.4). Since $a_n \in I \subseteq M$, $a_n \preceq \bigvee P$ for all $n \in \mathbb{N}$, and hence, $\bigvee P \in F$, contradicting $M \cap F = \emptyset$.

So we choose $x \in I \cap F$, since $x \in I$, then $x \preceq a_{N_0}$ for some N_0. Since x is also in F, $x \succeq a_{N_0}$, and hence $x = a_{N_0}$. So for all $n \geq N_0$, we have $a_n = a_{N_0}$.

□

Another well-known fact for lattices is that if a lattice L satisfies ACC, then L is complete. We will show that this statement can be proved within ACA_0.

Theorem 4.3.4. *(ACA_0) If a countable lattice $(L, \preceq, \vee, \wedge, 0, 1)$ satisfies ACC, then for every nonempty subset S of L, $\bigvee S$ exists. Furthermore, L is a complete lattice.*

Proof. Assume that S is nonempty, let

$$D = \{\bigvee F : F \text{ is a nonempty finite subset of S}\}.$$

D exists by Σ^0_1-comprehension. As L satisfies ACC, by Lemma 2.4.1, D has a maximal element, b say.

Claim: b is the least upper bound of S.

As $b \in D$, $b = \bigvee F_b$ for some finite subset F_b of S. For each $a \in S$, $\bigvee(F_b \bigcup \{a\})$ exists and $b = \bigvee F_b \preceq \bigvee(F_b \bigcup \{a\})$, which implies that $b = \bigvee(F_b \bigcup \{a\})$, as b is maximal in D. Hence $b \succeq a$. Therefore, b is an upper bound of S.

Suppose that c is another upper bound of S, then c is an upper bound of F_b, and $b = \bigvee F_b \preceq c$. This shows that b is the least upper bound of S. Thus, ACA_0 can prove the existence of $\bigvee S$.

To prove that L is complete, we need to show that for every nonempty subset S of L, $\bigwedge S$, the greatest lower bound of S, exists. Let

$$S' = \{x \in L : (\forall y \in S)(x \preceq y)\}.$$

S' exists by Π^0_1-comprehension, and $S' \neq \emptyset$ since $0 \in S'$.

The argument above ensures the existence of the least upper bound of S', d say.

We claim that d is the greatest lower bound of S. Note that any $x \in S$ is an upper bound of S', and hence $d \preceq x$, which implies that d is a lower bound of S. Suppose that d' is another lower bound of S, then $d' \in S'$, and $d' \preceq d$, as d is an upper bound of S'. This proves the claim.

Thus, L is a complete lattice.

□

4.4 Existence of $\mathcal{J}(L)$ and ACA$_0$

In the section on Birkhoff's Representation Theorem, we have seen that for a finite lattice L, $\mathcal{J}(L)$, the set of join-irreducible elements, exists and is join-dense in L, which can be proved within RCA$_0$. The following theorem says that if L is countably infinite, then the existence of $\mathcal{J}(L)$ is equivalent to ACA$_0$ over RCA$_0$.

Theorem 4.4.1. *The following are equivalent over RCA$_0$:*

(1) ACA$_0$;

(2) For every countable lattice L, $\mathcal{J}(L)$ exists.

Proof. We first assume ACA$_0$. Then

$$x \in \mathcal{J}(L) \iff x \neq 0 \text{ and } \forall a, b \in L (x = a \vee b \Rightarrow x = a \text{ or } x = b),$$

and $\mathcal{J}(L)$ exists, by Π^0_1-comprehension.

We now prove the other direction. We assume that $\mathcal{J}(L)$ exists for every countable lattice L, and prove that the range of any one-to-one function $f : \mathbb{N} \to \mathbb{N}$ exists.

