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Stated SL_n -skein Theory

Wang Zhihao

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Wang Zhihao

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

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University in partial fulfilment of the requirement for the
degree of Doctor of Philosophy

2025

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Andrew Kricker

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Andrew James Kricker

Authorship Attribution Statement

This thesis contains material from one paper published in the following peer-reviewed journal and five papers posted on arXiv in which I am in which listed as an author.

Chapter 2 is published as Z. Wang, The classical limit for stated SL_n -skein modules, 2024, arXiv:2401.14753.

Chapter 3 is published as Z. Wang, On stated $SL(n)$ -skein modules, 2023, arXiv:2307.10288.

Chapter 4 is published as Z. Wang, Stated SL_n -skein modules, roots of unity, and TQFT, to be published by Israel Journal of Mathematics.

Chapter 5 is published as Z. Wang, Finiteness and dimension of stated skein modules over Frobenius, 2023, arXiv:2212.01069.

Chapter 6 is published as Z. Wang, Representation-reduced stated skein modules and algebras, Journal of Algebra 661 (2025): 831-852.

Chapter 7 is published as Z. Wang, Kauffman bracket intertwiners and the volume conjecture, to be published by MSP in Algebraic & Geometric Topology.

Abstract

The stated SL_n -skein module $\mathcal{S}_n(M, \mathcal{N}, v)$ of a (marked) 3-manifold (M, \mathcal{N}) is a quantization of the regular function ring of the SL_n -representation variety of $\pi_1(M, \mathcal{N})$. It is a module over a commutative domain R with an invertible parameter v . This framework extends the Kauffman bracket skein theory (the SL_2 case) to a more general SL_n setting, and it is closely related to the quantum higher Teichmüller space, quantum cluster algebras, and quantum moduli algebras.

In chapter 2, we show that the classical limit of the stated skein module of a marked 3-manifold M corresponds to the ring of functions on the SL_n -representation variety of $\pi_1(M)$. Chapter 3 introduces a \mathbb{C} -linear map, called the Frobenius map, from $\mathcal{S}_n(M, \mathcal{N}, 1)$ to $\mathcal{S}_n(M, \mathcal{N}, v)$, where $R = \mathbb{C}$ and v is a root of unity. This map is pivotal in the representation theory of the stated SL_n -skein algebra, as its image lies within the center of the stated SL_n -skein algebra. In chapter 4, we explore the implications of the stated SL_n -TQFT theory.

Chapters 5 and 6 concentrate on the SL_2 case of stated skein theory. We prove that for compact marked 3-manifolds, the stated skein module at roots of unity is finitely generated over the image of the Frobenius map. Additionally, we define the representation-reduced stated skein module by scaling the action of the Frobenius image on the stated skein module. For closed 3-manifolds over \mathbb{C} , we establish that the dimension of the representation-reduced stated skein module is 1.

In chapter 7, we examine the Bonahon-Wong-Yang volume conjecture. This conjecture is formulated based on intertwiners between two isomorphic irreducible representations of the skein algebra. We compute these intertwiners for the closed torus and the once-punctured torus, showing that the trace of these intertwiners approaches zero in the limit for Seifert manifolds.

Acknowledgements

I am now sitting on my sofa, writing down my thoughts. My emotions at this moment are complex—both happy and sad. My happiness comes from the joy of graduating and my anticipation for the future. My sadness comes from the imminent farewells—to my friends, colleagues, and advisors, as well as to my PhD life itself. These four years of my PhD have been a rare and unforgettable time in my life. The people I have met and the experiences I have had during these four years are destined to be an irreplaceable part of my life. I believe that no matter how much time passes, I will always cherish these memories.

My PhD life was divided into two parts: the first two years in Singapore and the latter two years in the Netherlands. While in Singapore, I worked under the supervision of Professor Andrew James Kricker. During those two years, he helped me immensely. Especially at the beginning of my PhD, when I had no idea what to study or how to proceed, Andrew guided me by recommending papers and books, and sometimes we read papers together. Whenever I encountered concepts I did not understand, Andrew was always willing to discuss them with me, which was a tremendous help. His guidance during those two years laid a solid foundation for my PhD. I am deeply grateful to Professor Andrew James Kricker for his support and assistance throughout my PhD.

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To my parents

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Chapter 1

Introduction and overview

In this chapter, we provide the background for this thesis and briefly introduce chapters 2-7.

1.1 Introduction

We begin with an introduction of framed links inside oriented 3-manifolds [KRT97; CF12], which will be used to introduce the skein modules and algebras. A **link** in an oriented 3-manifold M is a finite collection of disjoint circles smoothly embedded in M . We need to consider normal vector fields on links in M . A link $L \subset M$ provided with a non-singular normal vector field is said to be **framed**. Two vector fields on L homotopic in the class of non-singular vector fields determine the same framing. Two framed links in M are said to be **isotopic** if one can be obtained from the other by the smooth deformation.

We use link diagrams, see Figure 1.1, to represent (framed) links. We consider the blackboard framing, which means that the vector field is orthogonal to the paper/blackboard where the part of links is drawn. When we draw (framed) links in M , we normally draw local pictures. Let $D \times [0, 1]$ be a smoothly embedded cube in M such that it intersects a (framed) link L in M , where D is the 2-dimensional square. We can isotope L such that the projection of $L \cap (D \times [0, 1])$ on D is generic, i.e., consists of immersed loops or lines with only double transversal crossings and the framing is given by the positive direction of $[0, 1]$. Over each such crossing, lie

exactly two points of L . One of these two points lies higher than the other one which allows to provide the crossing with an additional information: one of the two intersecting branches is “higher” and the other one is “lower”, see Figure 1.1 and pictures in (1.2), (1.3).

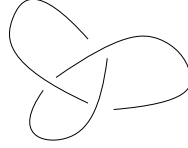


FIGURE 1.1: The trefoil knot.

Here we introduce the SL_n character variety whose quantization is the SL_n -skein algebra (section 2.1). For any topological space T such that $\pi_1(T)$ is a finitely generated group, we use

$$\mathrm{Hom}(\pi_1(T), \mathrm{SL}_n(\mathbb{C}))$$

to denote the set of group homomorphisms from $\pi_1(T)$ to $\mathrm{SL}_n(\mathbb{C})$ (one might want to choose a basepoint of T ; this choice will not matter later). We define an equivalence relation \simeq on $\mathrm{Hom}(\pi_1(T), \mathrm{SL}_n(\mathbb{C}))$. Let $\rho, \rho' \in \mathrm{Hom}(\pi_1(T), \mathrm{SL}_n(\mathbb{C}))$. Define $\rho \simeq \rho'$ if and only if $\mathrm{tr}(\rho(x)) = \mathrm{tr}(\rho'(x))$ for all $x \in \pi_1(T)$. Define

$$\mathfrak{X}_{\mathrm{SL}_n(\mathbb{C})}(T) = \mathrm{Hom}(\pi_1(T), \mathrm{SL}_n(\mathbb{C})) / \simeq, \quad (1.1)$$

called the SL_n **character variety**. Then $\mathfrak{X}_{\mathrm{SL}_n(\mathbb{C})}(T)$ is an algebraic set over the complex field [Sik01].

The Kauffman bracket skein module of an oriented 3-manifold M was introduced by Przytycki [Prz98] and Turaev [Tur91], which is a quantization of the regular function ring of $\mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(M)$ when M is the thickening of an oriented surface [Bul97; BFK99; PS00]. Let R be a commutative domain with an invertible element q . The **(Kauffman bracket) skein module** $\mathcal{S}_q(M)$ of M , as an R -module, is generated by all isotopy classes of framed links in M , subject to the following relations:

$$\begin{array}{c} \text{X} \\ \text{---} \end{array} = q \begin{array}{c} \text{X} \\ \text{---} \end{array} + q^{-1} \begin{array}{c} \text{X} \\ \text{---} \end{array} \quad (1.2)$$

$$\bigcirc = -(q^2 + q^{-2}) \square, \quad (1.3)$$

where each gray square represents a projection of an embedded cube in M and in each equation the parts of framed links outside the gray squares are identical.

Definition 1.1.1. *By a **surface** we mean a surface obtained from an oriented compact surface $\overline{\mathfrak{S}}$ possibly with boundary by removing a finite set \mathcal{P} of points. The members of \mathcal{P} are called **punctures** of \mathfrak{S} . If \mathcal{P} has a non-empty intersection with each boundary component of $\overline{\mathfrak{S}}$, then \mathfrak{S} is called a **pb surface**. If \mathfrak{S} has empty boundary and if \mathcal{P} has a non-empty intersection with each connected component of $\overline{\mathfrak{S}}$, we say that \mathfrak{S} is a **punctured surface**, which is of primary interest in this thesis. If $\mathcal{P} = \emptyset$, we say that \mathfrak{S} is a closed surface.*

*A pb surface \mathfrak{S} is called an **essentially bordered pb surface** if every component of \mathfrak{S} has non-empty boundary.*

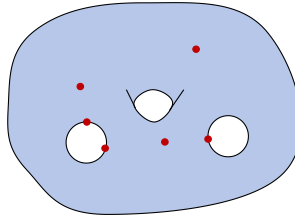


FIGURE 1.2: An example of the pb surface, where the red points are punctures.

When $M = \mathfrak{S} \times [-1, 1]$, where \mathfrak{S} is a surface, we use $\mathcal{S}_q(\mathfrak{S})$ to denote $\mathcal{S}_q(M)$. Then $\mathcal{S}_q(\mathfrak{S})$ has an algebra structure given by stacking framed links. For any two framed links $L_1, L_2 \in M$, define $L_1 \cdot L_2$ to be the framed link in M obtained by stacking L_1 above L_2 . We call $\mathcal{S}_q(\mathfrak{S})$ the **skein algebra** of \mathfrak{S} . It is well-known that the skein algebra is a domain and is finitely generated as an algebra [AF17; BW11; PS19].

Bonahon and Wong constructed an algebra embedding from the skein algebra to the balanced quantization of the Teichmüller space [BW11], called the **quantum trace map**. This algebra embedding relates the skein theory and the quantization of the Teichmüller theory [FG09] (or quantum cluster algebra [BZ05; Mul16]). This algebra embedding is helpful to study the center and the representation theory of the skein algebra [BW16a; BW17; BW19; FKL19].

Suppose that the ground ring R is the complex field \mathbb{C} and q^2 is a root of unity of order N . Set $\eta = q^{N^2}$. For an oriented surface \mathfrak{S} , Bonahon and Wong defined an

algebra embedding

$$\mathcal{F}: \mathcal{S}_\eta(\mathfrak{S}) \rightarrow \mathcal{S}_q(\mathfrak{S}), \quad (1.4)$$

called the **Chebyshev-Frobenius map** (or **Frobenius map**) [BW16a]. It has been shown that $\text{Im } \mathcal{F}$ or a subalgebra thereof is contained in the center of $\mathcal{S}_q(\mathfrak{S})$ [BW16a], highlighting the significance of the algebra embedding \mathcal{F} . Moreover, \mathcal{F} plays a crucial role in understanding the center of $\mathcal{S}_q(\mathfrak{S})$ and in establishing the Unicity Theorem for the skein algebra [FKL19], which classifies its irreducible representations.

The stated skein algebra was introduced by Bonahon and Wong [BW11] to construct the quantum trace map. Lê refined their definition and provided a clearer and easier way to understand the quantum trace map [Lê18]. The definition of the stated skein module involves marked 3-manifolds and stated tangles.

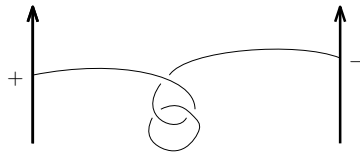


FIGURE 1.3: An example of stated tangles

Definition 1.1.2. A **marked 3-manifold** is a pair (M, \mathcal{N}) , where M is an oriented 3-manifold, and \mathcal{N} is a one dimensional submanifold of ∂M consisting of finitely many oriented open intervals such that there is no intersection between the closure of any two open intervals. Each component of \mathcal{N} is called the **marking** of (M, \mathcal{N}) .

A properly embedded one dimensional submanifold α of M is called an (M, \mathcal{N}) -**tangle** if $\partial\alpha \subset \mathcal{N}$ and α is equipped with a framing such that framings at $\partial\alpha$ respect to velocity vectors of \mathcal{N} . If there is a map $s: \partial\alpha \rightarrow \{-, +\}$, then we call α a **stated** (M, \mathcal{N}) -**tangle**.

Let R be a commutative domain with an invertible element $q^{\frac{1}{2}}$. For a marked 3-manifold (M, \mathcal{N}) , we use $\text{Tangle}(M, \mathcal{N})$ to denote the free R -module with all isotopy classes of stated (M, \mathcal{N}) -tangles as a basis. Then the stated skein module $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ of (M, \mathcal{N}) is $\text{Tangle}(M, \mathcal{N})$ quotiented by relations (1.2)-(1.3) and

the following relations:

$$\begin{array}{c} \text{[gray square]} \\ \text{[circle with dot]} \\ \text{[- +]} \end{array} = q^{-\frac{1}{2}} \begin{array}{c} \text{[gray square]} \\ \text{[dot]} \\ \text{[]} \end{array}, \quad \begin{array}{c} \text{[gray square]} \\ \text{[circle with dot]} \\ \text{[+ +]} \end{array} = \begin{array}{c} \text{[gray square]} \\ \text{[circle with dot]} \\ \text{[- -]} \end{array} = 0 \quad (1.5)$$

$$\begin{array}{c} \text{[gray square]} \\ \text{[V-shape]} \\ \text{[+ -]} \end{array} = q^2 \begin{array}{c} \text{[gray square]} \\ \text{[V-shape]} \\ \text{[- +]} \end{array} + q^{-\frac{1}{2}} \begin{array}{c} \text{[gray square]} \\ \text{[V-shape]} \\ \text{[dot]} \end{array} \quad (1.6)$$

where each black dot represents an oriented interval in \mathcal{N} with the orientation pointing towards readers, each gray square represents a projection of an embedded cube in M , the black lines are parts of stated (M, \mathcal{N}) -tangles, and in each equation the parts of (M, \mathcal{N}) -tangles outside the gray squares are identical.

Let \mathfrak{S} be a pb surface. For each component e of $\partial\mathfrak{S}$, we select a point $x_e \in e$. Set

$$M = \mathfrak{S} \times [-1, 1], \quad \mathcal{N} = \cup_e (\{x_e\} \times (-1, 1))$$

where the union takes over all components of $\partial\mathfrak{S}$. Then (M, \mathcal{N}) is a marked 3-manifold, and we say (M, \mathcal{N}) is the thickening of \mathfrak{S} . We define $\mathcal{S}_{q^{1/2}}(\mathfrak{S})$ to be $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$. For any two stated (M, \mathcal{N}) -tangles α, β , the product $\alpha\beta$ is obtained by stacking α above β . This makes $\mathcal{S}_{q^{1/2}}(\mathfrak{S})$ into an algebra, called the **stated skein algebra** of \mathfrak{S} .

The stated skein algebra has more interesting properties than the skein algebra, such as the splitting map, the comodule structure over $\mathcal{O}_q(\mathrm{SL}_2)$, and the gluing ideal triangle map [CL22a]. Costantino and Lê [CL22a] defined a quotient algebra of the stated skein algebra, called the reduced stated skein algebra. Lê and Yu proved that the reduced stated skein algebra is isomorphic to the quantum cluster algebra [BZ05; Mul16] when the corresponding pb surface has no interior punctures [LY22].

The (stated) SL_n -skein module or algebra was introduced in [LS21; Sik05], which is related to the quantum higher Teichmüller space [FG06; LY23], the quantum cluster algebra [LY22; Mul16; Kim24; KW24b; IY23], and the quantum moduli algebra [Fai20; BFR23]. The above stated skein theory is a special case for stated SL_n -skein when $n = 2$ [LS21]. The classical limit of the stated SL_n -skein module was discussed in [Wan24]; the Frobenius map of the stated SL_n -skein module was discussed in [Wan23c]; the center and the Unicity Theorem for the stated SL_n -skein

algebra were discussed in [KW24a; KW24c; Wan23c]; the quantum trace map for the stated SL_n -skein algebra is discussed in [LY23].

1.2 Overview

In chapters 2–4, we focus on general stated SL_n -skein theory [LS21]. The definition of the stated SL_n -skein module of a marked 3-manifold (M, \mathcal{N}) involves stated n -webs (see Definition 2.1.1). Let R be a commutative domain with an invertible element $v = q^{\frac{1}{2n}}$. The stated SL_n -skein module of a marked 3-manifold (M, \mathcal{N}) , denoted as $\mathcal{S}_n(M, \mathcal{N}, v)$, is generated by isotopy classes of stated n -webs in (M, \mathcal{N}) , subject to relations (2.4)–(2.11).

It is well known that $\mathcal{S}_2(M, \mathcal{N}, v)$ is isomorphic to $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ as R -modules [LS21]. Thus, the stated SL_n -skein module can be regarded as a generalization of the stated skein module (the SL_2 case) to all SL_n .

We investigate the classical limit of the stated SL_n -skein module in chapter 2, drawing on the author’s original work [Wan24]. In chapter 3, we focus on the Frobenius map of the stated SL_n -skein module, based on the author’s earlier results [Wan23c]. We explore the SL_n -TQFT theory in chapter 4, building on the author’s recent work [Wan25c].

In chapters 5–7, we explore aspects of the stated skein theory (the SL_2 case). In chapter 5, we investigate the finiteness property and the dimension of the stated skein module at roots of unity over the image of the Frobenius map (see (5.2)), based on the author’s previous work [Wan23b]. Chapter 6 demonstrates that the representation-reduced stated skein module of closed 3-manifolds is isomorphic to \mathbb{C} (section 6.2), based on the author’s recent work [Wan25b]. Finally, in chapter 7, we examine the volume conjecture in the context of the skein algebra, building on the author’s work [Wan25a].

For the convenience of the reader, we include below a more detailed chapter-wise description of the contents of this thesis. All the theorems presented in sections 1.2.1–1.2.6 are the author’s original contributions, derived from the following works: [Wan24] (chapter 2), [Wan23c] (chapter 3), [Wan25c] (chapter 4), [Wan23a] (chapter 5), [Wan25b] (chapter 6), and [Wan25a] (chapter 7).

1.2.1 Chapter 2

In this chapter, we concentrate on examining the classical limit of the stated SL_n -skein module. Understanding the classical limit is crucial for constructing the significant Frobenius map for the stated SL_n -skein module, which in turn plays a role in studying the representation theory of stated SL_n -skein algebras. We present a clear exposition of the algebraic generators and their relations in the classical limit, as detailed in Theorem 2.4.15. Using these generators and relations, we then proceed to construct the Frobenius map for stated SL_n -skein modules in chapter 3.

For the SL_2 -skein theory, Bullock established a surjective algebra homomorphism from $\mathcal{S}_2(M, \emptyset, v^2 = -1)$, which is isomorphic to $\mathcal{S}_2(M, \emptyset, 1)$ [Bar99], to the coordinate ring of some algebraic set, and showed the kernel of this map consists of all nilpotents [Bul97]. This is an important development since it showed the connection between the skein theory and the character variety. Moreover it offers a way to interpret $\mathcal{S}_2(M, \emptyset, 1)$. It is also useful to understand the representation theory for SL_2 -skein algebras [BW16a; BW17; BW19].

We generalize Bullock's work to stated SL_n -skein modules for marked 3-manifolds. We construct a surjective algebra homomorphism from $\mathcal{S}_n(M, \mathcal{N}, 1)$ to the coordinate ring of some algebraic set and calculate the kernel of this map. Costantino and Lê have constructed this map when $n = 2$ and (M, \mathcal{N}) is the thickening of an essentially bordered pb surface, and proved that it is an isomorphism [CL22a]. Motivated by Costantino and Lê, we choose the algebraic set to be the homomorphism from the groupoid $\pi_1(M, \mathcal{N})$ to $\mathrm{SL}_n(\mathbb{C})$, whose coordinate ring is denoted as $R_n(M, \mathcal{N})$.

Theorem 1.2.1. *Let (M, \mathcal{N}) be a marked 3-manifold. There exists a surjective algebra homomorphism*

$$\Phi^{(M, \mathcal{N})} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow R_n(M, \mathcal{N}).$$

We show that $\Phi^{(M, \mathcal{N})}$ commutes with the splitting map (section 2.1.4), see Theorem 2.2.12. Our construction is compatible with the construction by Costantino and Lê when $n = 2$ and (M, \mathcal{N}) is the thickening of an essentially bordered pb surface, see section 2.2.5.

In order to calculate $\text{Ker } \Phi^{(M, \mathcal{N})}$, we give an explicit presentation for $\mathcal{S}_n(M, \mathcal{N}, 1)$. We do this in two steps. First we consider the case when \mathcal{N} has only one component.

Theorem 1.2.2. *Let (M, \mathcal{N}) be a connected marked 3-manifold with \mathcal{N} consisting of one component. Then $\mathcal{S}_n(M, \mathcal{N}, 1) \simeq \Gamma_n(M)$, where $\Gamma_n(M)$ is the universal presentation algebra of $\pi_1(M)$, see Definition 2.2.4.*

We use $\mathcal{O}(\text{SL}_n)$ to denote the coordinate ring of $\text{SL}_n(\mathbb{C})$. Then

$$\mathcal{O}(\text{SL}_n) = \mathbb{C}[x_{i,j} \mid 1 \leq i, j \leq n] / (\det(X) = 1)$$

where X is an n by n matrix such that $X_{i,j} = x_{i,j}$, $1 \leq i, j \leq n$. We have that X^{-1} makes sense and is an n by n matrix in $\mathcal{O}(\text{SL}_n)$ because $\det(X) = 1$.

Theorem 1.2.3. *Suppose (M, \mathcal{N}) is a connected marked 3-manifold with $\mathcal{N} \neq \emptyset$, and \mathcal{N}' is obtained from \mathcal{N} by adding one extra marking (Definition 2.4.7). Then $\mathcal{S}_n(M, \mathcal{N}', 1) \simeq \mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\text{SL}_n)$ as algebras.*

Let (M, \mathcal{N}) be a connected marked 3-manifold with $\mathcal{N} \neq \emptyset$. Combining the above two Theorems we have

$$\mathcal{S}_n(M, \mathcal{N}, 1) \simeq \Gamma_n(M) \otimes \mathcal{O}(\text{SL}_n)^{\otimes (\#\mathcal{N}-1)}$$

as algebras, where $\#\mathcal{N}$ is the number of components of \mathcal{N} .

For a commutative algebra A , we use $\sqrt{0}_A$ to denote the ideal consisting of all nilpotents. We can omit the subscript for $\sqrt{0}_A$ when there is no confusion with the algebra A . There is a projection from $\Gamma_n(M) \otimes \mathcal{O}(\text{SL}_n)^{\otimes (\#\mathcal{N}-1)}$ to $R_n(M, \mathcal{N})$, whose kernel is $\sqrt{0}$. The isomorphism from $\mathcal{S}_n(M, \mathcal{N}, 1)$ to $\Gamma_n(M) \otimes \mathcal{O}(\text{SL}_n)^{\otimes (\#\mathcal{N}-1)}$ is compatible with $\Phi^{(M, \mathcal{N})}$, that is, the combination of this isomorphism and the projection from $\Gamma_n(M) \otimes \mathcal{O}(\text{SL}_n)^{\otimes (\#\mathcal{N}-1)}$ to $R_n(M, \mathcal{N})$ is $\Phi^{(M, \mathcal{N})}$. Thus we have the following theorem.

Theorem 1.2.4. *For a marked 3-manifold (M, \mathcal{N}) , we have $\text{Ker } \Phi^{(M, \mathcal{N})} = \sqrt{0}$.*

Theorems 1.2.1 and 1.2.4 are proved in [Sik01; Sik05] when $\mathcal{N} = \emptyset$.

Costantino and Lê also defined the **generalized marked 3-manifold** by allowing oriented closed circles in \mathcal{N} [CL22b]. A generalized marked 3-manifold is said

to be connected if the corresponding 3-manifold is connected. We can define the stated SL_n -skein module for the generalized marked 3-manifold as for the marked 3-manifold . For any generalized marked 3-manifold (M, \mathcal{N}) , we define $\Gamma_n(M, \mathcal{N})$ as a quotient algebra of $\Gamma_n(M)$ (Def. 2.5.2).

Theorem 1.2.5. *Let (M, \mathcal{N}) be a connected generalized marked 3-manifold with $\mathcal{N} \neq \emptyset$. Then $\mathcal{S}_n(M, \mathcal{N}, 1) \simeq \Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes (\#\mathcal{N}-1)}$.*

1.2.2 Chapter 3

There exists the Chebyshev-Frobenius homomorphism from $\mathcal{S}_2(M, \mathcal{N}, 1)$ to $\mathcal{S}_2(M, \mathcal{N}, v)$ when v is a root of unity of odd order [BL20; BW16a; KQ24]. When (M, \mathcal{N}) is the thickening of a punctured bordered surface, it is injective and its image lies in the center of the stated skein algebra. The Chebyshev-Frobenius homomorphism is important to understand the center of the stated skein algebra and its representation theory [BW16a; BW17; BW19; FKL19].

We will construct the Frobenius homomorphism from $\mathcal{S}_n(M, \mathcal{N}, 1)$ to $\mathcal{S}_n(M, \mathcal{N}, v)$ when v is a primitive m -th root of unity with m being coprime with $2n$ and every component of M contains at least one marking. The Frobenius homomorphism was built for $n = 2, 3$ in [BL20; BW16a; Hig23; Hig24; KQ24].

Theorem 1.2.6. *Let (M, \mathcal{N}) be a marked 3-manifold, where every component of M contains at least one marking, and v be a primitive m -th root of unity with m being coprime with $2n$. Then there exists a unique linear map $\mathcal{F} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)$ such that $\mathcal{F}(l) = l^{(m)}$ for any stated n -web l consisting of stated framed oriented arcs, where $l^{(m)}$ is obtained from l by taking m parallel copies along the framing direction for each stated framed oriented arc.*

We also show that the Frobenius homomorphism commutes with the splitting map (section 2.1.4).

Theorem 1.2.7. *Suppose D is a properly embedded disk in a marked 3-manifold (M, \mathcal{N}) and D contains an embedded oriented open interval β . Let (M', \mathcal{N}') be the*

result of splitting (M, \mathcal{N}) along (D, β) . Then the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_n(M, \mathcal{N}, 1) & \xrightarrow{\Theta} & \mathcal{S}_n(M', \mathcal{N}', 1) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \mathcal{S}_n(M, \mathcal{N}, v) & \xrightarrow{\Theta} & \mathcal{S}_n(M', \mathcal{N}', v) \end{array},$$

where Θ is the splitting map defined in section 2.1.4.

For a punctured bordered surface \mathfrak{S} , we use $\mathcal{S}_n(\mathfrak{S}, v)$ to denote the stated SL_n -skein algebra of \mathfrak{S} (section 2.1.3).

We show that the image of \mathcal{F} is transparent, which indicates that $\mathrm{Im}\mathcal{F}$ lies in the center of the stated SL_n -skein algebra when (M, \mathcal{N}) is the thickening of an essentially bordered pb surface. We also show that \mathcal{F} is injective when (M, \mathcal{N}) is the thickening of an essentially bordered pb surface.

Theorem 1.2.8. *Let \mathfrak{S} be an essentially bordered pb surface. Then $\mathcal{F} : \mathcal{S}_n(\mathfrak{S}, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)$ is an injective algebra homomorphism, and $\mathrm{Im}\mathcal{F}$ lies in the center of $\mathcal{S}_n(\mathfrak{S}, v)$.*

The Unicity Theorem, alongside the Frobenius map, plays a crucial role in studying the representation theory for skein algebras. The Unicity Theorem was conjectured by Bonahon and Wong [BW16a] and proved by Frohman, Kania-Bartoszyńska, Lê for SL_2 -skein algebras [FKL19]. Korinman generalized the Unicity Theorem to stated SL_2 -skein algebras [Kor21]. The key to proving the Unicity Theorem for skein algebras is to prove that the skein algebra is affine almost Azumaya (section 3.2).

Theorem 1.2.9. *Let \mathfrak{S} be an essentially bordered pb surface, and v be a primitive m -th root of unity with m being coprime with $2n$. Then $\mathcal{S}_n(\mathfrak{S}, v)$ is affine almost Azumaya (Def. 3.2.1).*

Because of the above Theorem, the Unicity Theorem applies for $\mathcal{S}_n(\mathfrak{S}, v)$ when \mathfrak{S} is an essentially bordered pb surface. Then we have the following Theorem.

Theorem 1.2.10. *Let \mathfrak{S} be an essentially bordered pb surface, and v be a primitive m -th root of unity with m being coprime with $2n$. Then $\mathcal{S}_n(\mathfrak{S}, v)$ is finitely*

generated as a module over its center. Suppose the rank of $\mathcal{S}_n(\mathfrak{S}, v)$ over its center is K (see section 3.2 for the definition of this rank). Let us use $C(\mathcal{S}_n(\mathfrak{S}, v))$ to denote the center of $\mathcal{S}_n(\mathfrak{S}, v)$. Then the followings hold:

(1) any irreducible representation of $\mathcal{S}_n(\mathfrak{S}, v)$ has dimension at most the square root of K ;

(2) the map $\mathcal{X} : \text{Irrep} \rightarrow \text{MaxSpec}(C(\mathcal{S}_n(\mathfrak{S}, v)))$, defined in Remark 3.1.6, is surjective.

(3) there exists a Zariski open dense subset $U \subset \text{MaxSpec}(C(\mathcal{S}_n(\mathfrak{S}, v)))$ such that for any two irreducible representations V_1, V_2 of $\mathcal{S}_n(\mathfrak{S}, v)$ with $\mathcal{X}(V_1) = \mathcal{X}(V_2) \in U$, then V_1 and V_2 are isomorphic and have dimension the square root of K . Moreover any representation sending $C(\mathcal{S}_n(\mathfrak{S}, v))$ to scalar operators and whose induced character lies in U is semi-simple.

