Robust Fuzzy Model Predictive Control of Discrete-Time Takagi-Sugeno Systems With Nonlinear Local Models

Long Teng, Youyi Wang, Wenjian Cai, and Hua Li

***Abstract*—Robust fuzzy model predictive control of discrete nonlinear systems is investigated in this work. A recently developed Takagi-Sugeno (T-S) fuzzy approach which uses nonlinear local models is adopted to approximate nonlinear systems. A critical issue that restricts the practical application of classical model predictive control is the online computational cost. For model predictive control of T-S fuzzy systems, the online computational burden is even worse. Especially for complex systems with severe nonlinearities, parametric uncertainties, and disturbances, existing model predictive control of T-S fuzzy systems usually leads to a very conservative solution or even no solution in some occasions. However, more relaxed results can be achieved by the proposed fuzzy model predictive control approach which adopts T-S systems with nonlinear local models. Another advantage is that, online computational cost of the optimization problem through solving matrix inequalities can be significantly reduced at the same time. Simulations on a numerical example and a two-tank system are presented to verify the effectiveness and advantages of the proposed method. Comparisons among several T-S fuzzy approaches are illustrated and show that the best settling time is achieved via the proposed method.**

***Index Terms*—T-S fuzzy systems, model predictive control, robust control, input-to-state stability.**

I. Introduction

Model predictive control (MPC) has attracted many research interests from academic community, and has been widely used in industrial process control for decades. Usually, at each sampling instant, an optimal control sequence are computed via minimization of a finite horizon cost function under some constraints, and stability and recursive feasibility are guaranteed [1]. Then the first control law is merely implemented, and the optimization is repeated at the next instant with the newly obtained system information. On the other hand, the T-S fuzzy model is firstly proposed in [2] and has been widely investigated for control of nonlinear systems [3]–[10].

The industrial processes are mostly nonlinear systems. Considering MPC of such systems, one approach is to deal with nonlinearities directly. However, when the nonlinearities are very complex, nonlinear model predictive control (NMPC) may not able to execute effectively [11]. In addition, the combination of MPC and T-S fuzzy systems has become a very good alternative. Several T-S fuzzy control techniques have been investigated for fuzzy model predictive control (FMPC) such as, common and piecewise Lyapunov functions [12], parallel distributed compensation (PDC) and non-PDC techniques [13], [14]. Especially, complex nonlinear systems with both structural uncertainties and persistent disturbances are considered in [15] that, both online and off-line robust FMPC (RFMPC) approaches are provided. In [16], FMPC of nonlinear systems modeled as stochastic T-S fuzzy models subjected to data package loss in a network scenario is investigated. Output feedback predictive control of T-S systems has been considered in [17], [18], in case that the state information cannot be measured. In [19], the cooperative fuzzy model predictive control (CFMPC) is investigated for input-coupled multi-agent systems. RFMPC of systems with multiple delays and time-varying delay are considered, respectively, in [20] and [21]. In addition, MPC of a hybrid T-S fuzzy model is investigated in [22], [23], and the results show its advantages over the approach which adopts a hybrid linear model. The motivations of FMPC are that, compared with traditional fuzzy control algorithms, the implementation of MPC to T-S fuzzy systems provides a systematic method to handle constraints, and optimal control inputs are calculated at each sampling instant by minimizing the cost function.

Many contributions have been achieved to reduce the conservativeness of traditional T-S fuzzy controllers. As it is mentioned above that, which has already been utilized in FMPC, the piecewise Lyapunov functions are developed in replace of common Lyapunov functions to obtain less conservative results [24]. Compared with the PDC technique for T-S fuzzy systems [25], the control performance can be improved by adopting non-PDC control laws and nonquadratic Lyapunov functions [26]. T-S fuzzy systems with imperfect premise matching are investigated in [27], the computational cost induced by the complexity of fuzzy controller under the PDC technique is alleviated. In addition, nonlinear systems are modeled as polynomial fuzzy systems other than the T-S method [28], and the sum-of-squares (SOS) techniques are utilized, which shows that stability analysis and control design in terms of SOS are more general and relaxed than those of T-S approaches using matrix inequalities. Moreover, in contrast to the conventional fuzzy control methods which adopt T-S fuzzy systems with linear local models (LLMs), a new T-S fuzzy modeling and

Corresponding author: Youyi Wang. Tel.: +65 67904537; Email: eyywang@ntu.edu.sg.

L. Teng is with the Energy Research Institute @ NTU, Interdisciplinary Graduate School, and with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, e-mail: (lteng001@e.ntu.edu.sg).

Y. Wang and W. Cai are with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, e-mail: (eyywang@ntu.edu.sg, ewjcai@ntu.edu.sg).

H. Li is with the School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore, e-mail: (lihua@ntu.edu.sg).

control approach using nonlinear local models (NLMs) and nonlinear local controllers is developed in [29] for continuous-time nonlinear systems and extended to the discrete case in [30]. Compared with the LLM-based T-S fuzzy approach, the number of fuzzy rules is reduced via the NLM-based fuzzy approach, as a consequence, a less conservative result can be obtained and the computational cost can be significantly reduced simultaneously. It is noted that the NLM-based fuzzy systems has been adopted for fault estimation and detection [31], [32].

Intrinsically, the control method provided in [29] for T-S fuzzy systems with NLMs belongs to the robust control, in which the details of the nonlinearity at each instant are ignored in the controller design. Instead, it is regarded as uncertainties within a convex hull and the robust control technique is adopted to deal with it. Therefore, such a kind of controller is more robust than the one acquired from traditional T-S fuzzy systems with LLMs.

For the context of FMPC, although some techniques have been discussed to reduce conservatism, such as the utilization of piecewise Lyapunov functions and non-PDC techniques for FMPC in [12] and [13], respectively, there still are some drawbacks for existing FMPC approaches that restrict their implementations in practical applications. It is noted that the aforementioned FMPC methods only adopt LLM-based T-S fuzzy systems. One problem is the conservatism induced by the LLM-based T-S fuzzy models especially if the systems contain severe nonlinearities. Moreover, it usually result in a heavy computational cost due to the structural complexity of the fuzzy controller by solving the matrix inequalities problem. In particular, if RFMPC of systems with disturbances and uncertainties are considered [15], the whole computational burden of the online optimization problem may become even worse. As a result, it may not be appropriate for practical implementation considering the sampling restrictions.

Inspired by the NLM-based fuzzy approach [29], [30], RFMPC of nonlinear systems modeled as NLM-based T-S fuzzy systems is investigated in this work. An optimal control law is obtained at each sampling instant by solving an online optimization problem. The main contributions of this work can be summarized as follows: (1) It is noted that Only H_∞ and H_2 control problems are investigated in [29] and [30], which require merely off-line computation to obtain the control law. Thus the lower computational cost advantage of NLM-based T-S systems is not fully utilized in these works. (2) Especially for complex systems with severe nonlinearities, parametric uncertainties, and disturbances, model predictive control of T-S fuzzy systems with LLMs usually leads to a very conservative solution or even no solution in some occasions. However, more relaxed and robust results can be achieved by the proposed method. (3) Compared with an existing RFMPC approach with LLMs, the computational cost which is an critical issue for the practical implementation of MPC, can be significantly reduced. (4) In addition, we show that a good control performance can be achieved by the proposed method among several typical fuzzy control and FMPC algorithms in stabilization control.

It is noted that an early version of this work which considers merely disturbances has been published in [33]. In this publication, both disturbances and parametric uncertainties are investigated. In addition, the detailed algorithm is not provided in the early version, however, it is summarized in this work to make the proposed method more readable and more reasonable. Moreover, a multi-input numerical system and a practical application are illustrated in the simulation. Furthermore, comparisons among different approaches are provided in more details.

The rest of the paper is structured as follows. In Section II, some preliminary knowledge about NLM-based T-S fuzzy systems and the control scheme of MPC are introduced. In Section III, based on the concept of robust positive invariance (RPI), the computation of terminal constraint set for NLM-based T-S fuzzy systems is provided, and the control algorithm is presented in sequence. In Section IV, the results are extended to systems with both disturbances and structural uncertainties. In Section V, simulations on a numerical example and a two-tank system are illustrated to verify the effectiveness of the proposed method. Finally, some conclusions are drawn in Section VI.

Notations: $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{Z}_{[m,n]}$ denote the set of real numbers, the set of non-negative reals, the set of integers, the set of non-negative integers, and the set of integers in the interval $[m,n]$, respectively. $\|x\|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$. $\mathbf{d}_{[c1,c2]} := \{d(l)\}_{l\in\mathbb{Z}_{[c1,c2]}}$, with $c1, c2 \in \mathbb{Z}$, denotes a sequence that is ordered monotonically with respect to the index $l \in \mathbb{Z}_{[c1,c2]}$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$, i.e., $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$. $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{s\to\infty} a(s) = \infty$. The function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{KL}$, i.e., $\beta \in \mathcal{KL}$, if for each fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{K}$ and for each fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is strictly deceasing and $\lim_{s\to\infty} \beta(r, s) = 0$. The symbol $*$ represents a symmetric structure in matrix inequalities. An element belonging to $co\{\cdot\}$ means that it is a convex combination of the elements in $\{\cdot\}$, and the combination coefficient is nonnegative and their sum is equal to 1.

II. Preliminaries

A. Useful Definitions and Lemmas

Definition 1 (Input-to-state Stability (ISS) [34]): Consider $x(k+1) \in F(x(k), d(k))$, it is called ISS if there exist $\beta \in \mathcal{KL}$ and $\delta \in \mathcal{K}$, such that for all $k \in \mathbb{Z}_+$ it holds that

$$\|x(k)\| \le \beta(x(0), k) + \delta(\|\mathbf{d}\|), \tag{1}$$

where $x(0)$ is the initial state, $\mathbf{d} = \{d(0), d(1), \ldots, d(k-1)\}$ is the disturbance sequence.