Consider $L = \{0, 1\} \cup \{a^i_k, b_i : k, i \in \mathbb{N}\}$ with the order $\preceq$ such that

- $a^i_k \preceq b_j \iff i = j$ and $(\exists m \leq k)\ (f(m) = i)$;

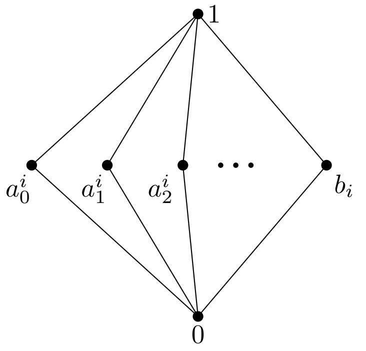


Figure 4.1: $i \notin rng(f)$

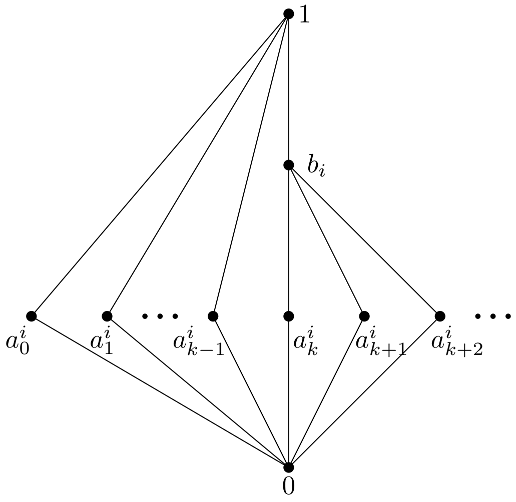


Figure 4.2: $i \in rng(f)$ and $i = f(k)$

- $a_m^i \preceq a_n^j \iff i = j$ and $m = n$;
- $b_i \preceq b_j \iff i = j$.

Figure 4.1 and Figure 4.2 give the graph of the i-th block for the lattice L according to whether i is in the range of f or not. L is a lattice which can be defined over RCA$_0$ because

$$a_k^i \vee b_j = \begin{cases} b_j & \text{if } i = j \text{ and } \exists m \leq k\ (f(m) = i), \\ 1 & \text{otherwise.} \end{cases}$$

$$a_k^i \wedge b_j = \begin{cases} a_k^i & \text{if } i = j \text{ and } \exists m \leq k(f(m) = i, \\ 0 & \text{otherwise.} \end{cases}$$

$$a_m^i \vee a_n^j = \begin{cases} a_m^i & \text{if } i = j, m = n, \\ b_i & \text{if } i = j, m \neq n \text{ and there exist } p \leq m, q \leq n\ (f(p) = f(q) = i), \\ 1 & \text{otherwise.} \end{cases}$$

$$a_m^i \wedge a_n^j = \begin{cases} a_m^i & \text{if } i = j, m = n, \\ 0 & \text{otherwise.} \end{cases}$$

$$b_i \vee b_j = \begin{cases} b_i & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

$$b_i \wedge b_j = \begin{cases} b_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

From the graph above, it's easy to see that

$$i \in rng(f) \iff b_i \text{ is not join-irreducible.}$$

By assumption, $\mathcal{J}(L)$ exists. Now let $X = \{i \in \mathbb{N} : b_i \notin \mathcal{J}(L)\}$ and X exists by Σ_0^0-comprehension. Then $\forall n(n \in X \iff \exists m(f(m) = n))$, and $rng(f)$ exists. Therefore, ACA$_0$ holds. $\square$

Chapter 5

Rudin's Lemma

5.1 Introduction

Domain theory formalizes the intuitive ideas of approximation and convergence in a very general way. It provides a fundamental tool in the study of computation theory. The basic idea was proposed independently by Scott in paper [28] and Ershov in paper [10] around the 1970's. Domain theory studies various topologies on partially ordered sets, such as the Alexandrov topology, the Scott topology, and the Lawson topology, etc. A domain refers to a continuous *dcpo*, i.e., a continuous directed complete partial orders.

