

Natural-ordered complex Hadamard transform

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Abstract

This paper presents a new transform known as natural-ordered complex Hadamard transform (NCHT) which is derived from the Walsh–Hadamard transform (WHT) through the direct block matrix operation. Some of its properties, including the exponential property of the NCHT and the shift invariant property of the NCHT power spectrum, are presented. The relationship of the NCHT with the sequency-ordered complex Hadamard transform (SCHT) is discussed. In fact, it is shown that NCHT is a natural-ordered version of complex Hadamard transform whereas SCHT shows the sequency ordering. This is parallel to their real-valued counterparts, the WHT and the sequency-ordered Walsh transform (SOWT). A fast algorithm for computing the NCHT is also developed using sparse matrix factorization and its computational complexity is examined.

Keywords: Discrete orthogonal transform, Complex Hadamard transform, Fast algorithm, Natural-ordered complex Hadamard transform (NCHT)

1. Introduction

Discrete orthogonal transforms have been used extensively in control and communication theories, and digital signal processing. The commonly used orthogonal transforms which are frequently found in the literature are the discrete Fourier transform (DFT), the discrete cosine transform (DCT), the discrete wavelet transform (DWT), the Karhunen–Loeve transform (KLT), etc. The DCT is popular for its superior energy compaction property, that is, a large fraction of the average energy of the signal is packed into a relatively few components of the DCT coefficients [1]. It is often used in data and image compression [2,3]. The DWT is well-known for its multiresolutional analysis for image signal processing [4]. The KLT is known to be a statistically optimal transform at the price of costly computational requirement [1,5]. The Hilbert transform is also well-known in signal processing, which provides the imaginary component for the real signal creating a so called strong analytic signal [6]. It is often used in designing modulated single sideband signal in order to remove the redundant information during the demodulation process. In [7], Huang et al. introduced a new method for analyzing

nonlinear and non-stationary data, which is known as empirical mode decomposition (EMD), by making use of the Hilbert transform based on the iterative extraction of intrinsic mode functions which yield instantaneous frequencies as functions of time. Gianfelici et al. [8] used the iterated application of the Hilbert transform in order to get accurate demodulation of multicomponent AM–FM signals with applications to both synthetic signals and natural speech.

The simplest form of the orthogonal transforms is the Walsh–Hadamard transform (WHT) which is widely used in the fields of digital signal processing and communications. The WHT is derived from the Walsh functions. Since the Walsh functions form an ordered set of rectangular waveforms, they consist of two amplitude values $\{\pm 1\}$. Each row vector of a WHT matrix represents a sampled version of the Walsh function over a normalized time base $0 \leq t \leq 1$. The Walsh functions can be arranged to form any of three main orderings which are in common use. They are Walsh or sequency order, dyadic order and natural or Hadamard order [9,10], each of which has its own particular advantages in applications. The choice of orderings depends on the particular applications. In general, dyadic and natural orderings are suited to be used in mathematical work, in image transmission and for computational efficiency, whereas sequency order is favored for communications and signal processing work such as spectral analysis and filtering [9,11].

The WHT matrix simply expresses that a sampled Walsh function series is arranged in natural or Hadamard order. Therefore, the arrangement of the row vectors of a Hadamard matrix describes the ordering of the matrix or a Walsh function series. If the Walsh function values are arranged in ascending values of sequencies, a sequency-ordered Walsh transform (SOWT) matrix will be obtained [12]. Sequency describes the number of times that the row vectors of a matrix crosses the imaginary axis in the unit circle per unit time interval. Actually, it measures how fast the elements of the row vectors of a matrix vary over a normalized time base $0 \leq t \leq 1$. Sequency is analogous to frequency in the discrete Fourier transform (DFT) in which the Fourier components are positioned in an increasing harmonic number. One can compute the sequences of different orderings from one known ordered sequence using their relationships. For example, an SOWT matrix can be obtained by rearranging the row vectors of a WHT matrix through a bit reversal and a Gray code conversion [9].