1.2.3 Chapter 4

For a pb surface \mathfrak{S} , positive integers m, n with $m \mid n$, and an element $\epsilon \in R$ with $\epsilon^{2m} = 1$, we construct an R -linear isomorphism $\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}, \epsilon v)$, which restricts to an algebra isomorphism from a subalgebra of $\mathcal{S}_n(\mathfrak{S}, v)$ to a subalgebra of $\mathcal{S}_n(\mathfrak{S}, \epsilon v)$.

Theorem 1.2.11. *Let \mathfrak{S} be a pb surface, let m, n be two positive integers with $m \mid n$, and let $\epsilon \in R$ with $\epsilon^{2m} = 1$. Then there exists an R -linear isomorphism $\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}, \epsilon v)$.*

For a 3-manifold M , the SL_n -skein module $\mathcal{S}_n(M, \emptyset, v)$ is quantized by v^2 . So we will use $\mathcal{S}_n(M, v^2)$ to denote $\mathcal{S}_n(M, \emptyset, v)$. Barrett built an R -linear isomorphism from $\mathcal{S}_2(M, v^2)$ to $\mathcal{S}_2(M, -v^2)$ with a choice of a spin structure of M [Bar99]. This linear isomorphism is an algebra isomorphism when M is the thickening of an oriented surface. In this chapter, we generalize Barrett's result to all n .

Theorem 1.2.12. *Let M be a 3-manifold with a spin structure. For each positive interger n , there exists an R -linear isomorphism $\Phi_n : \mathcal{S}_n(M, v^2) \rightarrow \mathcal{S}_n(M, -v^2)$. In particular Φ_n is an algebra isomorphism when M is the thickening of an oriented surface.*

We also prove the stated version for Theorem 1.2.12.

Theorem 1.2.13. *Let (M, \mathcal{N}) be a marked 3-manifold, and let n be a positive integer. There exists an R -linear isomorphism $\Psi_n : \mathcal{S}_n(M, \mathcal{N}, v) \rightarrow \mathcal{S}_n(M, \mathcal{N}, -v)$. In particular, Ψ_n is an algebra isomorphism when (M, \mathcal{N}) is the thickening of a pb surface and $\Psi_n(\alpha) = \alpha$ for any stated n -web α in (M, \mathcal{N}) without endpoints.*

Costantino and Lê formulated the stated SL_2 -TQFT theory [CL22b]. They constructed a symmetric monoidal functor from the category of decorated cobordisms to the Morita category, please refer to section 4.5 for the definitions of these two categories. The stated TQFT theory offers potential techniques to calculate the dimension of the skein modules of closed 3-manifolds and prove Witten's Finiteness Conjecture [Det21; GJS23]. The Conjecture was already proved in [GJS23], but the generalized Conjecture in [Det21] for compact 3-manifolds with boundaries is still open, please refer to [KW25] for related discussion. For any closed 3-manifold and a positive integer k , we use M_k to denote the marked 3-manifold obtained from M by cutting out k open balls and adding one marking to each newly created sphere boundary component. As an application for the stated SL_2 -TQFT theory, the author proved the dimension of the representation reduced stated skein module of M_k is one (see chapter 6).

We generalize the stated SL_2 -TQFT theory to SL_n . We assign the marked surface with a stated SL_n -skein algebra and the decorated manifold with a stated SL_n -skein module, a bimodule over two stated SL_n -skein algebras, please refer to section 4.5 for detailed discussions. This assignment is actually a symmetric monoidal functor from the category of decorated cobordisms to the Morita category, see Theorem 4.5.2. Using the stated SL_n -TQFT theory, we prove the following theorem.

Theorem 1.2.14. *Let (M, \mathcal{N}) be a connected marked 3-manifold with $\mathcal{N} \neq \emptyset$, and let (M, \mathcal{N}') be another marked 3-manifold obtained from (M, \mathcal{N}) by adding one extra marking e' . Suppose $\epsilon \in R$ such that $\epsilon^4 = 1$. Then:*

- (1) *The R -linear map from $\mathcal{S}_n(M, \mathcal{N}, \epsilon)$ to $\mathcal{S}_n(M, \mathcal{N}', \epsilon)$ induced by the embedding from (M, \mathcal{N}) to (M, \mathcal{N}') is injective.*
- (2) *$\mathcal{S}_n(M, \mathcal{N}', \epsilon) \simeq \mathcal{S}_n(M, \mathcal{N}, \epsilon) \otimes_R \mathcal{O}_{q_\epsilon}(\mathrm{SL}_n)$ (subsection 2.1.5).*

When $\epsilon = 1$, Theorem 1.2.14 is also proved in [Wan23c] using a quite different and complicated technique.

Lê and Sikora defined the splitting map for stated SL_n -skein modules and algebras and conjectured the injectivity of the splitting map for pb surfaces, Conjecture 7.12 in [LS21] (or refer to Conjecture 4.7.1). Lê proved the injectivity of the splitting map for pb surfaces when $n = 2$ [Lê18], Higgins proved the case when $n = 3$ [Hig23], Lê and Sikora proved the Conjecture when the pb surface is connected and has a non-empty boundary. The author proved the splitting map is injective for all marked 3-manifolds when the quantum parameter v is 1 [Wan23c].

There are very few non-trivial affirmative examples for Conjecture 4.7.1 when the pb surface has an empty boundary and $n > 3$ (the only non-trivial example is the case when the quantum parameter is 1 [Wan23c]).

As applications for the above Theorems, we prove the injectivity of the splitting map for some special cases.

Theorem 1.2.15. *Let \mathfrak{S} be a pb surface with an ideal arc c , let m, n be two positive integers with $m \mid n$, and let $\epsilon \in R$ such that $\epsilon^{2m} = 1$. Then the splitting map $\Theta_c : \mathcal{S}_n(\mathfrak{S}, \epsilon) \rightarrow \mathcal{S}_n(\text{Cut}_c(\mathfrak{S}), \epsilon)$ is injective, where $\text{Cut}_c(\mathfrak{S})$ is the pb surface obtained from \mathfrak{S} by cutting along c (subsection 2.1.4).*

Theorem 1.2.16. *Let (M, \mathcal{N}) be a connected marked 3-manifold with $\mathcal{N} \neq \emptyset$, let D be a properly embedded disk in M , and let e be an open oriented interval in D . Suppose $\epsilon \in R$ such that $\epsilon^4 = 1$. Then the splitting map*

$$\Theta_{(D,e)} : \mathcal{S}_n(M, \mathcal{N}, \epsilon) \rightarrow \mathcal{S}_n(\text{Cut}_{(D,e)}(M, \mathcal{N}), \epsilon)$$

is injective (subsection 2.1.4).

Theorem 1.2.17. *Let (M, \mathcal{N}) be marked 3-manifold, let D be a properly embedded disk in M , and let e be an open oriented interval in D . Then the splitting map $\Theta_{(D,e)} : \mathcal{S}_n(M, \mathcal{N}, -1) \rightarrow \mathcal{S}_n(\text{Cut}_{(D,e)}(M, \mathcal{N}), -1)$ is injective.*

1.2.4 Chapter 5

Recall that, when $q^{\frac{1}{2}}$ is a root of unity of order N with N odd, there is an important linear map, called the Frobenius map, $\mathcal{F} : \mathcal{S}_1(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ [BL20; BW16a; Wan23c]. This map is an algebra embedding when (M, \mathcal{N}) is the thickening of a pb surface.

The stated skein module $\mathcal{S}_1(M, \mathcal{N})$ has a commutative algebra structure given by taking the disjoint union of stated tangles in (M, \mathcal{N}) . Let $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})^{(N)} = \text{Im}\mathcal{F}$ when $q^{\frac{1}{2}}$ is a root of unity of order N with N odd. We show that $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})^{(N)}$ has a commutative algebra structure, which makes $\mathcal{F} : \mathcal{S}_1(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})^{(N)}$ a surjective algebra homomorphism (Thm. 5.1.4).

The stated skein module $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ has an $\mathcal{S}_1(M, \mathcal{N})$ -module structure, defined by equation (5.4). It was proved that $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ is finitely generated over $\mathcal{S}_1(M, \mathcal{N})$ when (M, \mathcal{N}) is the thickening of a pb surface, which is the crucial step to prove the Unicity Theorem for stated skein algebras [AF17; FKL19; Kor21; Wan23c]. In this chapter, we will prove the following more general result.

Theorem 1.2.18. *When $q^{\frac{1}{2}}$ is a root of unity of odd order, the stated skein module $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ is finitely generated over $\mathcal{S}_1(M, \mathcal{N})$ for any compact marked 3-manifold (M, \mathcal{N}) .*

We call the property that $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ is finitely generated over $\mathcal{S}_1(M, \mathcal{N})$ as finiteness of stated skein modules over Frobenius.

As mentioned before, the commutative algebra $\mathcal{S}_1(M, \mathcal{N})$ has an action on $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ when $q^{\frac{1}{2}}$ is a root of unity of odd order. Let I be a maximal ideal of $\mathcal{S}_1(M, \mathcal{N})$. Define

$$I\mathcal{S}_{q^{1/2}}(M, \mathcal{N}) = \{x_1 \cdot \alpha_1 + \cdots + x_k \cdot \alpha_k \mid k \in \mathbb{N}, x_i \in I, \alpha_i \in \mathcal{S}_{q^{1/2}}(M, \mathcal{N}), 1 \leq i \leq k\}.$$

We define the **representation-reduced stated skein module** of (M, \mathcal{N}) with respect to I to be

$$\mathcal{S}_{q^{1/2}}(M, \mathcal{N}) / (I\mathcal{S}_{q^{1/2}}(M, \mathcal{N})). \quad (1.7)$$

The parallel definition for this quotient vector space for the skein module (non-stated case) is defined in [FKL23].

Theorem 1.2.19. *When $q^{\frac{1}{2}}$ is a root of unity of odd order, the representation-reduced stated skein module of a compact marked 3-manifold is finite-dimensional over \mathbb{C} .*

The representation-reduced stated skein module is an important tool for studying the skein theory. Detcherry, Kalfagianni, and Sikora used the representation-reduced skein module to calculate the dimension of the skein module for certain closed 3-manifolds [DKS23]. The representation-reduced (stated) skein module of the handlebody H_g is helpful in constructing irreducible representations of the (stated) skein algebra of ∂H_g [FKL23; Wan25b].

For a 3-manifold M , the stated skein module of (M, \emptyset) is the (Kauffman bracket) skein module of M . We will use $\mathcal{S}_q(M)$ to denote the skein module of M . Let q^2 be a root of unity of order N with N odd. The Frobenius map $\mathcal{F} : \mathcal{S}_{q^N}(M) \rightarrow \mathcal{S}_q(M)$ for the skein module is defined by Bonahon and Wong (note that $q^N = \pm 1$) [BW16a].

Since $q^N = \pm 1$, then $\mathcal{S}_{q^N}(M)$ has a commutative algebra structure given by taking the disjoint union of framed links. Similarly, as the stated case, the skein module $\mathcal{S}_q(M)$ has an $\mathcal{S}_{q^N}(M)$ -module structure. So we have a stronger version for Theorem 1.2.18 when $\mathcal{N} = \emptyset$. That is, $\mathcal{S}_q(M)$ is finitely generated over $\mathcal{S}_{q^N}(M)$ when M is compact and q^2 is a root of unity of order N with N odd [FKL22].

For every 3-manifold M , the skein algebra $\mathcal{S}_q(\partial M)$ acts on $\mathcal{S}_q(M)$, see section 5.2.2. Using the stronger version for Theorem 1.2.18 when $\mathcal{N} = \emptyset$, we find out the finiteness of $\mathcal{S}_{q^N}(M)$ over $\mathcal{S}_{q^N}(\partial M)$ implies the finiteness of $\mathcal{S}_q(M)$ over $\mathcal{S}_q(\partial M)$ when M is compact and q^2 is a root of unity of order N with N odd.

Theorem 1.2.20. *Let M be a compact 3-manifold. Suppose q^2 is a root of unity of order N with N odd, and $\mathcal{S}_{q^N}(M)$ is finitely generated over $\mathcal{S}_{q^N}(\partial M)$. Then $\mathcal{S}_q(M)$ is finitely generated over $\mathcal{S}_q(\partial M)$.*

For a compact marked 3-manifold (M, \mathcal{N}) , we give an upper bound for the dimension of $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ over $\mathcal{S}_1(M, \mathcal{N})$, which is defined to be the minimal number of generators of $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ over $\mathcal{S}_1(M, \mathcal{N})$. This furthermore gives an upper bound for the dimension of the representation-reduced stated skein module of (M, \mathcal{N}) .

Theorem 1.2.21. *Suppose $q^{\frac{1}{2}}$ is a root of unity of order N with N odd. Let (M, \mathcal{N}) be a connected compact marked 3-manifold. Suppose the Heegaard genus of M is g , and \mathcal{N} has k components.*

(a) *When $k = 0$, the dimension of $\mathcal{S}_{q^{1/2}}(M, \emptyset)$ over $\mathcal{S}_1(M, \emptyset)$ is not more than N^{2g-1} .*

(b) When $k > 0$, the dimension of $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ over $\mathcal{S}_1(M, \mathcal{N})$ is not more than

$$\left(2N^3 - \frac{N(N+1)(2N+1)}{6}\right)^{2g+k-1}.$$

Let \mathfrak{S} be a pb surface, it is well-known that $\mathcal{S}_{q^{1/2}}(\mathfrak{S})$ is a domain [Lê18]. Then we have $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)} = \text{Im}\mathcal{F}$ is a commutative domain when $q^{\frac{1}{2}}$ is a root of unity of order N with N odd, where $\mathcal{F} : \mathcal{S}_1(\mathfrak{S}) \rightarrow \mathcal{S}_{q^{1/2}}(\mathfrak{S})$ is the Frobenius map. We use $\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})N}$ to denote the field of fractions of $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}$, and use $\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}$ to denote $\mathcal{S}_{q^{1/2}}(\mathfrak{S}) \otimes_{\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}} \widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})N}$. Then $\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}$ is a finite-dimensional vector space over the field $\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})N}$. We define $r(\mathfrak{S}) = -\chi(\mathfrak{S}) + \sharp\partial\mathfrak{S}$, where $\chi(\mathfrak{S})$ is the Euler characteristic of \mathfrak{S} and $\sharp\partial\mathfrak{S}$ is the number of boundary components of \mathfrak{S} .

Theorem 1.2.22. *Suppose $q^{\frac{1}{2}}$ is a root of unity of order N with N odd. Let \mathfrak{S} be a connected pb surface, we require $\chi(\mathfrak{S})$ is negative if $\partial\mathfrak{S} = \emptyset$. Then*

$$\dim_{\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})N}} \widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})} = N^{3r(\mathfrak{S})}.$$

When $\partial\mathfrak{S} = \emptyset$, Theorem 1.2.22 was proved in [FK18; FKL21]. Tao Yu discussed the center and the PI-dimension of the stated skein algebra in [Yu23]. When the center is the image of the Frobenius map, the PI-dimension should be the square root of $\dim_{\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})N}} \widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}$. It is a trivial check that Theorem 1.2.22 coincides with the result in [Yu23] when the center of the stated skein algebra is the image of the Frobenius map.

1.2.5 Chapter 6

We use $\text{MaxSpec}(\mathcal{S}_1(M, \mathcal{N}))$ to denote the set of all maximal ideals of $\mathcal{S}_1(M, \mathcal{N})$. Then $\text{MaxSpec}(\mathcal{S}_1(M, \mathcal{N}))$ is actually isomorphic to a representation variety when every component of M contains at least one marking, and is isomorphic to a character variety when $\mathcal{N} = \emptyset$ [Bul97; Wan23c].

Recall that when $q^{\frac{1}{2}}$ is a root of unity of odd order, the commutative algebra $\mathcal{S}_1(M, \mathcal{N})$ has an action on $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ via the Frobenius map. Then for any

$\rho \in \text{MaxSpec}(\mathcal{S}_1(M, \mathcal{N}))$, in (1.7) we define

$$\mathcal{S}_{q^{1/2}}(M, \mathcal{N})_\rho = \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \otimes_{\mathcal{S}_1(M, \mathcal{N})} (\mathcal{S}_1(M, \mathcal{N})/\rho) \simeq \mathcal{S}_{q^{1/2}}(M, \mathcal{N})/\rho \cdot \mathcal{S}_{q^{1/2}}(M, \mathcal{N}).$$

A **marked surface** is a pair (Σ, \mathcal{P}) , where Σ is a compact surface and \mathcal{P} is a set of finite points in $\partial\Sigma$, called **marked points**.

For a marked surface (Σ, \mathcal{P}) , we set $M = \Sigma \times [0, 1]$, and set $\mathcal{N} = \mathcal{P} \times (0, 1)$ where the orientation of every component of \mathcal{N} is given by the positive direction of $(0, 1)$. Then (M, \mathcal{N}) is a marked 3-manifold. We call (M, \mathcal{N}) the thickening of (Σ, \mathcal{P}) , and define $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P}) = \mathcal{S}_{q^{1/2}}(M, \mathcal{N})$. Then $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})$ has an algebra structure given by staking stated tangles, that is, for any two stated tangles α, β , the product $\alpha\beta$ is defined to be staking α above β . We call $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})$ the stated skein algebra of (Σ, \mathcal{P}) . When $\mathcal{P} = \emptyset$, the algebra $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})$ is the (Kauffman bracket) skein algebra.

When (M, \mathcal{N}) is the thickening of a marked surface (Σ, \mathcal{P}) , it is easy to show $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})_\rho$ is a quotient algebra of $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})$ for any $\rho \in \text{MaxSpec}(\mathcal{S}_1(\Sigma, \mathcal{P}))$, where $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})_\rho$ denotes $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})_\rho$ and $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})$ denotes $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$. This quotient algebra is very important to understand the representation theory of the stated skein algebra. Actually any irreducible representation of $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})$ reduces to an irreducible representation of $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})_\rho$ for some $\rho \in \text{MaxSpec}(\mathcal{S}_1(\Sigma, \mathcal{P}))$.

Let (M, \mathcal{N}) be a marked 3-manifold. For any properly embedded disk D in M and any embedded oriented open interval $u \subset D$, we use $\text{Cut}_{(D, u)}(M, \mathcal{N})$ to denote the marked 3-manifold obtained from (M, \mathcal{N}) by cutting along (D, u) , please refer to subsection 6.2.1 for more details. There exists a linear map

$$\Theta : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(\text{Cut}_{(D, u)}(M, \mathcal{N})) \quad (1.8)$$

called the splitting map. When $q^{\frac{1}{2}} = 1$, we have that $\Theta : \mathcal{S}_1(M, \mathcal{N}) \rightarrow \mathcal{S}_1(\text{Cut}_{(D, u)}(M, \mathcal{N}))$ is an algebra homomorphism. Then it induces a map

$$\Theta^* : \text{MaxSpec}(\mathcal{S}_1(\text{Cut}_{(D, u)}(M, \mathcal{N}))) \rightarrow \text{MaxSpec}(\mathcal{S}_1((M, \mathcal{N}))).$$

We prove that, for any $\rho \in \text{MaxSpec}(\mathcal{S}_1(\text{Cut}_{(D,u)}(M, \mathcal{N})))$, the splitting map in equation (1.8) induces a linear map

$$\Theta_\rho : \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_{\Theta^*(\rho)} \rightarrow \mathcal{S}_{q^{1/2}}(\text{Cut}_{(D,u)}(M, \mathcal{N}))_\rho.$$

Theorem 1.2.23. *Let (M, \mathcal{N}) be a marked 3-manifold, let D be a properly embedded disk in M , and let u be an embedded oriented open interval in D . Suppose the component V of ∂M contains ∂D and $V \cap \mathcal{N} \neq \emptyset$. Then, for any $\rho \in \text{MaxSpec}(\mathcal{S}_1(\text{Cut}_{(D,u)}(M, \mathcal{N})))$, we have Θ_ρ is injective.*

We use H_g to denote the genus g handlebody. For any positive integer k , we use $H_{g,k}$ to denote the marked 3-manifold obtained from H_g by adding k markings on ∂H_g . Then $H_{g,k}$ is defined up to isomorphism. We use Σ_g to denote the closed surface of genus g . For any positive integer k , we use $\Sigma_{g,k}$ to denote the marked surface obtained from Σ_g by removing k open disks and equipping each newly created boundary component with one marked point. Then the stated skein algebra $\mathcal{S}_{q^{1/2}}(\Sigma_{g,k})$ acts on $\mathcal{S}_{q^{1/2}}(H_{g,k})$, please refer to section 6.3 for more details.

Theorem 1.2.24. *Let k be a positive integer. For any $\rho \in \text{MaxSpec}(\mathcal{S}_1(H_{g,k}))$, we have $\mathcal{S}_{q^{1/2}}(H_{g,k})_\rho$ is an irreducible representation of $\mathcal{S}_{q^{1/2}}(\Sigma_{g,k})$. Meanwhile, it is an Azumaya representation of $\mathcal{S}_{q^{1/2}}(\Sigma_{g,k})$ (section 6.3).*

Let M be an oriented connected closed 3-manifold. For any positive integer k , we use M_k to denote the marked 3-manifold obtained from M by removing k open three dimensional balls and adding one marking to each newly created sphere boundary component.

Theorem 1.2.25. *Let M be any oriented connected closed 3-manifold, and let k be any positive integer. For any $\rho \in \text{MaxSpec}(\mathcal{S}_1(M_k))$, we have $\mathcal{S}_{q^{1/2}}(M_k)_\rho \simeq \mathbb{C}$ (as \mathbb{C} -vector spaces).*

Parallel results for Theorems 1.2.24 and 1.2.25 for the skein case (that is the non-stated case) are proved in [FKL23]. For the non-stated case, they have some restrictions for ρ . Theorems 1.2.24 and 1.2.25 hold for any $\rho \in \text{MaxSpec}(\mathcal{S}_1(H_{g,k}))$ or $\rho \in \text{MaxSpec}(\mathcal{S}_1(M_k))$.

1.2.6 Chapter 7

Let S be an oriented surface, let φ be a diffeomorphism for S , and let $q_n = e^{2\pi i/n}$ with $(q_n)^{1/2} = e^{\pi i/n}$ and n odd. Using these data, Bonahon-Wong-Yang built a sequence of intertwiners $\{\Lambda_{\varphi,\gamma}^{q_n}\}_{n \in 2\mathbb{N}+1}$ between irreducible representations of the skein algebra of S [BWY21; BWY22]. When S has negative Euler characteristic and φ is pseudo-Anosov, they formulated the volume conjecture using these intertwiners:

$$\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace} \Lambda_{\varphi,\gamma}^{q_n}| = \frac{1}{4\pi} \text{vol}_{hyp}(M_\varphi),$$

where $\text{vol}_{hyp}(M_\varphi)$ is the volume of the complete hyperbolic metric of the mapping torus M_φ .

We explicitly compute the intertwiners corresponding to all diffeomorphisms of the closed torus using an algebraic embedding from the skein algebra of the closed torus to a quantum torus [FG00], see section 7.2 for more details. The representation theory for this quantum torus is well-studied. We prove that almost all the irreducible representations of this quantum torus can be restricted to irreducible representations of the skein algebra of the closed torus. So intertwiners between two isomorphic irreducible representations of this quantum torus are also the intertwiners between irreducible representations for the skein algebra of the closed torus. These intertwiners are built when the quantum parameter q for the skein algebra is a primitive root of unity of odd order. We use Λ_n to denote the intertwiner obtained as above when the quantum parameter is $q_n = e^{2\pi i/n}$ with $(q_n)^{1/2} = e^{\pi i/n}$ and n odd. We also normalize Λ_n such that $|\det(\Lambda_n)| = 1$. Then we prove the following Theorem, please refer to Theorem 7.3.11 for a more detailed version.

Theorem 1.2.26. *Let Λ_n be defined as above, then*

$$\limsup_{\text{odd } n \rightarrow \infty} \frac{\log(|\text{Trace} \Lambda_n|)}{n} = 0.$$

The volume conjecture was first introduced by Kashaev [Kas97], and was rewritten and generalized to the non-hyperbolic case by Hitoshi Murakami and Jun Murakami [MM01] using the simplicial volume.

Bonahon-Wong-Yang only formulated the conjecture when the diffeomorphisms are pseudo-Anosov for surfaces with negative Euler characteristic. In this chapter, we

broaden the scope of the conjecture to include periodic diffeomorphisms. When φ is a periodic diffeomorphism for the surface S , the corresponding mapping torus M_φ is a Seifert manifold whose simplicial volume is zero. So we conjecture the limits are zero for periodic diffeomorphisms. We prove our conjecture for the once punctured torus, which serves as examples to confirm the limit is the simplicial volume of the corresponding mapping torus.

Let S be an oriented surface with negative Euler characteristic, and let φ be a periodic diffeomorphism for S . According to page 371 in [FM11], φ fixes a point in the Teichmüller space of S . This fixed point in the Teichmüller space offers a smooth φ -invariant character γ (that is γ is a group homomorphism from $\pi_1(S)$ to $SL(2, \mathbb{C})$ such that $\gamma\varphi_*$ and γ have the same character, where φ_* is the isomorphism from $\pi_1(S)$ to $\pi_1(S)$ induced by φ). Suppose the quantum parameter for the skein algebra is $q_n = e^{2\pi i/n}$ with $(q_n)^{1/2} = e^{\pi i/n}$ and n odd. For each puncture v of S , we choose a complex number p_v such that $p_v = p_{\varphi(v)}$ and $T_n(p_v) = -\text{Trace}(\gamma(\alpha_v))$, where T_n is the n -th Chebyshev polynomial of the first type and α_v is the element in $\pi_1(S)$ going around puncture v . According to Theorem 7.1.1, we know γ and p_v uniquely determine an irreducible representation ρ of the skein algebra. Let $\varphi_\#$ be the isomorphism from the skein algebra of S to itself induced by φ . Since both γ and p_v are φ -invariant, we have ρ and $\rho\varphi_\#$ are isomorphic according to Theorem 7.1.2. Thus there exists the intertwiner $\Lambda_{\varphi,\gamma}^{q_n}$ between these two isomorphic irreducible representations. We normalize it such that $|\det(\Lambda_{\varphi,\gamma}^{q_n})| = 1$. Then we formulate the following conjecture, please refer to Conjecture 7.4.4 for a more detailed version.

Conjecture 1.2.27. *Let S be a surface with negative Euler characteristic, let φ be a periodic diffeomorphism for S , and let $\Lambda_{\varphi,\gamma}^{q_n}$ be defined as above, then*

$$\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace} \Lambda_{\varphi,\gamma}^{q_n}| = 0.$$

In Theorem 7.4.5, we prove the limit in Conjecture 1.2.27 is less than or equal to zero if it exists by using the periodic property. It seems like that we are half way there to prove our conjecture. But proving that the limit is greater than or equal to zero is harder, which is actually related to an interesting question raised by Gerald Myerson [Mye86] and Terry Tao [Tao]. By direct calculations and using some conclusions in [LL00; Mye86], we prove the above conjecture for some special cases:

Theorem 1.2.28. *For any surface with negative Euler characteristic if φ is of order 2^m where m is any positive integer, we have*

$$\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace} \Lambda_{\varphi, \gamma_\varphi}^{q_n}| = 0.$$

Theorem 1.2.29. *Conjecture 1.2.27 holds if S is the once punctured torus.*

Conventions: We use \mathbb{N} to denote the set of nonnegative integers and use \mathbb{Z} to denote the set of integers. We set $\mathbb{N}^* = \mathbb{N} - \{0\}$, $\mathbb{Z}^* = \mathbb{Z} - \{0\}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Part I

General SL_n -skein theory

Chapter 2

The classical limit of the stated SL_n -skein module

In chapters 2-4, we discuss the classical limit, the Frobenius map, and the TQFT theory for the general stated SL_n -skein theory.

This chapter is based on the author's work in [Wan24]. Here, we review the definition of the stated SL_n -skein module $\mathcal{S}_n(M, \mathcal{N}, v)$ of a marked 3-manifold, as introduced in [LS21]. It is a module over a commutative domain, generated by isotopy classes of stated n -webs (Def. 2.1.1), subject to the relations (2.4)-(2.11). We will define a surjective algebra homomorphism

$$\mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \text{the function ring of a representation variety}$$

(Thm. 2.2.9). Furthermore, we will show that the kernel of this surjective algebra homomorphism consists precisely of all nilpotent elements of $\mathcal{S}_n(M, \mathcal{N}, 1)$ (Thm. 2.4.19). The results in this chapter generalize those in [Bul97; Sik05] to the stated SL_n -skein theory.

2.1 Stated SL_n -skein modules

In this section, we will recall some definitions and results about the stated SL_n -skein modules and introduce some conventions. We follow the definition in [LS21] for stated SL_n -skein modules. Here we briefly recall the definition.

Definition 2.1.1 ([LS21]). An n -web l in a marked 3-manifold (M, \mathcal{N}) is a disjoint union of oriented closed paths and a directed finite graph properly embedded into M . We also have the following requirements:

- (1) l only contains 1-valent or n -valent vertices. Each n -valent vertex is a source or a sink. The set of one valent vertices is denoted as ∂l , which are called endpoints of l .
- (2) Every edge e of the graph is an embedded oriented closed interval in M . We can regard e as an embedding from $[0, 1]$ to M . Then $e(0)$ (resp. $e(1)$) is called the **starting point** (resp. **ending point**).
- (3) l is equipped with a transversal framing.
- (4) The set of half-edges at each n -valent vertex is equipped with a cyclic order.
- (5) ∂l is contained in \mathcal{N} and the framing at these endpoints is the velocity vector of \mathcal{N} .

For any two points $a, b \in \partial l$, we say a is higher than b if they belong to a same component e of \mathcal{N} and the direction of e is going from b to a .

A state of an n -web l is a map $s : \partial l \rightarrow \{1, 2, \dots, n\}$. If there is such a map s for l , we say l is stated by s . For any point $a \in \partial l$, we say a is stated by $s(a)$.