In their work of generalizing the theory of *continuous dcpos* to general ordered structures, and also in the search of complete lattices on which the Lawson topology is Hausdorff, Gierz, Lawson and Stralka introduced quasicontinuous domains in [16], where many good properties of domains remain valid. In this theory, one of the milestone results is Rudin's Lemma, which enables us to find a "cross-section" of certain descending family of sets.

Rudin's Lemma [15]: Given a poset $(P, \preceq)$, and $\{F_i\}_{i \in I}$ a $\sqsubseteq$-directed family of nonempty finite subsets of P, there is a $\preceq$-directed subset $D \subseteq \bigcup_{i \in I} F_i$ that meets all F_i.

Rudin's Lemma implies that the Scott topology on a quasicontinuous domain is locally compact and sober, and that the Lawson topology on a quasicontinuous domain is regular and Hausdorff.

In this chapter, we study Rudin's Lemma from a reverse mathematics point of view and prove that Rudin's Lemma is equivalent to ACA_0 over RCA_0. We organize this chapter as follows. In Section 5.2, we will introduce basic definitions about domain theory. We will show in this section that RCA_0 can prove the equivalence between chain completeness and directed completeness. In Section 5.3, we will prove the equivalence between Rudin's Lemma and ACA_0 over RCA_0. In Section 5.4, we will study a variant of Rudin's Lemma, and prove that this variant is also equivalent to ACA_0 over RCA_0.

5.2 Preliminaries

We recall the following definitions:

Definition 5.2.1. *Let $(X, \preceq)$ be a countable poset. For $A, B \subseteq X$, define*

$$A \sqsubseteq B \iff \uparrow B \subseteq \uparrow A.$$

The relation $\sqsubseteq$ on the power set of X is called the Smyth pre-order.

It is obvious that $A \sqsubseteq B \iff (\forall b \in B)\ (\exists a \in A)\ [a \preceq b]$.

Definition 5.2.2. *Let $(X, \preceq)$ be a countable poset. A subset D of X is directed if for any x, y in D, there exists $z \in D$ such that $x, y \preceq z$.*

So, in Rudin's Lemma , "$\{F_i\}_{i\in I}$ is a $\sqsubseteq$-*directed family*" means that for all F_i and F_j, there exists an F_k such that $F_i, F_j \sqsubseteq F_k$.

Definition 5.2.3. *For a poset $(X, \preceq)$,*

(1) X is directed complete if for any directed subset $D \subseteq X$, the least upper bound of D exists in X.

(2) X is chain complete if for any chain $\{x_i\}_{i\in\mathbb{N}} \subseteq X$, the least upper bound of the chain exists in X.

In [23], Markowsky proved that assuming the well-ordering principle, directed completeness and chain completeness are equivalent. Here, for countable posets, we show that this equivalence can be proved in RCA_0.

Proposition 5.2.4. *(RCA_0) For a countable poset $(X, \preceq)$, X is directed complete if and only if X is chain complete.*

Proof. One direction is trivial, since chains are all directed.

For the other direction, let X be chain complete, and $D \subseteq X$ be a directed subset. Let $D = \{d_0, d_1, \ldots, d_n, \ldots\}$. Define a chain $E \subseteq D$ as follows: $e_0 = d_0$ and for $n \geq 1$, $e_n \in D$ with $e_{n-1}, d_n \preceq e_n$. e_n exists because D is directed. The chain $E = \{e_0, e_1, \ldots, e_n, \ldots\}$ exists by Σ^0_0-comprehension since

$$d_n \in E \iff \exists m \leq n[d_n = e_m].$$

It is easy to see that $\sup D = \sup E$. Since X is chain-complete, $\sup E$ exists, and hence $\sup D$ exists. □

The main part of this chapter is to prove the equivalence between Rudin's Lemma and ACA_0. Below is an easy fact about finite directed sets provable within RCA_0.