Ref. [13] introduced a concept of multipaired unitary transforms which explain the mathematical structure of Fourier transforms. Such paired transforms can be used to obtain the matrix decompositions of the Fourier and Hadamard transforms. On the other hand, the complex binary matrices can be obtained from similar decompositions (via the paired transforms) of the Fourier transforms by modifying the elements of the diagonal matrices which consists of the twiddle factors. The resultant matrices consists of the elements which are strictly confined to be $\{\pm 1, \pm j\}$, where $j = \sqrt{-1}$, and they represents the complex Hadamard matrices. But they do not exhibit any ordering which may correspond to their real-valued counterparts (i.e., the WHT and the SOWT). In [14], a new orthogonal transform known as the sequency-ordered complex Hadamard transform (SCHT), whose elements are $\{\pm 1, \pm j\}$, was introduced and some of its properties and potential applications were reported. This particular transform possesses sequency

property, that is, its row vectors are positioned in ascending order of the number of zero crossings in the unit circle of a complex plane. It can be shown that the SCHT is complex counterpart of the SOWT. Therein, the question arises whether complex Hadamard transform whose ordering may follow that of the WHT exists or not. This motivates us to find the complex Hadamard matrices which correspond to WHT matrices (natural order).

It is then the purpose of this paper to propose the natural-ordered complex Hadamard transform whose transform matrix is constructed based on the WHT and the direct block matrix operator. It is a new complex Hadamard transform and different from the SCHT since the SCHT matrices are generated based on the complex Rademacher matrices [14]. Generally, NCHT is a discrete orthogonal transform whose elements are confined to four complex values $\{\pm 1, \pm j\}$, and its row vectors are orthogonal to each other in the complex domain. It is a natural-ordered version of complex Hadamard matrix while SCHT is a complex Hadamard transform with sequency ordering. In this paper, we focus on the orderings of complex Hadamard transforms, which are actually in line with their real-valued counterparts, that is, the WHT and the SOWT. The organization of the paper is as follows. Section 2 describes the construction of the NCHT based on the WHT and some of its important properties. The direct block matrix operator is also defined in that section. The relationship of the NCHT with the SCHT is presented in Section 3. Using the sparse matrix factorization approach, a fast algorithm to compute the NCHT is derived in Section 4. Finally, this paper is concluded in Section 5.

2. Basic definitions and properties of the NCHT

In this section, we shall present the construction of the NCHT matrices and some of their properties. Before going into the generation of the NCHT, we first describe the WHT and its matrix.

2.1. WHT

Walsh-Hadamard transform is a linear transform which could represent a vector by the linear combination of the row vectors of a transform matrix. Its transformation matrix, \mathbf{W}_N , is a square and symmetric matrix defined as

$$\mathbf{W}_N = \begin{bmatrix} \mathbf{W}_{N/2} & \mathbf{W}_{N/2} \\ \mathbf{W}_{N/2} & -\mathbf{W}_{N/2} \end{bmatrix} \quad (1)$$

where $\mathbf{W}_1 = 1$. Particularly, if $N = 2^n$ where n is a positive integer,

$$\mathbf{W}_N = \mathbf{W}_2 \otimes \mathbf{W}_{N/2} = \mathbf{W}_2 \otimes^n \quad (2)$$

where \otimes^n denotes the right hand side Kronecker product being applied n times.

2.2. NCHT

A natural-ordered complex Hadamard transform matrix, denoted as \mathbf{H}_N , is constructed using a similar way of generating the WHT matrix but the multiplications via \mathbf{S} matrices are introduced in the lower half of \mathbf{H}_N . It can be expressed as follows:

$$\mathbf{H}_N = \left[\begin{array}{c|c} \mathbf{H}_{N/2} & \mathbf{H}_{N/2} \\ \hline \mathbf{H}_{N/2}\mathbf{S}_{N/2} & -\mathbf{H}_{N/2}\mathbf{S}_{N/2} \end{array} \right] \quad (3)$$

where $N = 2^n$ and

$$\mathbf{S}_{2^{n-1}} = \left[\begin{array}{cc} \mathbf{I}_{2^{n-2}} & \mathbf{0} \\ \mathbf{0} & j\mathbf{I}_{2^{n-2}} \end{array} \right] \quad (4)$$

where $\mathbf{I}_{2^{n-2}}$ is the identity matrix of order $2^{n-2} \times 2^{n-2}$.