Our ground ring is a commutative domain R with an invertible element $v \in R$. We set $q = v^{2n}$ such that $q^{\frac{1}{2n}} = v$, and define the following constants:

$$c_i = (-q)^{n-i} q^{\frac{n-1}{2n}}, \quad t = (-1)^{n-1} q^{\frac{n^2-1}{n}}, \quad t^{n/2} = (-1)^{\frac{(n-1)n}{2}} q^{\frac{n^2-1}{2}} \quad (2.1)$$

$$a = q^{\frac{n+1-2n^2}{4}}, \quad d_n = (-1)^{n-1}. \quad (2.2)$$

Note that

$$\prod_{i=1}^n c_i = t^{n/2} = (-1)^{\frac{(n-1)n}{2}} q^{\frac{n^2-1}{2}} \text{ and } c_i \cdot c_i = t, \text{ for } i = 1, \dots, n. \quad (2.3)$$

We use S_n to denote the permutation group on $\{1, 2, \dots, n\}$.

The stated SL_n -skein module of (M, \mathcal{N}) , denoted as $\mathcal{S}_n(M, \mathcal{N}, v)$, is obtained in two steps. We first use all isotopy classes of stated n -webs in (M, \mathcal{N}) as a basis to

generate a free R -module, then quotient the following relations.

$$q^{\frac{1}{n}} \begin{array}{|c|} \hline \diagup \diagdown \\ \hline \end{array} - q^{-\frac{1}{n}} \begin{array}{|c|} \hline \diagdown \diagup \\ \hline \end{array} = (q - q^{-1}) \begin{array}{|c|} \hline \rightarrow \\ \hline \rightarrow \\ \hline \end{array}, \quad (2.4)$$

$$\begin{array}{|c|} \hline \rightarrow \bigcirc \\ \hline \end{array} = t \begin{array}{|c|} \hline \rightarrow \\ \hline \end{array}, \quad (2.5)$$

$$\begin{array}{|c|} \hline \bigcirc \\ \hline \end{array} = (-1)^{n-1} [n] \begin{array}{|c|} \hline \\ \hline \end{array}, \quad \text{where } [n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad (2.6)$$

$$\begin{array}{|c|} \hline \vdots \diagup \diagdown \vdots \\ \hline \end{array} = (-q)^{\frac{n(n-1)}{2}} \cdot \sum_{\sigma \in S_n} (-q^{\frac{1-n}{n}})^{\ell(\sigma)} \begin{array}{|c|} \hline \vdots \diagup \diagdown \vdots \\ \hline \end{array} \sigma_+, \quad (2.7)$$

where the ellipse enclosing σ_+ is the minimum crossing positive braid representing a permutation $\sigma \in S_n$ and $\ell(\sigma) = |\{(i, j) \mid 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}|$ is the length of $\sigma \in S_n$.

$$\begin{array}{|c|} \hline \vdots \diagup \diagdown \vdots \\ \hline \end{array} = a \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \begin{array}{|c|} \hline \sigma(n) \\ \hline \vdots \\ \hline \sigma(2) \\ \hline \sigma(1) \\ \hline \end{array}, \quad (2.8)$$

$$\begin{array}{|c|} \hline \curvearrowright^i \\ \hline \end{array} = \delta_{\bar{j}, i} c_i \begin{array}{|c|} \hline \\ \hline \end{array}, \quad (2.9)$$

$$\begin{array}{|c|} \hline \curvearrowleft \\ \hline \end{array} = \sum_{i=1}^n (c_i)^{-1} \begin{array}{|c|} \hline \vdots \\ \hline \bullet \\ \hline \vdots \\ \hline \end{array} \begin{array}{|c|} \hline i \\ \hline \end{array}, \quad (2.10)$$

$$\begin{array}{|c|} \hline \vdots \diagup \diagdown \vdots \\ \hline \end{array} = q^{-\frac{1}{n}} \left(\delta_{j < i} (q - q^{-1}) \begin{array}{|c|} \hline \vdots \diagup \diagdown \vdots \\ \hline \end{array} + q^{\delta_{i,j}} \begin{array}{|c|} \hline \vdots \diagup \diagdown \vdots \\ \hline \end{array} \right), \quad (2.11)$$

where $\delta_{j < i} = \begin{cases} 1, & j < i \\ 0, & \text{otherwise} \end{cases}$, $\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$. Each shaded rectangle in the above relations is the projection of a cube in M . The lines contained in the shaded rectangle represent parts of stated n -webs with framing pointing to readers. The black or white dot represents the orientation of the stated n -web such that the orientation represented by the white dot is opposite to the orientation represented

by the black dot. The thick line in the edge of shaded rectangle is part of the marking. For detailed explanation for the above relations, please refer to [LS21].

2.1.1 Functoriality

For any two marked 3-manifolds $(M, \mathcal{N}), (M', \mathcal{N}')$, if an orientation preserving embedding $f : M \rightarrow M'$ maps \mathcal{N} to \mathcal{N}' and preserves orientations between \mathcal{N} and \mathcal{N}' , we call f an embedding from (M, \mathcal{N}) to (M', \mathcal{N}') . Clearly f induces a linear map $f_* : \mathcal{S}_n(M, \mathcal{N}, v) \rightarrow \mathcal{S}_n(M', \mathcal{N}', v)$ [LS21].

2.1.2 Reversing orientations of n -webs

Let $\bar{\alpha}$ denote an n -web α with its orientation reversed (and unchanged framing).

Corollary 2.1.2 ([LS21]).

$$\bar{\cdot} : \mathcal{S}_n(M, \mathcal{N}, v) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)$$

is a well defined linear automorphism.

2.1.3 Stated SL_n -skein algebras

The stated SL_n -skein algebra, denoted as $\mathcal{S}_n(\mathfrak{S}, v)$, of a pb surface \mathfrak{S} (Def. 1.1.1) is defined as following: For every component c of $\partial\mathfrak{S}$, we choose a point x_c . Let $M = \mathfrak{S} \times [-1, 1]$ and $\mathcal{N} = \cup_c (x_c \times (-1, 1))$ where c goes over all components of $\partial\mathfrak{S}$. Then we define $\mathcal{S}_n(\mathfrak{S}, v)$ to be $\mathcal{S}_n(M, \mathcal{N}, v)$. We will call (M, \mathcal{N}) the **thickening of the pb surface \mathfrak{S}** . Obviously $\mathcal{S}_n(\mathfrak{S}, v)$ admits an algebra structure. For any two stated n -webs l_1 and l_2 in the thickening of \mathfrak{S} , we define $l_1 l_2 \in \mathcal{S}_n(\mathfrak{S}, v)$ to be the result of stacking l_1 above l_2 .

Let $f : \mathfrak{S}_1 \rightarrow \mathfrak{S}_2$ be a proper embedding for pb surfaces. Note that it is possible that f maps more than one boundary components of \mathfrak{S}_1 into one boundary component of \mathfrak{S}_2 . For a boundary component c of \mathfrak{S}_2 , we can give a linear order on the set of boundary components of \mathfrak{S}_1 that are mapped into c under f . We call such a linear order for c a **c -order**. If for each boundary component c of \mathfrak{S}_2 , there

exists such a c -order, we call f a **height ordered embedding**, which induces a linear map $f_* : \mathcal{S}_n(\mathfrak{S}_1, v) \rightarrow \mathcal{S}_n(\mathfrak{S}_2, v)$ [LS21].

For every boundary component c of \mathfrak{S} , the orientation of c induced by the orientation of \mathfrak{S} is called the **positive orientation**. The orientation of c , which is opposite to the positive orientation of c , is called the **negative orientation**. The c -order induced by the positive (respectively negative) orientation of c is called the **positive c -order** (respectively **negative c -order**).

2.1.4 Splitting map

Let (M, \mathcal{N}) be any marked 3-manifold, and D be a properly embedded disk in M such that there is no intersection between D and the closure of \mathcal{N} . After removing a collar open neighborhood of D , we get a new 3-manifold M' . And $\partial M'$ contains two copies D_1 and D_2 of D such that we can get M from M' by gluing D_1 and D_2 . We use pr to denote the obvious projection from M' to M .

Let $\beta \subset D$ be an oriented open interval. Suppose $\text{pr}^{-1}(\beta) = \beta_1 \cup \beta_2$ with $\beta_1 \subset D_1$ and $\beta_2 \subset D_2$. We cut (M, \mathcal{N}) along (D, β) to obtain a new marked 3-manifold (M', \mathcal{N}') , where $\mathcal{N}' = \mathcal{N} \cup \beta_1 \cup \beta_2$. We will denote (M', \mathcal{N}') as $\text{Cut}_{(D, \beta)}(M, \mathcal{N})$. It is easy to see that $\text{Cut}_{(D, \beta)}(M, \mathcal{N})$ is defined up to isomorphism. If β' is another oriented open interval in D , obviously we have $\text{Cut}_{(D, \beta)}(M, \mathcal{N})$ is isomorphic to $\text{Cut}_{(D, \beta')}(M, \mathcal{N})$.

For a stated n -web l in (M, \mathcal{N}) , we say l is **(D, β) -transverse** if the vertices of l are not in D , $l \cap D = l \cap \beta$, and the framing of l at each point in $l \cap \beta$ is given by the velocity vector of β . For any map $s : l \cap \beta \rightarrow \{1, 2, \dots, n\}$, let l_s , which is a stated n -web in $\text{Cut}_{(D, \beta)}(M, \mathcal{N})$, be the lift of l such that for every $P \in l \cap \beta$ the two newly created boundary points corresponding to P both have the state $s(P)$. There is a linear homomorphism [LS21], called the splitting map, $\Theta_{(D, \beta)} : \mathcal{S}_n(M, \mathcal{N}, v) \rightarrow \mathcal{S}_n(M', \mathcal{N}', v)$ defined by

$$\Theta_{(D, \beta)}(l) = \sum_{s: l \cap \beta \rightarrow \{1, 2, \dots, n\}} l_s. \quad (2.12)$$

When there is no confusion we can omit the subscript for $\Theta_{(D, \beta)}$.

Let c be an ideal arc of a pb surface \mathfrak{S} such that it is contained in the interior of \mathfrak{S} . After cutting \mathfrak{S} along c , we get a new pb surface $\text{Cut}_c \mathfrak{S}$, which has two copies c_1, c_2 for c such that $\mathfrak{S} = \text{Cut}_c \mathfrak{S} / (c_1 = c_2)$. We use pr to denote the projection from $\text{Cut}_c \mathfrak{S}$ to \mathfrak{S} . Suppose α is a diagram for a stated n -web in \mathfrak{S} , which is transverse to c . Let s be a map from $c \cap \alpha$ to $\{1, 2, \dots, n\}$, and let h be a linear order on $c \cap \alpha$. Then there is lift diagram $\alpha(h, s)$ for a stated n -web in $\text{Cut}_c \mathfrak{S}$. The heights of the newly created endpoints of $\alpha(h, s)$ are induced by h (via pr), and the states of the newly created endpoints of $\alpha(h, s)$ are induced by s (via pr). Then the splitting map for the stated SL_n -skein algebra is defined by

$$\Theta_c(\alpha) = \sum_{s: \alpha \cap c \rightarrow \{1, \dots, n\}} \alpha(h, s),$$

furthermore Θ_c is an algebra homomorphism [LS21]. When there is no confusion we can omit the subscript for Θ_c .

Proposition 2.1.3 ([LS21]). *Let \mathfrak{S} be an essentially bordered pb surface (Def. 1.1.1), and c be any ideal arc in the interior of \mathfrak{S} . Then the splitting map $\Theta_c : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\text{Cut}_c \mathfrak{S}, v)$ is injective.*

An **ideal triangulation** \mathcal{E} of a pb surface \mathfrak{S} is (1) a collection of ideal arcs in \mathfrak{S} , (2) any arc in this collection does not bound a disk, and any two arcs are pairwise non-isotopic and pairwise disjoint, (3) this collection is maximal under condition (2). The ideal arcs in \mathcal{E} not isotopic to boundary components split \mathfrak{S} into ideal triangles. We use $\text{tri}(\mathcal{E})$ to denote the set of these ideal triangles. Define $\text{Int}(\mathcal{E}) = \{e \in \mathcal{E} \mid e \text{ is not isotopic to a boundary component}\}$.

Suppose \mathfrak{S} is a pb surface, and \mathcal{E} is an ideal triangulation of \mathfrak{S} . After cutting \mathfrak{S} into a collection of ideal triangles using \mathcal{E} , for each $e \in \text{Int}(\mathcal{E})$, $\text{Cut}_e(\mathfrak{S})$ has a left coaction ${}_e\Delta$ and a right coaction Δ_e over the Hopf algebra $\mathcal{S}_n(\mathfrak{B}, v)$, see subsection 7.1 in [LS21]. Because of the commutativity of splitting maps, we can combine ${}_e\Delta$ (respectively Δ_e) for all $e \in \text{Int}(\mathcal{E})$ in any order, which we denote as $\text{Int}(\mathcal{E})\Delta$ (respectively $\Delta_{(\text{Int}(\mathcal{E}))}$).

We use fl to denote the transposition

$$fl : (\otimes_{e \in \text{Int}(\mathcal{E})} \mathcal{S}_n(\mathfrak{B}, v)) \otimes (\otimes_{\mathfrak{T} \in \text{tri}(\mathcal{E})} \mathcal{S}_n(\mathfrak{T}, v)) \rightarrow (\otimes_{\mathfrak{T} \in \text{tri}(\mathcal{E})} \mathcal{S}_n(\mathfrak{T}, v)) \otimes (\otimes_{e \in \text{Int}(\mathcal{E})} \mathcal{S}_n(\mathfrak{B}, v))$$

defined by $fl(a \otimes b) = b \otimes a$ where $a \in \otimes_{e \in \text{Int}(\mathcal{E})} \mathcal{S}_n(\mathfrak{B}, v)$ and $b \in \otimes_{\mathfrak{T} \in \text{tri}(\mathcal{E})} \mathcal{S}_n(\mathfrak{T}, v)$.

Lemma 2.1.4 ($n = 2$ [KQ24], $n = 3$ [Hig23]). *Suppose \mathfrak{S} is an essentially bordered pb surface, and \mathcal{E} is an ideal triangulation of \mathfrak{S} . Then we have the following exact sequence:*

$$\mathcal{S}_n(\mathfrak{S}, v) \xrightarrow{\Theta} \otimes_{\mathfrak{I} \in \mathrm{tri}(\mathcal{E})} \mathcal{S}_n(\mathfrak{I}, v) \xrightarrow{T_{\mathcal{E}}} (\otimes_{\mathfrak{I} \in \mathrm{tri}(\mathcal{E})} \mathcal{S}_n(\mathfrak{I}, v)) \otimes (\otimes_{e \in \mathrm{Int}(\mathcal{E})} \mathcal{S}_n(\mathfrak{B}, v))$$

where $T_{\mathcal{E}} = \Delta_{\mathrm{Int}(\mathcal{E})} - \mathrm{fl} \circ_{\mathrm{Int}(\mathcal{E})} \Delta$. The arrow with a tail means the corresponding map is injective.

Proof. Theorem 8.6 in [LS21], and Proposition 2.1.3. □

Suppose c is an ideal arc of a pb surface \mathfrak{S} such that it is contained in the interior of \mathfrak{S} . Let $\mathcal{V} \subset c$ be a subset of c consisting of finite points, and let $\hat{\mathfrak{S}} = \mathfrak{S} \setminus \mathcal{V}$. Then $c \setminus \mathcal{V} = \cup_{i=1}^k c_i$ is the disjoint union of ideal arcs c_i of $\hat{\mathfrak{S}}$. Let \mathfrak{S}' be the result of cutting \mathfrak{S} along c , and $\hat{\mathfrak{S}}'$ be the result of splitting $\hat{\mathfrak{S}}$ along all c_i . We have natural embeddings $\iota : \hat{\mathfrak{S}} \rightarrow \mathfrak{S}$ and $\iota : \hat{\mathfrak{S}}' \rightarrow \mathfrak{S}'$. A linear order h on set $\{c_i \mid 1 \leq i \leq k\}$ induces a height ordered embedding $\iota_h : \hat{\mathfrak{S}}' \rightarrow \mathfrak{S}'$.

From the commutativity of the splitting maps, the composition of all the splitting homomorphisms Θ_{c_i} can be taken in any order. We also denote this composition by Θ_c .

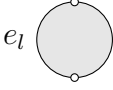
Lemma 2.1.5 ([BL20]). *With the above notations, we have the following commutative diagram:*




$$\begin{array}{ccc} \mathcal{S}_n(\mathfrak{S}, v) & \xleftarrow{\iota_*} & \mathcal{S}_n(\hat{\mathfrak{S}}, v) \\ \Theta_c \downarrow & & \downarrow \Theta_c \\ \mathcal{S}_n(\mathfrak{S}', v) & \xleftarrow{(\iota_h)_*} & \mathcal{S}_n(\hat{\mathfrak{S}}', v) \end{array}$$

The above Lemma is a generalization for Lemma 3.6 in [BL20], in which h is the linear order induced by an orientation of c .

2.1.5 The bigon and $\mathcal{O}_q(\mathrm{SL}_n)$

We refer to [LS21; LY23] for the definitions for the bigon and $\mathcal{O}_q(\mathrm{SL}_n)$.

The **bigon** \mathfrak{B} is obtained from a closed disk D by removing two points in ∂D . We can label the two boundary components of a bigon by e_l and e_r . A bigon with this labeling is called a **directed bigon**, see an example . We can draw

\mathfrak{B} like . We use a_j^i to denote  j , and use \bar{a}_j^i to denote  j , where $i, j \in \{1, 2, \dots, n\}$.

For $i, j, k, l \in \{1, 2, \dots, n\}$, we have the following coefficients

$$\mathcal{R}_{lk}^{ij} = q^{-\frac{1}{n}} \left(q^{\delta_{i,j}} \delta_{j,k} \delta_{i,l} + (q - q^{-1}) \delta_{j < i} \delta_{j,l} \delta_{i,k} \right), \quad (2.13)$$

where $\delta_{j < i} = 1$ if $j < i$ and $\delta_{j < i} = 0$ otherwise.

Let $\mathcal{O}_q(M_n)$ be the associated R -algebra generated by $u_{i,j}$, $i, j \in \{1, 2, \dots, n\}$, subject to the relations

$$(\mathbf{u} \otimes \mathbf{u})\mathcal{R} = \mathcal{R}(\mathbf{u} \otimes \mathbf{u}), \quad (2.14)$$

where \mathcal{R} is the $n^2 \times n^2$ matrix given by equation (2.13), and $\mathbf{u} \otimes \mathbf{u}$ is the $n^2 \times n^2$ matrix with entries $(\mathbf{u} \otimes \mathbf{u})_{jl}^{ik} = u_{i,j} u_{k,l}$ for $i, j, k, l \in \{1, 2, \dots, n\}$. Define the element

$$\det_q(\mathbf{u}) := \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} u_{1,\sigma(1)} \cdots u_{n,\sigma(n)} = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} u_{\sigma(1),1} \cdots u_{\sigma(n),n}.$$

Define $\mathcal{O}_q(SL_n)$ to be $\mathcal{O}_q(M_n)/(\det_q \mathbf{u} - 1)$. Then $\mathcal{O}_q(SL_n)$ is a Hopf algebra with the Hopf algebra structure given by

$$\begin{aligned} \Delta(u_{i,j}) &= \sum_{k=1}^n u_{i,k} \otimes u_{k,j}, & \epsilon(u_{i,j}) &= \delta_{i,j}. \\ S(u_{i,j}) &= (\mathbf{u}^1)_{i,j} = (-q)^{i-j} \det_q(\mathbf{u}^{j,i}). \end{aligned}$$

Here $\mathbf{u}^{j,i}$ is the result of removing the j -th row and i -th column from \mathbf{u} .

Theorem 2.1.6 ([LS21]). (a) $\mathcal{S}_n(\mathfrak{B}, v)$ is a Hopf algebra over R .

(b) We have a unique Hopf algebra isomorphism $g_{\text{big}} : \mathcal{O}_q(SL_n) \rightarrow \mathcal{S}_n(\mathfrak{B}, v)$ defined by $g_{\text{big}}(u_{i,j}) = a_j^i$.

Lemma 2.1.7. The map $\bar{\cdot} : \mathcal{S}_n(\mathfrak{B}, v) \rightarrow \mathcal{S}_n(\mathfrak{B}, v)$ is a Hopf algebra isomorphism.

Proof. Since $\bar{\tau}$ is already an algebra homomorphism. We only need to show $\bar{\tau}$ respects comultiplication, counit, and antipode. We only need to check this for a_j^i because $\bar{\tau}, \Delta, \epsilon$ are algebra homomorphisms and S is an anti-algebra homomorphism.

From equations (63) and (65) in [LS21], we know

$$\Delta \circ \bar{\tau} = \bar{\tau} \circ \Delta, \quad \epsilon \circ \bar{\tau} = \bar{\tau} \circ \epsilon.$$

For any $i, j \in \{1, 2, \dots, n\}$, we have

$$\bar{\tau}(S(a_j^i)) = (-q)^{i-j}(\bar{\tau}(\bar{a}_i^j)) = (-q)^{i-j}(a_i^j), \quad S(\bar{\tau}(a_j^i)) = S(\bar{a}_j^i) = (-q)^{i-j}(a_i^j).$$

□

For any $i, j \in \{1, 2, \dots, n\}$, We use $b_{i,j}$ to denote \bar{a}_j^i . Combining Theorem 2.1.6 and Lemma 2.1.7, we have the following Theorem. The reason why we prefer $b_{i,j}$ than a_j^i is because $b_{i,j}$ coincides with our notation, which requires the orientation of $b_{i,j}$ to be from j to i .

Theorem 2.1.8. *We have a unique Hopf algebra isomorphism $f_{big} : \mathcal{O}_q(SL_n) \rightarrow \mathcal{S}_n(\mathfrak{B}, v)$ defined by $f_{big}(u_{i,j}) = b_{i,j}$.*

Lemma 2.1.9 ([PW91]). *Suppose q is a primitive m -th root of unity with m being odd. Then in $\mathcal{O}_q(SL_n)$, for any $1 \leq i, j \leq n$, we have*

(a)

$$\begin{aligned} & \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} (u_{1,\sigma(1)})^m (u_{2,\sigma(2)})^m \dots (u_{n,\sigma(n)})^m \\ &= \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} (u_{\sigma(1),1})^m (u_{\sigma(2),2})^m \dots (u_{\sigma(n),n})^m = 1, \end{aligned}$$

(b)

$$\Delta((u_{i,j})^m) = \sum_{1 \leq k \leq n} (u_{i,k})^m \otimes (u_{k,j})^m,$$

(c) $(u_{i,j})^m$ is in the center of $\mathcal{O}_q(SL_n)$.

2.1.6 Conventions

An **oriented arc** α in (M, \mathcal{N}) is a smooth embedding from $[0, 1]$ to M with the orientation given by the positive direction of $[0, 1]$ such that $\alpha \cap \partial M = \{\alpha(0), \alpha(1)\} \cap \mathcal{N}$. A **framed oriented arc** is an oriented arc with transversal framing such that the framings at $\{\alpha(0), \alpha(1)\} \cap \mathcal{N}$, if not empty, are given by the velocity vectors of \mathcal{N} . An **oriented circle** in (M, \mathcal{N}) is a smooth embedding $\beta : S^1 \rightarrow M$ with a chosen orientation such that $\beta \subset \text{int}(M)$. A **framed oriented knot** is an oriented circle with transversal framing.

If both two ends of a framed oriented arc lie in \mathcal{N} we call it a framed oriented boundary arc of (M, \mathcal{N}) , or just **framed oriented boundary arc** when there is no confusion with (M, \mathcal{N}) . If the two ends of a framed oriented boundary arc are both stated, we call it a **stated framed oriented boundary arc**.

For a framed oriented boundary arc α , we use $\alpha_{i,j}$ to denote α with two ends stated by $s(\alpha(0)) = j$ and $s(\alpha(1)) = i$. For a stated framed oriented boundary arc α , suppose $s(\alpha(0)) = j$ and $s(\alpha(1)) = i$, we can also use $\alpha_{i,j}$ to denote α to indicate the information that $s(\alpha(0)) = j$ and $s(\alpha(1)) = i$.

For two framed oriented arcs α, β , we say $\alpha * \beta$ is well-defined if (1) $\alpha \cap \beta = \{\alpha(0)\} = \{\beta(1)\}$ and they have the same framing and velocity vector at point $\beta(1) = \alpha(0)$, or (2) $\alpha \cap \beta = \{\alpha(0), \alpha(1)\} = \{\beta(0), \beta(1)\}$, where $\alpha(0) = \beta(1)$ and $\alpha(1) = \beta(0)$, and they have the same framings and velocity vectors at their intersecting points. We use $\alpha * \beta$ to denote the new framed oriented arc (or framed oriented knot) obtained by connecting α and β at their intersecting points. Note that it is possible that $\alpha * \beta$ is not a well-defined framed oriented arc (or framed oriented knot) because the intersecting points could be contained in \mathcal{N} . If this happens, we just isotopically push the parts nearby intersecting points to the inside of M . Then we obtain a well-defined framed oriented arc (or framed oriented knot), which is still denoted as $\alpha * \beta$. Note that for case (2), we have both $\alpha * \beta$ and $\beta * \alpha$ are well-defined and they represent the same framed oriented knot.

Suppose R_1, R_2 are two algebras, and f is map from R_1 to R_2 . Let $A = (a_{i,j})$ be a k_1 by k_2 matrix in R_1 where k_1, k_2 are two positive integers. We define $f(A)$ to be a k_1 by k_2 matrix in R_2 with $[f(A)]_{i,j} = f(a_{i,j})$. If f is an algebra homomorphism, we have $f(I) = I, f(A_1 A_2) = f(A_1) f(A_2), f(A_3 + A_4) = f(A_3) + f(A_4), f(A_5^{-1}) =$

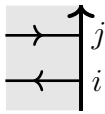
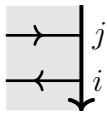
$(f(A_5))^{-1}$ where I is the identity matrix in any size and $A_t, 1 \leq t \leq 5$, are matrices in R_1 such that the above operations for A_t make sense.

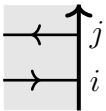
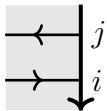
In this thesis, when we talk about 3-manifold, we always mean a 3-manifold with a chosen orientation and a chosen Riemannian metric.

2.2 The commutative algebra structure for $\mathcal{S}_n(M, \mathcal{N}, 1)$ and the coordinate ring

In this section we will give a commutative algebra structure to $\mathcal{S}_n(M, \mathcal{N}, 1)$, and introduce an algebraic set related to (M, \mathcal{N}) . The main goal of this section is to construct a surjective algebra homomorphism from $\mathcal{S}_n(M, \mathcal{N}, 1)$ to the coordinate ring of this algebraic set. In next section, we will prove the well-definedness and surjectivity of this algebra homomorphism.

2.2.1 Height exchange relations for boundary arcs with opposite orientations

In this subsection, we try to discuss the relation between  and .

(between  and ).

Proposition 2.2.1 ([Wan24]). *Let (M, \mathcal{N}) be any marked 3-manifold with $\mathcal{N} \neq \emptyset$. In $\mathcal{S}_n(M, \mathcal{N}, v)$, we have*

$$\begin{aligned}
 \begin{array}{|c|} \hline \bullet \\ \hline \circ \\ \hline \end{array} \begin{array}{|c|} \hline \uparrow j \\ \hline \downarrow i \\ \hline \end{array} &= \begin{array}{|c|} \hline \bullet \\ \hline \circ \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow i \\ \hline \uparrow j \\ \hline \end{array} = \begin{cases} q^{\frac{1-n}{n}} \left(\begin{array}{|c|} \hline \bullet \\ \hline \circ \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow j \\ \hline \uparrow i \\ \hline \end{array} + c_i(1 - q^2) \sum_{j < k \leq n} c_k^{-1} \begin{array}{|c|} \hline \bullet \\ \hline \circ \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow k \\ \hline \uparrow \bar{k} \\ \hline \end{array} \right), & \text{if } j = \bar{i}, \\ q^{\frac{1}{n}} \begin{array}{|c|} \hline \bullet \\ \hline \circ \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow j \\ \hline \uparrow i \\ \hline \end{array}, & \text{if } j \neq \bar{i}. \end{cases}
 \end{aligned}$$

2.2.2 Commutative algebra structure for $\mathcal{S}_n(M, \mathcal{N}, 1)$

In this subsection, we will give a commutative algebra structure for $\mathcal{S}_n(M, \mathcal{N}, 1)$. To do so, first we define the product of two stated n -webs as the disjoint union, that is, we first isotope two stated n -webs such that they have no intersection, then take their union as the product. To prove this product is well defined, we have to show the product is independent of how we take the union of these two webs.

Corollary 2.2.2. *For any marked 3-manifold (M, \mathcal{N}) , we have $\mathcal{S}_n(M, \mathcal{N}, 1)$ is a commutative algebra under the above defined multiplication.*

Proof. Because of relations (2.4) and (2.11), it suffices to show for any two states i, j , we have

$$\begin{array}{c} \begin{array}{|c|} \hline \rightarrow \\ \hline \leftarrow \\ \hline \end{array} \begin{array}{|c|} \hline \uparrow \\ \hline \downarrow \\ \hline \end{array} \begin{array}{|c|} \hline j \\ \hline i \\ \hline \end{array} \\ \hline \end{array} = \begin{array}{c} \begin{array}{|c|} \hline \rightarrow \\ \hline \leftarrow \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \uparrow \\ \hline \end{array} \begin{array}{|c|} \hline j \\ \hline i \\ \hline \end{array} \\ \hline \end{array}, \begin{array}{c} \begin{array}{|c|} \hline \leftarrow \\ \hline \rightarrow \\ \hline \end{array} \begin{array}{|c|} \hline \uparrow \\ \hline \downarrow \\ \hline \end{array} \begin{array}{|c|} \hline j \\ \hline i \\ \hline \end{array} \\ \hline \end{array} = \begin{array}{c} \begin{array}{|c|} \hline \leftarrow \\ \hline \rightarrow \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \uparrow \\ \hline \end{array} \begin{array}{|c|} \hline j \\ \hline i \\ \hline \end{array} \\ \hline \end{array}$$

when $v = 1$, which can be easily derived from Proposition 2.2.1.