Proposition 5.2.5. *(RCA_0) Let $(X, \preceq)$ be a countable poset and A be a finite subset of X. Then A is directed if and only if A contains a greatest element of A.*

Proof. Suppose $A = \{a_0, a_1, \cdots, a_n\}$. If A is directed, then let $b_0 = a_0$ and for $0 < k \leq n$, let $b_k \in A$ with $b_{k-1}, a_k \preceq b_k$, so $b_n \in A$ is the greatest element.

For the other direction, if A contains a greatest element b, then certainly any two elements of A are less than or equal to b, and hence A is directed. □

In Proposition 5.2.5, the finiteness of A is necessary for the forward direction. A simple counterexample is when A is the set of all finite subsets of $\mathbb{N}$ with set inclusion $\subseteq$ as its order. Obviously, A is directed but has no greatest element.

5.3 Rudin's Lemma and ACA_0

In this section, we show the equivalence between Rudin's Lemma and ACA_0 over RCA_0.

Theorem 5.3.1. *The following are equivalent over RCA_0:*

(1) ACA_0;

(2) Rudin's Lemma: Given a countable poset $(X,\preceq)$, and $\mathcal{F} = \langle F_i : i \in \mathbb{N}\rangle$ a $\sqsubseteq$-directed family of nonempty finite subsets of X, there exists a $\preceq$-directed subset $D \subseteq \bigcup\limits_{i\in\mathbb{N}} F_i$ that meets all F_i.

Proof. We first prove Rudin's Lemma from ACA_0. Let $X \subseteq \mathbb{N}$. We define a function $f : \mathbb{N} \to \mathbb{N}$ as follows:

$$\begin{aligned} f(0) &= 0, \\ f(n+1) &= \mu k\ [F_{n+1} \sqsubseteq F_k,\ F_{f(n)} \sqsubseteq F_k] \end{aligned}$$

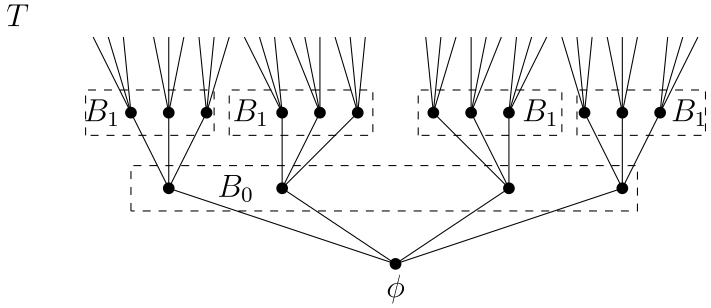
f is well-defined since $\mathcal{F}$ is $\sqsubseteq$-directed. Now, consider a new family $\mathcal{B} = \langle B_i : i \in \mathbb{N}\rangle$ where $B_{2n} = F_{f(n)}$ and $B_{2n+1} = F_{n+1}$.

$\mathcal{B}$ exists and is $\sqsubseteq$-directed. Obviously, $\mathcal{B}$ contains all F_i and $\bigcup\limits_{i\in\mathbb{N}} B_i = \bigcup\limits_{i\in\mathbb{N}} F_i$. So it is sufficient to find a $\preceq$-directed subset $D \subseteq \bigcup\limits_{i\in\mathbb{N}} B_i$ that meets all B_i.

Define a tree $T \subseteq \omega^{<\omega}$ by letting

$$\sigma \in T \iff \forall n < lh(\sigma)[\sigma(n) \in B_n].$$

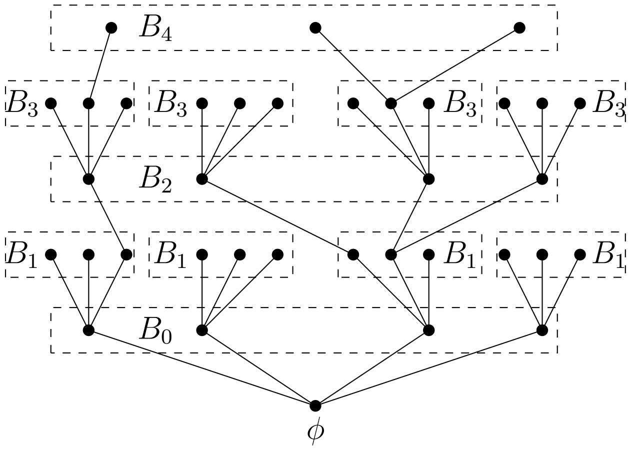
T is an infinite, finitely branching tree, because every B_n is a finite set.