Definition 2 (ISS Lyapunov Function [34]): Suppose that there exists a positive definite function $V(x(k))$ such that satisfies the following conditions:

(i) There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying,

$$\alpha_1(\|x(k)\|) \le V(x(k)) \le \alpha_2(\|x(k)\|) \tag{2}$$

for all $(x(k), d(k), x(1)) \in \mathbb{R}^n \times \mathbb{R}^c \times F(x(0), d(0))$;

(ii) For any initial condition, there exists $\alpha_3 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ satisfying

$$V(x(k+1)) - V(x(k)) \leq -\alpha_3 \|x(k)\| + \sigma \|d(k)\| \quad (3)$$

then $V(x(k))$ is called an ISS Lyapunov Function for $x(k+1) \in F(x(k), d(k))$.

Lemma 1 ([30]): Let $x(k) = \begin{bmatrix} x_1(k) & x_2(k) & \cdots & x_n(k) \end{bmatrix}$, $1 \leq p \leq n$. For $\Lambda^{-1} = diag\begin{bmatrix} \rho_1 & \rho_2 & \cdots & \rho_q \end{bmatrix}^{-1}$, where ρ_m, $1 \leq m \leq q$, are arbitrary positive scalars, if there exist $W = \begin{bmatrix} W_1{}^T & W_2{}^T & \cdots & W_q{}^T \end{bmatrix}^T$, where $W_m = \begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{p-1} & w_{mp} & \underbrace{0 \ \cdots \ 0}_{n-p} \end{bmatrix}$, w_{mp} are positive scalars, and $\phi(k) = \begin{bmatrix} \phi_1(k) & \phi_2(k) & \cdots & \phi_q(k) \end{bmatrix}^T$, where $\phi_m(k) = F_m(x_p(k))$ with $F_m(0) = 0$, such that $\phi_m(k) \in co\{0, W_m x(k)\}$, then

$$\phi^T \Lambda^{-1} W x - \phi^T \Lambda^{-1} \phi \geq 0 \quad (4)$$

Lemma 2 ([35]): Let N, F, S be real matrices of appropriate dimensions with $F(k)$ being a matrix function. For any $\sigma > 0$ and $F^T(k)F(k) \leq I$,

$$NF(k)S + S^T F^T(k) N^T \leq \tfrac{1}{\sigma} N N^T + \sigma S^T S \quad (5)$$

B. NLM-based T-S Fuzzy System

Firstly, to simplify the problem description, the parametric uncertainties are not involved. Thus we consider the following discrete nonlinear system with disturbances:

$$x(k+1) = f(x(k)) + g(x(k))u(k) + w(x(k))d(k) \quad (6)$$

where $x(k+1)$, $x(k) \in \mathbb{R}^n$ are the states, $u(k) \in \mathbb{R}^m$ and $d(k) \in \mathbb{R}^c$ are the control input and external disturbance, respectively. $f(\cdot)$, $g(\cdot)$, and $w(\cdot)$ are functions of $x(k)$. $f(\cdot)$ often contains a lot of nonlinear terms especially for systems with severe nonlinearities. Then the aforementioned system can be described by the following NLM-based T-S fuzzy system:

Plant Rule l:
IF: z_1 is F_1^l and $\ldots$ z_v is F_v^l
THEN:

$$x(k+1) = A_l x(k) + B_l u(k) + G_l \phi(k) + E_l d(k) \quad (7)$$

where, $z := [z_1, \ldots, z_v]$ are premise variables. $F_1^l, \ldots, F_v^l$ are fuzzy sets. $l \in \mathbb{Z}_{[1,L]}$, and L is the number of fuzzy rules. $\phi(k) \in co\{0, Wx(k)\}$ is a nonlinear term which is separated from $f(x(k))$,

$$f(x(k)) = \sum_{l=1}^{L} \mu_l(z) \left[A_l x(k) + G_l \phi(k)\right] \quad (8)$$

where $\mu_l(z)$ is a normalized membership function. The following notations are introduced,

$$A_\mu := \sum_{l=1}^{L} \mu_l(z) A_l$$
$$G_\mu := \sum_{l=1}^{L} \mu_l(z) G_l$$
$$B_\mu := \sum_{l=1}^{L} \mu_l(z) B_l$$
$$E_\mu := \sum_{l=1}^{L} \mu_l(z) E_l$$

Then the system can be rewritten as follows,

$$x^+ = A_\mu x + B_\mu u + G_\mu \phi + E_\mu d \quad (9)$$

where x^+ denotes the system state in the next instant for simplicity purpose.

The nonlinear local controller corresponding to each NLM is given as follows [30]:

Control Rule l:
IF: z_1 is F_1^l and $\ldots$ z_v is F_v^l
THEN: $u_l(k) = K_{al} x(k) + K_{bl} \phi(k)$

By using the fuzzy inference method with a singleton fuzzifier, product inference and center average defuzzifiers, the final control output can be obtained as follows,

$$u(k) = K_{a\mu} x(k) + K_{b\mu} \phi(k) \quad (10)$$

where

$$K_{a\mu} = \sum_{l=1}^{L} \mu_l K_{al}$$
$$K_{b\mu} = \sum_{l=1}^{L} \mu_l K_{bl}$$

C. Robust Fuzzy Model Predictive Control

The prediction model for the above T-S fuzzy system is

$$\begin{aligned} &x(k+i+1|k) = A_\mu x(k+i|k) + B_\mu u(k+i|k) \\ &+ G_\mu \phi(k+i|k) + E_\mu d(k+i|k) \end{aligned} \quad (11)$$

A finite horizon cost function is adopted and minimized in this work,

$$J(k) = \sum_{i=0}^{N-1} \ell(k+i|k) + V_t(x(k+N|k)) \quad (12)$$

where $\ell(k+i|k)$ and $V_t(x(k+N|k))$ are called the stage cost and the terminal cost, respectively. The stage cost is chosen as

$$\begin{aligned} &\ell(k+i|k) = x^T(k+i|k) Q x(k+i|k) \\ &+ u^T(k+i|k) R u(k+i|k) - \tau d^T(k+i|k) d(k+i|k) \end{aligned} \quad (13)$$

where Q, R are positive matrices, τ is a positive scalar. One can see that the disturbance is involved in the stage cost. It is noted that the cost function is inspired by the H_∞ control, see Section 4.7 in [36] for the so-called "H_∞ model predictive control". And similar settings of cost functions can be found in [15], [36], [37]. However, the cost function cannot be optimized directly as it contains disturbance. Instead, a min-max approach is adopted that the worst-case cost function is minimized [38]. In addition, in the presence of persistent disturbance, asymptotic stability to the origin cannot be reached, instead, asymptotic convergence to an invariant set can be achieved [39], [40]. As such, a robust positively invariant set is employed as the terminal constraint set that, the system state is guaranteed to enter the terminal set at the end of prediction. The online optimization problem can be summarized as follows,

$$\begin{aligned} &\min_{u(k+i|k)} \max_{d(k+i|k)} \quad J(k), \\ &u(k+i|k) \in U, \\ &d(k+i|k) \in D, \\ &x(k+N|k) \in \Omega_t, \end{aligned}$$

where Ω_t is the terminal constraint set. The disturbance and control input satisfy that

$$d \in D := \{d | d^T d \leq \gamma\}$$

$$u \in U := \{u | \, |u_s| \leq u_{s,\max}\}$$

where γ is a positive scalar, u_s is the s-th element of the inputs, $s \in \mathbb{Z}_{[1,\,m]}$.

III. Main Results

In this section, the RPI property and terminal constraint set are introduced firstly, based on which the whole algorithm of FMPC is summarized. Then Analysis of the recursive feasibility and the ISS are provided.

A. RPI Set

Denote Ω as an RPI set and $h(x)$ as the corresponding control law. The RPI property can be described as, $\forall x \in \Omega$, the control effort will guarantee $x^+ \in \Omega$ for all allowable uncertainties and disturbances. Define Ω as follows:

$$\Omega := \{x | x^T P_\mu x \leq \xi\} \tag{14}$$

where $P_\mu = \sum_{l=1}^{L} \mu_l(x) P_l$. The controller for T-S fuzzy systems with NLMs is selected as

$$h(x) := K_{a\mu} x + K_{b\mu} \phi \tag{15}$$

Lemma 3 ([15]): The set Ω is an RPI set if there exists a positive scalar $\lambda \in \mathbb{R}_{(0,1)}$ such that

$$\tfrac{1}{\xi} {x^+}^T P_\mu^+ x^+ - \tfrac{1-\lambda}{\xi} x^T P_\mu x - \tfrac{\lambda}{\gamma} d^T d \leq 0 \tag{16}$$

with $P_\mu^+ = \sum_{l=1}^{L} \mu_l(x^+) P_l$, for all $x^+ \in A_\mu x + B_\mu u + G_\mu \phi + E_\mu d$, $u \in U$, and $d \in D$.

Proof: A simple proof is also given here. Move the last two items in (16) to the right side. As $x^T P_\mu x \leq \xi$ and $d^T d \leq \gamma$, then $\frac{1}{\xi} {x^+}^T P_\mu^+ x^+ \leq (1-\lambda) + \lambda = 1$, thus ${x^+}^T P_\mu^+ x^+ \leq \xi$, which implies $x^+ \in \Omega$.