□

2.2.3 Relative spin structure and character variety

Let (M, \mathcal{N}) be any marked 3-manifold, and $\zeta : UM \rightarrow M$ be the unit tangent bundle. We know the fiber of this bundle is $SO(3)$, whose fundamental group is \mathbb{Z}_2 . For any point $P \in M$, we use ϑ_P to denote the nontrivial element in the fundamental group of $\zeta^{-1}(P)$. We have ϑ_P is homotopic to ϑ_Q for any two points P, Q in a same component of M . For a component Y of M , we use ϑ_Y to denote this homotopy type. When M is connected, we will use ϑ to denote this unique homotopy type.

For any component $e \in \mathcal{N}$, e has a unique lift \tilde{e} in UM . For any point $x \in e$, let u_x be the unit velocity vector at point x , let w_x be the unit tangent vector at x such that w_x is orthogonal to ∂M and pointing inside of M . Then the orientation of M determines the second unit tangent vector v_x such that (u_x, v_x, w_x) is the orientation of M . Obviously $\tilde{e} = \{(x, u_x, v_x, w_x) \mid x \in e\}$ is a smooth path in UM , which is diffeomorphic to $(0, 1)$. Let $\tilde{\mathcal{N}} = \cup_e \tilde{e}$ where the union takes over all components e of \mathcal{N} . Note that $\tilde{\mathcal{N}} = \emptyset$ when $\mathcal{N} = \emptyset$.

From now on, for a topological space X , we will use $Com(X)$ (respectively $PCom(X)$) to denote the set of components of X (the set of path connected components of X).

Definition 2.2.3. *A relative spin structure of a marked 3-manifold (M, \mathcal{N}) is defined to be a group homomorphism $h : H_1(UM, \tilde{\mathcal{N}}) \rightarrow \mathbb{Z}_2$ such that $h(\vartheta_Y) = 1$ for $Y \in Com(M)$.*

Note that when $\mathcal{N} = \emptyset$, h is just the usual spin structure.

Suppose (M, \mathcal{N}) is the disjoint union of (M_1, \mathcal{N}_1) and (M_2, \mathcal{N}_2) . Then we have $H_1(UM, \tilde{\mathcal{N}}) = H_1(UM_1, \tilde{\mathcal{N}}_1) \oplus H_1(UM_2, \tilde{\mathcal{N}}_2)$. For each $i = 1, 2$, let h_i be a relative spin structure for (M_i, \mathcal{N}_i) . Then $(h_1, h_2) : H_1(UM, \tilde{\mathcal{N}}) \rightarrow \mathbb{Z}_2$, defined by $(h_1, h_2)(x_1, x_2) = h_1(x_1) + h_2(x_2)$, is a relative spin structure for (M, \mathcal{N}) . Clearly every relative spin structure of (M, \mathcal{N}) is of this form.

rem Let X be any path connected topological space, and P be a set of finite points in X . We suppose $P = \{p_0, p_1, \dots, p_{m-1}\}$ where m is a positive integer. For each $1 \leq i \leq m-1$, let α_i be a path connecting p_0 and p_i . Then $H_1(X, P) = H_1(P) \oplus \mathbb{Z}([\alpha_1]) \oplus \dots \oplus \mathbb{Z}([\alpha_{m-1}])$.

Suppose M is connected. When \mathcal{N} has only one component, we have $H_1(UM, \tilde{\mathcal{N}}) = H_1(UM)$. Thus in this case the relative spin structure for (M, \mathcal{N}) is just the usual spin structure for M . When $\#\mathcal{N} > 1$, suppose the set of components of \mathcal{N} is $\{e_0, e_1, \dots, e_{m-1}\}$. For each $1 \leq i \leq m-1$, let α_i be a path connecting \tilde{e}_0 and \tilde{e}_i . Then we have $H_1(UM, \tilde{\mathcal{N}}) = H_1(UM) \oplus \mathbb{Z}([\alpha_1]) \oplus \dots \oplus \mathbb{Z}([\alpha_{m-1}])$. For any spin structure h for M , we can extend h to a relative spin structure for (M, \mathcal{N}) by defining $h([\alpha_i]) = r_i \in \mathbb{Z}_2, 1 \leq i \leq m-1$, where $r_i, 1 \leq i \leq m-1$, are $m-1$ arbitrary elements in \mathbb{Z}_2 . Reversely, for any relative spin structure, we can restrict h to $H_1(UM)$ to obtain a spin structure for M .

Suppose \mathcal{N}' is obtained from \mathcal{N} ($\mathcal{N} \neq \emptyset$) by adding one extra marking e such that $cl(e) \cap cl(\mathcal{N}) = \emptyset$. Let α be a path connecting $\tilde{\mathcal{N}}$ and \tilde{e} such that $\alpha(0) \in \tilde{\mathcal{N}}$ and $\alpha(1) \in \tilde{e}$. Then $H_1(UM, \tilde{\mathcal{N}}') = H_1(UM, \tilde{\mathcal{N}}) \oplus \mathbb{Z}([\alpha])$. Any relative spin structure h for (M, \mathcal{N}) can be extended to a relative spin structure for (M, \mathcal{N}') by defining $h([\alpha]) = r$ where r is an arbitrary element in \mathbb{Z}_2 . Reversely any relative spin structure h for (M, \mathcal{N}') can be restricted to $H_1(UM, \tilde{\mathcal{N}})$ to obtain a relative spin structure for (M, \mathcal{N}) .

38.2. The commutative algebra structure for $\mathcal{S}_n(M, \mathcal{N}, 1)$ and the coordinate ring

For a path connected topological space X , we use $\pi_1(X)$ to denote the fundamental group for X . For $[\alpha], [\beta] \in \pi_1(X)$, $[\alpha][\beta] = [\alpha * \beta]$ where $\alpha * \beta$ is obtained by first going through β , then going through α . Note that here $\alpha * \beta$ is different with conventional definition.

Definition 2.2.4. For any path connected topological space X , define

$$\Gamma_n(X) = \mathbb{C}[[\alpha]_{i,j} \mid [\alpha] \in \pi_1(X), 1 \leq i, j \leq n] / (Q_{[\alpha]}Q_{[\beta]} = Q_{[\alpha*\beta]}, \det(Q_{[\alpha]}) = 1, Q_{[o]} = I)$$

where $[\alpha], [\beta]$ go through all elements in $\pi_1(X)$, $[o]$ is the trivial loop in $\pi_1(X)$, $Q_{[\eta]} = ([\eta]_{i,j})_{1 \leq i, j \leq n}$ for any element $[\eta] \in \pi_1(X)$.

Note that $\pi_1(X)$ has an action on $\Gamma_n(X)$, defined by $[\alpha]([\beta]_{i,j}) = [\alpha * \beta * \alpha^{-1}]_{i,j}$ for any $[\alpha], [\beta], 1 \leq i, j \leq n$. We use $G_n(X)$ to denote the subalgebra of $\Gamma_n(X)$ fixed by this action.

Note that $\text{Trace}(Q_{[\alpha]}) \in G_n(X)$ for any $[\alpha] \in \pi_1(X)$, and $G_n(X)$ is generated by $\text{Trace}(Q_{[\alpha]})$, $[\alpha] \in \pi_1(X)$ as an algebra [Sik01].

rem We can generalize Definition 2.2.4 to general topological space. Let X be a topological space. Suppose $PCom(X) = \{X_1, \dots, X_m\}$, then define

$$\Gamma_n(X) = \Gamma_n(X_1) \otimes \cdots \otimes \Gamma_n(X_m), \quad G_n(X) = G_n(X_1) \otimes \cdots \otimes G_n(X_m).$$

But $\Gamma_n(X), G_n(X)$ are only well-defined up to isomorphism, since different order of X_i give different algebras.

Definition 2.2.5 ([CL22a]). Let X be a topological space and $\{E_j\}_{j \in J}$ be disjoint contractible subspaces of X . The fundamental groupoid $\pi_1(X, \cup_{j \in J} E_j)$ is the groupoid (i.e. a category with invertible morphisms) whose objects are $\{E_j\}_{j \in J}$ and whose morphisms are the homotopy classes of oriented paths in X with end points in $\cup_{j \in J} E_j$. A morphism of groupoids is a functor of the corresponding categories. We can regard the group as the groupoid consisting of only one object and all group elements being all morphisms.

For a marked 3-manifold (M, \mathcal{N}) , we use $\pi_1^{Mor}(M, \mathcal{N})$ to denote the set of morphisms in $\pi_1(M, \mathcal{N})$.

Let \mathcal{A}, \mathcal{B} be two categories. We try to define a new category $\mathcal{A} \cup \mathcal{B}$. The objects of $\mathcal{A} \cup \mathcal{B}$ is the union of objects in \mathcal{A} and objects in \mathcal{B} . Let U, V be any two objects in $\mathcal{A} \cup \mathcal{B}$. If U, V both belong to \mathcal{A} (respectively \mathcal{B}), then we define $\mathrm{Hom}_{\mathcal{A} \cup \mathcal{B}}(U, V)$ to be $\mathrm{Hom}_{\mathcal{A}}(U, V)$ (respectively $\mathrm{Hom}_{\mathcal{B}}(U, V)$). Otherwise we define $\mathrm{Hom}_{\mathcal{A} \cup \mathcal{B}}(U, V) = \emptyset$.

Let X be a topological space. Suppose $PCom(X) = \{X_1, \dots, X_m\}$. For each $1 \leq t \leq m$, let $\{E_j\}_{j \in J_t}$ be disjoint contractible subspaces of X_t . Obviously we have

$$\pi_1(X, \cup_{1 \leq t \leq m} (\cup_{j \in J_t} E_j)) = \pi_1(X_1, \cup_{j \in J_1} E_j) \cup \dots \cup \pi_1(X_m, \cup_{j \in J_m} E_j).$$

Definition 2.2.6. For a marked 3-manifold (M, \mathcal{N}) with every component of M containing at least one marking, define

$$\chi_n(M, \mathcal{N}) = \mathrm{Hom}(\pi_1(M, \mathcal{N}), \mathrm{SL}_n(\mathbb{C})),$$

and

$$\tilde{\chi}_n(M, \mathcal{N}) = \{\tilde{\rho} \in \mathrm{Hom}(\pi_1(UM, \tilde{\mathcal{N}}), \mathrm{SL}_n(\mathbb{C})) \mid \tilde{\rho}(\vartheta_Y) = d_n I, \text{ for all } Y \in \mathrm{Com}(M)\}$$

where I is the identity matrix.

Suppose (M, \mathcal{N}) is the disjoint union of (M_1, \mathcal{N}_1) and (M_2, \mathcal{N}_2) , then we have

$$\chi_n(M, \mathcal{N}) \simeq \chi_n(M_1, \mathcal{N}_1) \times \chi_n(M_2, \mathcal{N}_2), \quad \tilde{\chi}_n(M, \mathcal{N}) \simeq \tilde{\chi}_n(M_1, \mathcal{N}_1) \times \tilde{\chi}_n(M_2, \mathcal{N}_2).$$

From Lemma 8.1 in [CL22a], if M is connected we have

$$\chi_n(M, \mathcal{N}) \simeq \mathrm{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C})) \times \mathrm{SL}_n(\mathbb{C})^{\#\mathcal{N}-1}.$$

Then $\chi_n(M, \mathcal{N})$ is an affine algebraic set, whose coordinate ring is denoted as $R_n(M, \mathcal{N})$.

Definition 2.2.7. When M is connected and \mathcal{N} is empty, we define $\chi_n(M, \mathcal{N})$ to be the $\mathrm{SL}_n(\mathbb{C})$ -character variety of M . That is

$$\chi_n(M, \mathcal{N}) = \mathfrak{X}_{\mathrm{SL}_n(\mathbb{C})}(M)$$

where $\mathfrak{X}_{\mathrm{SL}_n(\mathbb{C})}(M)$ is defined in (1.1).

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Similarly we define

$$\tilde{\chi}_n(M, \emptyset) = \{\tilde{\rho} \in \text{Hom}(\pi_1(UM), \text{SL}(n, \mathbb{C})) \mid \tilde{\rho}(\vartheta) = d_n I\} / \simeq$$

where the definition for \simeq is the same as the one in (1.1) (that is two elements are considered to be equivalent if and only if they have the same “trace”).

From [Sik01], we know $\chi_n(M, \emptyset)$ is an affine algebraic set. We also denote its coordinate ring as $R_n(M, \emptyset)$. There is a surjective algebra homomorphism $\mathcal{Y} : G_n(M) \rightarrow R_n(M, \emptyset)$ defined by

$$\mathcal{Y}(\text{tr}(Q_{[\alpha]}))(\rho) = \text{tr}(\rho([\alpha])) \text{ where } [\alpha] \in \pi_1(M), \rho \in \chi_n(M, \emptyset),$$

and $\text{Ker } \mathcal{Y} = \sqrt{0}$.

We can simply generalize definitions for $\chi_n(M, \mathcal{N}), \tilde{\chi}_n(M, \mathcal{N}), R_n(M, \mathcal{N})$ to all marked 3-manifolds by taking product (or tensor product) for disjoint union.

Proposition 2.2.8. *For any marked 3-manifold (M, \mathcal{N}) , we have $\chi_n(M, \mathcal{N}) \simeq \tilde{\chi}_n(M, \mathcal{N})$.*

Proof. We can assume M is connected. Here we only consider the case when $\mathcal{N} \neq \emptyset$ since we can prove the case when $\mathcal{N} = \emptyset$ by using the same technique.

Let h be a relative spin structure for (M, \mathcal{N}) . We use h to establish an isomorphism $f_h : \chi_n(M, \mathcal{N}) \rightarrow \tilde{\chi}_n(M, \mathcal{N})$. For any $\rho \in \chi_n(M, \mathcal{N})$, define

$$f_h(\rho)(\tilde{\alpha}) = d_n^{h(\tilde{\alpha})} \rho(\alpha) \text{ where } \tilde{\alpha} \in \pi_1(UM, \tilde{\mathcal{N}}), \alpha = \zeta(\tilde{\alpha}).$$

Clearly $f_h(\rho)$ is a homomorphism from $\pi_1(UM, \tilde{\mathcal{N}})$ to $\text{SL}_n(\mathbb{C})$, and $f_h(\rho)(\vartheta) = d_n^{h(\vartheta)} \rho(\zeta(\vartheta)) = d_n I$, thus $f_h \in \tilde{\chi}_n(M, \mathcal{N})$.

Then we try to define $g_h : \tilde{\chi}_n(M, \mathcal{N}) \rightarrow \chi_n(M, \mathcal{N})$. For any $\tilde{\rho} \in \tilde{\chi}_n(M, \mathcal{N})$ and $\alpha \in \pi_1(M, \mathcal{N})$

$$g_h(\tilde{\rho})(\alpha) = d_n^{h(\tilde{\alpha})} \tilde{\rho}(\tilde{\alpha}) \text{ where } \tilde{\alpha} \in \pi_1(UM, \tilde{\mathcal{N}}) \text{ such that } \zeta(\tilde{\alpha}) = \alpha.$$

Suppose $\zeta(\tilde{\alpha}) = \zeta(\tilde{\beta}) = \alpha$. Since $\tilde{\rho}(\vartheta) = d_n I$ and $h(\vartheta) = 1$, we have

$$\tilde{\rho}(\tilde{\alpha}^{-1} \tilde{\beta}) = d_n^{h(\tilde{\alpha}^{-1}) + h(\tilde{\beta})} I = d_n^{h(\tilde{\alpha}) + h(\tilde{\beta})} I.$$

Then $d_n^{h(\tilde{\beta})}\tilde{\rho}(\tilde{\beta}) = d_n^{h(\tilde{\alpha})}\tilde{\rho}(\tilde{\alpha})$, which shows $g_h(\tilde{\rho})$ is well-defined. Obviously we have $g_h(\tilde{\rho}) \in \chi(M, \mathcal{N})$.

We have

$$\begin{aligned} f_h(g_h(\tilde{\rho}))(\tilde{\alpha}) &= d_n^{h(\tilde{\alpha})}g_h(\tilde{\rho})(\alpha) = d_n^{h(\tilde{\alpha})}d_n^{h(\tilde{\alpha})}\tilde{\rho}(\tilde{\alpha}) = \tilde{\rho}(\tilde{\alpha}), \\ g_h(f_h(\rho))(\alpha) &= d_n^{h(\tilde{\alpha})}f_h(\rho)(\tilde{\alpha}) = d_n^{h(\tilde{\alpha})}d_n^{h(\tilde{\alpha})}\rho(\alpha) = \rho(\alpha). \end{aligned}$$

Thus g_h and f_h are inverse to each other. \square

When there is a fixed relative spin structure, we don't have to distinguish between $\chi_n(M, \mathcal{N})$ and $\tilde{\chi}_n(M, \mathcal{N})$, and we also regard $R_n(M, \mathcal{N})$ as the coordinate ring for $\tilde{\chi}_n(M, \mathcal{N})$ using Proposition 2.2.8.

rem For any two relative spin structures h_1, h_2 , we have $F_{h_2-h_1} \circ f_{h_1} = f_{h_2}$ where $F_{h_2-h_1}$ is an isomorphism from $\tilde{\chi}_n(M, \mathcal{N})$ to $\tilde{\chi}_n(M, \mathcal{N})$ defined as

$$F_{h_2-h_1}(\tilde{\rho})(\tilde{\alpha}) = d_n^{h_2(\tilde{\alpha})-h_1(\tilde{\alpha})}\tilde{\rho}(\tilde{\alpha}).$$

rem From Remark 2.2.3, we know there is a surjective algebra homomorphism $\mathcal{Y} : G_n(M) \rightarrow R_n(M, \emptyset)$, and $\text{Ker } \mathcal{Y} = \sqrt{0}$. When there is a spin structure h , we can regard $R_n(M, \emptyset)$ as the coordinate ring for $\tilde{\chi}_n(M, \emptyset)$ using Proposition 2.2.8. Then \mathcal{Y} is given by

$$\mathcal{Y}(\text{tr}(Q_{[\alpha]}))(\tilde{\rho}) = d_n^{h([\tilde{\alpha}]})\text{tr}(\tilde{\rho}([\tilde{\alpha}])) \quad (2.15)$$

where $[\alpha] \in \pi_1(M)$ and $[\tilde{\alpha}] \in \pi_1(UM)$ is any lift of $[\alpha]$. Since the definition of \mathcal{Y} in equation (2.15) is related to h , we will use \mathcal{Y}_h , instead of \mathcal{Y} , to denote the map defined by equation (2.15).

2.2.4 Surjective algebra homomorphism from $\mathcal{S}_n(M, \mathcal{N}, 1)$ to the coordinate ring

For any marked 3-manifold (M, \mathcal{N}) , we are trying to define a surjective algebra homomorphism $\Phi_h^{(M, \mathcal{N})} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow R_n(M, \mathcal{N})$ (here we regard $R_n(M, \mathcal{N})$ as the coordinate ring for $\tilde{\chi}_n(M, \mathcal{N})$).

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Recall that $\tilde{\mathcal{N}}$ is lifted by \mathcal{N} . For any component $e \in \mathcal{N}$, we have $\tilde{e} = \{(x, u_x, v_x, w_x) \mid x \in e\}$, where u_x is the unit velocity vector at x , w_x is the unit tangent vector at x orthogonal to ∂M pointing into M and (u_x, v_x, w_x) is the orientation of M .

For any n -web l and a component e of \mathcal{N} , we can isotope l such that the velocity vector of l at each its end point x contained in e is parallel to v_x . Then we say l is in **good position with respect to e** . If l is in good position with respect to every component of \mathcal{N} , we say it is in good position with respect to (M, \mathcal{N}) , or just l is in **good position** when there is no confusion with (M, \mathcal{N}) .

Let α be any stated framed oriented boundary arc in (M, \mathcal{N}) , then we can lift α to an element $\tilde{\alpha}$ in $\pi_1(UM, \tilde{\mathcal{N}})$. We first isotope α such that α is in good position and the framing is normal everywhere. Then α lifts to $\tilde{\alpha}$ where the first vector is the framing, the second vector is the velocity vector, and the third vector is determined by the orientation of M . Suppose $s(\alpha(0)) = j$ and $s(\alpha(1)) = i$. Then for any $\tilde{\rho} \in \tilde{\chi}_n(M, \mathcal{N})$, define

$$tr_\alpha(\tilde{\rho}) = [A\tilde{\rho}(\tilde{\alpha})]_{\tilde{i}, \tilde{j}}, \text{ where } A_{i,j} = (-1)^{i+1} \delta_{i,j}, 1 \leq i, j \leq n.$$

Note that $\det A = 1$ and $A^2 = d_n I$.

Let α be any framed oriented knot in $\mathcal{S}_n(M, \mathcal{N})$, then we lift α to a closed path $\tilde{\alpha}$ in UM as above (first isotope α such that the framing is normal everywhere, then use framing as the first vector and use velocity vector as the second vector). We use a path to connect $\tilde{\mathcal{N}}$ (respectively the base point for $\pi_1(UM)$) and $\tilde{\alpha}$ when $\mathcal{N} \neq \emptyset$ (respectively $\mathcal{N} = \emptyset$). This gives an element in $\pi_1(UM, \tilde{\mathcal{N}})$ or $\pi_1(UM)$, which is still denoted as $\tilde{\alpha}$. For any $\tilde{\rho} \in \tilde{\chi}_n(M, \mathcal{N})$ define

$$tr_\alpha(\tilde{\rho}) = \text{tr}(\tilde{\rho}(\tilde{\alpha})).$$

Since Trace is invariant under the same conjugacy class, we have $tr_\alpha(\tilde{\rho})$ is well-defined.

Theorem 2.2.9. *Let (M, \mathcal{N}) be a marked 3-manifold with M being connected. Then there exists a surjective algebra homomorphism $\Phi^{(M, \mathcal{N})} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow R_n(M, \mathcal{N})$ defined as following: For any stated n -web l in (M, \mathcal{N}) , we use relation (2.8) to kill all the sinks and sources to obtain l' consisting of arcs and knots if $\mathcal{N} \neq \emptyset$ (we use relation (2.7) to kill all the sinks and sources to obtain l' consisting*

of knots if $\mathcal{N} = \emptyset$). Suppose $l' = \cup_{\alpha} \alpha$ where each α is a stated framed oriented boundary arc or a framed oriented knot, then define

$$\Phi^{(M, \mathcal{N})}(l)(\tilde{\rho}) = \prod_{\alpha} \mathrm{tr}_{\alpha}(\tilde{\rho})$$

where $\tilde{\rho} \in \tilde{\chi}_n(M, \mathcal{N})$.

Although we assume M is connected in Theorem 2.2.9 for simplicity, we can easily generalize Theorem 2.2.9 to general marked 3-manifolds.

When there is no confusion with the marked 3-manifold (M, \mathcal{N}) , we can omit the superscript for $\Phi^{(M, \mathcal{N})}$. We will prove Theorem 2.2.9 in next section.

2.2.5 Compatibility with the construction by Costantino and Lê for essentially bordered pb surfaces when $n = 2$.

Let \mathfrak{S} be an essentially bordered pb surface, and (M, \mathcal{N}) be the thickening of \mathfrak{S} . Recall that for every boundary component e , we select a point $x_e \in e$, and set $\mathcal{N} = \cup_e (\{x_e\} \times (-1, 1))$ where e is taken over all the boundary components of \mathfrak{S} . The orientation and the Riemannian metric of M are the product orientation and the product Riemannian metric respectively. For simplicity, we can assume \mathfrak{S} is connected.

If we regard the state "2" as the state "+", and the state "1" as the state "-". Then $\mathcal{S}_2(\mathfrak{S}, 1)$ is the same as the commutative stated skein algebra, mentioned in Section 8 in [CL22a], as shown in [LS21]. When $n = 2$, Costantino and Lê established an isomorphism $tr : \mathcal{S}_2(\mathfrak{S}, 1) \rightarrow \chi(\mathfrak{S})$ where $\chi(\mathfrak{S})$ is the coordinate ring of the so called flat twisted $\mathrm{SL}_2(\mathbb{C})$ -bundle. Here we briefly recall the definition of the flat twisted $\mathrm{SL}_2(\mathbb{C})$ -bundle, please refer to Section 8 in [CL22a] for more details. We follow their notation, and use $\pi_1(U\mathfrak{S}, \tilde{\partial}\mathfrak{S})$ to denote their groupoid, where $U\mathfrak{S}$ is the unit tangent bundle over \mathfrak{S} and $\tilde{\partial}\mathfrak{S}$ is the lift of $\partial\mathfrak{S}$. Then every point in $U\mathfrak{S}$ is a pair (x, v_x) where $x \in \mathfrak{S}$ and v_x is a unit tangent vector at point x . For a point $x \in \mathfrak{S}$ the fiber \mathbb{O} is a circle, and we orient it according to the orientation of

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\mathfrak{S} . Then the flat twisted $\mathrm{SL}_2(\mathbb{C})$ -bundle is defined to be

$$\{\rho \in \mathrm{Hom}(\pi_1(U\mathfrak{S}, \widetilde{\partial\mathfrak{S}}), \mathrm{SL}_2(\mathbb{C})) \mid \rho(\mathbb{O}) = -I\}.$$

Let $pr : M \rightarrow \mathfrak{S}$ be the projection given by $pr(x, t) = x$ for all $x \in \mathfrak{S}, t \in [-1, 1]$, and $l : \mathfrak{S} \rightarrow M$ be the embedding given by $l(x) = (x, 0)$. We use pr_* (respectively l_*) to denote the induced map from $T(M)$ to $T(\mathfrak{S})$ (respectively from $T(\mathfrak{S})$ to $T(M)$), where $T(\cdot)$ is the tangent bundle. For every point $y \in M$, we use u_y to denote the unit vertical tangent vector at y such that u_y points from -1 to 1 . Then l induces an embedding

$$\begin{aligned} l_{\sharp} : U\mathfrak{S} &\rightarrow UM \\ (x, v_x) &\mapsto (l(x), u_{l(x)}, l_*(v_x), w_{l(x)}) \end{aligned}$$

where $w_{l(x)}$ is determined by the orientation of M . Let $VM = \{(y, a_y, b_y, c_y) \in UM \mid a_y = u_y\}$ be a submanifold of UM with one dimension less. Then pr induces a projection

$$\begin{aligned} pr_{\sharp} : VM &\rightarrow \mathfrak{S} \\ (y, a_y, b_y, c_y) &\mapsto (pr(y), pr_*(b_y)). \end{aligned}$$

Clearly $\mathrm{Im}l_{\sharp} \subset VM$, and $pr_{\sharp} \circ l_{\sharp} = \mathrm{Id}_{U\mathfrak{S}}$.

For each boundary component e of \mathfrak{S} , we use \tilde{x}_e to denote a point in \tilde{e} whose projection on e is the point x_e .

Define

$$\begin{aligned} f_l : \pi_1(U\mathfrak{S}, \widetilde{\partial\mathfrak{S}}) &\rightarrow \pi_1(UM, \tilde{\mathcal{N}}) \\ [\alpha] &\mapsto [l_{\sharp} \circ \alpha] \end{aligned}$$

where α is a representative of $[\alpha]$ such that the two endpoints of α belong to $\cup_e \{\tilde{x}_e\}$. And define

$$\begin{aligned} f_{pr} : \pi_1(UM, \tilde{\mathcal{N}}) &\rightarrow \pi_1(\mathfrak{S}, \widetilde{\partial\mathfrak{S}}) \\ [\beta] &\mapsto [pr_{\sharp} \circ \beta] \end{aligned}$$

where β is a representative of $[\beta]$ such that $\text{Im}\beta \subset VM$. It is easy to show f_l and f_{pr} are inverse to each other, and $f_l(\mathbb{O}) = \vartheta$. Then f_l induces isomorphism from $\tilde{\chi}_2(M, \mathcal{N})$ to $\{\rho \in \text{Hom}(\pi_1(U\mathfrak{S}, \widetilde{\partial\mathfrak{S}}), \text{SL}_2(\mathbb{C})) \mid \rho(\mathbb{O}) = -I\}$, which further induces an isomorphism $f_* : \chi(\mathfrak{S}) \rightarrow R_2(M, \mathcal{N})$. Then it is a trivial check that

$$f_* \circ tr = \Phi.$$

2.2.6 Compatibility with the splitting map

In this subsection we discuss the splitting maps for both $R_n(M, \mathcal{N})$ and $\mathcal{S}_n(M, \mathcal{N}, 1)$ and the commutativity for these two splitting maps.

Recall that when D is a properly embedded disk in M and $\beta \subset D$ is an embedded oriented open interval, there exists a splitting map $\Theta_{(D, \beta)} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(\text{Cut}_{(D, \beta)}(M, \mathcal{N}), 1)$.

Lemma 2.2.10. *The above linear map $\Theta_{(D, \beta)}$ is an algebra homomorphism.*

Proof. It follows easily from the definition of $\Theta_{(D, \beta)}$. □

We will use (M', \mathcal{N}') to denote $\text{Cut}_{(D, \beta)}(M, \mathcal{N})$. Then there is a projection $\text{pr} : (M', \mathcal{N}') \rightarrow (M, \mathcal{N})$. If we orient ∂D , the orientations of ∂D and M give a way to distinguish between D_1 and D_2 such that the orientation pointing from D_2 to D_1 and the orientation of ∂D coincide with the orientation of M , see Figure 2.1.