Now, we construct a subtree S of T such that for any infinite path g of S, the nodes on g form a $\preceq$-directed set, i.e., $\{g(n) : n \in \mathbb{N}\}$ is $\preceq$-directed. S is constructed as follows: $\emptyset \in S$, and for $\sigma \in T$, if $lh(\sigma) = 1$, put σ on S; If $lh(\sigma) = 2n (n \geq 1)$, put σ on S if each $\eta \subset \sigma$ is on S; if $lh(\sigma) = 2n + 1 (n \geq 1)$, put σ on S if each $\eta \subset \sigma$ is on S and $\sigma(2n-2), \sigma(2n-1) \preceq \sigma(2n)$.

Thus, if $\sigma \in S$, then $\sigma(2k-2), \sigma(2k-1) \preceq \sigma(2k)$ for any k with $2k < lh(\sigma)$. So $\sigma(2k)$ is the greatest element among $\{\sigma(n) : n \leq 2k\}$, implying that $\{\sigma(n) : n \leq 2k\}$ is $\preceq$-directed by Proposition 5.2.5.

S



Obviously, S is a finitely branching tree. We have the following claims for S:

1. For each x in B_{2n}, $n \in \mathbb{N}$, there exists a $\sigma \in S$ with $lh(\sigma) = 2n + 1$ and $\sigma(2n) = x$.
2. S is an infinite tree.
3. For every infinite path g of S, the set $\{g(n) : n \in \mathbb{N}\}$ is $\preceq$-directed.

By Claim 2, using König lemma (which is equivalent to ACA$_0$), S contains at least one infinite path, g say. By Claim 3, using Σ^0_1-comprehension,

$$D = \{x : \exists n[x = g(n)]\}$$

exists and forms a $\preceq$-directed set. As D meets all B_i, D is the desired set.

We prove Claim 1 by induction. Let

$$\phi(n) \iff \forall x\ [x \in B_{2n} \to \exists \sigma \in S[lh(\sigma) = 2n+1 \text{ and } \sigma(2n) = x]].$$

It is obvious that $\phi(0)$ is true. Suppose that $\phi(k)$ is true for all $k \leq n$. To show that $\phi(n+1)$ is true, we consider for each $x \in B_{2n+2} = F_{f(n+1)}$, by $B_{2n+1} = F_{n+1} \sqsubseteq F_{f(n+1)}$ and $B_{2n} = F_{f(n)} \sqsubseteq F_{f(n+1)}$, there is some $y \in B_{2n+1}$, and $z \in B_{2n}$ such that $y \preceq x$ and $z \preceq x$. By the induction hypothesis, $\phi(n)$ holds, and hence there is an $\eta \in S$ such that $lh(\eta) = 2n+1$ and $\eta(2n) = z$. By the construction of S, $\eta^\frown y \in S$, and hence $\eta^\frown y^\frown x \in S$. So we choose $\sigma = \eta^\frown y^\frown x$, and $\phi(n+1)$ holds.

Claim 2 follows directly from Claim 1. To prove Claim 3, suppose that g is an infinite path of S. Let

$$D = \{x : \exists n[x = g(n)]\}.$$

We will prove that D is $\preceq$-directed. For any $x, y \in D$, there exists an $l \in \mathbb{N}$ such that $x, y \in \{g(0), g(1), \cdots, g(l)\}$. Without loss of generality, assume that l is even, then $g(l)$ is the greatest element among $\{g(0), g(1), \cdots, g(l)\}$, and hence $x, y \preceq g(l)$. This shows that D is directed.