In this way, an $N \times N$ NCHT matrix can be expressed using the NCHT matrices of order $\frac{N}{2} \times \frac{N}{2}$, and the smallest dimension of the NCHT matrix will be the size of 4×4 as shown below.

$$\mathbf{H}_4 = \left[\begin{array}{cc} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2\mathbf{S}_2 & -\mathbf{H}_2\mathbf{S}_2 \end{array} \right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \\ 1 & -j & -1 & j \end{bmatrix} \quad (5)$$

where

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_2 = \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}.$$

As a result, the NCHT matrix can be expressed using the WHT matrix and the direct block matrix operator as follows:

$$\mathbf{H}_N = \mathbf{W}_N \diamond \mathbf{A}_{n-1,n-1} \diamond \cdots \diamond \mathbf{A}_{1,1} \quad (6)$$

where \mathbf{W}_N is the $N \times N$ WHT matrix,

$$\begin{aligned} \mathbf{A}_{1,1} &= [\mathbf{I}_{2^{n-1}}, \mathbf{S}_{2^{n-1}}]^T \\ \mathbf{A}_{2,2} &= [\mathbf{I}_{2^{n-2}}, \mathbf{S}_{2^{n-2}}, \mathbf{I}_{2^{n-2}}, \mathbf{S}_{2^{n-2}}]^T; \\ \mathbf{A}_{n-1,n-1} &= [\mathbf{I}_2, \mathbf{S}_2, \dots, \mathbf{I}_2, \mathbf{S}_2]^T \end{aligned} \quad (7)$$

and \diamond denotes the direct block matrix operator which is defined as follows.

Let \mathbf{B} be a block matrix such that

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1N} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \cdots & \mathbf{B}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{M1} & \mathbf{B}_{M2} & \cdots & \mathbf{B}_{MN} \end{bmatrix},$$

\mathbf{R} be a row block matrix, that is,

$$\mathbf{R} = [\mathbf{R}_1 \quad \mathbf{R}_2 \quad \cdots \quad \mathbf{R}_N]$$

And \mathbf{C} be a column block matrix which is

$$\mathbf{C} = [\mathbf{C}_1 \quad \mathbf{C}_2 \quad \cdots \quad \mathbf{C}_M]^T.$$

Then the direct block matrix operator \diamond is defined as

$$\mathbf{B} \diamond \mathbf{R} = \begin{bmatrix} \mathbf{B}_{11}\mathbf{R}_1 & \mathbf{B}_{12}\mathbf{R}_2 & \cdots & \mathbf{B}_{1N}\mathbf{R}_N \\ \mathbf{B}_{21}\mathbf{R}_1 & \mathbf{B}_{22}\mathbf{R}_2 & \cdots & \mathbf{B}_{2N}\mathbf{R}_N \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{M1}\mathbf{R}_1 & \mathbf{B}_{M2}\mathbf{R}_2 & \cdots & \mathbf{B}_{MN}\mathbf{R}_N \end{bmatrix} \quad (8)$$

and

$$\mathbf{B} \diamond \mathbf{C} = \begin{bmatrix} \mathbf{B}_{11}\mathbf{C}_1 & \mathbf{B}_{12}\mathbf{C}_1 & \cdots & \mathbf{B}_{1N}\mathbf{C}_1 \\ \mathbf{B}_{21}\mathbf{C}_2 & \mathbf{B}_{22}\mathbf{C}_2 & \cdots & \mathbf{B}_{2N}\mathbf{C}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{M1}\mathbf{C}_M & \mathbf{B}_{M2}\mathbf{C}_M & \cdots & \mathbf{B}_{MN}\mathbf{C}_M \end{bmatrix} \quad (9)$$

where the submatrices \mathbf{B}_{mn} , \mathbf{R}_k and \mathbf{C}_l are the square matrices which have the same dimensions, and m, n, k and l are positive integers such that $1 \leq m, l \leq M$ and $1 \leq n, k \leq N$. If the dimensions of the submatrices are 1×1 , the operation of the direct block matrix operator is similar to that of the matrix operator defined in [15].

If $N = 8$, for example, with $n = \log_2 8 = 3$, then (6) will become

$$\mathbf{H}_8 = \mathbf{W}_8 \diamond \mathbf{A}_{2,2} \diamond \mathbf{A}_{1,1}. \quad (10)$$