Theorem 1: Consider the T-S fuzzy system (9), if there exist positive definite matrices X_i (or X_j, X_l), matrices L_j, Y_{aj}, Y_{bj}, Z, $\Lambda^{-1} = diag\begin{bmatrix} \rho_1 & \rho_2 & \cdots & \rho_q \end{bmatrix}^{-1}$ with $\rho_1, \rho_2, \ldots, \rho_q > 0$, and positive scalar $\lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \Phi_{11} & * & * & * \\ W L_j & -2\Lambda & * & * \\ 0 & 0 & -\lambda/\gamma & * \\ A_i L_j + B_i Y_{aj} & G_i \Lambda + B_i Y_{bj} & E_i & -X_l \end{bmatrix} \leq 0 \tag{17}$$

$$\begin{bmatrix} Z & * & * \\ {Y_{aj}}^T & \Phi_{22} & * \\ {Y_{bj}}^T & -W L_j & 2\Lambda \end{bmatrix} \geq 0, Z_{ss} \leq u_{s,\max}^2, s \in \mathbb{Z}_{[1,\,m]} \tag{18}$$

where $\Phi_{11} = (-1+\lambda)(L_j + {L_j}^T - X_i)$, $\Phi_{22} = L_j + {L_j}^T - X_j$. Z_{ss} is the s-th diagonal element of matrix Z, and $i, j, l \in \mathbb{Z}_{[1,\,L]}$, then the set $\Omega = \{x | x^T P_\mu x \leq \xi\}$, with $P_\mu = \sum_{i=1}^{L} \mu_i(x) P_i$ and $P_i = \xi X_i^{-1}$, is an RPI set for the fuzzy system (9) corresponding to the feedback control law $h(x) = K_{a\mu} x + K_{b\mu} \phi$, with $K_{a\mu} = \sum_{i=1}^{L} \mu_i K_{ai} = \sum_{i=1}^{L} \mu_i Y_{ai} {L_i}^{-1}$, $K_{b\mu} = \sum_{i=1}^{L} \mu_i K_{bi} = \sum_{i=1}^{L} \mu_i Y_{bi} \Lambda^{-1}$.

Proof: Resorting to the dilation lemma [41],

$$-L_j^T X_i^{-1} L_j \leq X_i - L_j^T - L_j \tag{19}$$

then multiplying $diag\left\{L_j^{-T}, \Lambda^{-1}, I, I\right\}$ and its transpose from both sides of (17), respectively, substituting $Y_{ai} {L_i}^{-1}$ with K_{ai}, and $Y_{bi} \Lambda^{-1}$ with K_{bi}, yields that

$$\begin{bmatrix} (-1+\lambda) X_i^{-1} & * & * & * \\ \Lambda^{-1} W & -2\Lambda^{-1} & * & * \\ 0 & 0 & -\lambda/\gamma & * \\ A_i + B_i K_{aj} & G_i + B_i K_{bj} & E_i & -X_l \end{bmatrix} \leq 0 \tag{20}$$

Substituting X_i^{-1} with P_i/ξ, then rewrite (20) as follows,

$$\begin{bmatrix} (-1+\lambda) P_\mu/\xi & * & * & * \\ \Lambda^{-1} W & -2\Lambda^{-1} & * & * \\ 0 & 0 & -\lambda/\gamma & * \\ A_\mu + B_\mu K_{a\mu} & G_\mu + B_\mu K_{b\mu} & E_\mu & -X_l \end{bmatrix} \leq 0 \tag{21}$$

Applying schur complement to (21), then

$$\begin{bmatrix} (-1+\lambda) P_\mu/\xi & * & * \\ \Lambda^{-1} W & -2\Lambda^{-1} & * \\ 0 & 0 & -\lambda/\gamma \end{bmatrix} + \Psi^T X_l^{-1} \Psi \leq 0 \tag{22}$$

where $\Psi = \begin{bmatrix} A_\mu + B_\mu K_{a\mu} & G_\mu + B_\mu K_{b\mu} & E_\mu \end{bmatrix}$. Multiplying $\begin{bmatrix} x^T & \phi^T & d^T \end{bmatrix}$ and its transpose from both sides of (22), respectively, it holds that

$$\begin{aligned} &{x^+}^T X_l^{-1} x^+ - \tfrac{1}{\xi} x^T P_\mu x - \lambda(\tfrac{1}{\gamma} d^T d - \tfrac{1}{\xi} x^T P_\mu x) \\ &+ 2(\phi^T \Lambda^{-1} W x - \phi^T \Lambda^{-1} \phi) \leq 0 \end{aligned} \tag{23}$$

where $x^+ = (A_\mu + B_\mu K_{a\mu}) x + (G_\mu + B_\mu K_{b\mu}) \phi + E_\mu d$. Substituting X_l^{-1} with P_l/ξ into the above inequality, with $2(\phi^T \Lambda^{-1} W x - \phi^T \Lambda^{-1} \phi) \geq 0$ from Lemma 1, then

$$\tfrac{1}{\xi} (x^+)^T P_\mu^+ x^+ - \tfrac{1}{\xi} x^T P_\mu x - \lambda(\tfrac{1}{\gamma} d^T d - \tfrac{1}{\xi} x^T P_\mu x) \leq 0 \tag{24}$$

Thus the RPI property of the set Ω can be established if (17) holds. Moreover, the input constraint can be satisfied by (18), and proof is given below.

From (19), the following inequality can be got if (18) holds,

$$\begin{bmatrix} Z & * & * \\ {Y_{aj}}^T & L_j^T X_j^{-1} L_j & * \\ {Y_{bj}}^T & -WL_j & 2\Lambda \end{bmatrix} \geq 0 \quad (25)$$

Multiplying $diag\left\{I, L_j^{-1}, \Lambda^{-1}\right\}$ and its transpose from both sides of (25), respectively, then substituting $Y_{ai}{L_i}^{-1}$ with K_{ai}, and $Y_{bi}\Lambda^{-1}$ with K_{ai}, yields that

$$\begin{bmatrix} Z & * & * \\ {K_{aj}}^T & X_j^{-1} & * \\ {K_{bj}}^T & -\Lambda^{-1}W & 2\Lambda^{-1} \end{bmatrix} \geq 0 \quad (26)$$

(26) can be written as

$$\begin{bmatrix} Z & * & * \\ {K_{a\mu}}^T & X_\mu^{-1} & * \\ {K_{b\mu}}^T & -\Lambda^{-1}W & 2\Lambda^{-1} \end{bmatrix} \geq 0 \quad (27)$$

Applying schur complement to (27), then

$$\begin{bmatrix} X_\mu^{-1} & * \\ -\Lambda^{-1}W & 2\Lambda^{-1} \end{bmatrix} - \frac{1}{Z}\begin{bmatrix} {K_{a\mu}}^T \\ {K_{b\mu}}^T \end{bmatrix}\begin{bmatrix} K_{a\mu} & K_{b\mu} \end{bmatrix} \geq 0 \quad (28)$$

Multiplying $\begin{bmatrix} x^T & \phi^T \end{bmatrix}$ and its transpose from both sides of (28), respectively, and substituting X_μ^{-1} with P_μ/ξ, yields that

$$\begin{aligned} &\tfrac{1}{Z}(K_{a\mu}x + K_{b\mu}\phi)^T(K_{a\mu}x + K_{b\mu}\phi) - \tfrac{1}{\xi}x^T P_\mu x + \\ &2(\phi^T\Lambda^{-1}Wx - \phi^T\Lambda^{-1}\phi) \leq 0 \end{aligned} \quad (29)$$

Thus it holds that

$$\tfrac{1}{Z}(K_{a\mu}x + K_{b\mu}\phi)^T(K_{a\mu}x + K_{b\mu}\phi) - \tfrac{1}{\xi}x^T P_\mu x \leq 0 \quad (30)$$

As $u = K_{a\mu}x + K_{b\mu}\phi$, thus the following inequality is got,

$$\tfrac{1}{Z}u^T u \leq \tfrac{1}{\xi}x^T P_\mu x \quad (31)$$

Due to $\frac{1}{\xi}x^T P_\mu x \leq 1$, thus $u^T u \leq Z$. Proof is complete.

B. Terminal Constraint Set

The terminal constraint set Ω_t should satisfy two requirements. Firstly, it should be an RPI set. Secondly, there exists a positive definite function (terminal cost function) $V(x)$ such that $\forall\, x \in \Omega_t$,

$$\alpha_3(\|x\|) \leq V(x) \leq \alpha_4(\|x\|) \quad (32)$$

$$V(x^+) - V(x) < -(x^T Qx + h(x)^T Rh(x) - \tau d^T d) \quad (33)$$

where α_3 and α_4 are K_∞ functions, $V(x)$ is given as

$$V(x) := \sum_{l=1}^{L} \mu_l(x) x^T P_l x \quad (34)$$

Theorem 2: Consider the T-S fuzzy system (9), if (17), (18) and the following matrix inequality is feasible

$$\begin{bmatrix} \Xi_{11} & * & * & * & * & * \\ WL_j & -2\Lambda & * & * & * & * \\ 0 & 0 & -\tau\xi & * & * & * \\ \Xi_{41} & \Xi_{42} & \xi E_i & -X_l & * & * \\ QL_j & 0 & 0 & 0 & -\xi Q & * \\ RY_{aj} & RY_{bj} & 0 & 0 & 0 & -\xi R \end{bmatrix} < 0 \quad (35)$$

where $\Xi_{11} = X_i - L_j - {L_j}^T$, $\Xi_{41} = A_i L_j + B_i Y_{aj}$, $\Xi_{42} = G_i\Lambda + B_i Y_{bj}$. X_i (or X_j, X_l), L_j, Y_{aj}, Y_{bj} are the same as mentioned in Theorem 1, then Ω_t is a terminal constraint set corresponding to the terminal cost function $V(x)$.