For the other direction, we assume Rudin's Lemma, and let $f : \mathbb{N} \to \mathbb{N}$ be a one-to-one function. Consider $(\mathbb{N}, \leq)$ with the standard order, and define

$$A_{\langle i,j\rangle} = \begin{cases} \{2i\}, & \text{if there is some } k \leq j \text{ such that } f(k) = i; \\ \\ \{1\}, & \text{otherwise.} \end{cases}$$

where $\langle\cdot,\ \cdot\rangle$ is a standard coding from $\mathbb{N}^2$ to $\mathbb{N}$. $A_{\langle i,j\rangle}$ exists by Σ^0_0-comprehension:

$$x \in A_{\langle i,j\rangle} \iff \big[x = 2i \text{ and } \exists k \leq j(f(k) = i)\big] \text{ or } \big[x = 1 \text{ and } \forall k \leq j(f(k) \neq i)\big].$$

$\langle A_{\langle i,j\rangle} : i, j \in \mathbb{N}\rangle$ is a $\sqsubseteq$-directed family because any two $A_{\langle i_1,j_1\rangle}, A_{\langle i_2,j_2\rangle}$ are $\sqsubseteq$-comparable. By Rudin's Lemma, there exists a set $D \subseteq \bigcup\limits_{i,j\in\mathbb{N}} A_{\langle i,j\rangle}$ such that D meets all $A_{\langle i,j\rangle}$. Let $X = \{i \in \mathbb{N} : 2i \in D\}$, and X exists by Σ^0_0-comprehension. Then $\forall n(n \in X \iff \exists m(f(m) = n))$, so ACA_0 holds. □

5.4 A variant of Rudin's Lemma

One may ask the following question: in Rudin's Lemma, is it possible to find a directed subset D of $\bigcup_{i\in I} F_i$ which intersects each F_i at exactly one point? Heckmann and Keimel pointed out in [17] that in general, we cannot have a positive solution. For example, if the family contains the subsets $\{1\}$, $\{2\}$ and $\{1,2\}$, then the directed set D must intersect $\{1,2\}$ at two points. However, we can give a positive solution to this question if $\{F_i\}_{i\in I}$ is a disjoint family. In the following, we prove that this variant is equivalent to ACA_0 over RCA_0.

Theorem 5.4.1. *The following are equivalent over RCA_0:*

(1) ACA_0;

(2) Given a countable poset $(X,\preceq)$, and $\mathcal{F}=\langle F_i : i\in\mathbb{N}\rangle$ a $\sqsubseteq$-directed family of disjoint nonempty finite subsets of X, then there exists a $\preceq$-directed subset $E\subseteq\bigcup_{i\in\mathbb{N}} F_i$ such that each $E\cap F_i$ is a singleton.

Proof. We first assume ACA_0. The proof of (2) is quite similar to the one given in the former theorem. We will construct B_n, $n\in\mathbb{N}$, and D such that each $B_n\cap D$ is $\preceq$-linearly ordered. We will then select the greatest element in $B_n\cap D$ as the unique witness.

First, we define functions $f,g:\mathbb{N}\to\mathbb{N}$ simultaneously as follows:

$$f(0)=0, g(0)=1,$$

$$f(n+1)=\mu k[F_{f(n)},F_{g(n)}\sqsubseteq F_k],$$

$$g(n+1)=\mu k[k\notin\{f(0),g(0),\cdots,f(n),g(n),f(n+1)\}].$$

Now, consider a new family $\mathcal{B}=\langle B_i : i\in\mathbb{N}\rangle$ where $B_{2n}=F_{f(n)}, B_{2n+1}=F_{g(n)}$ Obviously, $\mathcal{B}$ is also a $\sqsubseteq$-directed family and contains all F_i, with $\bigcup_{i\in\mathbb{N}} B_i=\bigcup_{i\in\mathbb{N}} F_i$, so it is sufficient to find a $\preceq$-directed subset $E\subseteq\bigcup_{i\in\mathbb{N}} B_i$ that meets each B_i at exactly one element, and hence meets each F_i also at exactly one element.