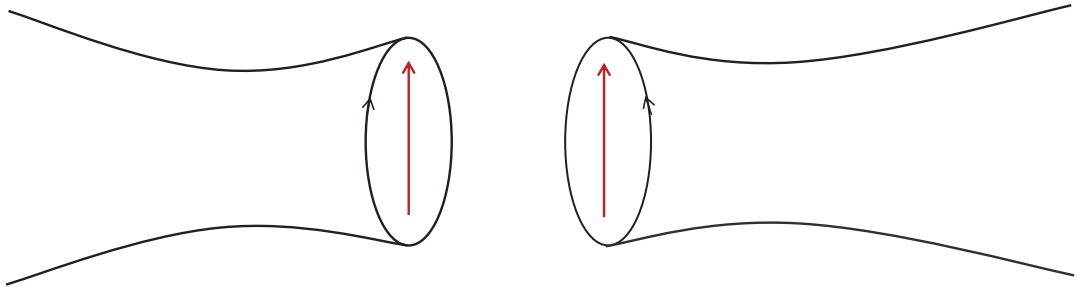


FIGURE 2.1: The orientation of D is indicated by the arrow, the orientation of M is right handed. The left (respectively right) disk copy is D_1 (respectively D_2). The left (respectively right) red arrow is β_1 (respectively β_2).

In the following discussion, we fix an orientation for ∂D . Note that $\beta \in D$ lifts to an element in UM . For every point P in β , the velocity vector gives the first

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unit tangent vector, the orientation of ∂D and the orientation of M give the third unit tangent vector (since the orientation of ∂D and the orientation of M give an orientation of D , which determines a unit tangent vector at P), then the orientation of M gives the second unit tangent vector. We use $\tilde{\beta}$ to denote this lift. Recall that $\zeta : UM \rightarrow M$ is the projection, we define $\tilde{D} = \zeta^{-1}(D)$. The projection $\text{pr} : (M', \mathcal{N}') \rightarrow (M, \mathcal{N})$ induces a projection $\tilde{\text{pr}} : UM' \rightarrow UM$. Then $\tilde{\text{pr}}^{-1}(\tilde{\beta}) = \tilde{\beta}'_1 \cup \tilde{\beta}'_2$ where $\tilde{\beta}'_1 \in \tilde{D}_1, \tilde{\beta}'_2 \in \tilde{D}_2$ (\tilde{D}_1 and \tilde{D}_2 are defined in the same way with \tilde{D}). Note that $\tilde{\beta}_1 = \tilde{\beta}'_1$ and $\tilde{\beta}_2 \neq \tilde{\beta}'_2$. The orientation of β determines a path a_β from $\tilde{\beta}_2$ to $\tilde{\beta}'_2$ and a path b_β from $\tilde{\beta}'_2$ to $\tilde{\beta}_2$ such that both $a_\beta * b_\beta$ and $b_\beta * a_\beta$ are in the same homotopy type with ϑ .

rem According to Lemma 8.1 in [CL22a], any $\rho' \in \tilde{\chi}_n(M', \mathcal{N}')$ can be extended to a homomorphism $\rho'' : \pi_1(UM', \tilde{\mathcal{N}}' \cup \{\tilde{\beta}'_2\}) \rightarrow \text{SL}_n(\mathbb{C})$ by setting $\rho''(a_\beta) = d_n A$.

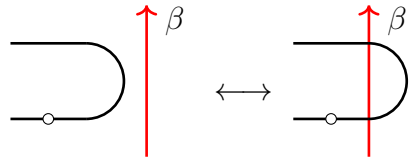
For any $\alpha \in \pi_1(UM, \tilde{\mathcal{N}})$, we can isotope α such that $\alpha \cap \tilde{D} = \alpha \cap \tilde{\beta}$ and $\alpha \cap \tilde{\beta}$ consists of finite points. Then $\alpha = \alpha_k * \alpha_{k-1} * \dots * \alpha_1$ where each $\alpha_i \in \pi_1(UM, \tilde{\mathcal{N}} \cup \{\tilde{\beta}\})$ intersects \tilde{D} at most in its endpoints and exactly along $\tilde{\beta}$. For any $\rho' \in \tilde{\chi}_n(M', \mathcal{N}')$, define

$$\nu^*(\rho')(\alpha) = \rho''(\alpha'_k) \rho''(\alpha'_{k-1}) \dots \rho''(\alpha'_1)$$

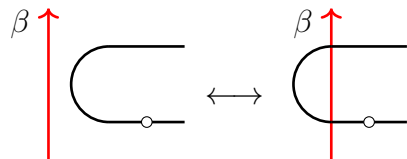
where $\alpha'_i = \tilde{\text{pr}}^{-1}(\alpha_i) \in \pi_1(UM', \tilde{\mathcal{N}}' \cup \{\tilde{\beta}'_2\})$.

Proposition 2.2.11. *Let (M, \mathcal{N}) be a marked 3-manifold, D be any properly embedded disk with an embedded oriented open interval $\beta \in D$. Then there is a surjective homomorphism $\nu^* : \tilde{\chi}_n(M', \mathcal{N}') \rightarrow \tilde{\chi}_n(M, \mathcal{N})$ where $(M', \mathcal{N}') = \text{Cut}_{(D, \beta)}(M, \mathcal{N})$. Especially ν^* induces an injective algebra homomorphism $\nu : R_n(M, \mathcal{N}) \rightarrow R_n(M', \mathcal{N}')$.*

Proof. First we show $\nu^*(\rho') \in \tilde{\chi}_n(M, \mathcal{N})$. Clearly we have $\rho''(\vartheta_Y) = d_n I$ for any component Y of M' . Since ρ'' preserves height exchange and crossing exchange, to show $\nu^*(\rho')$ is well-defined, it suffices to show $\nu^*(\rho')$ preserves the following two moves:



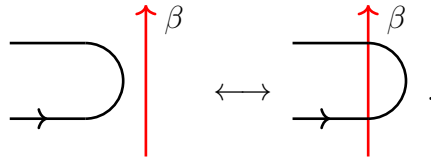
$$(2.16)$$



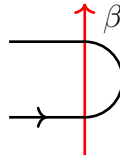
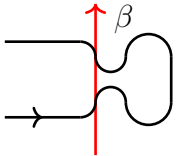
$$(2.17)$$

The red arrow in equations (2.16) and (2.17) is the projection of β , to get the original β , we just rotate the red arrow in equations (2.16) and (2.17) 90 degrees such that it points towards readers. The black line represents part of path in UM , and the white dot represents the direction of the path. The first unit tangent vector of the path is the one pointing towards readers and the second one is given by the velocity vector of the black line.

Here we only prove $\nu^*(\rho')$ preserves



The same proving technique applies for other three cases.

We isotope  to . From the definition of $\nu^*(\rho')$, we know

$$\begin{aligned} \nu^*(\rho')(\text{Diagram 1}) &= \rho''(\text{Diagram 2}) \rho''(\text{Diagram 3}) \rho''(\text{Diagram 4}) \\ &= \rho''(\text{Diagram 5}) \rho''(\text{Diagram 6}) = \rho''(\text{Diagram 7}) \\ &= \nu^*(\rho')(\text{Diagram 1}). \end{aligned}$$

Then, trivially we have $\nu^*(\rho') \in \tilde{\chi}_n(M, \mathcal{N})$.

Then we want to show ν^* is surjective. We use $-\tilde{\beta}$ to denote $\tilde{\text{pr}}(\tilde{\beta}_2)$, and use $\overline{a_\beta}$ to denote $\tilde{\text{pr}}(a_\beta)$. Then $\overline{a_\beta}$ is a path from $-\tilde{\beta}$ to $\tilde{\beta}$. For any $\rho \in \tilde{\chi}_n(M, \mathcal{N})$, we use Lemma 8.1 in [CL22a] to extend ρ to $\rho'' : \pi_1(UM, \tilde{\mathcal{N}} \cup \{\tilde{\beta}, -\tilde{\beta}\})$ setting in particular $\rho''(\overline{a_\beta}) = d_n A$. The projection $\tilde{\text{pr}} : UM' \rightarrow UM$ induces a homomorphism $\text{pr}_* : \pi_1(UM', \tilde{\mathcal{N}}' \cup \{\tilde{\beta}', -\tilde{\beta}'\}) \rightarrow \pi_1(UM, \tilde{\mathcal{N}} \cup \{\tilde{\beta}, -\tilde{\beta}\})$. Then $\rho'' \circ \text{pr}_*$ is a homomorphism

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from $\pi_1(UM', \tilde{\mathcal{N}}' \cup \{\tilde{\beta}_2'\})$ to $\mathrm{SL}_n(\mathbb{C})$. Set ρ' to be the restriction of $\rho'' \circ \mathrm{pr}_*$ on $\pi_1(UM', \tilde{\mathcal{N}}')$. Then it is easy to show we have $\rho' \in \tilde{\chi}_n(M', \mathcal{N}')$ and $\nu^*(\rho') = \rho$.

□

Theorem 2.2.12. *Let (M, \mathcal{N}) be any marked 3-manifold, D be any properly embedded disk with an embedded oriented open interval $\beta \in D$. Then we have*

$$\Phi^{(M', \mathcal{N}')} \circ \Theta_{(D, \beta)} = \nu \circ \Phi^{(M, \mathcal{N})}.$$

Proof. We can assume M is connected. Note that M' may not be connected.

Since both $\Phi^{(M', \mathcal{N}')} \circ \Theta_{(D, \beta)}$ and $\nu \circ \Phi^{(M, \mathcal{N})}$ are algebra homomorphisms, it suffices to show $\Phi^{(M', \mathcal{N}')}(\Theta_{(D, \beta)}(\alpha)) = \nu(\Phi^{(M, \mathcal{N})}(\alpha))$ for any framed oriented knot or stated framed oriented boundary arc α . If there is no intersection between α and D , it is obvious. Then we look at the case when α intersects D . We isotope α such that α is transverse to D , $\alpha \cap D \subset \beta$, and the framing at each point of $\alpha \cap \beta$ is the velocity vector of β .

If α is a stated framed oriented boundary arc with $s(\alpha(0)) = i$ and $s(\alpha(1)) = j$. Then $\alpha = \alpha_m * \alpha_{m-1} * \cdots * \alpha_1$ where all α_i are framed oriented arcs and are parts of α such that each α_t has two ends on β for $2 \leq t \leq m-1$ and $\alpha_1(1), \alpha_m(0) \in \beta$ and the interior of each α_t has no intersection with D . Let $\alpha'_t = \mathrm{pr}^{-1}(\alpha_t)$, $1 \leq t \leq m$, then

$$\Theta_{(D, \beta)}(\alpha) = \sum_{1 \leq k_1, \dots, k_{m-1} \leq n} (\alpha'_m)_{j, k_{m-1}} (\alpha'_{m-1})_{k_{m-1}, k_{m-2}} \cdots (\alpha'_1)_{k_1, i}.$$

For any $\rho' \in \tilde{\chi}_n(M', \mathcal{N}')$, we have

$$\begin{aligned} \Phi^{(M', \mathcal{N}')}(\Theta_{(D, \beta)}(\alpha))(\rho') &= \sum_{1 \leq k_1, \dots, k_{m-1} \leq n} (A\rho'(\widetilde{\alpha'_m}))_{\bar{j}, \bar{k}_{m-1}} (A\rho'(\widetilde{\alpha'_{m-1}}))_{\bar{k}_{m-1}, \bar{k}_{m-2}} \cdots (A\rho'(\widetilde{\alpha'_1}))_{\bar{k}_1, \bar{i}} \\ &= (A\rho'(\widetilde{\alpha'_m}) A\rho'(\widetilde{\alpha'_{m-1}}) \cdots A\rho'(\widetilde{\alpha'_1}))_{\bar{j}, \bar{i}} \end{aligned}$$

and

$$\nu(\Phi^{(M, \mathcal{N})}(\alpha))(\rho') = \Phi^{(M, \mathcal{N})}(\alpha)(\nu^*(\rho')) = (A\nu^*(\rho')(\tilde{\alpha}))_{\bar{j}, \bar{i}}.$$

According to the definition of ν^* , we know

$$\nu^*(\rho')(\tilde{\alpha}) = \rho''(\tilde{\alpha}'_m) \rho''(\tilde{\alpha}'_{m-1}) \cdots \rho''(\tilde{\alpha}'_1).$$

It is easy to see

$$\rho''(\tilde{\alpha}'_m)\rho''(\tilde{\alpha}'_{m-1})\cdots\rho''(\tilde{\alpha}'_1) = \rho''(\widetilde{\alpha}'_m)\rho''(\widetilde{\gamma}_{m-1})\rho''(\widetilde{\alpha}'_{m-1})\rho''(\widetilde{\gamma}_{m-2})\cdots\rho''(\gamma_1)\rho''(\widetilde{\alpha}'_1)$$

where $\gamma_t = a_\beta^{-1}$ or b_β^{-1} for $1 \leq t \leq m-1$. Since $\rho''(a_\beta^{-1}) = \rho''(b_\beta^{-1}) = A$, we have

$$\begin{aligned} \nu^*(\rho')(\tilde{\alpha}) &= \rho''(\widetilde{\alpha}'_m)\rho''(\widetilde{\gamma}_{m-1})\rho''(\widetilde{\alpha}'_{m-1})\rho''(\widetilde{\gamma}_{m-2})\cdots\rho''(\gamma_1)\rho''(\widetilde{\alpha}'_1) \\ &= \rho'(\widetilde{\alpha}'_m)A\rho'(\widetilde{\alpha}'_{m-1})A\cdots A\rho'(\widetilde{\alpha}'_1). \end{aligned}$$

Thus we get

$$\nu(\Phi^{(M,\mathcal{N})}(\alpha))(\rho') = (A\rho'(\widetilde{\alpha}'_m)A\rho'(\widetilde{\alpha}'_{m-1})\cdots A\rho'(\widetilde{\alpha}'_1))_{\bar{j},\bar{i}} = \Phi^{(M,\mathcal{N}')}(\Theta_{(D,\beta)}(\alpha))(\rho').$$

If α is a framed knot. Let η be a path in UM connecting the base point of $\pi_1(UM)$ to $\tilde{\alpha}$ when $\mathcal{N} = \emptyset$ or a path connecting $\tilde{\mathcal{N}}$ to $\tilde{\alpha}$ when $\mathcal{N} \neq \emptyset$ such that $\eta \cap \tilde{D} = \emptyset$. Similarly suppose $\alpha = \alpha_m * \alpha_{m-1} * \cdots * \alpha_1$ where all α_i are framed oriented arcs and parts of α such that each α_t has two ends on β and does not intersect with D on its interior. Let $\alpha'_t = \text{pr}^{-1}(\alpha_t)$, $1 \leq t \leq m$, then

$$\Theta_{(D,\beta)}(\alpha) = \sum_{1 \leq i, k_1, \dots, k_{m-1} \leq n} (\alpha'_m)_{i, k_{m-1}} (\alpha'_{m-1})_{k_{m-1}, k_{m-2}} \cdots (\alpha'_1)_{k_1, i}.$$

For any $\rho' \in \tilde{\chi}_n(M', \mathcal{N}')$, we have

$$\begin{aligned} \Phi^{(M',\mathcal{N}')}(\Theta_{(D,\beta)}(\alpha))(\rho') &= \sum_{1 \leq i, k_1, \dots, k_{m-1} \leq n} (A\rho'(\widetilde{\alpha}'_m))_{\bar{i}, \overline{k_{m-1}}} (A\rho'(\widetilde{\alpha}'_{m-1}))_{\overline{k_{m-1}}, \overline{k_{m-2}}} \cdots (A\rho'(\widetilde{\alpha}'_1))_{\overline{k_1}, \bar{i}} \\ &= \text{tr}(A\rho'(\widetilde{\alpha}'_m)A\rho'(\widetilde{\alpha}'_{m-1})\cdots A\rho'(\widetilde{\alpha}'_1)) \end{aligned}$$

and

$$\nu(\Phi^{(M,\mathcal{N})}(\alpha))(\rho') = \Phi^{(M,\mathcal{N})}(\alpha)(\nu^*(\rho')) = \text{tr}(\nu^*(\rho')(\eta^{-1} * \tilde{\alpha} * \eta)).$$

We assume $\eta(1) \in \widetilde{\alpha}'_1$, otherwise we can relabel α_i to make this happen, and $\eta(1)$ divides $\widetilde{\alpha}'_1$ into two parts $\widetilde{\alpha}'_1, \widetilde{\alpha}''_1$ such that $\widetilde{\alpha}'_1 = \widetilde{\alpha}'_1 * \widetilde{\alpha}''_1$. Using the same technique as α being an arc, we get

$$\nu^*(\rho')(\eta^{-1} * \tilde{\alpha} * \eta) = \rho''(\eta^{-1} * \widetilde{\alpha}'_1'')A\rho''(\widetilde{\alpha}'_m)A\rho''(\widetilde{\alpha}'_{m-1})A\cdots A\rho''(\widetilde{\alpha}'_1' * \eta).$$

Then we have

$$\begin{aligned}
\nu(\Phi^{(M,\mathcal{N})}(\alpha))(\rho') &= \text{tr}(\rho''(\eta^{-1} * \widetilde{\alpha}_1'') A \rho''(\widetilde{\alpha}_m') A \rho''(\widetilde{\alpha}_{m-1}') A \dots A \rho''(\widetilde{\alpha}_1' * \eta)) \\
&= \text{tr}(A \rho''(\widetilde{\alpha}_m') A \rho''(\widetilde{\alpha}_{m-1}') A \dots A \rho''(\widetilde{\alpha}_1' * \eta) \rho''(\eta^{-1} * \widetilde{\alpha}_1'')) \\
&= \text{tr}(A \rho''(\widetilde{\alpha}_m') A \rho''(\widetilde{\alpha}_{m-1}') A \dots A \rho''(\widetilde{\alpha}_1' * \widetilde{\alpha}_1'')) \\
&= \text{tr}(A \rho''(\widetilde{\alpha}_m') A \rho''(\widetilde{\alpha}_{m-1}') A \dots A \rho''(\widetilde{\alpha}_1')) \\
&= \text{tr}(A \rho'(\widetilde{\alpha}_m') A \rho'(\widetilde{\alpha}_{m-1}') A \dots A \rho'(\widetilde{\alpha}_1')) \\
&= \Phi^{(M',\mathcal{N}')}(\Theta_{(D,\beta)}(\alpha))(\rho').
\end{aligned}$$

□

2.3 Proof for Theorem 2.2.9

For any m by k matrix A , $C_t(A), 1 \leq t \leq k$, denotes the t -th column of A , $R_t(A), 1 \leq t \leq m$, denotes the t -th row of A .

To simplify the notation, we will omit the superscript for $\Phi^{(M,\mathcal{N})}$ when there is no confusion for (M,\mathcal{N}) .

2.3.1 The case when $\mathcal{N} = \emptyset$

Sikora proved $\mathcal{S}_n(M; \mathbb{C}, 1) \simeq G_n(M)$ [Sik05]. Lê and Sikora proved $\mathcal{S}_n(M, \emptyset, 1) \simeq \mathcal{S}_n(M; \mathbb{C}, 1)$, which is related to the spin structure h [LS21]. Then it is easy to show the combination of $\mathcal{S}_n(M, \emptyset, 1) \simeq \mathcal{S}_n(M; \mathbb{C}, 1) \simeq G_n(M) \rightarrow R_n(M, \emptyset)$ is Φ , where the third map $G_n(M) \rightarrow R_n(M, \emptyset)$ is \mathcal{Y}_h in Remark 2.2.3. Thus Φ is a well-defined surjective algebra homomorphism. Especially $\text{Ker } \Phi = \sqrt{0}$ since $\text{Ker } \mathcal{Y}_h = \sqrt{0}$.

2.3.2 Independence of how to kill sinks and sources ($\mathcal{N} \neq \emptyset$)

When we try to use relation (2.8) to kill all the sinks and sources, we first drag all the sinks and sources close enough to some component of \mathcal{N} , then use relation (2.8). In this subsection, we want to show Φ is independent of how to kill sinks

and sources, that is, to show Φ is independent of how we drag sinks and sources close to \mathcal{N} .

Let l be a stated n -web. Suppose l' and l'' are obtained from l by killing all the sinks and sources. Note that, for each sink or source of l , we may use different ways to kill this sink or source to get l' and l'' . First assume we kill all the sinks and sources in the same way to obtain l' and l'' except one source or sink, which is denoted as \mathfrak{S} . Let l_1 be obtained from l by first killing all the sinks and sources except \mathfrak{S} using the same way as l' and l'' then eliminating the component containing \mathfrak{S} .

Suppose \mathfrak{S} is a sink. Then

$$\begin{aligned} l' &= \left[\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} (\eta_n * \alpha_n)_{\sigma(n), u_n} (\eta_{n-1} * \alpha_{n-1})_{\sigma(n-1), u_{n-1}} \cdots (\eta_1 * \alpha_1)_{\sigma(1), u_1} \right] l_1 \\ l'' &= \left[\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} (\gamma_n * \alpha_n)_{\sigma(n), u_n} (\gamma_{n-1} * \alpha_{n-1})_{\sigma(n-1), u_{n-1}} \cdots (\gamma_1 * \alpha_1)_{\sigma(1), u_1} \right] l_1 \end{aligned}$$

where $\alpha_t, \eta_t, \gamma_t, 1 \leq t \leq n$, are framed oriented arcs such that $\eta_t * \alpha_t, \gamma_t * \alpha_t, 1 \leq t \leq n$, are well-defined framed oriented boundary arcs in (M, \mathcal{N}) . Note that $\eta_t(1), 1 \leq t \leq n$, belong to a same component of \mathcal{N} (the same with $\gamma_t(1)$), and $\eta_t, 1 \leq t \leq n$ are isotopic to each other (the same with γ_t).

For any element $\rho \in \tilde{\chi}_n(M, \mathcal{N})$, we have

$$\begin{aligned} & \left(\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \text{tr}_{(\eta_n * \alpha_n)_{\sigma(n), u_n}} \text{tr}_{(\eta_{n-1} * \alpha_{n-1})_{\sigma(n-1), u_{n-1}}} \cdots \text{tr}_{(\eta_1 * \alpha_1)_{\sigma(1), u_1}} \right) (\rho) \\ &= \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} [A\rho(\widetilde{\eta_n * \alpha_n})]_{\sigma(n), \bar{u}_n} [A\rho(\widetilde{\eta_{n-1} * \alpha_{n-1}})]_{\sigma(n-1), \bar{u}_{n-1}} \cdots [A\rho(\widetilde{\eta_1 * \alpha_1})]_{\sigma(1), \bar{u}_1} \\ &= (-1)^{\frac{n(n-1)}{2}} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} [A\rho(\widetilde{\eta_n * \alpha_n})]_{\sigma(n), \bar{u}_n} [A\rho(\widetilde{\eta_{n-1} * \alpha_{n-1}})]_{\sigma(n-1), \bar{u}_{n-1}} \cdots [A\rho(\widetilde{\eta_1 * \alpha_1})]_{\sigma(1), \bar{u}_1} \\ &= (-1)^{\frac{n(n-1)}{2}} \det \left(C_{\bar{u}_1}(A\rho(\widetilde{\eta_1 * \alpha_1})) \quad \cdots \quad C_{\bar{u}_{n-1}}(A\rho(\widetilde{\eta_{n-1} * \alpha_{n-1}})) \quad C_{\bar{u}_n}(A\rho(\widetilde{\eta_n * \alpha_n})) \right) \\ &= (-1)^{\frac{n(n-1)}{2}} \det(A) \det \left(C_{\bar{u}_1}(\rho(\widetilde{\eta_1 * \alpha_1})) \quad \cdots \quad C_{\bar{u}_{n-1}}(\rho(\widetilde{\eta_{n-1} * \alpha_{n-1}})) \quad C_{\bar{u}_n}(\rho(\widetilde{\eta_n * \alpha_n})) \right) \\ &= (-1)^{\frac{n(n-1)}{2}} \det \left(C_{\bar{u}_1}(\rho(\widetilde{\eta_1 * \alpha_1})) \quad \cdots \quad C_{\bar{u}_{n-1}}(\rho(\widetilde{\eta_{n-1} * \alpha_{n-1}})) \quad C_{\bar{u}_n}(\rho(\widetilde{\eta_n * \alpha_n})) \right). \end{aligned}$$

Similarly we have

$$\begin{aligned} & \left(\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \text{tr}_{(\gamma_n * \alpha_n)_{\sigma(n), u_n}} \text{tr}_{(\gamma_{n-1} * \alpha_{n-1})_{\sigma(n-1), u_{n-1}}} \cdots \text{tr}_{(\gamma_1 * \alpha_1)_{\sigma(1), u_1}} \right) (\rho) \\ &= (-1)^{\frac{n(n-1)}{2}} \det \left(C_{\bar{u}_1}(\rho(\widetilde{\gamma_1 * \alpha_1})) \quad \cdots \quad C_{\bar{u}_{n-1}}(\rho(\widetilde{\gamma_{n-1} * \alpha_{n-1}})) \quad C_{\bar{u}_n}(\rho(\widetilde{\gamma_n * \alpha_n})) \right) \end{aligned}$$

$$\begin{aligned}
&= (-1)^{\frac{n(n-1)}{2}} \det \left(C_{\overline{u_1}}(\rho(\tilde{\gamma}_1 * \tilde{\eta}_1^{-1})\rho(\tilde{\eta}_1 * \tilde{\alpha}_1)) \quad \dots \quad C_{\overline{u_n}}(\rho(\tilde{\gamma}_n * \tilde{\eta}_n^{-1})\rho(\tilde{\eta}_n * \tilde{\alpha}_n)) \right) \\
&= (-1)^{\frac{n(n-1)}{2}} \det \left(C_{\overline{u_1}}(\rho(\tilde{\eta}_1 * \tilde{\alpha}_1)) \quad \dots \quad C_{\overline{u_{n-1}}}(\rho(\tilde{\eta}_{n-1} * \tilde{\alpha}_{n-1})) \quad C_{\overline{u_n}}(\rho(\tilde{\eta}_n * \tilde{\alpha}_n)) \right).
\end{aligned}$$

The last equality is because $\rho(\tilde{\gamma}_1 * \tilde{\eta}_1^{-1}) = \dots = \rho(\tilde{\gamma}_n * \tilde{\eta}_n^{-1}) \in \mathrm{SL}_n(\mathbb{C})$.

Suppose \mathfrak{S} is a source. Similarly we have

$$\begin{aligned}
l' &= \left[\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} (\beta_n * \epsilon_n)_{v_n, \sigma(n)} (\beta_{n-1} * \epsilon_{n-1})_{v_{n-1}, \sigma(n-1)} \dots (\beta_1 * \epsilon_1)_{v_1, \sigma(1)} \right] l_1 \\
l'' &= \left[\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} (\beta_n * \zeta_n)_{v_n, \sigma(n)} (\beta_{n-1} * \zeta_{n-1})_{v_{n-1}, \sigma(n-1)} \dots (\beta_1 * \zeta_1)_{v_1, \sigma(1)} \right] l_1
\end{aligned}$$

where $\beta_t, \epsilon_t, \zeta_t, 1 \leq t \leq n$, are framed oriented arcs such that $\beta_t * \epsilon_t, \beta_t * \zeta_t, 1 \leq t \leq n$, are well-defined framed oriented boundary arcs in (M, \mathcal{N}) .

For any element $\rho \in \tilde{\chi}_n(M, \mathcal{N})$, similarly we can get

$$\begin{aligned}
&\left(\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \mathrm{tr}_{(\beta_n * \epsilon_n)_{v_n, \sigma(n)}} \mathrm{tr}_{((\beta_{n-1} * \epsilon_{n-1})_{v_{n-1}, \sigma(n-1)} \dots \mathrm{tr}_{(\beta_1 * \epsilon_1)_{v_1, \sigma(1)}}) (\rho) \right) \\
&= (-1)^{\frac{n(n-1)}{2}} \det \begin{pmatrix} R_{\overline{v_1}}(A\rho(\tilde{\beta}_1 * \tilde{\epsilon}_1)) \\ \vdots \\ R_{\overline{v_{n-1}}}(\rho(\tilde{\beta}_{n-1} * \tilde{\epsilon}_{n-1})) \\ R_{\overline{v_n}}(\rho(\tilde{\beta}_n * \tilde{\epsilon}_n)) \end{pmatrix}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\left(\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \mathrm{tr}_{(\beta_n * \zeta_n)_{v_n, \sigma(n)}} \mathrm{tr}_{((\beta_{n-1} * \zeta_{n-1})_{v_{n-1}, \sigma(n-1)} \dots \mathrm{tr}_{(\beta_1 * \zeta_1)_{v_1, \sigma(1)}}) (\rho) \right) \\
&= (-1)^{\frac{n(n-1)}{2}} \det \begin{pmatrix} R_{\overline{v_1}}(A\rho(\tilde{\beta}_1 * \tilde{\zeta}_1)) \\ \vdots \\ R_{\overline{v_{n-1}}}(\rho(\tilde{\beta}_{n-1} * \tilde{\zeta}_{n-1})) \\ R_{\overline{v_n}}(\rho(\tilde{\beta}_n * \tilde{\zeta}_n)) \end{pmatrix} \\
&= (-1)^{\frac{n(n-1)}{2}} \det \begin{pmatrix} R_{\overline{v_1}}(A\rho(\tilde{\beta}_1 * \tilde{\epsilon}_1)\rho(\tilde{\epsilon}_1^{-1} * \tilde{\zeta}_1)) \\ \vdots \\ R_{\overline{v_{n-1}}}(\rho(\tilde{\beta}_{n-1} * \tilde{\epsilon}_{n-1})\rho(\tilde{\epsilon}_{n-1}^{-1} * \tilde{\zeta}_{n-1})) \\ R_{\overline{v_n}}(\rho(\tilde{\beta}_n * \tilde{\epsilon}_n)\rho(\tilde{\epsilon}_n^{-1} * \tilde{\zeta}_n)) \end{pmatrix}
\end{aligned}$$

$$= (-1)^{\frac{n(n-1)}{2}} \det \begin{pmatrix} R_{v_1}(A\rho(\tilde{\beta}_1 * \tilde{\epsilon}_1)) \\ \vdots \\ R_{v_{n-1}}(\rho(\widetilde{\beta_{n-1}} * \widetilde{\epsilon_{n-1}})) \\ R_{v_n}(\rho(\widetilde{\beta_n} * \widetilde{\epsilon_n})) \end{pmatrix}.$$

Thus we have $\Phi(l') = \Phi(l'')$. In the general case, we have a sequence $l' = l^{(1)}, l^{(2)}, \dots, l^{(k)} = l''$ such that each $l^{(t)}$ (where $1 \leq t \leq k$) is obtained from l by eliminating all sources and sinks using relation (2.8). Additionally, each $l^{(t)}$ and $l^{(t+1)}$ (where $1 \leq t \leq k-1$) are obtained from l using the same method to eliminate all but one sink or source. Then $\Phi(l') = \Phi(l^{(1)}) = \Phi(l^{(2)}) = \dots = \Phi(l^{(k)}) = \Phi(l'')$. Then Φ is well-defined on the set of framed n -webs.