Proof: Since (32) can be easily got by resorting to the eigenvalues. The main concern is focused on the condition for (33).

Considering (19), substituting Y_{ai} with $K_{ai}L_i$, and Y_{bi} with $K_{bi}\Lambda$, then multiplying $diag\{{L_j}^{-1}, \Lambda^{-1}, I, I, I, I\}$ and its transpose to both sides of (35), respectively, yields that

$$\begin{bmatrix} -X_i^{-1} & * & * & * & * & * \\ \Lambda^{-1}W & -2\Lambda^{-1} & * & * & * & * \\ 0 & 0 & -\tau\xi & * & * & * \\ \Upsilon_{41} & \Upsilon_{42} & \xi E_i & -X_l & * & * \\ Q & 0 & 0 & 0 & -\xi Q & * \\ RK_{aj} & RK_{bj} & 0 & 0 & 0 & -\xi R \end{bmatrix} < 0 \quad (36)$$

where $\Upsilon_{41} = A_i + B_i K_{aj}$, $\Upsilon_{42} = G_i + B_i K_{bj}$. Substituting X_i^{-1} with $\xi^{-1}P_i$ and X_l^{-1} with $\xi^{-1}P_l$, (36) is equivalent to

$$\begin{bmatrix} -P_i & * & * & * & * & * \\ \Lambda^{-1}W & -2\Lambda^{-1} & * & * & * & * \\ 0 & 0 & -\tau & * & * & * \\ \Upsilon_{41} & \Upsilon_{42} & E_i & -P_l^{-1} & * & * \\ Q & 0 & 0 & 0 & -Q & * \\ RK_{aj} & RK_{bj} & 0 & 0 & 0 & -R \end{bmatrix} < 0 \quad (37)$$

(37) can be written as

$$\begin{bmatrix} -P_\mu & * & * & * & * & * \\ \Lambda^{-1}W & -2\Lambda^{-1} & * & * & * & * \\ 0 & 0 & -\tau & * & * & * \\ \Pi_{41} & \Pi_{42} & E_\mu & -P_l^{-1} & * & * \\ Q & 0 & 0 & 0 & -Q & * \\ RK_{a\mu} & RK_{b\mu} & 0 & 0 & 0 & -R \end{bmatrix} < 0 \quad (38)$$

where $\Pi_{41} = A_\mu + B_\mu K_{a\mu}$, $\Pi_{42} = G_\mu + B_\mu K_{b\mu}$. Denoting P_l as P_μ^+, and then applying schur complement, (38) is equivalent to

$$\begin{aligned} &\begin{bmatrix} -P_\mu + Q & * & * \\ \xi\Lambda^{-1}W & -2\xi\Lambda^{-1} & * \\ 0 & 0 & -\tau \end{bmatrix} + \Theta^T P_\mu^+ \Theta \\ &+ \begin{bmatrix} {K_{a\mu}}^T \\ {K_{b\mu}}^T \\ 0 \end{bmatrix} R \begin{bmatrix} K_{a\mu} & K_{b\mu} & 0 \end{bmatrix} < 0 \end{aligned} \quad (39)$$

where $\Theta = \begin{bmatrix} \Pi_{41} & \Pi_{42} & E_\mu \end{bmatrix}$. Multiplying $\begin{bmatrix} x^T & \phi^T & d^T \end{bmatrix}$ and its transpose to both sides of (39), respectively, yields that

$$\begin{aligned} &{x^+}^T P_\mu^+ x^+ - x^T P_\mu x + x^T Qx \\ &+(K_{a\mu}x + K_{b\mu}\phi)^T R(K_{a\mu}x + K_{b\mu}\phi) \\ &-\tau d^T d + 2\xi(\phi^T\Lambda^{-1}Wx - \phi^T\Lambda^{-1}\phi) < 0 \end{aligned} \quad (40)$$

where $x^+ = (A_\mu + B_\mu K_{a\mu})x + (G_\mu + B_\mu K_{b\mu})\phi + E_\mu d$. As $2\xi(\phi^T\Lambda^{-1}Wx - \phi^T\Lambda^{-1}\phi) \leq 0$, $h(x) = K_{a\mu}x + K_{b\mu}\phi$, then (33) is got. Proof is thus complete.

C. Control Algorithm

Based on the above results, online control algorithm is discussed in this subsection to complete the FMPC design.

From the aforementioned computation of terminal constraint set, the terminal cost function $V(x(k))$, see (34), should satisfy the following condition,

$$V(x(k)) = \sum_{l=1}^{L} \mu_l(x(k))x^T(k)P_l x(k) \leq \xi \quad (41)$$

Then the following optimization problem is considered to minimize $V(x(k))$,

$$\min \ \xi, \quad \textit{subjected to} \ V(x(k)) \leq \xi \quad (42)$$

Moreover, a sufficient condition for $V(x(k)) \leq \xi$ is

$$\begin{bmatrix} 1 & x^T(k) \\ x(k) & X_l \end{bmatrix} \geq 0 \quad (43)$$

In addition, consider the condition for terminal constraint set in (35), τ also can be optimized. Taking the same idea as stated in [15], a new variable ε is defined as: $\varepsilon = \tau\xi$, thus ε is minimized instead. Then (35) is rewritten as

$$\begin{bmatrix} \Xi & * & * & * & * & * \\ WL_j & -2\Lambda & * & * & * & * \\ 0 & 0 & -\varepsilon & * & * & * \\ \Omega & \Theta & \xi E_i & -X_l & * & * \\ QL_j & 0 & 0 & 0 & -\xi Q & * \\ RY_{aj} & RY_{bj} & 0 & 0 & 0 & -\xi R \end{bmatrix} < 0 \quad (44)$$

In summary, the MPC algorithm is given as follows. It can be seen that bilinear matrix inequalities (BMIs) are introduced in (17) due to the variable λ. To reduce the computational burden, firstly, we try different values of λ and find a feasible one, then it is utilized all the time. Although it may be conservative to set λ as a constant value, the optimization problem is reduced to solving linear matrix inequalities (LMIs).

Algorithm:

Step 1: Obtain the system state $x(k)$;

Step 2: Solve the optimization problem

$$\min_{\xi,\varepsilon,X_i,L_j,\Lambda,Y_{aj},Y_{bj},Z} \quad \varepsilon, \quad (45)$$

subject to (17), (18), (43), (44). Go to Step 3;

Step 3: Implement the control input $u(k) = K_{a\mu}x(k) + K_{b\mu}\phi(k)$, where $K_{a\mu} = \sum_{l=1}^{L} \mu_l(x(k))Y_{al}L_l^{-1}$, and $K_{b\mu} = \sum_{l=1}^{L} \mu_l(x(k))Y_{bl}\Lambda^{-1}$. Move the time instant from k to $k+1$ and go to Step 1.

Theorem 3: For system (9), if the optimization problem has a solution at time 0, it will always be solvable. Furthermore, recursive feasibility can be achieved.

Proof: Assume that (45) is satisfied at time k, then $x(k) \in \Omega(k)$, where $\Omega(k)$ represents the terminal set Ω obtained at time k. As $\Omega(k)$ is an RPI set, thus $x(k+1) \in \Omega(k)$ is got. Then it can be concluded that (45) is solvable at time $k+1$. Moreover, the solution obtained at time k is feasible at time $k+1$. Applying the same process, it is observed that the optimization problem is feasible at all time.

Theorem 4: If the optimization problem is feasible at the initial time 0, system (9) is ISS with respect to disturbance d.

Proof: Consider the Lyapunov function $V(x(k)) = x^T(k){P_\mu}^*(k)x(k)$, where ${P_\mu}^*(k) = \sum_{l=1}^{L} \mu_l(z){P_l}^*(k)$, ${P_l}^*(k)$ is an optimal value of $P_l(k)$ at time k , then it can be easily got

$$\rho^*_{\min}\|x(k)\|^2 \leq V(k,x) \leq \rho^*_{\max}\|x(k)\|^2 \quad (46)$$

where

$$\rho^*_{\min} := \min\left\{\rho_{\min}({P_l}^*(k)) | l \in \mathbb{Z}_{[1,L]}, k \in \mathbb{N}\right\}$$

$$\rho^*_{\max} := \max\left\{\rho_{\max}({P_l}^*(k)) | l \in \mathbb{Z}_{[1,L]}, k \in \mathbb{N}\right\}$$

where $\rho_{\max}(\cdot)$ and $\rho_{\min}(\cdot)$ represent the maximal and minimal eigenvalues, respectively.