Substituting the corresponding values of (7) into (10), we have

$$\mathbf{H}_8 = \left[\begin{array}{cc|cc} \mathbf{W}_2 & \mathbf{W}_2 & \mathbf{W}_2 & \mathbf{W}_2 \\ \mathbf{W}_2 & -\mathbf{W}_2 & \mathbf{W}_2 & -\mathbf{W}_2 \\ \hline \mathbf{W}_2 & \mathbf{W}_2 & -\mathbf{W}_2 & -\mathbf{W}_2 \\ \mathbf{W}_2 & -\mathbf{W}_2 & -\mathbf{W}_2 & \mathbf{W}_2 \end{array} \right] \diamond \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{S}_2 \\ \mathbf{I}_2 \\ \mathbf{S}_2 \end{bmatrix} \diamond \begin{bmatrix} \mathbf{I}_4 \\ \mathbf{S}_4 \end{bmatrix} \quad (11)$$

where

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is a 2×2 WHT matrix. Thus,

$$\mathbf{H}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & 1 & j & j & -1 & -1 & -j & -j \\ 1 & -1 & j & -j & -1 & 1 & -j & j \\ 1 & j & -j & 1 & -1 & -j & j & -1 \\ 1 & -j & -j & -1 & -1 & j & j & 1 \end{bmatrix}. \quad (12)$$

Since the NCHT is corresponding to the WHT, its construction is based on the WHT and recursive operation as well. The NCHT, however, is a complex Hadamard transform which is useful to deal with the complex-valued functions as compared to the WHT. With the transform matrix developed, the NCHT of a complex signal vector $\mathbf{x}_N = [x(0), x(1), \dots, x(N-1)]^T$ is defined as

$$\mathbf{X}_N = \frac{1}{N} \bar{\mathbf{H}}_N \mathbf{x}_N \quad (13)$$

where $N = 2^n$, $\mathbf{X}_N = [X(0), X(1), \dots, X(N-1)]^T$ is the transformed complex column vector and $\bar{\mathbf{H}}_N$ is the complex conjugate of \mathbf{H}_N which is defined in (6). The data sequence can be retrieved uniquely from the inverse transform which is given by

$$\mathbf{x}_N = \mathbf{H}_N^T \mathbf{X}_N \quad (14)$$

where \mathbf{H}_N^T is the transpose of \mathbf{H}_N and $\mathbf{H}_N^T \bar{\mathbf{H}}_N = \mathbf{H}_N^* \mathbf{H}_N = N \mathbf{I}_N$ (unitary property) where \mathbf{H}_N^* is the complex conjugate transpose of \mathbf{H}_N .

2.3. Properties of the NCHT

This subsection presents the properties of the NCHT.

Property 1. *Exponential property of the NCHT.* Let \mathbf{H}_N be any NCHT matrix of the dimension $N \times N$ where $N = 2^n$ and $h(p, k)$ be an element of \mathbf{H}_N at p th row and k th column, where $0 \leq p, k \leq N-1$. Then,

$$h(p, k) = (-1)^{\sum_{r=0}^{n-1} p_r k_r} (j)^{\sum_{r=0}^{n-1} \hat{p}_r k_r} \quad (15)$$

where $n = \log_2 N$ and p_r, k_r are the r th bits of the binary representations of the decimal integers p and k respectively, i.e., $(p)_{10} = (p_{n-1}, p_{n-2}, \dots, p_0)_2$ and $(k)_{10} = (k_{n-1}, k_{n-2}, \dots, k_0)_2$. \hat{p}_r represents the $(n-1)$ least significant bits (LSB) of the bit reversal of the decimal integer p in a bit-reversed order. Table 1 displays the binary representations of \hat{p}_r for $N = 8$. The complex conjugate of \mathbf{H}_N can be generated by just changing j to $-j$ in (15).

Property 2. *Unitary property.* NCHT is a unitary transform whose row vectors of the transform matrix are orthogonal in the complex domain. Let \mathbf{H}_N be any NCHT matrix of order $N \times N$ where $N = 2^n$, then \mathbf{H}_N is said to be orthogonal in the complex domain as

$$|\det \mathbf{H}_N| = N^{N/2} \quad (16)$$

and

$$\mathbf{H}_N \mathbf{H}_N^* = \mathbf{H}_N^* \mathbf{H}_N = N \mathbf{I}_N \quad (17)$$

where \mathbf{H}_N^* denotes the complex conjugate transpose of the matrix \mathbf{H}_N and \mathbf{I}_N is the identity matrix of order $N \times N$.

Property 3. *Linearity.* NCHT is a linear operator just like the SCHT.