Suppose the stated n -webs l_1 and l_2 are isotopic. From the definition of Φ , we first use relation (2.8) to kill all the sinks and sources to obtain $(l_1)'$. According to the isotopy between l_1 and l_2 and how we kill sinks and sources in l_1 , we can pick a way to kill all the sinks and sources in l_2 to obtain $(l_2)'$ such that $(l_1)'$ and $(l_2)'$ are homotopic relative to \mathcal{N} . Thus $\Phi(l_1) = \Phi(l_2)$, that is, Φ is well-defined on the set of isotopy classes of stated n -webs.

2.3.3 Checking for relations ($\mathcal{N} \neq \emptyset$)

From the definition of Φ , we know Φ respects relations (2.4), (2.8) and (2.11). Since, for any $\rho \in \tilde{\chi}_n(M, \mathcal{N})$, $\rho(\vartheta) = d_n I$, then Φ respects relations (2.5) and (2.6).

We use l' (respectively l'') to denote the stated n -webs on the left (respectively right) hand side of "=" in relations (2.7), (2.9) and (2.10).

Relation (2.7): From the definition of Φ and Subsection 2.3.2, we can suppose the parts outside of the box are $2n$ framed oriented arcs connecting to the box. We label the framed oriented arcs connecting to the box on the left edge as $\alpha_n, \dots, \alpha_2, \alpha_1$ from top to bottom, label the framed oriented arcs connecting to the box on the right edge as $\beta_n, \dots, \beta_2, \beta_1$ from top to bottom. Suppose $s(\alpha_t(0)) = u_t$, $s(\beta_t(1)) = v_t$, $1 \leq t \leq n$. To kill the sink and source in l' using relation (2.8), we use the same path to drag them close to \mathcal{N} .

For $\rho \in \tilde{\chi}_n(M, \mathcal{N})$, we have

$$\begin{aligned}
\Phi(l')(\rho) &= \det \begin{pmatrix} R_{\overline{v_n}}(A\rho(\widetilde{\beta}_n * \widetilde{\epsilon}_n)) \\ \vdots \\ R_{\overline{v_1}}(A\rho(\widetilde{\beta}_1 * \widetilde{\epsilon}_1)) \end{pmatrix} \det \left(C_{\overline{u_1}}(\rho(\widetilde{\eta}_1 * \widetilde{\alpha}_1)) \quad \dots \quad C_{\overline{u_n}}(\rho(\widetilde{\eta}_n * \widetilde{\alpha}_n)) \right) \\
&= \det \begin{pmatrix} R_{\overline{v_n}}(A\rho(\widetilde{\beta}_n * \widetilde{\epsilon}_n))C_{\overline{u_1}}(\rho(\widetilde{\eta}_1 * \widetilde{\alpha}_1)) & \dots & R_{\overline{v_n}}(A\rho(\widetilde{\beta}_n * \widetilde{\epsilon}_n))C_{\overline{u_n}}(\rho(\widetilde{\eta}_n * \widetilde{\alpha}_n)) \\ \vdots & & \vdots \\ R_{\overline{v_1}}(A\rho(\widetilde{\beta}_1 * \widetilde{\epsilon}_1))C_{\overline{u_1}}(\rho(\widetilde{\eta}_1 * \widetilde{\alpha}_1)) & \dots & R_{\overline{v_1}}(A\rho(\widetilde{\beta}_1 * \widetilde{\epsilon}_1))C_{\overline{u_n}}(\rho(\widetilde{\eta}_n * \widetilde{\alpha}_n)) \end{pmatrix} \\
&= \det \begin{pmatrix} [A\rho(\widetilde{\beta}_n * \widetilde{\epsilon}_n)\rho(\widetilde{\eta}_1 * \widetilde{\alpha}_1)]_{\overline{v_n}, \overline{u_1}} & \dots & [A\rho(\widetilde{\beta}_n * \widetilde{\epsilon}_n)\rho(\widetilde{\eta}_n * \widetilde{\alpha}_n)]_{\overline{v_n}, \overline{u_n}} \\ \vdots & & \vdots \\ [A\rho(\widetilde{\beta}_1 * \widetilde{\epsilon}_1)\rho(\widetilde{\eta}_1 * \widetilde{\alpha}_1)]_{\overline{v_1}, \overline{u_1}} & \dots & [A\rho(\widetilde{\beta}_1 * \widetilde{\epsilon}_1)\rho(\widetilde{\eta}_n * \widetilde{\alpha}_n)]_{\overline{v_1}, \overline{u_n}} \end{pmatrix} \\
&= (-1)^{\frac{n(n-1)}{2}} \det \begin{pmatrix} [A\rho(\widetilde{\beta}_1 * \widetilde{\epsilon}_1 * \widetilde{\eta}_1 * \widetilde{\alpha}_1)]_{\overline{v_1}, \overline{u_1}} & \dots & [A\rho(\widetilde{\beta}_1 * \widetilde{\epsilon}_1 * \widetilde{\eta}_n * \widetilde{\alpha}_n)]_{\overline{v_1}, \overline{u_n}} \\ \vdots & & \vdots \\ [A\rho(\widetilde{\beta}_n * \widetilde{\epsilon}_n * \widetilde{\eta}_1 * \widetilde{\alpha}_1)]_{\overline{v_n}, \overline{u_1}} & \dots & [A\rho(\widetilde{\beta}_n * \widetilde{\epsilon}_n * \widetilde{\eta}_n * \widetilde{\alpha}_n)]_{\overline{v_n}, \overline{u_n}} \end{pmatrix}
\end{aligned}$$

For each pair $1 \leq i, j \leq n$, we use $a_{i,j}$ to denote the oriented straight line in the shaded box (its framing is the one pointing to readers) connecting $\alpha_j(1)$ and $\beta_i(0)$ such that $\beta_i * a_{i,j} * \alpha_j$ is a well-defined stated framed oriented boundary arc. Because we use the same path to drag the source and the sink, then

$$\rho(\widetilde{\beta}_i * \widetilde{\epsilon}_i * \widetilde{\eta}_j * \widetilde{\alpha}_j) = d_n \rho(\widetilde{\beta}_i * \widetilde{a}_{i,j} * \widetilde{\alpha}_j)$$

for all $1 \leq i, j \leq n$, or

$$\rho(\widetilde{\beta}_i * \widetilde{\epsilon}_i * \widetilde{\eta}_j * \widetilde{\alpha}_j) = \rho(\widetilde{\beta}_i * \widetilde{a}_{i,j} * \widetilde{\alpha}_j)$$

for all $1 \leq i, j \leq n$.

For any $\rho \in \tilde{\chi}_n(M, \mathcal{N})$, since $(d_n)^n = 1$, we have

$$\begin{aligned}
&\Phi(l'')(\rho) \\
&= (-1)^{\frac{n(n-1)}{2}} \sum_{\sigma \in S_n} (-1)^{l(\sigma)} [A\rho(\widetilde{\beta}_{\sigma(1)} * \widetilde{\epsilon}_{\sigma(1)} * \widetilde{\eta}_1 * \widetilde{\alpha}_1)]_{\overline{v_{\sigma(1)}}, \overline{u_1}} \dots [A\rho(\widetilde{\beta}_{\sigma(n)} * \widetilde{\epsilon}_{\sigma(n)} * \widetilde{\eta}_n * \widetilde{\alpha}_n)]_{\overline{v_{\sigma(n)}}, \overline{u_n}}
\end{aligned}$$

$$= (-1)^{\frac{n(n-1)}{2}} \det \begin{pmatrix} [A\rho(\tilde{\beta}_1 * \tilde{\epsilon}_1 * \tilde{\eta}_1 * \tilde{\alpha}_1)]_{\overline{v_1}, \overline{u_1}} & \cdots & [A\rho(\tilde{\beta}_1 * \tilde{\epsilon}_1 * \tilde{\eta}_n * \tilde{\alpha}_n)]_{\overline{v_1}, \overline{u_n}} \\ \vdots & & \vdots \\ [A\rho(\tilde{\beta}_n * \tilde{\epsilon}_n * \tilde{\eta}_1 * \tilde{\alpha}_1)]_{\overline{v_n}, \overline{u_1}} & \cdots & [A\rho(\tilde{\beta}_n * \tilde{\epsilon}_n * \tilde{\eta}_n * \tilde{\alpha}_n)]_{\overline{v_n}, \overline{u_n}} \end{pmatrix}$$

Thus $\Phi(l') = \Phi(l'')$.

Relation (2.9): We need to show

$$\Phi\left(\begin{array}{c} \downarrow \\ \curvearrowright^i \\ \downarrow \\ j \end{array}\right) = \Phi\left(\begin{array}{c} \downarrow \\ \curvearrowleft^i \\ \downarrow \\ j \end{array}\right) = \delta_{j,i} (-1)^{n-i}.$$

We use α_1 to denote $\begin{array}{c} \downarrow \\ \curvearrowright^i \\ \downarrow \\ j \end{array}$ and α_2 to denote $\begin{array}{c} \downarrow \\ \curvearrowleft^i \\ \downarrow \\ j \end{array}$, we have

$$\Phi(\alpha_1)(\rho) = [A\rho(\tilde{\alpha}_1)]_{\overline{j}, \overline{i}} = d_n A_{\overline{j}, \overline{i}} = \delta_{\overline{j}, \overline{i}} (-1)^{n-i}, \Phi(\alpha_2)(\rho) = [A\rho(\tilde{\alpha}_2)]_{\overline{i}, \overline{j}} = A_{\overline{i}, \overline{j}} = \delta_{\overline{j}, \overline{i}} (-1)^{n-i}.$$

Relation (2.10): We only prove the case when the white dot represents an arrow going from right to left. From the definition of Φ and Subsection 2.3.2, we only have two cases to consider: (1) the left hand side of "=" is a knot, (2) the left hand side of "=" is an arc.

When the left hand side of "=" is a knot, the right hand side of "=" is a framed oriented boundary arc, which is denoted as α . Then for $\rho \in \tilde{\chi}_n(M, \mathcal{N})$, we have

$$\Phi(l'')(\rho) = \sum_{1 \leq i \leq n} (-1)^{i+1} [A\rho(\tilde{\alpha})]_{\overline{i}, \overline{i}} = \sum_{1 \leq i \leq n} [\rho(\tilde{\alpha})]_{\overline{i}, \overline{i}} = \text{tr}(\rho(\tilde{\alpha})) = \Phi(l')(\rho).$$

When the left hand side of "=" is an arc, the right hand side of "=" consists two framed oriented boundary arcs, which are denoted as γ_2 and γ_1 such that γ_2 is above γ_1 is the box. Suppose $s(\gamma_1(0)) = v$, $s(\gamma_2(1)) = u$. Then for $\rho \in \tilde{\chi}_n(M, \mathcal{N})$, we have

$$\begin{aligned} \Phi(l'')(\rho) &= \sum_{1 \leq i \leq n} (-1)^{i+1} [A\rho(\tilde{\gamma}_2)]_{\overline{u}, \overline{i}} [A\rho(\tilde{\gamma}_1)]_{\overline{i}, \overline{v}} = \sum_{1 \leq i \leq n} [A\rho(\tilde{\gamma}_2)]_{\overline{u}, \overline{i}} [\rho(\tilde{\gamma}_1)]_{\overline{i}, \overline{v}} \\ &= [A\rho(\tilde{\gamma}_2)\rho(\tilde{\gamma}_1)]_{\overline{u}, \overline{v}} = [A\rho(\tilde{\gamma}_2 * \tilde{\gamma}_1)]_{\overline{u}, \overline{v}} = \Phi(l')(\rho). \end{aligned}$$

2.3.4 Algebra homomorphism and surjectivity

The definition of Φ implies it is an algebra homomorphism.

When \mathcal{N} is empty, we already proved Φ is surjective. Assume $\mathcal{N} \neq \emptyset$. For any element $[\alpha] \in \pi_1^{Mor}(M, \mathcal{N})$, we choose a representative α for $[\alpha]$ such that α has no self-intersection and only intersects ∂M at its endpoints. Then we can give a framing for α to make α a framed oriented boundary arc for (M, \mathcal{N}) . Then $R_n(M, \mathcal{N})$ is generated by $\Phi(\alpha_{j,i}), 1 \leq j, i \leq n, [\alpha] \in \pi_1^{Mor}(M, \mathcal{N})$, as an algebra. Thus Φ is surjective.

In Section 2.4, we will give a unique way to lift $[\alpha]$ to a framed oriented boundary arc.

2.4 Classical limit and Ker Φ

In this section we try to understand the classical limit of the stated SL_n -skein module of the marked 3-manifold. Then we will use the classical limit to show the kernel of Φ is $\sqrt{0}$. Using Lemma 2.4.2, we can reduce the general marked 3-manifold to the connected marked 3-manifold.

Lemma 2.4.1 ([Bly18; Prz98]). *Suppose $0 \rightarrow A_1 \rightarrow B_1 \rightarrow C_1 \rightarrow 0$, and $0 \rightarrow A_2 \rightarrow B_2 \rightarrow C_2 \rightarrow 0$ are two short exact sequences, then*

$$0 \rightarrow A_1 \otimes B_2 + B_1 \otimes A_2 \rightarrow B_1 \otimes B_2 \rightarrow C_1 \otimes C_2 \rightarrow 0$$

is an exact sequence. All A_i, B_i, C_i are vector spaces over \mathbb{C} and all the involved maps are linear maps.

Lemma 2.4.2. *Suppose (M, \mathcal{N}) is the disjoint union of (M_1, \mathcal{N}_1) and (M_2, \mathcal{N}_2) . If $\text{Ker } \Phi^{(M_i, \mathcal{N}_i)} = \sqrt{0}_{\mathcal{S}_n(M_i, \mathcal{N}_i, 1)}$ for $i = 1, 2$, then $\text{Ker } \Phi^{(M, \mathcal{N})} = \sqrt{0}_{\mathcal{S}_n(M, \mathcal{N}, 1)}$.*

Proof. Since $R_n(M, \mathcal{N})$ contains no nonzero nilpotents, we have $\sqrt{0}_{\mathcal{S}_n(M, \mathcal{N}, 1)} \subset \text{Ker } \Phi^{(M, \mathcal{N})}$. We know $\mathcal{S}_n(M, \mathcal{N}, 1) = \mathcal{S}_n(M_1, \mathcal{N}_1, 1) \otimes \mathcal{S}_n(M_2, \mathcal{N}_2, 1)$, $R_n(M, \mathcal{N}) = R_n(M_1, \mathcal{N}_1) \otimes R_n(M_2, \mathcal{N}_2)$. From the assumption, we have the following two exact sequences:

$$0 \rightarrow \sqrt{0}_{\mathcal{S}_n(M_i, \mathcal{N}_i, 1)} \rightarrow \mathcal{S}_n(M_i, \mathcal{N}_i, 1) \rightarrow R_n(M_i, \mathcal{N}_i) \rightarrow 0$$

for $i = 1, 2$. Then Lemma 2.4.1 implies the following exact sequence:

$$0 \rightarrow \sqrt{0}_{\mathcal{S}_n(M_1, \mathcal{N}_1, 1)} \otimes \mathcal{S}_n(M_2, \mathcal{N}_2, 1) + \mathcal{S}_n(M_1, \mathcal{N}_1, 1) \otimes \sqrt{0}_{\mathcal{S}_n(M_2, \mathcal{N}_2, 1)} \rightarrow \\ \mathcal{S}_n(M_1, \mathcal{N}_1, 1) \otimes \mathcal{S}_n(M_2, \mathcal{N}_2, 1) \rightarrow R_n(M_1, \mathcal{N}_1) \otimes R_n(M_2, \mathcal{N}_2) \rightarrow 0.$$

Thus we have

$$\text{Ker}\Phi^{(M, \mathcal{N})} = \sqrt{0}_{\mathcal{S}_n(M_1, \mathcal{N}_1, 1)} \otimes \mathcal{S}_n(M_2, \mathcal{N}_2, 1) + \mathcal{S}_n(M_1, \mathcal{N}_1, 1) \otimes \sqrt{0}_{\mathcal{S}_n(M_2, \mathcal{N}_2, 1)} \subset \sqrt{0}_{\mathcal{S}_n(M, \mathcal{N}, 1)}.$$

Then

$$\text{Ker}\Phi^{(M, \mathcal{N})} = \sqrt{0}_{\mathcal{S}_n(M_1, \mathcal{N}_1, 1)} \otimes \mathcal{S}_n(M_2, \mathcal{N}_2, 1) + \mathcal{S}_n(M_1, \mathcal{N}_1, 1) \otimes \sqrt{0}_{\mathcal{S}_n(M_2, \mathcal{N}_2, 1)} = \sqrt{0}_{\mathcal{S}_n(M, \mathcal{N}, 1)}.$$

□

In the remaining of this section, we will assume all the marked 3-manifolds involved are connected. We also fix a relative spin structure h for (M, \mathcal{N}) . For any (stated) framed oriented boundary arc α in (M, \mathcal{N}) , we consider $[\alpha]$ as an element in $\pi_1^{Mor}(M, \mathcal{N})$ by forgetting the framing of α .

rem A morphism $[\alpha] \in \pi_1^{Mor}(M, \mathcal{N})$ and two integers $1 \leq i, j \leq n$ uniquely determine an element in $\mathcal{S}_n(M, \mathcal{N}, 1)$ in the following way: We choose a good representative α such that α is a properly embedded arc in M . Then we give a framing to α respecting \mathcal{N} , that is, the framing at endpoints are given by the velocity vectors of \mathcal{N} . We denote this framed oriented boundary arc as $\hat{\alpha}$. We choose the framing such that $h(\hat{\alpha}) = 0$, then we obtain an element $\hat{\alpha}_{i,j} \in \mathcal{S}_n(M, \mathcal{N}, 1)$. Suppose we choose a different good representative α' . We have $[\alpha] = [\alpha'] \in \pi_1^{Mor}(M, \mathcal{N})$ and $h(\hat{\alpha}') = h(\hat{\alpha}) = 0$. Then $\hat{\alpha}'_{i,j} = \hat{\alpha}_{i,j}$ because of relations (2.4), (2.5), (2.11) and Corollary 2.2.2 (here we use a standard fact that two embeddings of a compact graph in M are homotopic if and only if one can be obtained from the other by crossing changes, height changes, and isotopies [PS00]).

We use $S_{i,j}^{[\alpha]}$ to denote $\hat{\alpha}_{i,j}$, and use $S^{[\alpha]}$ to denote an n by n matrix in $\mathcal{S}_n(M, \mathcal{N}, 1)$ such that $(S^{[\alpha]})_{i,j} = S_{i,j}^{[\alpha]}$, $1 \leq i, j \leq n$.

For any two stated oriented framed boundary arcs α_1, α_2 , suppose $s(\alpha_1(0)) = s(\alpha_2(0))$ and $s(\alpha_1(1)) = s(\alpha_2(1))$. If $h(\hat{\alpha}_1) = h(\hat{\alpha}_2)$ and $[\alpha_1] = [\alpha_2] \in \pi_1^{Mor}(M, \mathcal{N})$,

then $\alpha_1 = \alpha_2 \in \mathcal{S}_n(M, \mathcal{N}, 1)$ because of relations (2.4), (2.5), (2.11) and Corollary 2.2.2.

The for any stated oriented framed boundary arc $\alpha_{i,j}$, we have

$$\alpha_{i,j} = d_n^{h(\tilde{\alpha})} S_{i,j}^{[\alpha]} \in \mathcal{S}_n(M, \mathcal{N}, 1). \quad (2.18)$$

Proposition 2.4.3. (a) For any two morphisms $[\alpha], [\beta] \in \pi_1^{Mor}(M, \mathcal{N})$, if $[\beta][\alpha]$ makes sense, then $AS^{[\beta*\alpha]} = AS^{[\beta]}AS^{[\alpha]}$.

(b) For any $[\eta] \in \pi_1^{Mor}(M, \mathcal{N})$, we have $\det(S^{[\eta]}) = 1$. Especially $\det(AS^{[\eta]}) = 1$.

(c) Suppose $[o] \in \pi_1(M, \mathcal{N})$ is the identity morphism for an object, then $S^{[o]} = d_n A$. Especially $AS^{[o]} = I$.

Proof. (a) We have

$$(S^{[\beta]}AS^{[\alpha]})_{i,j} = \sum_{1 \leq k \leq n} (-1)^{k+1} S_{i,k}^{[\beta]} S_{k,j}^{[\alpha]} = S_{i,j}^{[\beta*\alpha]} = (S^{[\beta*\alpha]})_{i,j}$$

where the second equality is because of relation (2.10). Thus $AS^{[\beta*\alpha]} = AS^{[\beta]}AS^{[\alpha]}$.

(b) We have

$$\det(S^{[\eta]}) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} S_{1,\sigma(1)}^{[\eta]} S_{2,\sigma(2)}^{[\eta]} \cdots S_{n,\sigma(n)}^{[\eta]} = \begin{array}{c} \begin{array}{|c} \hline \uparrow 1 \\ \hline \vdots \\ \hline \uparrow n-1 \\ \hline \uparrow n \\ \hline \end{array} \\ \hline \end{array} = 1$$

where the second equality is from relation (2.8) and the last equality is because of equation (54) in [LS21].

(c) For $1 \leq i, j \leq n$, we have

$$S_{i,j}^{[o]} = \begin{array}{c} \begin{array}{|c} \hline \uparrow i \\ \hline \leftarrow \\ \hline \uparrow j \\ \hline \end{array} \\ \hline \end{array} = d_n A_{i,j}.$$

Thus $S^{[o]} = d_n A$.

□

2.4.1 Isomorphism between $\mathcal{S}_n(M, \mathcal{N}, 1)$ and $\Gamma_n(M)$ when \mathcal{N} has one component

In this subsection, \mathcal{N} always contains one component unless specified. If \mathcal{N} has only one component, then $\pi_1^{Mor}(M, \mathcal{N}) = \pi_1(M)$ (we choose the base point for $\pi_1(M)$ to be a point in \mathcal{N}).

Lemma 2.4.4. *Let (M, \mathcal{N}) be a marked 3-manifold with \mathcal{N} consisting of one component. There exists an algebra homomorphism $F : \Gamma_n(M) \rightarrow \mathcal{S}_n(M, \mathcal{N}, 1)$ defined by*

$$F([\alpha]_{i,j}) = F((Q_{[\alpha]})_{i,j}) = (AS^{[\alpha]})_{i,j}$$

where $[\alpha] \in \pi_1^{Mor}(M, \mathcal{N})$, $1 \leq i, j \leq n$.

Proof. Lemma 2.4.3 shows F respects all the relations defined for $\Gamma_n(M)$. Thus F is a well-defined algebra homomorphism. \square

Let α be a framed oriented arc in $\mathcal{S}_n(M, \mathcal{N}, 1)$. Then $[\alpha]$ is an element in $\pi_1^{Mor}(M, \mathcal{N})$ by forgetting the framing of α . We define $G(\alpha_{i,j}) = d_n^{h(\tilde{\alpha})+1}(-1)^{i+1}[\alpha]_{i,j} \in \Gamma_n(M)$. For a framed oriented knot α , first we forget the framing of α and then we use a path β to connect α and \mathcal{N} . Then we obtain an element in $\pi_1^{Mor}(M, \mathcal{N})$, which is denoted as $[\alpha_\beta]$. We define $G(\alpha) = d_n^{h(\tilde{\alpha})} \text{tr}(Q_{[\alpha_\beta]}) \in \Gamma_n(M)$. It is easy to show $G(\alpha)$ is independent of the choice of β .

For any stated n -web l , we use relation (2.8) to kill all the sinks and sources to obtain a new stated n -web l' . Suppose $l' = \cup_\alpha \alpha$ where each α is a stated framed oriented boundary arc or a framed oriented knot, define $G(l) = \prod_\alpha G(\alpha)$.

Lemma 2.4.5. *The above map $G : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \Gamma_n(M)$ is a well-defined algebra homomorphism.*

Proof. We prove G is well-defined in two steps. First we prove the definition of G is independent of the choice of how we kill sinks and sources, then we prove G respects all the relations defined for $\mathcal{S}_n(M, \mathcal{N}, 1)$. Note that these two steps appeared when we tried to prove Theorem 2.2.9. Actually the proving techniques here are the same with the techniques used in Subsections 2.3.2 and 2.3.3. So here we omit all the details. \square

Theorem 2.4.6. *Let (M, \mathcal{N}) be a marked 3-manifold with \mathcal{N} consisting of one component. There exist algebra homomorphisms $F : \Gamma_n(M) \rightarrow \mathcal{S}_n(M, \mathcal{N}, 1)$, $G : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \Gamma_n(M)$ such that $F \circ G = Id_{\mathcal{S}_n(M, \mathcal{N}, 1)}$ and $G \circ F = Id_{\Gamma_n(M)}$. Especially $\Gamma_n(M) \simeq \mathcal{S}_n(M, \mathcal{N}, 1)$.*

Proof. Lemmas 2.4.4 and 2.4.5 show the existence of F and G . It remains to show they are inverse to each other.

For any $[\alpha] \in \pi_1^{Mor}(M, \mathcal{N})$, $1 \leq i, j \leq n$, we have

$$G(F([\alpha]_{i,j})) = G((-1)^{i+1} S_{i,j}^{[\alpha]}) = (-1)^{i+1} d_n (-1)^{\bar{i}+1} [\alpha]_{i,j} = [\alpha]_{i,j}.$$

Thus $G \circ F = Id_{\Gamma_n(M)}$ since $[\alpha]_{i,j}$, $[\alpha] \in \pi_1^{Mor}(M, \mathcal{N})$, $1 \leq i, j \leq n$, generate $\Gamma_n(M)$ as an algebra.

Obviously all the stated framed oriented boundary arcs generate $\mathcal{S}_n(M, \mathcal{N}, 1)$ as an algebra. For any stated oriented framed boundary arc $\alpha_{i,j} \in \mathcal{S}_n(M, \mathcal{N}, 1)$, we have

$$F(G(\alpha_{i,j})) = d_n^{h(\bar{\alpha})+1} (-1)^{i+1} F([\alpha]_{i,j}) = d_n^{h(\bar{\alpha})+1} (-1)^{i+1} (-1)^{\bar{i}+1} S_{i,j}^{[\alpha]} = d_n^{h(\bar{\alpha})} S_{i,j}^{[\alpha]} = \alpha_{i,j}.$$

Thus $F \circ G = Id_{\mathcal{S}_n(M, \mathcal{N}, 1)}$.

□

rem Korinman and Murakami proved the isomorphism between $\mathcal{S}_2(M, \mathcal{N}, 1)$ and $\Gamma_2(M)$ using a different technique [KM22].

2.4.2 Adding one extra marking to marked 3-manifold

In this subsection, we will investigate the effects on $\mathcal{S}_n(M, \mathcal{N}, 1)$ when we put one extra marking on ∂M .

Definition 2.4.7. *Let (M, \mathcal{N}) be a marked 3-manifold. We say that \mathcal{N}' is obtained from \mathcal{N} by adding one extra marking if $\mathcal{N}' = \mathcal{N} \cup e$ where e is an embedded oriented open interval in ∂M such that $cl(e) \cap cl(\mathcal{N}) = \emptyset$. We call the linear map $\mathcal{S}_n(M, \mathcal{N}, v) \rightarrow \mathcal{S}_n(M, \mathcal{N}', v)$, induced by embedding $(M, \mathcal{N}) \rightarrow (M, \mathcal{N}')$, adding marking map. Obviously this map is an algebra homomorphism when $v = 1$. We*

will use λ_{ad}^e to denote the adding marking map. We can omit the superscript when there is no confusion with marking e .

rem Suppose (M, \mathcal{N}) is a marked 3-manifold with $\mathcal{N} \neq \emptyset$, and \mathcal{N}' is obtained from \mathcal{N} by adding one extra marking e . As in Remark 2.2.3, we can extend the relative spin structure h for (M, \mathcal{N}) to a relative spin structure for (M, \mathcal{N}') , which is still denoted as h .

Let α be an oriented path connecting \mathcal{N} and e . We require α does not intersect itself, α only intersects ∂M at its endpoints, and $\alpha(0)$ belongs to a component $e_1 \subset \mathcal{N}$, and $\alpha(1) \in e$. Then we give a framing to α to obtain a framed oriented boundary arc in (M, \mathcal{N}') , which is still denoted as α , such that $h(\tilde{\alpha}) = 0$. Similarly we give a framing to α^{-1} such that $h(\widetilde{\alpha^{-1}}) = 0$. Then we have $\alpha_{i,j} = S_{i,j}^{[\alpha]}, \alpha_{i,j}^{-1} = S_{i,j}^{[\alpha^{-1}]}, 1 \leq i, j \leq n$. From Proposition 2.4.3, we know $AS^{[\alpha]}AS^{[\alpha^{-1}]} = AS^{[\alpha^{-1}]}AS^{[\alpha]} = AS^{[e]} = I$.

We can regard $\mathcal{S}_n(M, \mathcal{N}', 1)$ as an $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra because of the adding marking map.