Moreover, (33) implies that

$$\begin{aligned} &V_k(x(k+1)) - V(x(k)) < -(x(k)^T Qx(k) + u^T(k)Ru(k) \\ &-\tau d^T(k)d(k)) \end{aligned} \quad (47)$$

where $V_k(x(k+1)) = x(k+1)^T {P_\mu}^*(k)x(k+1)$. Thus one has

$$V_k(x(k+1)) - V(k,x) < -x(k)^T Qx(k) + \tau d^T(k)d(k) \quad (48)$$

$$\begin{bmatrix} (-1+\lambda)(L_j + {L_j}^T - X_i) & * & * & * & * & * \\ WL_j & -2\Lambda & * & * & * & * \\ 0 & 0 & -\lambda/\gamma^2 & * & * & * \\ A_iL_j + B_iY_{aj} & G_i\Lambda + B_iY_{bj} & E_i & -X_l & * & * \\ N_{Ai}L_j + N_{Bi}Y_{aj} & N_{Gi}\Lambda + N_{Bi}Y_{bj} & 0 & 0 & -\eta_{ijl}I & * \\ 0 & 0 & 0 & \eta_{ijl}M^T & 0 & -\eta_{ijl}I \end{bmatrix} \leq 0 \quad (52)$$

$$\begin{bmatrix} X_i - L_j - {L_j}^T & * & * & * & * & * & * & * \\ WL_j & -2\Lambda & * & * & * & * & * & * \\ 0 & 0 & -\varepsilon & * & * & * & * & * \\ A_iL_j + B_iY_{aj} & G_i\Lambda + B_iY_{bj} & \xi E_i & -X_l & * & * & * & * \\ QL_j & 0 & 0 & 0 & -\xi Q & * & * & * \\ RY_{aj} & RY_{bj} & 0 & 0 & 0 & -\xi R & * & * \\ N_{Ai}L_j + N_{Bi}Y_{aj} & N_{Gi}\Lambda + N_{Bi}Y_{bj} & 0 & 0 & 0 & 0 & -\nu_{ijl}I & * \\ 0 & 0 & 0 & \nu_{ijl}M^T & 0 & 0 & 0 & \nu_{ijl}I \end{bmatrix} < 0 \quad (53)$$

Owing to the optimality of (45) at time $k+1$, it holds

$$V_{k+1}(x(k+1)) \leq V_k(x(k+1)) \tag{49}$$

Then it holds that

$$V_{k+1}(x(k+1)) - V(k,x) < -x(k)^T Qx(k) + \tau d^T(k)d(k) \tag{50}$$

From Definition 2, (46) and (50) together lead to that $V(x(k))$ is an ISS Lyapunov function. Hence the closed-loop system is ISS with respect to disturbances. The proof is complete.

IV. Extension To Systems With Parametric Uncertainties

In the aforementioned algorithm for FMPC of T-S fuzzy systems with NLMs, to simplify the synthesis, the parametric uncertainties have not been investigated that only the disturbances and input constraints are considered. However, it is noted that parametric uncertainties can be easily dealt with. And it is shown in this subsection.

Considering T-S fuzzy systems with structured uncertainties, then the T-S fuzzy system (7) is rewritten as

$$\begin{aligned} x(k+1) &= (A_l + \Delta A_l)x(k) + (B_l + \Delta B_l)u(k) \\ &+ (G_l + \Delta G_l)\phi(k) + E_l d(k) \end{aligned} \tag{51}$$

where the structural uncertainties are defined as follows:

$$\begin{bmatrix} \Delta A_l & \Delta B_l & \Delta G_l \end{bmatrix} = MH \begin{bmatrix} N_{Al} & N_{Bl} & N_{Gl} \end{bmatrix}$$

where H is an unknown matrix function satisfying $H^T H \leq I$, M, N_{Al}, N_{Bl}, and N_{Gl} are known real constant matrices that structure the uncertainty.

Under such circumstances, the control law at each instant can be computed by solving an optimization problem where (17) and (44) are replaced by (52) and (53), respectively. In (52) and (53), η_{ijl} and ν_{ijl} are positive scalars. Proof can be easily got via following a similar procedure for (17) and (44). And Lemma 2 is utilized to deal with the structural uncertainties. For simplicity, the proof is omitted.

V. Illustrative Example

In this section, two examples are introduced to show the effectiveness of the proposed method. The simulations are implemented with MATLAB and all LMIs are solved with MATLAB LMI toolbox. The simulations are executed on HP desktop (3.40 GHz Intel Core i7, 8GB RAM).

Two typical fuzzy control and FMPC methods, and the proposed method will be executed:

1. ***NFC:*** The NLM-based fuzzy control (NFC) method proposed in [30] considering disturbances and input constraints.

2. ***LFMPC:*** The LLM-based fuzzy model predictive control (LFMPC) method proposed in [15];

3. ***NFMPC:*** The proposed NLM-based fuzzy model predictive control (NFMPC) method;

Example 1: The model is given by

$$\begin{cases} x_1(k+1) = (1 - 0.2\sin^2(x_1(k)))x_1(k) - 0.3x_2(k) \\ \qquad\qquad\quad + u_1(k) \\ x_2(k+1) = 0.4x_1(k) + (1 - 1.2\sin^2(x_1(k)))\sin(x_2(k)) \\ \qquad\qquad\quad + 0.6x_2(k) + u_2(k) + d(k) \end{cases}$$

The input constraints are $|u_1(k)| \leq 1.5$, and $|u_2(k)| \leq 1$. And the disturbance is a white noise satisfying $|d(k)| \leq 0.01$.

It can be seen that nonlinearities are caused by $z_1(k) = \sin^2(x_1(k))$, $z_2(k) = \sin(x_2(k))$. To utilize the proposed FMPC method, the model is rewritten as follows

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} f_1(x_1(k)) & -0.3 \\ 0.4 & f_2(x_1(k)) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$
$$+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ f_3(x_1(k)) \end{bmatrix} \phi(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(k)$$

where $f_1(x_1(k)) = 1 - 0.2\sin^2(x_1(k))$, $f_2(x_1(k)) = 0.6 + \frac{2}{\pi}(1 - 1.2\sin^2(x_1(k)))$, $f_3(x_1(k)) = (1 - 1.2\sin^2(x_1(k)))$, $\phi(k) = \sin(x_2(k)) - \frac{2}{\pi}x_2(k)$, $x_1(k),\ x_2(k) \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$. From Fig. 1, it can be find that $\phi_0(k) = \sin(x_2(k)) \in co\{\frac{2}{\pi}x_2(k), x_2(k)\}$, thus $\phi(k) \in co\left\{0, (1-\frac{2}{\pi})x_2(k)\right\}$.

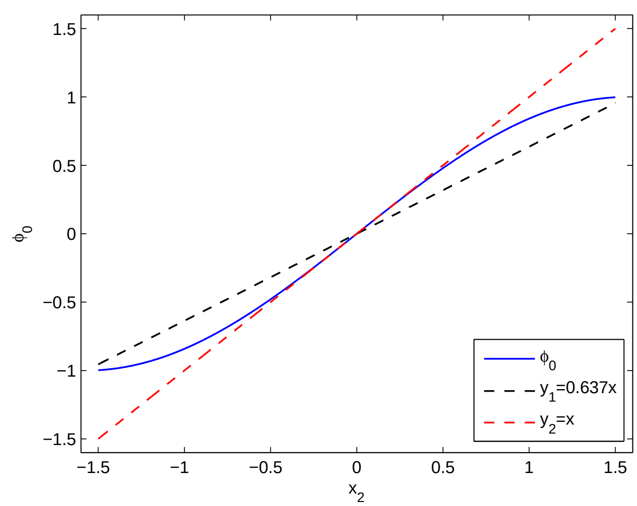

Fig. 1. The bound of ϕ_0.

Then the following T-S fuzzy model is constructed:
Rule 1:
IF $z_1(k)$ is 0,
THEN $x(k+1) = A_1 x(k) + B_1 u(k) + G_1\phi(k) + E_1 d(k)$
Rule 2:
IF $z_1(k)$ is 1,
THEN $x(k+1) = A_2 x(k) + B_2 u(k) + G_2\phi(k) + E_2 d(k)$
where $A_1 = \begin{bmatrix} 1 & -0.3 \\ 0.4 & 0.6+\frac{2}{\pi} \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.8 & -0.3 \\ 0.4 & 0.6-\frac{0.4}{\pi} \end{bmatrix}$, $B_1 = B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $G_2 = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}$, $E_1 = E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The membership functions are defined as follows,

$$\begin{cases} F_1 = \cos^2(x_2(k)) \\ F_2 = 1 - F_1 \end{cases}$$

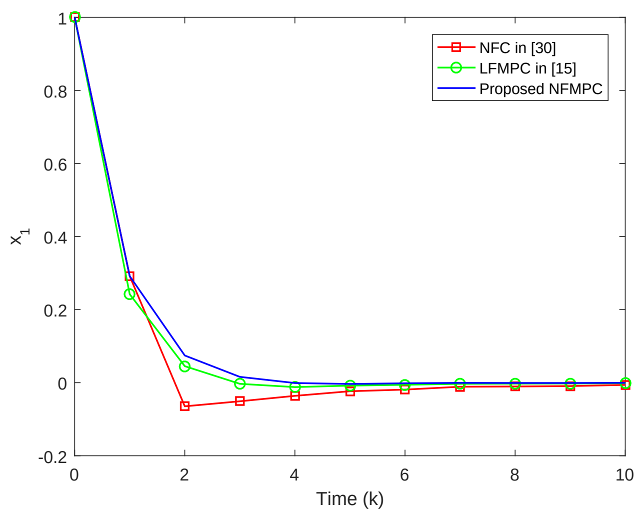

Fig. 2. Trajectories of x_1.

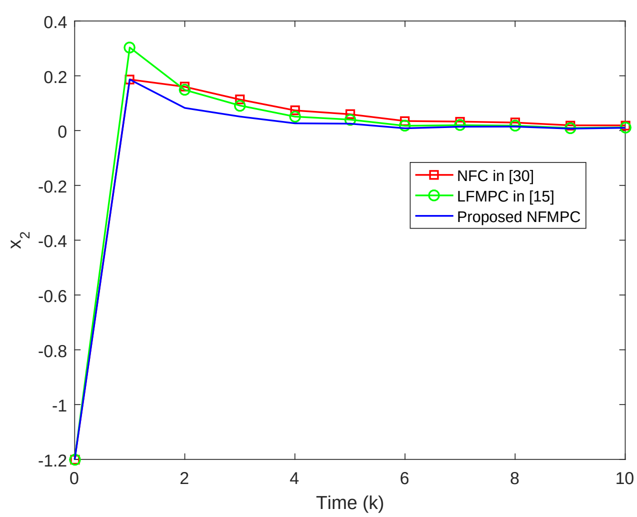

Fig. 3. Trajectories of x_2.