Property 4. *Parseval's Theorem.* NCHT obeys the Parseval's Theorem. The energy of the signal in time and transform domains is related as

$$\frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |X(k)|^2. \quad (18)$$

Proof. From Eq. (13),

$$\mathbf{X}_N^* \mathbf{X}_N = \frac{1}{N} [\overline{\mathbf{H}_N \mathbf{X}_N}]^* \frac{1}{N} \overline{\mathbf{H}_N \mathbf{X}_N} = \frac{1}{N^2} \mathbf{X}_N^* \mathbf{H}_N^T \overline{\mathbf{H}_N \mathbf{X}_N} = \frac{1}{N} \mathbf{X}_N^* \mathbf{X}_N$$

where $\mathbf{H}_N^T \overline{\mathbf{H}_N} = N \mathbf{I}_N$, and $(\cdot)^*$ denotes the complex conjugate transpose. The proof is completed here.

Property 5. *Shift invariant power spectrum.* The NCHT is invariant to the cyclic shift of the data sequence $\{x(n)\}_0^{N-1}$ of length N . Let \mathbf{x}_N^m be the data sequence, $x(n)$, which is cyclically shifted to the right by m places, that is, $\mathbf{x}_N^m = [x(N-m), \dots, x(N-1), x(0), \dots, x(N-m-1)]^T$ where $m = 1, 2, \dots, N-1$. Then, the NCHT of the data vector, \mathbf{x}_N^m , is related to \mathbf{X}_N as

$$\mathbf{X}_N^m = \frac{1}{N} \overline{\mathbf{H}_N} \mathbf{x}_N^m = \mathbf{D}_N^m \mathbf{X}_N \quad (19)$$

where

$$\mathbf{D}_N^m = \frac{1}{N} \overline{\mathbf{H}_N} \mathbf{P}_N^m \mathbf{H}_N^T$$

and \mathbf{P}_N^m is the identity matrix of order $N \times N$, \mathbf{I}_N , whose rows are cyclically shifted by m places. \mathbf{D}_N^m is the cyclic transformation matrix which express the relation of \mathbf{X}_N^m with \mathbf{X}_N . It has a block diagonal unitary structure which leads to the shift invariant NCHT power spectrum as follows:

$$P(k) = |X(k)|^2 = |X^m(k)|^2$$

where $k = 0, 1, 2, 3$ and

$$\begin{aligned}
 P(2l+2) &= \sum_{k=2^{l+1}}^{3 \cdot 2^l - 1} |X(k)|^2 = \sum_{k=2^{l+1}}^{3 \cdot 2^l - 1} |X^m(k)|^2, \\
 P(2l+3) &= \sum_{k=3 \cdot 2^l}^{2^{l+2} - 1} |X(k)|^2 = \sum_{k=3 \cdot 2^l}^{2^{l+2} - 1} |X^m(k)|^2
 \end{aligned} \tag{20}$$

where $l = 1, 2, \dots, n-2$ and $n = \log_2 N$. Therefore, the NCHT has $2n$ power spectral points. Besides, it is also observed that the total energy of the signal is conserved, i.e.,

$$\sum_{k=0}^{N-1} |X^m(k)|^2 = \sum_{k=0}^{N-1} |X(k)|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2. \tag{21}$$

3. Relationship of the NCHT with the SCHT

In [14], it was pointed out that the SCHT matrices can be constructed from the complex Rademacher functions, and their row vectors are arranged in ascending order of sequencies. The row vectors of both SCHT and NCHT matrices are the same except that the way they are arranged in the transform matrices is different. In fact, this is similar to what is happening in the WHT and the SOWT where the row vectors of both transform matrices are the same but they are positioned in different ordered manners.

The natural order and sequency order of complex Hadamard matrices are related through a bit reversal conversion. For example, given a complex Hadamard matrix in sequency ordering, \mathbf{H}_N^{SCHT} , a natural-ordered complex Hadamard matrix, \mathbf{H}_N^{NCHT} , is obtained by

$$\mathbf{H}_N^{NCHT}(p, k) = \mathbf{H}_N^{SCHT}(b(p), k) \tag{22}$$

where $b(p)$ is a decimal number obtained by the bit reversal conversion of the decimal p . A list of converted values for $N = 8$ is given in Fig. 1. An SCHT matrix also can be obtained by rearranging the row vectors of an NCHT matrix in bit-reversed order and vice versa.

The SCHT matrices are constructed based on the complex Rademacher functions while the generation of the NCHT is based on the WHT and the direct block matrix operation. Even though they are generated in separate ways, their row vector are the same and only the arrangements they are positioned in the transform matrices are different. Since, the NCHT is a bit-reversed version of the SCHT, the applications which are suited for the SCHT [14,16] could possibly employ the NCHT as well.