Lemma 2.4.8. *Suppose (M, \mathcal{N}) is a marked 3-manifold with $\mathcal{N} \neq \emptyset$, and \mathcal{N}' is obtained from \mathcal{N} by adding one extra marking e . Then as an $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra, $\mathcal{S}_n(M, \mathcal{N}', 1)$ is generated by $\alpha_{i,j}, 1 \leq i, j \leq n$.*

Proof. Let T be the $\mathcal{S}_n(M, \mathcal{N}, 1)$ -subalgebra of $\mathcal{S}_n(M, \mathcal{N}', 1)$ generated by $\alpha_{i,j}, 1 \leq i, j \leq n$. Since $\det(S^{[\alpha]}) = 1 \in \mathcal{S}_n(M, \mathcal{N}', 1)$, we have $(S^{[\alpha]})^{-1}$ is well-defined and $[(S^{[\alpha]})^{-1}]_{i,j}, 1 \leq i, j \leq n$, are polynomials in $\alpha_{i,j}, 1 \leq i, j \leq n$. Especially $[(S^{[\alpha]})^{-1}]_{i,j} \in T, 1 \leq i, j \leq n$. We know $S^{[\alpha^{-1}]} = A^{-1}(S^{[\alpha]})^{-1}A^{-1}$, which implies $\alpha_{i,j}^{-1} = S_{i,j}^{[\alpha^{-1}]} \in T, 1 \leq i, j \leq n$.

From relation (2.8), we know, as an $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra, $\mathcal{S}_n(M, \mathcal{N}', 1)$ is generated by stated framed oriented boundary arcs with at least one end point in e . Suppose $\beta_{i,j}$ is such an arc in $\mathcal{S}_n(M, \mathcal{N}', 1)$. Recall that $\beta_{i,j} = d_n^{h(\tilde{\beta})} S_{i,j}^{[\beta]}$.

For the case when $\beta(0), \beta(1) \in e$, we have

$$AS^{[\beta]} = AS^{[\alpha]}AS^{[\alpha^{-1}*\beta*\alpha]}AS^{[\alpha^{-1}]}$$

where $[\alpha^{-1} * \beta * \alpha]$ is a path with two end points in \mathcal{N} . Especially we get

$$S^{[\beta]} = S^{[\alpha]} AS^{[\alpha^{-1} * \beta * \alpha]} AS^{[\alpha^{-1}]}.$$

Then $S_{i,j}^{[\beta]} \in T$, $1 \leq i, j \leq n$, because $S_{i,j}^{[\alpha]}, S_{i,j}^{[\alpha^{-1} * \beta * \alpha]}, S_{i,j}^{[\alpha^{-1}]} \in T$, $1 \leq i, j \leq n$. Thus $\beta_{i,j} = d_n^{h(\beta)} S_{i,j}^{[\beta]} \in T$.

For the other cases, we can use the same way to show $\beta_{i,j} = d_n^{h(\beta)} S_{i,j}^{[\beta]} \in T$. Thus $T = \mathcal{S}_n(M, \mathcal{N}', 1)$.

□

Recall that

$$\mathcal{O}(\mathrm{SL}_n) = \mathbb{C}[x_{i,j} \mid 1 \leq i, j \leq n] / (\det(X) = 1)$$

where X is an n by n matrix such that $X_{i,j} = x_{i,j}$, $1 \leq i, j \leq n$. We have X^{-1} makes sense and is an n by n matrix in $\mathcal{O}(\mathrm{SL}_n)$ because $\det(X) = 1$. For $1 \leq i, j \leq n$, We use $x_{i,j}^{-1}$ to denote $(X^{-1})_{i,j}$. Obviously $\mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n)$ is an $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra, and as an $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra,

$$\mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n) = \mathcal{S}_n(M, \mathcal{N}, 1)[x_{i,j} \mid 1 \leq i, j \leq n] / (\det(X) = 1)$$

by regarding $1 \otimes x_{i,j}$ as $x_{i,j}$.

Lemma 2.4.9. *Suppose (M, \mathcal{N}) is a marked 3-manifold with $\mathcal{N} \neq \emptyset$, and \mathcal{N}' is obtained from \mathcal{N} by adding one extra marking e . Then there exists an $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra homomorphism*

$$\begin{aligned} \iota : \mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n) &\rightarrow \mathcal{S}_n(M, \mathcal{N}', 1) \\ 1 \otimes x_{i,j} &\mapsto (AS^{[\alpha]})_{i,j}. \end{aligned}$$

Proof. Since $\mathcal{S}_n(M, \mathcal{N}', 1)$ is a commutative $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra and $\det(AS^{[\alpha]}) = 1 \in \mathcal{S}_n(M, \mathcal{N}', 1)$, then ι is a well-defined $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra homomorphism. □

Next we try to define an $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra homomorphism

$$j : \mathcal{S}_n(M, \mathcal{N}', 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n).$$

Let l be a stated n -web in (M, \mathcal{N}') , and s_l be the state of l . If $l \cap e = \emptyset$, define $j(l) = l \otimes 1 \in \mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(SL_n)$.

If $l \cap e \neq \emptyset$, we suppose $|l \cap e| = m$, and label the ends of l on e from 1 to m . We use E_k to denote the end of l at e labeled by the number $k, 1 \leq k \leq m$. Define

$$\begin{aligned} f_k &= \begin{cases} -1, & \text{if } E_k \text{ points towards } e, \\ 1, & \text{if } E_k \text{ points out of } e, \end{cases} \\ g_k &= \begin{cases} Id \in \mathcal{S}_n, & \text{if } E_k \text{ points towards } e, \\ \delta \in \mathcal{S}_n, & \text{if } E_k \text{ points out of } e, \end{cases} \\ h_k(i, j) &= \begin{cases} i, j, & \text{if } E_k \text{ points towards } e, \\ j, i, & \text{if } E_k \text{ points out of } e, \end{cases} \end{aligned}$$

where $\delta(\lambda) = \bar{\lambda}, 1 \leq \lambda \leq n, 1 \leq i, j \leq n, 1 \leq k \leq m$.

We can connect E_k with α^{-1} or α by the following way: Suppose E_k points towards e . First we isotope α^{-1} by moving $\alpha^{-1}(0)$ along e to meet the end E_k . Then we isotope l, α^{-1} nearby their endpoints at e such that they are both in good position with respect to e . Then we connect E_k with α^{-1} . When E_k points out of e , we can use the same way to connect E_k with α .

Then we try to define an element $l(\alpha_{j_1}^{f_1}, \alpha_{j_2}^{f_2}, \dots, \alpha_{j_m}^{f_m}) \in \mathcal{S}_n(M, \mathcal{N}, 1)$ by the following way: For each $1 \leq k \leq m$ we connect E_k with α^{f_k} , and assign the state j_k to the other end of α^{f_k} that is not used to connect E_k . During the process of connecting each E_k and α^{f_k} , we can isotope α^{f_k} such that $l(\alpha_{j_1}^{f_1}, \alpha_{j_2}^{f_2}, \dots, \alpha_{j_m}^{f_m})$ does not intersect itself. After connecting each E_k and α^{f_k} , we can isotope the parts nearby the connecting points such that $l(\alpha_{j_1}^{f_1}, \alpha_{j_2}^{f_2}, \dots, \alpha_{j_m}^{f_m})$ only intersects ∂M at its endpoints. Then $l(\alpha_{j_1}^{f_1}, \alpha_{j_2}^{f_2}, \dots, \alpha_{j_m}^{f_m}) \in \mathcal{S}_n(M, \mathcal{N}, 1)$. Obviously $l(\alpha_{j_1}^{f_1}, \alpha_{j_2}^{f_2}, \dots, \alpha_{j_m}^{f_m})$ is a well-defined element in $\mathcal{S}_n(M, \mathcal{N}, 1)$. We define

$$j(l) = \sum_{1 \leq j_1, \dots, j_m \leq n} c_{g_1(j_1)} \dots c_{g_m(j_m)} l(\alpha_{j_1}^{f_1}, \dots, \alpha_{j_m}^{f_m}) \otimes \mu(\alpha_{h_1(i_1, j_1)}^{-f_1}) \dots \mu(\alpha_{h_m(i_m, j_m)}^{-f_m})$$

where $\mu(\alpha_{i,j}) = d_n(-1)^{i+1} x_{i,j}^{-1}, \mu(\alpha_{i,j}^{-1}) = d_n(-1)^{i+1} x_{i,j}^{-1}, i_k = s_l(E_k), 1 \leq k \leq m, c_t = (-1)^{n-t}, 1 \leq t \leq n$.

Note that if l_1 and l_2 are isotopic to each other, we have

$$l_1(\alpha_{j_1}^{f_1}, \alpha_{j_2}^{f_2}, \dots, \alpha_{j_m}^{f_m}) = l_2(\alpha_{j_1}^{f_1}, \alpha_{j_2}^{f_2}, \dots, \alpha_{j_m}^{f_m}) \in \mathcal{S}_n(M, \mathcal{N}, 1)$$

where the labelings of endpoints of l_i , $i = 1, 2$, on e are preserved by the isotopy. Then j respects isotopy classes, that is, j is defined on the set of isotopy classes of stated n -webs. For any two stated n -webs l_1, l_2 , we isotope l_t , $t = 1, 2$, such that $l_1 \cap l_2 = \emptyset$, then we have $j(l_1 \cup l_2) = j(l_1)j(l_2)$.

Lemma 2.4.10. *Suppose (M, \mathcal{N}) is a marked 3-manifold with $\mathcal{N} \neq \emptyset$, and \mathcal{N}' is obtained from \mathcal{N} by adding one extra marking e . Then $j : \mathcal{S}_n(M, \mathcal{N}', 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n)$ is a well-defined $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra homomorphism.*

Proof. From the above discussion, it suffices to show j preserves relations (2.4)-(2.11) for well-definedness.

It is obvious that j preserves relations (2.4)-(2.7), and (2.11).

It is obvious that j preserves relations (2.8)-(2.10) if the boundary component in the picture is not e . Then we suppose the boundary component in these pictures is e . We use l (respectively l') to denote the left handside (respectively right handside) of "=" in these relations.

Relation (2.8): We only prove the case where the white dot represents an arrow going from left to right, that is, all the arrows point towards e . We choose a labeling for endpoints of l on e . From bottom to top, we label the endpoints in the right picture from $m + 1$ to $m + n$. The other endpoints of l' not in the picture are labeled in the same way as l .

Then we have

$$\begin{aligned}
j(l') &= \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sum_{\substack{1 \leq j_1, \dots, j_m \leq n \\ 1 \leq k_1, \dots, k_n \leq n}} c_{g_1(j_1)} \cdots c_{g_m(j_m)} c_{k_1} \cdots c_{k_n} \\
& l'(\alpha_{j_1}^{f_1}, \dots, \alpha_{j_m}^{f_m}, \alpha_{k_1}^{-1}, \dots, \alpha_{k_n}^{-1}) \otimes \mu(\alpha_{h_1(i_1, \bar{j}_1)}^{-f_1}, \dots, \mu(\alpha_{h_m(i_m, \bar{j}_m)}^{-f_m}) x_{\sigma(1), \bar{k}_1} \cdots x_{\sigma(n), \bar{k}_n} \\
&= \sum_{\substack{1 \leq j_1, \dots, j_m \leq n \\ 1 \leq k_1, \dots, k_n \leq n}} c_{g_1(j_1)} \cdots c_{g_m(j_m)} c_{k_1} \cdots c_{k_n} \\
& l'(\alpha_{j_1}^{f_1}, \dots, \alpha_{j_m}^{f_m}, \alpha_{k_1}^{-1}, \dots, \alpha_{k_n}^{-1}) \otimes \mu(\alpha_{h_1(i_1, \bar{j}_1)}^{-f_1}, \dots, \mu(\alpha_{h_m(i_m, \bar{j}_m)}^{-f_m}) \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} x_{\sigma(1), \bar{k}_1} \cdots x_{\sigma(n), \bar{k}_n} \\
&= \sum_{\substack{1 \leq j_1, \dots, j_m \leq n \\ 1 \leq k_1, \dots, k_n \leq n}} c_{g_1(j_1)} \cdots c_{g_m(j_m)} c_{k_1} \cdots c_{k_n}
\end{aligned}$$

as l . Then we have

$$\begin{aligned}
j(l') &= \sum_{1 \leq i \leq n} (-1)^{i+1} \sum_{\substack{1 \leq j_1, \dots, j_m \leq n \\ 1 \leq k_1, k_2 \leq n}} c_{g_1(j_1)} \cdots c_{g_m(j_m)} c_{k_1}^{-1} c_{k_2} (-1)^{\bar{i} + \bar{k}_1} \\
&\quad l'(\alpha_{j_1}^{f_1}, \dots, \alpha_{j_m}^{f_m}, \alpha_{k_1}, \alpha_{k_2}^{-1}) \otimes \mu(\alpha_{h_1(i_1, \bar{j}_1)}^{-f_1}) \cdots \mu(\alpha_{h_m(i_m, \bar{j}_m)}^{-f_m}) x_{i, \bar{k}_2} x_{k_1, i}^{-1} \\
&= \sum_{\substack{1 \leq j_1, \dots, j_m \leq n \\ 1 \leq k_1, k_2 \leq n}} c_{g_1(j_1)} \cdots c_{g_m(j_m)} c_{k_2} \\
&\quad l'(\alpha_{j_1}^{f_1}, \dots, \alpha_{j_m}^{f_m}, \alpha_{k_1}, \alpha_{k_2}^{-1}) \otimes \mu(\alpha_{h_1(i_1, \bar{j}_1)}^{-f_1}) \cdots \mu(\alpha_{h_m(i_m, \bar{j}_m)}^{-f_m}) \sum_{1 \leq i \leq n} x_{i, \bar{k}_2} x_{k_1, i}^{-1} \\
&= \sum_{\substack{1 \leq j_1, \dots, j_m \leq n \\ 1 \leq k_1 \leq n}} c_{g_1(j_1)} \cdots c_{g_m(j_m)} (-1)^{k_1+1} \\
&\quad l'(\alpha_{j_1}^{f_1}, \dots, \alpha_{j_m}^{f_m}, \alpha_{k_1}, \alpha_{k_1}^{-1}) \otimes \mu(\alpha_{h_1(i_1, \bar{j}_1)}^{-f_1}) \cdots \mu(\alpha_{h_m(i_m, \bar{j}_m)}^{-f_m}) \\
&= \sum_{1 \leq j_1, \dots, j_m \leq n} c_{g_1(j_1)} \cdots c_{g_m(j_m)} l'(\alpha_{j_1}^{f_1}, \dots, \alpha_{j_m}^{f_m}) \otimes \mu(\alpha_{h_1(i_1, \bar{j}_1)}^{-f_1}) \cdots \mu(\alpha_{h_m(i_m, \bar{j}_m)}^{-f_m}) \\
&= j(l).
\end{aligned}$$

Then j is well-defined. Trivially it is an algebra homomorphism. For any $\alpha \in \mathcal{S}_n(M, \mathcal{N}, 1)$, we have $j(\lambda_{ad}(\alpha)) = \alpha \otimes 1$. Thus j is an $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra homomorphism. □

Theorem 2.4.11. *Suppose (M, \mathcal{N}) is a marked 3-manifold with $\mathcal{N} \neq \emptyset$, and \mathcal{N}' is obtained from \mathcal{N} by adding one extra marking. Then there exist $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra homomorphisms $\iota : \mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n) \rightarrow \mathcal{S}_n(M, \mathcal{N}', 1)$ and $j : \mathcal{S}_n(M, \mathcal{N}', 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n)$ such that*

$$j \circ \iota = \mathrm{Id}_{\mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n)}, \quad \iota \circ j = \mathrm{Id}_{\mathcal{S}_n(M, \mathcal{N}', 1)}.$$

Especially $\mathcal{S}_n(M, \mathcal{N}', 1) \simeq \mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n)$.

Proof. The existence of ι and j are given by Lemmas 2.4.9 and 2.4.10.

Let $i, j \in \{1, 2, \dots, n\}$. Then we have

$$j(\iota(1 \otimes x_{i,j})) = (-1)^{i+1} j(\alpha_{i,j}^{-1}) = (-1)^{i+1} \sum_{1 \leq k \leq n} C_k d_n (-1)^{\bar{i}+1} \alpha(\alpha_k^{-1}) \otimes x_{i, \bar{k}}$$

$$= \sum_{1 \leq k \leq n} (-1)^{n-k} \left(\begin{array}{c} \uparrow k \\ \leftarrow \text{ } \\ \downarrow j \end{array} \right) \otimes x_{i,\bar{k}} = 1 \otimes x_{i,j}.$$

Since $1 \otimes x_{i,j}$ are $\mathcal{S}_n(M, \mathcal{N}, 1)$ -algebra generators, we have $j \circ \iota = \mathrm{Id}_{\mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n)}$.

We also have

$$\iota(j(\alpha_{i,j})) = (-1)^{\bar{i}+1} \iota(1 \otimes x_{i,j}) = \alpha_{i,j}.$$

From Lemma 2.4.8, we get $\iota \circ j = \mathrm{Id}_{\mathcal{S}_n(M, \mathcal{N}', 1)}$. \square

Theorem 2.4.12. *Let (M, \mathcal{N}) be a marked 3-manifold. If $\mathcal{N} = \emptyset$, we have $\mathcal{S}_n(M, \mathcal{N}, 1) \simeq G_n(M)$. If $\mathcal{N} \neq \emptyset$, we have $\mathcal{S}_n(M, \mathcal{N}, 1) \simeq \Gamma_n(M) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes (\#\mathcal{N}-1)}$.*

Proof. Subsection 2.3.1, Theorems 2.4.6 and 2.4.11. \square

Corollary 2.4.13. *Let (M, \mathcal{N}) be a marked 3-manifold with $\mathcal{N} \neq \emptyset$. Suppose $\pi_1(M)$ is a free group generated by m elements. Then we have $\mathcal{S}_n(M, \mathcal{N}, 1) \simeq \mathcal{O}(\mathrm{SL}_n)^{\otimes (m+\#\mathcal{N}-1)}$.*

The second conclusion in Theorem 7.13 in [LS21] indicates the classical limit for essentially bordered pb surfaces, which coincides with Corollary 2.4.13.

Corollary 2.4.14. *Suppose (M, \mathcal{N}) is a marked 3-manifold, and \mathcal{N}' is obtained from \mathcal{N} by adding one extra marking. Then the adding mark map $\lambda_{ad} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}', 1)$ is injective.*

Proof. If \mathcal{N} is empty. We look at the following diagram:

$$\begin{array}{ccc} \mathcal{S}_n(M, \emptyset, 1) & \xrightarrow{\lambda_{ad}} & \mathcal{S}_n(M, \mathcal{N}', 1) \\ \downarrow \simeq & & \downarrow G \\ G_n(M) & \xrightarrow{\lambda} & \Gamma_n(M) \end{array}$$

where the isomorphism from $\mathcal{S}_n(M, \emptyset, 1)$ to $G_n(M)$ is the one introduced in Subsection 2.3.1 (the spin structure used for this isomorphism is the restriction of the relative spin structure for (M, \mathcal{N}')), and λ is the embedding. It is easy to check the above diagram is commutative. Then λ_{ad} is injective because G is an isomorphism.

If \mathcal{N} is not empty. For any $\alpha \in \mathcal{S}_n(M, \mathcal{N}, 1)$, we have $j(\lambda_{ad}(\alpha)) = \alpha \otimes 1$. Then λ_{ad} is injective because j is an isomorphism and the map from $\mathcal{S}_n(M, \mathcal{N}, 1)$ to $\mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n)$ given by $\alpha \mapsto \alpha \otimes 1$ is injective. \square

We can regard $\mathcal{S}_n(M, \mathcal{N}, 1)$ as a subalgebra of $\mathcal{S}_n(M, \mathcal{N}', 1)$ because of Corollary 2.4.14 and the adding marking map.

Suppose the components of \mathcal{N} consist of e_0, e_1, \dots, e_{k-1} where k is a positive integer. If $k \geq 2$, for each $1 \leq t \leq k-1$, let α_t be an oriented path connecting e_0 and e_t with $\alpha_t(0) \in e_0$ and $\alpha_t(1) \in e_t$. We use $[o]$ to denote the identity element in $\pi_1(M, e_0)$. The following Theorem offers algebraic generators and relations among these generators for the commutative algebra $\mathcal{S}_n(M, \mathcal{N}, 1)$.

Theorem 2.4.15. *The commutative algebra $\mathcal{S}_n(M, \mathcal{N}, 1)$ is generated by*

$$S_{i,j}^{[\alpha]}, [\alpha] \in \pi_1(M, e_0) \cup \{[\alpha_1], \dots, [\alpha_{k-1}]\}, 1 \leq i, j \leq n,$$

subject to the following relations.

$$\begin{aligned} \det(S^{[\alpha]}) &= 1 \text{ for all } [\alpha] \in \pi_1(M, e_0) \cup \{[\alpha_1], \dots, [\alpha_{k-1}]\}, \quad AS^{[o]} = I, \\ AS^{[\beta]}AS^{[\eta]} &= AS^{[\beta*\eta]} \text{ for all } [\beta], [\eta] \in \pi_1(M, e_0). \end{aligned} \quad (2.19)$$

Note that if $k = 1$, the set $\{[\alpha_1], \dots, [\alpha_{k-1}]\}$ is empty.

Proof. Theorems 2.4.6 and 2.4.11. \square

2.4.3 Ker $\Phi = \sqrt{0}$

Suppose \mathcal{N} has only one component. Then we define an algebra isomorphism $H : \Gamma_n(M) \rightarrow \Gamma_n(M)$ and a surjective algebra homomorphism $\tau : \Gamma_n(M) \rightarrow R_n(M, \mathcal{N})$. Let $[\alpha]$ be an element in $\pi_1^{Mor}(M, \mathcal{N})$ and i, j be two integers between 1 and n . Define $H([\alpha]_{i,j}) = [\alpha]_{\bar{i}, \bar{j}}$. It is easy to show H is a well-defined algebra isomorphism. For any $\rho \in \tilde{\chi}_n(M, \mathcal{N})$, define $\tau([\alpha]_{i,j})(\rho) = [\rho(\tilde{\alpha})]_{i,j}$ where $\tilde{\alpha} \in \pi_1(UM, \tilde{\mathcal{N}})$ is a lift for α such that $h(\tilde{\alpha}) = 0$. From the proof of Proposition 2.2.8, we know the definition of τ is independent of the choice of the lift for α . It is also obvious to show τ is a well-defined surjective algebra homomorphism.

Especially from Proposition 2.2.8 and definitions for $\Gamma_n(M)$ and $R_n(M, \mathcal{N})$, we have $\mathrm{Ker} \tau = \sqrt{0}_{\Gamma_n(M)}$.

Lemma 2.4.16. *Let (M, \mathcal{N}) be marked 3-manifold with \mathcal{N} consisting of one open oriented interval. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \Gamma_n(M) & \xrightarrow{F} & \mathcal{S}_n(M, \mathcal{N}, 1) \\ \downarrow H & & \downarrow \Phi \\ \Gamma_n(M) & \xrightarrow{\tau} & R_n(M, \mathcal{N}) \end{array} .$$

Especially $\mathrm{Ker} \Phi = \sqrt{0}_{\mathcal{S}_n(M, \mathcal{N}, 1)}$.

Proof. For any $[\alpha] \in \pi_1^{\mathrm{Mor}}(M, \mathcal{N})$, $1 \leq i, j \leq n$, we know $F([\alpha]_{i,j}) = (-1)^{i+1} \hat{\alpha}_{i,j}$ where $\hat{\alpha}$ is a framed oriented boundary arc such that $h(\hat{\alpha}) = 0$ and $[\hat{\alpha}] = [\alpha] \in \pi_1^{\mathrm{Mor}}(M, \mathcal{N})$. Then for any $\rho \in \tilde{\chi}_n(M, \mathcal{N})$, we have

$$\Phi(F([\alpha]_{i,j}))(\rho) = (-1)^{i+1} \Phi(\hat{\alpha}_{i,j})(\rho) = (-1)^{i+1} [A\rho(\hat{\alpha})]_{i,j} = [\rho(\hat{\alpha})]_{i,j}.$$

Since $\tilde{\alpha} \in \pi_1(UM, \tilde{\mathcal{N}})$ is a lift for α and $h(\tilde{\alpha}) = 0$, we have

$$\tau(H([\alpha]_{i,j}))(\rho) = \tau([\alpha]_{i,j})(\rho) = [\rho(\tilde{\alpha})]_{i,j}.$$

Thus the diagram commutes.

Since both F and H are isomorphisms and $\mathrm{Ker} \tau = \sqrt{0}_{\Gamma_n(M)}$, we get $\mathrm{Ker} \Phi = \sqrt{0}_{\mathcal{S}_n(M, \mathcal{N}, 1)}$. □

Lemma 2.4.17. *Suppose (M, \mathcal{N}) is a marked 3-manifold with $\mathcal{N} \neq \emptyset$, and \mathcal{N}' is obtained from \mathcal{N} by adding one extra marking. Then $\mathrm{Ker} \Phi^{(M, \mathcal{N}')}$ is the ideal generated by $\mathrm{Ker} \Phi^{(M, \mathcal{N})}$ (here we regard $\mathcal{S}_n(M, \mathcal{N}, 1)$ as a subalgebra of $\mathcal{S}_n(M, \mathcal{N}', 1)$).*

Proof. Here we use the notations in Remark 2.4.2.

From Proposition 2.2.8 and Lemma 8.1 in [CL22a], we know there is an algebra isomorphism $h : R_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n) \rightarrow R_n(M, \mathcal{N}')$ defined by

$$h(r \otimes x_{i,j})(\rho) = r(\rho|_{\pi_1(UM, \tilde{\mathcal{N}})})[\rho(\tilde{\alpha})]_{i,j}$$

where $r \in R_n(M, \mathcal{N})$, $1 \leq i, j \leq n$, $\rho \in \tilde{\chi}_n(M, \mathcal{N}')$ and $\rho|_{\pi_1(UM, \tilde{\mathcal{N}})}$ is the restriction of ρ on $\pi_1(UM, \tilde{\mathcal{N}})$. We have another algebra isomorphism $f: \mathcal{O}(\mathrm{SL}_n) \rightarrow \mathcal{O}(\mathrm{SL}_n)$ given by $x_{i,j} \rightarrow x_{\bar{i}, \bar{j}}$.

Then we want to show the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n) & \xrightarrow{\iota} & \mathcal{S}_n(M, \mathcal{N}', 1) \\ \downarrow \Phi^{(M, \mathcal{N})} \otimes f & & \downarrow \Phi^{(M, \mathcal{N}')} \\ R_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n) & \xrightarrow{h} & R_n(M, \mathcal{N}') \end{array}.$$

Let ρ be element in $\tilde{\chi}_n(M, \mathcal{N}')$, i, j be two integers between 1 and n , $\beta_{k,t}$ be a stated framed oriented boundary arc in (M, \mathcal{N}) . We have

$$\Phi^{(M, \mathcal{N}')}(\iota(\beta_{k,t} \otimes 1))(\rho) = \Phi^{(M, \mathcal{N}')}(\beta_{k,t})(\rho) = [A\rho(\tilde{\beta})]_{\bar{k}, \bar{t}},$$

and

$$\begin{aligned} (h \circ (\Phi^{(M, \mathcal{N})} \otimes f))(\beta_{k,t} \otimes 1)(\rho) &= h(\Phi^{(M, \mathcal{N})}(\beta_{k,t}) \otimes 1)(\rho) = \Phi^{(M, \mathcal{N})}(\beta_{k,t})(\rho|_{\pi_1(UM, \tilde{\mathcal{N}})}) \\ &= [A\rho|_{\pi_1(UM, \tilde{\mathcal{N}})}(\tilde{\beta})]_{\bar{k}, \bar{t}} = [A\rho(\tilde{\beta})]_{\bar{k}, \bar{t}}. \end{aligned}$$

We also have

$$\Phi^{(M, \mathcal{N}')}(\iota(1 \otimes x_{i,j}))(\rho) = (-1)^{i+1} \Phi^{(M, \mathcal{N}')}(\alpha_{\bar{i}, \bar{j}})(\rho) = (-1)^{i+1} [A\rho(\tilde{\alpha})]_{\bar{i}, \bar{j}} = [\rho(\tilde{\alpha})]_{\bar{i}, \bar{j}},$$

and

$$(h \circ (\Phi^{(M, \mathcal{N})} \otimes f))(1 \otimes x_{i,j})(\rho) = h(1 \otimes x_{\bar{i}, \bar{j}})(\rho) = [\rho(\tilde{\alpha})]_{\bar{i}, \bar{j}}.$$

Thus the above diagram is commutative because $\beta_{k,t} \otimes 1, 1 \otimes x_{i,j}$ generate $\mathcal{S}_n(M, \mathcal{N}, 1) \otimes \mathcal{O}(\mathrm{SL}_n)$ as an algebra and all the maps in the diagram are algebra homomorphisms.

We have $\mathrm{Ker}(\Phi^{(M, \mathcal{N})} \otimes f) = \mathrm{Ker}(\Phi^{(M, \mathcal{N})} \otimes \mathrm{Id}_{\mathcal{O}(\mathrm{SL}_n)}) = (\mathrm{Ker} \Phi^{(M, \mathcal{N})}) \otimes \mathcal{O}(\mathrm{SL}_n)$, where $(\mathrm{Ker} \Phi^{(M, \mathcal{N})}) \otimes \mathcal{O}(\mathrm{SL}_n)$ is an ideal generated by $(\mathrm{Ker} \Phi^{(M, \mathcal{N})}) \otimes 1$. Then $\mathrm{Ker} \Phi^{(M, \mathcal{N}')}$ is the ideal generated by $\mathrm{Ker} \Phi^{(M, \mathcal{N})}$ since $\iota((\mathrm{Ker} \Phi^{(M, \mathcal{N})}) \otimes 1) = \mathrm{Ker} \Phi^{(M, \mathcal{N}')}$.

□

Lemma 2.4.18. *Let (M, \mathcal{N}) be a marked 3-manifold with $\mathcal{N} \neq \emptyset$. We have (a) $\mathrm{Ker} \Phi^{(M, \mathcal{N}')} = \sqrt{0}_{\mathcal{S}_n(M, \mathcal{N}, 1)}$, and (b) $\sqrt{0}_{\mathcal{S}_n(M, \mathcal{N}, 1)}$ is the ideal generated by $\sqrt{0}_{\mathcal{S}_n(M, e, 1)}$*

where e is a component of \mathcal{N} (here we regard $\mathcal{S}_n(M, e, 1)$ as a subalgebra of $\mathcal{S}_n(M, \mathcal{N}, 1)$).