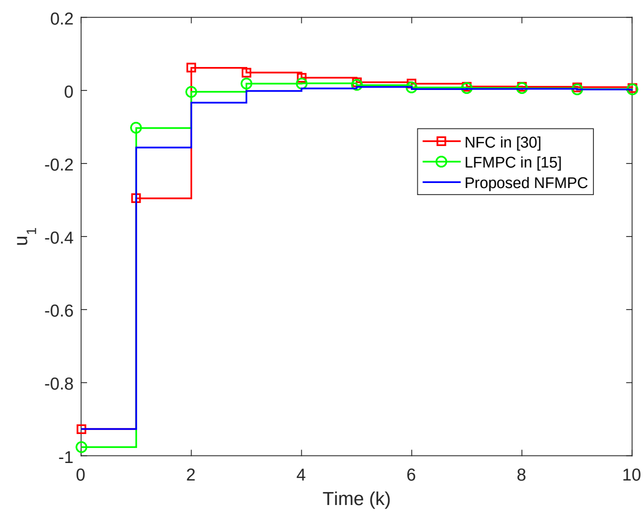

Fig. 4. Trajectories of u_1.

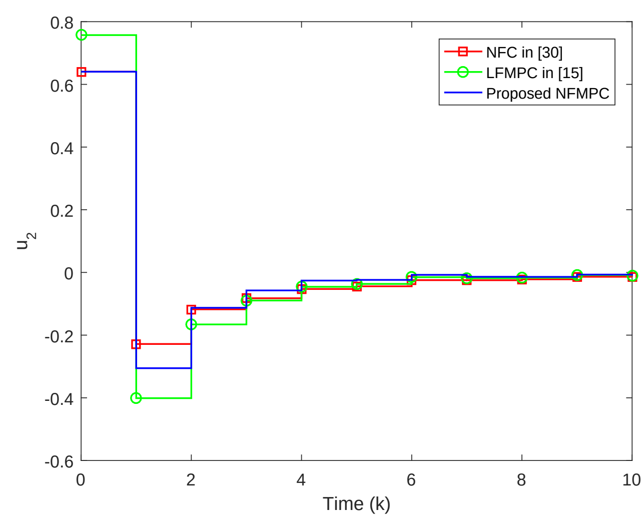

Fig. 5. Trajectories of u_2.

Let $Q = diag\{1, 1\}$, $R = diag\{1, 1\}$, $\lambda = 0.4688$.

Simulation results of the aforementioned three methods are shown in Figs. 2-5. From Figs. 2-5, it can be seen that, among all approaches, the best control performance is achieved via the NFMPC. Fig. 4 and Fig. 5 also show that the input constraints are guaranteed with all methods. On the other hand, Table 1 shows the computational cost of different approaches. The control law for NLM-based fuzzy approach in [30] is off-line computed, while the other two approaches are based on online optimization. In addition, compared with the LFMPC in [15], the cputime in simulation is reduced significantly by adopting the proposed NFMPC. Moreover, the numerical complexity, which can reflect the computational cost, represented by the number of decision variables and the number of lines in LMI, are also shown.

TABLE I
COMPUTATIONAL COMPLEXITY OF DIFFERENT METHODS

Methods	NFC in [30]	LFMPC in [15]	Proposed NFMPC
The number of fuzzy rules	2	4	2
The number of decision variables ($\mathcal{D}$)	29	44	29
Lines of the LMI ($\mathcal{L}$)	142	870	142
Numerical complexity ($\mathcal{C} = \mathcal{D}^3\mathcal{L}$)	3463238	74110080	3463238
Total cputime	off-line	20.3694s	4.3366s
Average cputime	–	1.8518s	0.3942s

Example 2:

Consider the two-tank system shown in Fig. 6, where h_1 and h_2 denote the water level of the two tanks, respectively. The control objective is to keep the water level of the two tanks in the desired operation point.

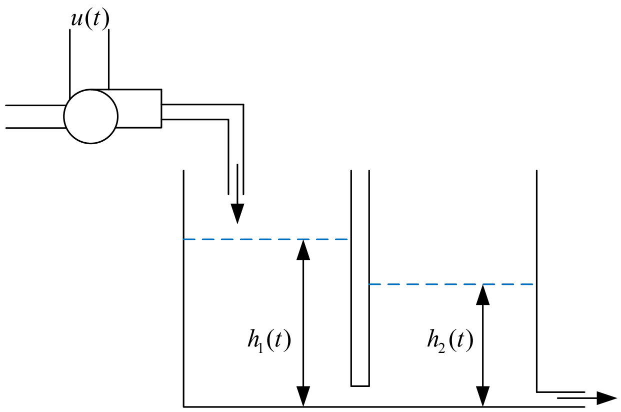

Fig. 6. The two-tank system.

The dynamic equations of the system are given as follows [42]:

$$\dot{h}_1(t) = \frac{1}{A}\left(ku(t) - a_1\sqrt{2g\left(h_1(t) - h_2(t)\right)}\right),$$
$$\dot{h}_2(t) = \frac{1}{A}\left(a_1\sqrt{2g\left(h_1(t) - h_2(t)\right)} - a_2\sqrt{2gh_2(t)}\right).$$

where the horizontal section is $A = 100cm^2$, the section of the valve connecting the two tanks is $a_1 = 1cm^2$, and the section of the outlet valve is $a_2 = 0.7cm^2$. Let $g = 981cm/s^2$. From the aforementioned dynamic equations, a desired setpoint is chosen as $[1.49; 1]$, with the reference input being 0.3101.

An incremental model of the dynamic system can be acquired as follows:

$$\dot{x}_1 = -\frac{a_1}{A}\frac{2g(x_1 - x_2)}{\sqrt{2g(x_{1s} - x_{2s} + x_1 - x_2)} + \sqrt{2g(x_{1s} - x_{2s})}} + \frac{k}{A}\Delta u$$
$$\dot{x}_2 = \frac{a_1}{A}\frac{2g(x_1 - x_2)}{\sqrt{2g(x_{1s} - x_{2s} + x_1 - x_2)} + \sqrt{2g(x_{1s} - x_{2s})}} - \frac{a_2}{A}\frac{2gx_2}{\sqrt{2g(x_{2s} + x_2)} + \sqrt{2gx_{2s}}}$$

where $x_s = [x_{1s}; x_{2s}]$ is the setpoint, Δu is the relative input corresponding to the reference input. The model is rewritten as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{a_1}{A}\frac{\sqrt{2g}}{f(x_1,x_2)} & \frac{a_1}{A}\frac{\sqrt{2g}}{f(x_1,x_2)} \\ \frac{a_1}{A}\frac{\sqrt{2g}}{f(x_1,x_2)} & -\frac{a_1}{A}\frac{\sqrt{2g}}{f(x_1,x_2)} - \frac{a_2}{A}\frac{\sqrt{2g}}{t(x_2)} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{k}{A} \\ 0 \end{bmatrix}\Delta u$$

where $f(x_1, x_2) = \sqrt{2g(x_{1s} - x_{2s} + x_1 - x_2)} + \sqrt{2g(x_{1s} - x_{2s})}$, $t(x_2) = \sqrt{2g(x_{2s} + x_2)} + \sqrt{2gx_{2s}}$.

It can be seen that the nonlinearities are caused by $z_1 = x_1 - x_2$, $z_2 = x_2$. Suppose $z_1 \in [-0.3, 0.3]$, $z_2 \in [-0.5, 0.5]$. To build the NLM-based fuzzy model, denote $\phi(t) = \left(\frac{1}{t(x_2)} - \frac{1}{\sqrt{x_{2s}+0.5}+\sqrt{x_{2s}}}\right)x_2$. Fig. 7 shows the bound of $\phi(t)$, which indicates $\phi(t) \in co\{0, 0.1363x_2\}$.

Then the continuous-time T-S fuzzy system can be built. The discrete-time system model can be got by setting the sampling time as $T_s = 5$s, and the following discretization method is adopted: $A_l = e^{A_{lc}T_s}$, $B_l = (\int_0^{T_s} e^{A_{lc}\tau}d\tau)B_{lc}$, $G_l = (\int_0^{T_s} e^{A_{lc}\tau}d\tau)G_{lc}$, $E_l = (\int_0^{T_s} e^{A_{lc}\tau}d\tau)E_{lc}$. The discrete T-S fuzzy system is shown as follows, where disturbance and structural uncertainties are introduced.

Rule 1:
IF z_1 is -0.3,
THEN $x(k+1) = (A_1 + MHN_{A1})x(k) + (B_1 + MHN_{B1})\Delta u(k) + (G_1 + MHN_{G1})\phi(k) + E_1 d(k)$
Rule 2:
IF z_1 is 0.3,
THEN $x(k+1) = (A_2 + MHN_{A2})x(k) + (B_2 + MHN_{B2})\Delta u(k) + (G_2 + MHN_{G2})\phi(k) + E_2 d(k)$
where

$$A_1 = \begin{bmatrix} 0.4337 & 0.3515 \\ 0.3515 & 0.3081 \end{bmatrix}, A_2 = \begin{bmatrix} 0.4735 & 0.3372 \\ 0.3372 & 0.3049 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 2.9936 \\ 1.5414 \end{bmatrix}, B_2 = \begin{bmatrix} 3.2474 \\ 1.3587 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} -0.4779 \\ -0.7574 \end{bmatrix}, G_2 = \begin{bmatrix} -0.4213 \\ -0.7963 \end{bmatrix},$$
$$E_1 = E_2 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$N_{A1} = N_{A2} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, N_{B1} = N_{B2} = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix},$$
$$N_{G1} = N_{G2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The membership functions are defined as follows,

$$\begin{cases} F_1 = \dfrac{0.3 - (x_1 - x_2)}{0.6} \\ F_2 = 1 - F_1 \end{cases}$$

Let $Q = diag\{0.1, 0.1\}$, $R = 0.1$, $\lambda = 0.3135$, $|\Delta u(k)| \le 0.05$, $|d(k)| \le 0.1$.