4. Fast algorithm

In this section, an efficient algorithm for computing the NCHT is developed using the sparse matrix factorization approach. In order to conceptualize the factoring method, let us, for example, consider $N = 8$. Then, (3) will become

$$\mathbf{H}_8 = \begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_4 \\ \mathbf{H}_4\mathbf{S}_4 & -\mathbf{H}_4\mathbf{S}_4 \end{bmatrix} \quad (23)$$

where

$$\mathbf{S}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & j\mathbf{I}_2 \end{bmatrix}.$$

Eq. 23 can be factorized as

$$\mathbf{H}_8 = \begin{bmatrix} \mathbf{H}_4 & \\ & \mathbf{H}_4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_4 & \\ & \mathbf{S}_4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_4 & \mathbf{I}_4 \\ \mathbf{I}_4 & -\mathbf{I}_4 \end{bmatrix}. \quad (24)$$

\mathbf{H}_4 can be further factorized and expressed in terms of \mathbf{H}_2 . That is,

$$\mathbf{H}_8 = \begin{bmatrix} \mathbf{H}_2 & & & \\ & \mathbf{H}_2 & & \\ & & \mathbf{H}_2 & \\ & & & \mathbf{H}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_2 & & & \\ & \mathbf{S}_2 & & \\ & & \mathbf{I}_2 & \\ & & & \mathbf{S}_2 \end{bmatrix} \times \begin{bmatrix} \mathbf{I}_2 & \mathbf{I}_2 & & \\ \mathbf{I}_2 & -\mathbf{I}_2 & & \\ & & \mathbf{I}_2 & \mathbf{I}_2 \\ & & \mathbf{I}_2 & -\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_4 & \\ & \mathbf{S}_4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_4 & \mathbf{I}_4 \\ \mathbf{I}_4 & -\mathbf{I}_4 \end{bmatrix} \quad (25)$$

where

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_2 = \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}.$$

The factorization of $\bar{\mathbf{H}}_8$ can be obtained by changing j to $-j$ in the \mathbf{S} matrices. This results in fast forward NCHT algorithm, since $\bar{\mathbf{H}}_N$ is the forward NCHT transformation matrix as defined in (13). The scaling factor of $(1/N)$ can be added at the start of the computation. The decomposition of fast forward 8-point NCHT is illustrated by the signal flow graph in Fig. 2.

Fig. 2 shows the signal flow graph of fast NCHT algorithm for $N = 8$. The multiplication by the scaling factor $(1/N)$ can be performed by the binary shifting operation without requiring any arithmetic computation. For simplicity, the scaling factor is omitted in the signal flow diagram. Fig. 2 has three stages and each stage performs eight addition/subtractions. In each stage, two trivial multiplications by $-j$ are required except for the first stage. In general, for an N -point NCHT, the fast algorithm needs $n = \log_2 N$ stages and each stage requires N addition/subtractions. Hence, the total computational requirements are $N \log_2 N$ complex addition/subtractions and

$(N/4) \log_2(N/2)$ complex multiplications for the trivial twiddle factors. Similar to fast SCHAT algorithm reported in [14], fast NCHAT algorithm reduces the computational cost significantly compared to the direct computation which requires $N(N - 1)$ complex addition/subtractions.

Since the NCHAT matrix is constructed through the recursive operation like that of the WHT, the transform is simple and easy to derive the fast algorithm as well. This sparse factorization is different from the matrix decomposition method for fast inverse SCHAT algorithm which was reported in [14]. But it is similar to the factorization of the decimation-in-frequency (DIS) fast forward SCHAT algorithm which was mentioned in [17] except that the latter needs the bit reversal ordering of the output $X(k)$ where $k = 0, 1, \dots, N - 1$. Consequently, the signal flow graph of the fast NCHAT algorithm is similar to that of DIS fast SCHAT algorithm. Hence, this fast algorithm is suited to be implemented in the hardware using the pipelined hardware structure [17], which can achieve a full hardware utilization as well as less complex data stores.