$\mathcal{B}$ has the following properties: for any i, j, either $B_i = B_j$ or $B_i \cap B_j = \varnothing$, but for each n, $B_{2n+1} \neq B_0, B_1, \cdots, B_{2n}$.

We now construct a tree $T \subseteq \omega^{<\omega}$ and $S \subseteq T$ as in the proof of Theorem 5.3.1. By the same argument, we can find a directed set D which meets all B_i. We then construct a subset $E \subseteq D$ such that:

$$x \in E \iff (x \in D) \text{ and } \big[\forall i \; \forall y \; (x, y \in B_i \cap D \to y \preceq x)\big],$$

i.e., we choose the greatest element from $B_i \cap D$ as a witness. We have the following three claims.

1. For all i, $B_i \cap D$ is $\preceq$-linearly ordered. So the greatest element of $B_i \cap D$ exists in $B_i \cap D$.
2. E intersects each B_i at exactly one element.
3. E is directed.

For Claim 1, we let $D = \{h(n) : n \in \mathbb{N}\}$, where h is an infinite path in the tree S. Fix i, we consider $B_i \cap D$. Assume that B_i repeats, i.e., $B_i = B_{j_1} = B_{j_2} = B_{j_3} = \cdots$ with $i < j_1 < j_2 < j_3 < \cdots$, then by the construction, we have all $j_1, j_2, \cdots$ are even number. Hence $h(i) \preceq h(j_1) \preceq h(j_2) \preceq h(j_3) \preceq \cdots$. So $B_i \cap D$ is $\preceq$-linearly ordered. Because B_i is finite, the greatest element of $B_i \cap D$ exists.

Claim 2 follows directly from Claim 1 since the greatest element is unique.

For Claim 3, note that for all $x, y \in E$, $x, y \in D$. As D is directed, there exists some $z \in D$ such that $x, y \preceq z$. Assume that $z \in B_i$ and w is the greatest element in $B_i \cap D$. Then $z \preceq w$ and $w \in E$, and hence w is an upper bound of x and y in E. Thus, E is directed.

We now prove the other direction. Assume that $f : \mathbb{N} \to \mathbb{N}$ is a one-to-one function. Consider $(\mathbb{N}, \leq)$ with the standard order, and define

$$A_{\langle i,j \rangle} = \begin{cases} \{5^{i+1}\} & \text{if } f(j) = i, \\ \\ \{2^i \cdot 3^j\} & \text{otherwise.} \end{cases}$$

$A_{\langle i,j\rangle}$ exists by Σ^0_0-comprehension, and they are disjoint from each other because f is one-to-one. $\{A_{\langle i,j\rangle} : i, j \in \mathbb{N}\}$ is also a $\sqsubseteq$-directed family because any two $A_{\langle i_1,j_1\rangle}, A_{\langle i_2,j_2\rangle}$ are $\sqsubseteq$-comparable. By our assumption, the variant of Rudin's Lemma, there exists a set $E \subseteq \bigcup_{i,j\in\mathbb{N}} A_{\langle i,j\rangle}$ such that E meets all $A_{\langle i,j\rangle}$. In fact, $E = \bigcup_{i,j\in\mathbb{N}} A_{\langle i,j\rangle}$ since each $A_{\langle i,j\rangle}$ has only one element. Let $X = \{i \in \mathbb{N} : 5^{i+1} \in E\}$. X exists by Σ^0_0-comprehension. Then $\forall n(n \in X \iff \exists m(f(m) = n))$, so the range of f exists. Therefore ACA_0 holds.

$\square$

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