Simulation results of aforementioned three methods are shown in Figs. 8-10. Fig. 8 and Fig. 9 show that the proposed NFMPC achieves best control performance among all methods. As persistent disturbances are involved, the system states are converged to a small region near the origin. The disturbance is eliminated after 60s, then it can be seen that asymptotical convergence to the origin can be achieved. From Fig. 10, it can be seen that the input constraint is satisfied with all methods. And computational cost for all methods are shown in Table II. Both the numerical complexity and the cputime cost for simulation are demonstrated. It shows that the computational cost can be significantly reduced by adopting the NFMPC when compared to the LFMPC. In addition, the average cputime of LFMPC for each instant is 4.3233s, which

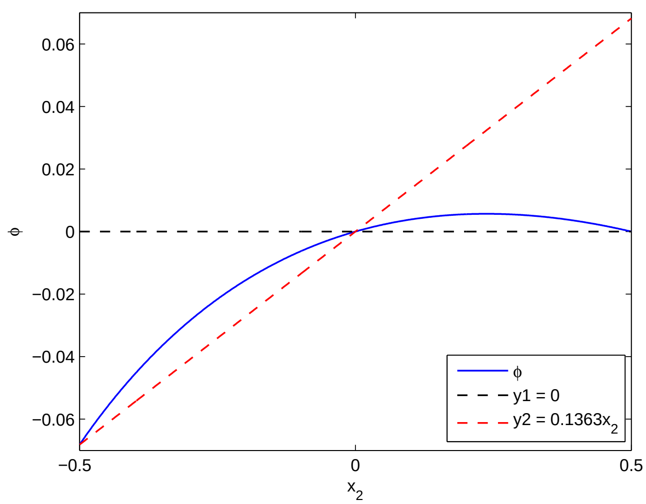

Fig. 7. The bound of ϕ.

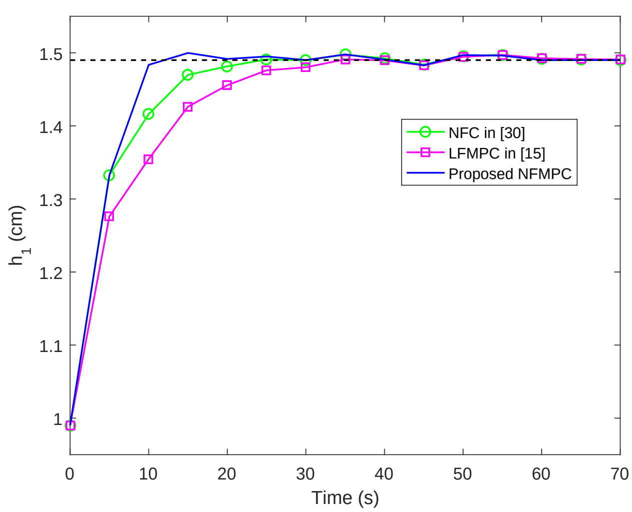

Fig. 8. Trajectories of x_1.

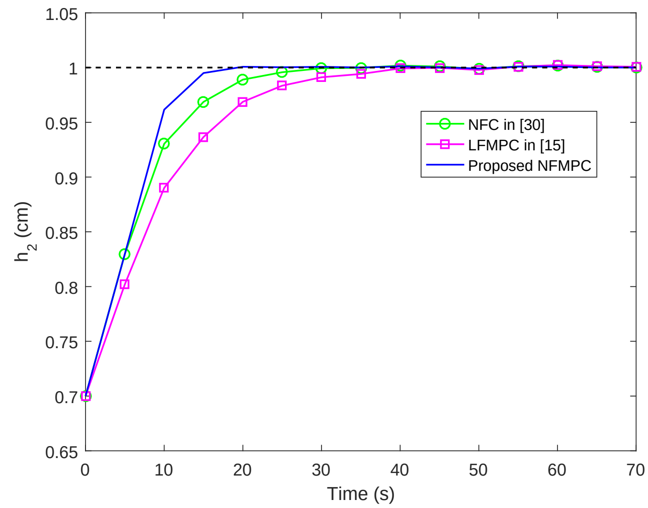

Fig. 9. Trajectories of x_2.

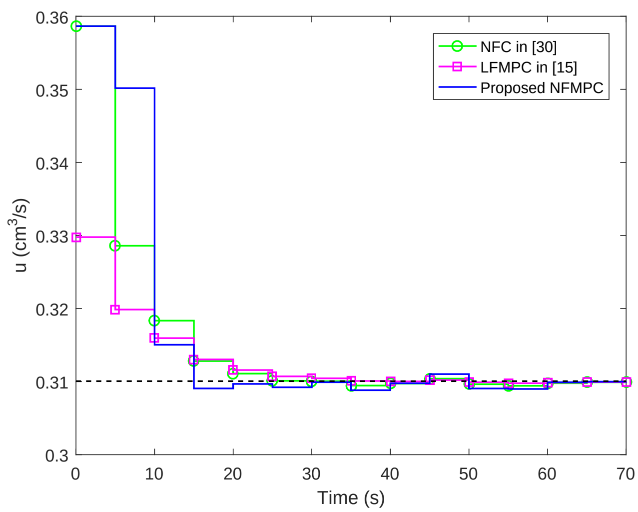

Fig. 10. Trajectories of u.

is very closed to the sampling time. As a result, the LFMPC may not be used effectively for the stabilization control of the two-tank system. Moreover, in Example 2, one can see that the cputime cost increases a lot compared with that of Example 1 due to the structural uncertainties.

In addition, the region of attraction is illustrated in Fig. 11 to further verify the relaxation provided by the NFMPC when compared to the LFMPC. One can see that a larger region of attraction is achieved via the NFMPC. It is noted that uncertainties are not involved to simplify the plant model for comparison.

Moreover, as λ is determined off-line, to find a best value, we run simulations with different values of λ. The results are shown in Figs. 12 and 13. It can be seen that the best settling

TABLE II
COMPUTATIONAL COMPLEXITY OF DIFFERENT METHODS

Methods	NFC in [30]	LFMPC in [15]	Proposed NFMPC
The number of fuzzy rules	2	4	2
The number of decision variables ($\mathcal{D}$)	38	163	38
Lines of the LMI ($\mathcal{L}$)	222	1508	222
Numerical complexity ($\mathcal{C} = \mathcal{D}^3\mathcal{L}$)	12181584	6530766476	12181584
Total cputime	off-line	64.8501s	14.0058s
Average cputime	–	4.3233s	0.9337s

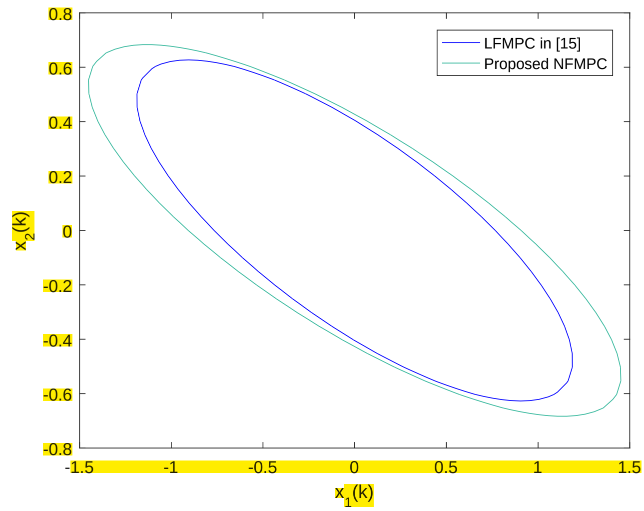

Fig. 11. Region of attraction.

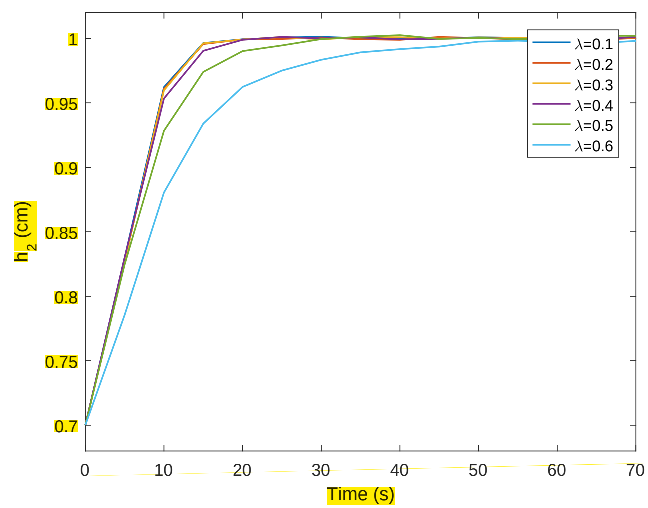

Fig. 13. Trajectories of x_2.

time is achieved when $\lambda = 0.1$.