5. Conclusion

Complex Hadamard transform with natural order is presented in this paper and some of its important properties are outlined. It has been shown that the NCHAT matrices can be generated based on the Walsh–Hadamard transform and through the direct block matrix operator. The exponential form of NCHAT has also been presented. It has been found that the NCHAT power spectrum is invariant to the cyclic shift of the sampled data $x(n)$. The NCHAT is related with the SCHAT through a bit reversal conversion. Similar to the WHT, the sparse matrix factorization of the complex Hadamard matrix leads to fast algorithm for computing the NCHAT. To compute an N -point NCHAT, the fast algorithm requires $N \log_2 N$ complex addition/subtractions and reduces $(N - 1)/\log_2 N$ times of complex addition/subtractions as compared to the direct computation. Since NCHAT is a bit reversal ordered version of SCHAT, it has the same potential to be employed in various applications as the SCHAT, besides, the applications can be further explored in the areas where the natural order is particularly suited.

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Table 1 Binary Notations for \hat{p}_r ($N = 8$).

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Fig. 1. Relationship between natural and sequency orderings of the complex Hadamard matrix.

Fig. 2. Flow graph of fast forward NCHT decomposition ($N = 8$).

p	Binary	Bit reversal	$(n - 1)$ LSB	Bit reversal	\hat{p}_r
0	000	000	00	00	000
1	001	100	00	00	000
2	010	010	10	01	001
3	011	110	10	01	001
4	100	001	01	10	010
5	101	101	01	10	010
6	110	011	11	11	011
7	111	111	11	11	011

Table 1

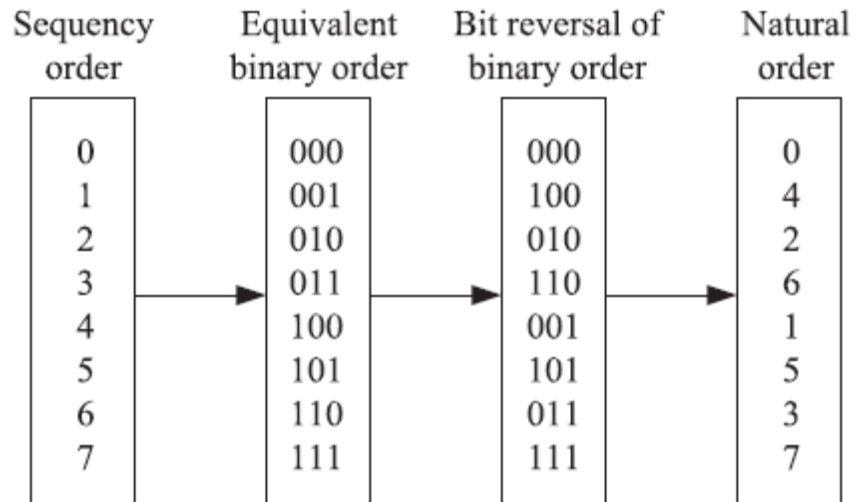


Fig. 1

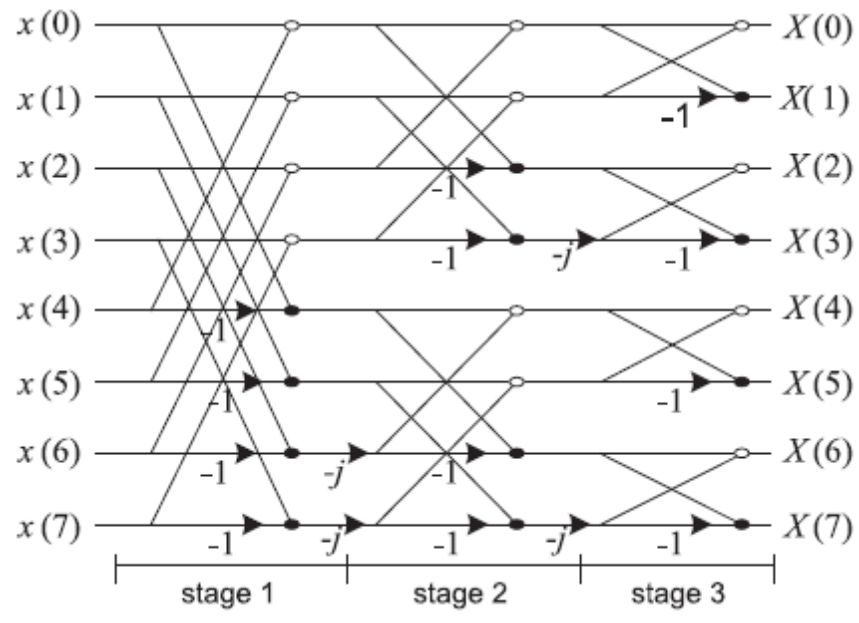


Fig. 2