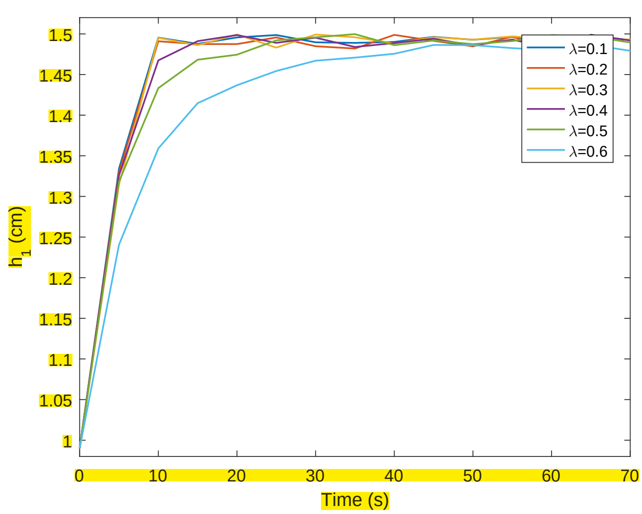

Fig. 12. Trajectories of x_1.

VI. Conclusions

Fuzzy model predictive control of discrete nonlinear systems is investigated. The T-S fuzzy systems with NLMs are adopted to represent nonlinear systems. Compared with the LLM-based FMPC, especially for systems with complex nonlinearities, parametric uncertainties and disturbances, which usually result in a very complex optimization problem, less conservative results and better control performance can be achieved through the proposed NLM-based FMPC. In particular, the computational cost of the online optimization problem by solving LMIs can be significantly reduced at the same time. Therefore, both control performance and computational cost are considered in the proposed method. Simulation results show the effectiveness and advantages of proposed FMPC approach. However, we find that the tracking control of NLM-based T-S fuzzy systems without a reference dynamic system is still a problem which deserves further attention. It can be regarded as one of our future work.

References

[1] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[2] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. Syst., Man, Cybern.*, no. 1, pp. 116–132, Jan.-Feb. 1985.

[3] G. Feng, "A survey on analysis and design of model-based fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 5, pp. 676–697, Oct. 2006.

[4] J. Dong, G.-H. Yang, and H. Zhang, "Stability analysis of T-S fuzzy control systems by using set theory," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 4, pp. 827–841, Aug. 2015.

[5] Q. Gao, G. Feng, D. Dong, and L. Liu, "Universal fuzzy models and universal fuzzy controllers for discrete-time nonlinear systems," *IEEE Trans. Cybern.*, vol. 45, no. 5, pp. 880–887, May 2015.

[6] T. Wang, Y. Zhang, J. Qiu, and H. Gao, "Adaptive fuzzy backstepping control for a class of nonlinear systems with sampled and delayed measurements," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 2, pp. 302–312, Apr. 2015.

[7] X. Xie, D. Yue, T. Ma, and X. Zhu, "Further studies on control synthesis of discrete-time TS fuzzy systems via augmented multi-indexed matrix approach," *IEEE Trans. Cybern.*, vol. 44, no. 12, pp. 2784–2791, Dec. 2014.

[8] W. Chang and W.-J. Wang, "H_∞ fuzzy control synthesis for a large-scale system with a reduced number of lmis," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 4, pp. 1197–1210, Aug. 2015.

[9] D. Zhang, Q.-L. Han, and X. Jia, "Network-based output tracking control for a class of TS fuzzy systems that can not be stabilized by nondelayed output feedback controllers," *IEEE Trans. Cybern.*, vol. 45, no. 8, pp. 1511–1524, Aug. 2015.

[10] J. Zhang, P. Shi, J. Qiu, and S. K. Nguang, "A novel observer-based output feedback controller design for discrete-time fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 1, pp. 223–229, Feb. 2015.

[11] Y. L. Huang, H. H. Lou, J. P. Gong, and T. F. Edgar, "Fuzzy model predictive control," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 6, pp. 665–678, Dec. 2000.

[12] T. Zhang, G. Feng, and J. Lu, "Fuzzy constrained min-max model predictive control based on piecewise Lyapunov functions," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 4, pp. 686–698, Aug. 2007.

[13] Y. Xia, H. Yang, P. Shi, and M. Fu, "Constrained infinite-horizon model predictive control for fuzzy-discrete-time systems," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 2, pp. 429–436, Apr. 2010.

[14] X.-H. Chang and G.-H. Yang, "Fuzzy robust constrained model predictive control for nonlinear systems," *Asian J. Control*, vol. 13, no. 6, pp. 947–955, 2011.

[15] W. Yang, G. Feng, and T. Zhang, "Robust model predictive control for discrete-time takagi–sugeno fuzzy systems with structured uncertainties and persistent disturbances," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 5, pp. 1213–1228, Oct. 2014.

[16] Y. Zhao, H. Gao, and T. Chen, "Fuzzy constrained predictive control of non-linear systems with packet dropouts," *IET Contr. Theory Appl.*, vol. 4, no. 9, pp. 1665–1677, 2010.

[17] B. Ding, "Dynamic output feedback predictive control for nonlinear systems represented by a Takagi–Sugeno model," *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 5, pp. 831–843, Oct. 2011.

[18] B. Ding and X. Ping, "Output feedback predictive control with one free control move for nonlinear systems represented by a Takagi–Sugeno model," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 2, pp. 249–263, Apr. 2014.

[19] M. Killian, B. Mayer, A. Schirrer, and M. Kozek, "Cooperative fuzzy model-predictive control," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 2, pp. 471–482, Apr. 2016.

[20] L. Teng, Y. Wang, W. Cai, and H. Li, "Robust model predictive control of discrete nonlinear systems with time delays and disturbances via T-S fuzzy approach," *J. Process Control*, vol. 53, pp. 70–79, 2017.

[21] ——, "Fuzzy model predictive control of discrete systems with time-varying delay and disturbances," *IEEE Trans. Fuzzy Syst.*, vol. PP, no. 99, pp. 1–1, 2017.

[22] J. Causa, G. Karer, A. Núnez, D. Sáez, I. Škrjanc, and B. Zupančič, "Hybrid fuzzy predictive control based on genetic algorithms for the temperature control of a batch reactor," *Comput. Chem. Eng.*, vol. 32, no. 12, pp. 3254–3263, 2008.

[23] G. Karer, G. Mušič, I. Škrjanc, and B. Zupančič, "Model predictive control of nonlinear hybrid systems with discrete inputs employing a hybrid fuzzy model," *Nonlinear Anal.-Hybrid Syst.*, vol. 2, no. 2, pp. 491–509, 2008.

[24] M. Johansson, A. Rantzer, and K.-E. Arzen, "Piecewise quadratic stability of fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 7, no. 6, pp. 713–722, Dec. 1999.

[25] H. O. Wang, K. Tanaka, and M. F. Griffin, "An approach to fuzzy control of nonlinear systems: stability and design issues," *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 1, pp. 14–23, Feb. 1996.

[26] T. M. Guerra and L. Vermeiren, "LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi–Sugeno's form," *Automatica*, vol. 40, no. 5, pp. 823–829, 2004.

[27] M. Narimani, H.-K. Lam, R. Dilmaghani, and C. Wolfe, "LMI-based stability analysis of fuzzy-model-based control systems using approximated polynomial membership functions," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 41, no. 3, pp. 713–724, Jun. 2011.

[28] K. Tanaka, H. Yoshida, H. Ohtake, and H. O. Wang, "A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 911–922, Aug. 2009.

[29] J. Dong, Y. Wang, and G.-H. Yang, "Control synthesis of continuous-time T-S fuzzy systems with local nonlinear models," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 39, no. 5, pp. 1245–1258, Oct. 2009.

[30] ——, "H_∞ and mixed H_2/H_∞ control of discrete-time T-S fuzzy systems with local nonlinear models," *Fuzzy Sets Syst.*, vol. 164, no. 1, pp. 1–24, 2011.

[31] H. Shengjuan and G.-H. Yang, "Input-output based fault estimation for ts fuzzy systems with local nonlinear parts," *IEEE Trans. Fuzzy Syst.*, 2016.

[32] G.-H. Yang and H. Wang, "Fault detection and isolation for a class of uncertain state-feedback fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 1, pp. 139–151, 2015.

[33] L. Teng, Y. Wang, W. Cai, and H. Li, "Robust model predictive control for discrete ts fuzzy systems with nonlinear local models," in *12th IEEE International Conference on Control and Automation*, 2016, pp. 74–79.

[34] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, 2001.

[35] Y. Wang, L. Xie, and C. E. de Souza, "Robust control of a class of uncertain nonlinear systems," *Syst. Control Lett.*, vol. 19, no. 2, pp. 139–149, 1992.

[36] L. Magni and R. Scattolini, "Robustness and robust design of MPC for nonlinear discrete-time systems," in *Assessment and future directions of nonlinear model predictive control*. Springer, 2007.

[37] D. M. Raimondo, D. Limon, M. Lazar, L. Magni, and E. F. Camacho, "Min-max model predictive control of nonlinear systems: A unifying overview on stability," *Eur. J. Control*, vol. 15, no. 1, pp. 5–21, 2009.

[38] D. Limon, T. Alamo, D. M. Raimondo, D. M. de la Peña, J. M. Bravo, A. Ferramosca, and E. F. Camacho, "Input-to-state stability: a unifying framework for robust model predictive control," in *Nonlinear model predictive control*. Springer, 2009.

[39] D. Q. Mayne, M. M. Seron, and S. V. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219–224, 2005.

[40] R. Ghaemi, J. Sun, and I. V. Kolmanovsky, "Robust control of constrained linear systems with bounded disturbances," *IEEE Trans. Autom. Control*, vol. 57, no. 10, pp. 2683–2688, Oct. 2012.

[41] J. C. Geromel, P. D. Peres, and J. Bernussou, "On a convex parameter space method for linear control design of uncertain systems," *SIAM J. Control Optim.*, vol. 29, no. 2, pp. 381–402, 1991.

[42] M. V. Sy and P. X. Minh, "Fuzzy model predictive control using Takagi-Sugeno model," in *International Conference on Control, Automation and Systems*, 2008, pp. 632–637.