Proof. For any marked 3-manifold (M_1, \mathcal{N}_1) with $\mathcal{N}_1 \neq \emptyset$, suppose \mathcal{N}'_1 is obtained from \mathcal{N}_1 by adding one extra marking. Then $\text{Ker } \Phi^{(M_1, \mathcal{N}'_1)} = (\text{Ker } \Phi^{(M_1, \mathcal{N}_1)})$ from Lemma 2.4.17. Thus if $\text{Ker } \Phi^{(M_1, \mathcal{N}_1)} = \sqrt{0}_{\mathcal{S}_n(M_1, \mathcal{N}_1, 1)}$, then $\text{Ker } \Phi^{(M_1, \mathcal{N}'_1)} = (\text{Ker } \Phi^{(M_1, \mathcal{N}_1)}) \subset \sqrt{0}_{\mathcal{S}_n(M_1, \mathcal{N}'_1, 1)}$. We also have $\sqrt{0}_{\mathcal{S}_n(M_1, \mathcal{N}'_1, 1)} \subset \text{Ker } \Phi^{(M_1, \mathcal{N}'_1)}$ since the coordinate ring has no nonzero nilpotents. Then $\text{Ker } \Phi^{(M_1, \mathcal{N}'_1)} = \sqrt{0}_{\mathcal{S}_n(M_1, \mathcal{N}'_1, 1)}$. Thus $\text{Ker } \Phi^{(M_1, \mathcal{N}_1)} = \sqrt{0}_{\mathcal{S}_n(M_1, \mathcal{N}_1, 1)}$ implies $\text{Ker } \Phi^{(M_1, \mathcal{N}'_1)} = \sqrt{0}_{\mathcal{S}_n(M_1, \mathcal{N}'_1, 1)}$. Combine with the fact that (a) is true if \mathcal{N} consists of only one oriented open interval (Lemma 2.4.16), we get (a) is true for general marked 3-manifold (M, \mathcal{N}) with $\mathcal{N} \neq \emptyset$.

If \mathcal{N} consists of one component, clearly (b) holds. If $\#\mathcal{N} > 1$, suppose $\text{Com}(\mathcal{N}) = \{e_1, e_2, \dots, e_m\}$. For any $1 \leq i \leq m$, define $\mathcal{N}_{(i)} = e_1 \cup \dots \cup e_i$. Then we have

$$\mathcal{S}_n(M, \mathcal{N}_{(1)}, 1) \subset \mathcal{S}_n(M, \mathcal{N}_{(2)}, 1) \subset \dots \subset \mathcal{S}_n(M, \mathcal{N}_{(m)}, 1).$$

Since $\text{Ker } \Phi^{(M, \mathcal{N}_{(i+1)})}$ is an ideal of $\mathcal{S}_n(M, \mathcal{N}_{(i+1)}, 1)$ generated by $\text{Ker } \Phi^{(M, \mathcal{N}_{(i)})}$, then we have

$$\text{Ker } \Phi^{(M, \mathcal{N})} = \text{Ker } \Phi^{(M, \mathcal{N}_{(m)})}$$

is an ideal of $\mathcal{S}_n(M, \mathcal{N}, 1)$ generated by $\text{Ker } \Phi_{(M, \mathcal{N}_{(1)})}$, which actually is $\text{Ker } \Phi_{(M, e_1)}$. From (a), we know $\text{Ker } \Phi^{(M, \mathcal{N})} = \sqrt{0}_{\mathcal{S}_n(M, \mathcal{N}, 1)}$ and $\text{Ker } \Phi^{(M, \{e_1\})} = \sqrt{0}_{\mathcal{S}_n(M, e_1, 1)}$. Since we can label any component of \mathcal{N} as e_1 , then (b) is true. \square

Theorem 2.4.19. *For any marked 3-manifold (M, \mathcal{N}) , we have $\text{Ker } \Phi^{(M, \mathcal{N})} = \sqrt{0}_{\mathcal{S}_n(M, \mathcal{N}, 1)}$.*

Proof. Subsection 2.3.1 and (a) in Lemma 2.4.18. \square

2.5 Generalized marked 3-manifold

Costantino and Lê defined the generalized marked 3-manifold in [CL22b], in which they allow \mathcal{N} contains oriented closed circles. For a generalized marked 3-manifold (M, \mathcal{N}) , obviously $\mathcal{S}_n(M, \mathcal{N}, 1)$ has a commutative algebra structure, given by

taking the disjoint union of the stated n -webs in (M, \mathcal{N}) . In this section, we will focus on the classical limit of the stated SL_n -skein module of the generalized marked 3-manifold.

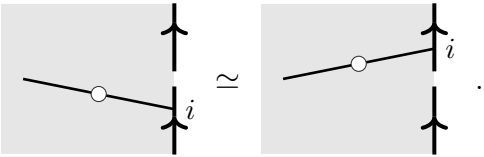
2.5.1 Cutting out the closure of a small open interval from \mathcal{N}

Let (M, \mathcal{N}) be a generalized marked 3-manifold with $\mathcal{N} \neq \emptyset$. Suppose U is a small open interval contained in e such that $cl(U) \subset e$, where e is a component of \mathcal{N} . Let $\mathcal{N}' = (\mathcal{N} \setminus e) \cup e'$ where $e' = e \setminus cl(U)$. Let $l_U : \mathcal{S}_n(M, \mathcal{N}', v) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)$ be the linear map induced by the embedding $(M, \mathcal{N}') \rightarrow (M, \mathcal{N})$. Clearly l_U is surjective, and is an algebra homomorphism when $v = 1$.

Proposition 2.5.1. *The above map l_U induces an isomorphism*

$$\bar{l}_U : \mathcal{S}_n(M, \mathcal{N}', v) / \simeq \rightarrow \mathcal{S}_n(M, \mathcal{N}, v),$$

where \simeq is the equivalence relation given by the following picture:


(2.20)

The missing part between two arrows is $cl(U)$.

Proof. Clearly l_U induces a linear map $\bar{l}_U : \mathcal{S}_n(M, \mathcal{N}', v) / \simeq \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)$. For a stated n -web α in (M, \mathcal{N}') , we use $\mathrm{cls}(\alpha)$ to denote the element in $\mathcal{S}_n(M, \mathcal{N}', v) / \simeq$ determined by α . Let β be any stated n -web in (M, \mathcal{N}) . We can isotope β such that $cl(U) \cap \beta = \emptyset$, and define $j_U(\beta) = \mathrm{cls}(\beta) \in \mathcal{S}_n(M, \mathcal{N}', v) / \simeq$. We have $j_U(\beta)$ is independent of how we isotope β because of relation (2.20). If β and β' are isotopic stated n -webs in (M, \mathcal{N}) , clearly we have $j_U(\beta) = j_U(\beta')$ because of relation (2.20). Trivially j_U preserves the defining skein relations for $\mathcal{S}_n(M, \mathcal{N}, v)$. Thus j_U is a well-defined linear map from $\mathcal{S}_n(M, \mathcal{N}, v)$ to $\mathcal{S}_n(M, \mathcal{N}', v) / \simeq$. It is easy to check l_U and j_U are inverse to each other. \square

2.5.2 Classical limit for stated SL_n -skein modules for generalized marked 3-manifolds

In this subsection, we will try to find out the classical limit for generalized marked 3-manifolds using results in Section 2.4 and Proposition 2.5.1. In this subsection, we will assume the 3-manifold is connected (the corresponding results can be easily generalized to general 3-manifolds). Let (M, \mathcal{N}) be a generalized marked 3-manifold. Suppose \mathcal{N} contains k ($k \geq 1$) closed oriented circles, which are denoted as e_0, e_1, \dots, e_{k-1} . We denote other oriented open intervals in \mathcal{N} , if any, as e_k, \dots, e_{m-1} . For each $0 \leq i \leq k-1$, we pick a small open interval U_i contained in e_i such that $cl(U_i) \subset e_i$, and set $e'_i = e_i \setminus cl(U_i)$. Set $e'_i = e_i$ for $k \leq i \leq m-1$. Let $\mathcal{N}' = \{e'_0, e'_1, \dots, e'_{m-1}\}$, then (M, \mathcal{N}') is a circle free marked 3-manifold. We choose a relative spin structure h for (M, \mathcal{N}') . For each $1 \leq i \leq m-1$, let α_i be an oriented path connecting e'_0 and e'_i such that $\alpha_i(0) \in e'_0$ and $\alpha_i(1) \in e'_i$. We still use l_U to denote the algebra homomorphism from $\mathcal{S}_n(M, \mathcal{N}', 1)$ to $\mathcal{S}_n(M, \mathcal{N}, 1)$ induced by the embedding from (M, \mathcal{N}') to (M, \mathcal{N}) .

We know there is an isomorphism L from $\Gamma_n(M) \otimes \mathcal{O}(SL_n)^{\otimes(m-1)}$ to $\mathcal{S}_n(M, \mathcal{N}', 1)$. For any element $y \in \Gamma_n(M)$, we will use y_{\otimes} to denote $y \otimes 1 \otimes \dots \otimes 1 \in \Gamma_n(M) \otimes \mathcal{O}(SL_n)^{\otimes(m-1)}$. For any $1 \leq i, j \leq n, 1 \leq t \leq m-1$, we use $x_{i,j}^t$ to denote $1 \otimes 1 \otimes \dots \otimes x_{i,j} \otimes \dots \otimes 1 \in \Gamma_n(M) \otimes \mathcal{O}(SL_n)^{\otimes(m-1)}$ where $x_{i,j}$ is in the t -th tensor factor for $\mathcal{O}(SL_n)^{\otimes(m-1)}$. Then the isomorphism L is given by:

$$([\alpha]_{i,j})_{\otimes} \rightarrow (AS^{[\alpha]})_{i,j} \text{ and } x_{i,j}^t \rightarrow (AS^{[\alpha]})_{i,j}$$

where $[\alpha] \in \pi_1(M, e'_0), 1 \leq i, j \leq n, 1 \leq t \leq m-1$. For each $0 \leq t \leq m-1$, set $X_t = (x_{i,j}^t)_{n \times n}$. For each element $[\alpha] \in \pi_1(M, e'_0)$, set $Q_{[\alpha], \otimes} = (([\alpha]_{i,j})_{\otimes})_{n \times n}$. Then $L(X_t) = AS^{[\alpha]t}, L(Q_{[\alpha], \otimes}) = AS^{[\alpha]}$.

As in Subsection 2.2.3, any component $e \in \mathcal{N}$ can be lifted to $\tilde{e} \subset UM$. Note that when e is an oriented closed circle, the lifting \tilde{e} is also an oriented closed circle in UM , which means \tilde{e} is an element in $H_1(UM)$.

Definition 2.5.2. *Let (M, \mathcal{N}) be a generalized marked 3-manifold, and h_s be a spin structure for M . Suppose \mathcal{N} contains k oriented closed circles.*

If $k = 0$, we define $\Gamma_n(M, \mathcal{N}) = \Gamma_n(M)$.

If $k \geq 1$, we denote all the oriented closed circles in \mathcal{N} as e_0, \dots, e_{k-1} . For each e_t , we use a path β_t ($\beta_t(0)$ is the base point for $\pi_1(M)$ and $\beta_t(1) \in e_t$) to connect the base point for $\pi_1(M)$ and e_t to obtain an element in $\pi_1(M)$, denoted as $[e_t^{\beta_t}]$. Define $\Gamma_n(M, \mathcal{N}) = \Gamma_n(M)/(D)$ where $D = \{(Q_{[e_t^{\beta_t}]} - d_n^{h_s(\tilde{e}_t)} I)_{i,j} \mid 0 \leq t \leq k-1, 1 \leq i, j \leq n\}$ and (D) is the ideal of $\Gamma_n(M)$ generated by D . For any element $x \in \Gamma_n(M)$, we use \bar{x} to denote $x + (D) \in \Gamma_n(M, \mathcal{N})$.

Note that the definition of $\Gamma_n(M, \mathcal{N})$ is independent of the choice of $\beta_t, 0 \leq t \leq k-1$. Suppose for each $0 \leq t \leq k-1$, we make another choice γ_t . Then the relation $Q_{[e_t^{\beta_t}]} = d_n^{h_s(\tilde{e}_t)} I$ becomes $Q_{[e_t^{\gamma_t}]} = d_n^{h_s(\tilde{e}_t)} I$. Since $Q_{[e_t^{\beta_t}]}$ and $Q_{[e_t^{\gamma_t}]}$ are conjugate to each other, then the relation $Q_{[e_t^{\beta_t}]} = d_n^{h_s(\tilde{e}_t)} I$ is the same with the relation $Q_{[e_t^{\gamma_t}]} = d_n^{h_s(\tilde{e}_t)} I$.

Note that we do not distinguish $\pi_1(M)$ and $\pi_1^{Mor}(M, e_0)$ where e_0 is an embedded open interval in ∂M . The definition for $\Gamma_n(M, \mathcal{N})$ is related to the spin structure h_s for M . Here we make a convention that the spin structure used for the definition of $\Gamma_n(M, \mathcal{N})$ is obtained by restricting the relative spin structure when the relative spin structure is given.

Lemma 2.5.3. *With the conventions and notations in Remark 2.5.2, we have*

$$\Gamma_n(M) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)} \xrightarrow{L} \mathcal{S}_n(M, \mathcal{N}', 1) \xrightarrow{l_U} \mathcal{S}_n(M, \mathcal{N}, 1)$$

induces a surjective algebra homomorphism $\bar{L} : \Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)} \rightarrow \mathcal{S}_n(M, \mathcal{N}, 1)$. Here we regard $\pi_1(M)$ as $\pi_1^{Mor}(M, e'_0)$.

Proof. We have the exact sequence:

$$(D) \twoheadrightarrow \Gamma_n(M) \twoheadrightarrow \Gamma_n(M, \mathcal{N})$$

where D and (D) are defined in Definition 2.5.2 (the arrow with two heads means the corresponding map is surjective). After using functor $-\otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)}$ acting on the above exact sequence, we get the following new exact sequence:

$$(D) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)} \twoheadrightarrow \Gamma_n(M) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)} \twoheadrightarrow \Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)} .$$

Note that $(D) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)}$ is the ideal of $\Gamma_n(M) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)}$ generated by $d_{\otimes}, d \in D$. Thus to show $l_U \circ L$ induces \bar{L} , it suffices to show $l_U(L(d_{\otimes})) = 0$ for all $d \in D$.

Let i, j be any two integers between 1 and n , and t be an integer between 0 and $k - 1$. Then we have

$$l_U(L([e_t^{\beta_t}]_{i,j} \otimes)) = (-1)^{i+1} l_U(S_{i,j}^{[e_t^{\beta_t}]}).$$

Thus we need to show $(-1)^{i+1} l_U(S_{i,j}^{[e_t^{\beta_t}]}) = d_n^{h(\tilde{e}_t)} \delta_{i,j}$, that is, to show $Al_U(S^{[e_t^{\beta_t}]}) = d_n^{h(\tilde{e}_t)} I$. From the definition of $[e_t^{\beta_t}]$, we know $[e_t^{\beta_t}] = [\beta_t^{-1} * e_t * \beta_t]$. Then we have

$$Al_U(S^{[\beta_t]}) Al_U(S^{[e_t^{\beta_t}]}) = l_U(AS^{[\beta_t]} AS^{[\beta_t^{-1} * e_t * \beta_t]}) = l_U(AS^{[e_t * \beta_t]}) = Al_U(S^{[e_t * \beta_t]}).$$

Note that in $\mathcal{S}_n(M, \mathcal{N}, 1)$, we have $l_U(S^{[e_t * \beta_t]}) = d_n^{h(\tilde{e}_t)} l_U(S^{[\beta_t]})$. Then we get

$$l_U(S^{[\beta_t]}) Al_U(S^{[e_t^{\beta_t}]}) = l_U(S^{[e_t * \beta_t]}) = d_n^{h(\tilde{e}_t)} l_U(S^{[\beta_t]}).$$

Then we have $Al_U(S^{[e_t^{\beta_t}]}) = d_n^{h(\tilde{e}_t)} I$ because $l_U(S^{[\beta_t]})$ is invertible.

The above discussion shows $l_U \circ L$ induces \bar{L} . The algebra homomorphism \bar{L} is surjective since $l_U \circ L$ is surjective. \square

Note that for any $x \in \Gamma_n(M), y \in \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)}$, we have $\bar{L}(\bar{x} \otimes y) = l_U(L(x \otimes y))$. We use π to denote the projection from $\Gamma_n(M) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)}$ to $\Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)}$. Then $\bar{L} \circ \pi = l_U \circ L$.

Lemma 2.5.4. *With the conventions and notations in Remark 2.5.2, we have*

$$\mathcal{S}_n(M, \mathcal{N}', 1) \xrightarrow{L^{-1}} \Gamma_n(M) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)} \xrightarrow{\pi} \Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)}$$

induces a surjective algebra homomorphism $\bar{L}^{-1} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes(m-1)}$. Here we regard $\pi_1(M)$ as $\pi_1^{Mor}(M, e'_0)$.

Proof. From Proposition 2.5.1, it suffices to show $\pi \circ L^{-1}$ preserves the equivalence relation (2.20) for every $U_t, 0 \leq t \leq k - 1$. Let α be any stated n -web for (M, \mathcal{N}') . Suppose there exists $0 \leq t \leq k - 1$ such that, nearby U_t , the stated n -web α looks like the left picture in the equivalence relation (2.20). Let α' be the same stated

n -web as α except, nearby U_t , α' looks like the right picture in equivalence relation (2.20). Then we want to show $\pi(L^{-1}(\alpha)) = \pi(L^{-1}(\alpha'))$.

We can use the same way to kill all the sinks and sources in α and α' . Then the resulting two stated n -webs only differ on a single stated arc. Since $\pi \circ L^{-1}$ is an algebra homomorphism, we can just assume α is a stated framed oriented boundary arc. Without loss of generality, we assume the white dot in equivalence relation (2.20) represents an arrow pointing from left to right, that is, pointing towards the boundary. It is easy to show $h(\tilde{\alpha}') = h(\tilde{\alpha}) + h(\tilde{e}_t)$. Suppose $s(\alpha(0)) = s(\alpha'(0)) = j$. We have $\alpha = d_n^{h(\tilde{\alpha})} S_{i,j}^{[\alpha]}$ and $\alpha' = d_n^{h(\tilde{\alpha}')} S_{i,j}^{[\alpha']}$.

Suppose $\alpha(0) \in e'_{t_1}$ where $0 \leq t_1 \leq m-1$. We have four cases to consider: (1) $t = t_1 = 0$, (2) $t = 0$ and $t_1 \neq 0$, (3) $t \neq 0$ and $t_1 = 0$, (4) $t \neq 0$ and $t_1 \neq 0$.

Here we only prove the case when $t \neq 0$ and $t_1 \neq 0$. We have

$$\begin{aligned} \pi(L^{-1}(S^{[\alpha]})) &= A^{-1}\pi(L^{-1}(AS^{[\alpha_t][\alpha_t^{-1}*\alpha*\alpha_{t_1}][\alpha_{t_1}^{-1}]})) = A^{-1}\pi(L^{-1}(AS^{[\alpha_t]}AS^{[\alpha_t^{-1}*\alpha*\alpha_{t_1}]}AS^{[\alpha_{t_1}^{-1}]})) \\ &= A^{-1}\pi(X_t Q_{[\alpha_t^{-1}*\alpha*\alpha_{t_1}],\otimes} X_{t_1}^{-1}) = A^{-1}\pi(X_t)\pi(Q_{[\alpha_t^{-1}*\alpha*\alpha_{t_1}],\otimes})\pi(X_{t_1}^{-1}). \end{aligned}$$

Similarly we have

$$\pi(L^{-1}(S^{[\alpha']})) = A^{-1}\pi(X_t)\pi(Q_{[\alpha_t^{-1}*\alpha'*\alpha_{t_1}],\otimes})\pi(X_{t_1}^{-1}).$$

We also have

$$Q_{[\alpha_t^{-1}*\alpha'*\alpha_{t_1}],\otimes} = Q_{[\alpha_t^{-1}*\alpha*\alpha_{t_1}],\otimes} Q_{[\alpha_{t_1}^{-1}*\alpha^{-1}*\alpha'*\alpha_{t_1}],\otimes}$$

where $[\alpha_{t_1}^{-1} * \alpha^{-1} * \alpha' * \alpha_{t_1}] = [(\alpha' * \alpha_{t_1})^{-1} * e_t * \alpha' * \alpha_{t_1}]$. Thus

$$\pi(Q_{[\alpha_t^{-1}*\alpha'*\alpha_{t_1}],\otimes}) = \pi(Q_{[\alpha_t^{-1}*\alpha*\alpha_{t_1}],\otimes})\pi(Q_{[\alpha_{t_1}^{-1}*\alpha^{-1}*\alpha'*\alpha_{t_1}],\otimes}) = d_n^{h(\tilde{e}_t)}\pi(Q_{[\alpha_t^{-1}*\alpha*\alpha_{t_1}],\otimes}).$$

Then

$$\begin{aligned} \pi(L^{-1}(\alpha')) &= d_n^{h(\tilde{\alpha}')} \pi L^{-1}(S_{i,j}^{[\alpha']}) = d_n^{h(\tilde{\alpha})+h(\tilde{e}_t)} [A^{-1}\pi(X_t)\pi(Q_{[\alpha_t^{-1}*\alpha'*\alpha_{t_1}],\otimes})\pi(X_{t_1}^{-1})]_{i,j} \\ &= d_n^{h(\tilde{\alpha})+h(\tilde{e}_t)} d_n^{h(\tilde{e}_t)} [A^{-1}\pi(X_t)\pi(Q_{[\alpha_t^{-1}*\alpha*\alpha_{t_1}],\otimes})\pi(X_{t_1}^{-1})]_{i,j} \\ &= d_n^{h(\tilde{\alpha})} [A^{-1}\pi(X_t)\pi(Q_{[\alpha_t^{-1}*\alpha*\alpha_{t_1}],\otimes})\pi(X_{t_1}^{-1})]_{i,j} = \pi(L^{-1}(\alpha)). \end{aligned}$$

□

For any stated n -web α in $\mathcal{S}_n(M, \mathcal{N}, 1)$, we can isotope α such that $cl(U_t) \cap \alpha = \emptyset$ for all $0 \leq t \leq k-1$. Then α is also a stated n -web α in $\mathcal{S}_n(M, \mathcal{N}', 1)$, we still use α to denote this element in $\mathcal{S}_n(M, \mathcal{N}', 1)$. Then $\overline{L^{-1}}(\alpha) = \pi(L^{-1}(\alpha))$, that is, we have $\overline{L^{-1}} \circ l_U = \pi \circ L^{-1}$.

Lemma 2.5.5. *The algebra homomorphism \overline{L} obtained in Lemma 2.5.3 and the algebra homomorphism $\overline{L^{-1}}$ obtained in Lemma 2.5.4 are inverse to each other. Especially for any generalized marked 3-manifold (M, \mathcal{N}) with \mathcal{N} containing at least one closed oriented circle, we have $\Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes (\#\mathcal{N}-1)} \simeq \mathcal{S}_n(M, \mathcal{N}, 1)$.*

Proof. For any stated n -web α in $\mathcal{S}_n(M, \mathcal{N}, 1)$, we isotope α such that $cl(U_t) \cap \alpha = \emptyset$ for all $0 \leq t \leq k-1$. Then

$$\overline{L}(\overline{L^{-1}}(\alpha)) = \overline{L}(\pi(L^{-1}(\alpha))) = (l_U \circ L)(L^{-1}(\alpha)) = l_U(\alpha) = \alpha.$$

For any $x \in \Gamma_n(M), y \in \mathcal{O}(\mathrm{SL}_n)^{\otimes (m-1)}$, we have

$$\overline{L^{-1}}(\overline{L}(\bar{x} \otimes y)) = \overline{L^{-1}}(l_U(L(x \otimes y))) = (\pi \circ L^{-1})(L(x \otimes y)) = \pi(x \otimes y) = \bar{x} \otimes y.$$

□

Theorem 2.5.6. *Let (M, \mathcal{N}) be a generalized marked 3-manifold with $\mathcal{N} \neq \emptyset$. Then $\mathcal{S}_n(M, \mathcal{N}, 1) \simeq \Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes (\#\mathcal{N}-1)}$.*

Proof. If \mathcal{N} is circle free, then $\Gamma_n(M, \mathcal{N}) = \Gamma_n(M)$. From Theorem 2.4.12, we have $\mathcal{S}_n(M, \mathcal{N}, 1) \simeq \Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes (\#\mathcal{N}-1)}$.

If \mathcal{N} contains at least one oriented closed circle, then Lemma 2.5.5 shows $\mathcal{S}_n(M, \mathcal{N}, 1) \simeq \Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes (\#\mathcal{N}-1)}$. □

For generalized marked 3-manifold (M, \mathcal{N}) , we can also define the corresponding adding marking map. Suppose $\mathcal{N}_{ad} = \mathcal{N} \cup e$ where e is an oriented open interval or an oriented closed circle such that there is no intersection between the closure of \mathcal{N} and the closure of e . We also say \mathcal{N}_{ad} is obtained from \mathcal{N} by adding one extra marking. The linear map from $\mathcal{S}_n(M, \mathcal{N}, v)$ to $\mathcal{S}_n(M, \mathcal{N}_{ad}, v)$ induced by the embedding $(M, \mathcal{N}) \rightarrow (M, \mathcal{N}_{ad})$ is also denoted as λ_{ad} . Clearly when $v = 1$, we have λ_{ad} is an algebra homomorphism.

Corollary 2.5.7. *Let (M, \mathcal{N}) be a generalized marked 3-manifold. Suppose \mathcal{N}_{ad} is obtained from \mathcal{N} by adding one extra oriented open interval. Then $\lambda_{ad} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}_{ad}, 1)$ is injective.*

Proof. We already proved the injectivity for λ_{ad} when \mathcal{N} is circle free in Corollary 2.4.14.

Then we suppose \mathcal{N} contains at least one oriented circle. When we cut the closure of small open intervals as in Remark 2.5.2, we choose the same way to cut them for (M, \mathcal{N}) and (M, \mathcal{N}_{ad}) , and the choices for α_i as in Remark 2.5.2 are compatible between (M, \mathcal{N}) and $(M, (\mathcal{N}_{ad})')$. The relative spin structure used for (M, \mathcal{N}) is the restriction of the relative spin structure used for $(M, (\mathcal{N}_{ad})')$. Since \mathcal{N}_{ad} is obtained from \mathcal{N} by adding one extra oriented open interval, we have $\Gamma_n(M, \mathcal{N}_{ad}) = \Gamma_n(M, \mathcal{N})$. Then it is easy to check we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes (\#\mathcal{N}-1)} & \xrightarrow{\bar{L}_{(M, \mathcal{N})}} & \mathcal{S}_n(M, \mathcal{N}, 1) \\ \downarrow J & & \downarrow \lambda_{ad} \\ \Gamma_n(M, \mathcal{N}) \otimes \mathcal{O}(\mathrm{SL}_n)^{\otimes (\#\mathcal{N})} & \xrightarrow{\bar{L}_{(M, \mathcal{N}_{ad})}} & \mathcal{S}_n(M, \mathcal{N}_{ad}, 1) \end{array}$$

where J is the obvious embedding. Since both $\bar{L}_{(M, \mathcal{N}_{ad})}$ and $\bar{L}_{(M, \mathcal{N})}$ are isomorphisms, we have λ_{ad} is injective. \square

Chapter 3

The Frobenius map for the stated SL_n -skein module

In this chapter, we assume that the ground ring R is the complex field \mathbb{C} . We construct the Frobenius map for the stated SL_n -skein module, which is a \mathbb{C} -linear map

$$\mathcal{F}: \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v),$$

where (M, \mathcal{N}) is a marked 3-manifold with $\mathcal{N} \neq \emptyset$, and v is a root of unity of odd order m such that $\gcd(m, 2n) = 1$. This chapter is based on the author's work in [\[Wan23c\]](#).

3.1 The Frobenius homomorphism for SL_n

The main goal of this section is to construct the Frobenius homomorphism

$$\mathcal{F}: \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)$$

when v is a primitive m -th root of unit with m being coprime with $2n$ and every component of M contains at least one marking.

We know the generators for algebra $\mathcal{S}_n(M, \mathcal{N}, 1)$ and relations for these generators (Thm. [2.4.15](#)). It seems like we can define the Frobenius homomorphism on these

generators and then check all the relations. But $\mathcal{S}_n(M, \mathcal{N}, v)$ does not have algebra structure unless (M, \mathcal{N}) is the thickening of a pb surface.

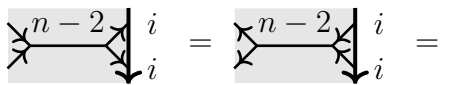
We use $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ to denote the set of linear maps from $\mathcal{S}_n(M, \mathcal{N}, v)$ to $\mathcal{S}_n(M, \mathcal{N}, v)$, then $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ has a natural algebra structure given by the combination of maps. Then we can define an algebra homomorphism $\tilde{\mathcal{F}} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ by defining $\tilde{\mathcal{F}}$ on the generators of $\mathcal{S}_n(M, \mathcal{N}, 1)$. We show the definition of $\tilde{\mathcal{F}}$ is independent of the choice of generators of $\mathcal{S}_n(M, \mathcal{N}, 1)$. Then we define \mathcal{F} to be the combination between $\tilde{\mathcal{F}} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ and an obvious linear map from $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ to $\mathcal{S}_n(M, \mathcal{N}, v)$, defined by sending $f \in \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ to $f(\emptyset)$ (\emptyset is the empty stated n -web). Furthermore $\tilde{\mathcal{F}}$ gives an $\mathcal{S}_n(M, \mathcal{N}, 1)$ -module structure for $\mathcal{S}_n(M, \mathcal{N}, v)$.

We will show that \mathcal{F} commutes with the splitting map Θ . Furthermore if (M, \mathcal{N}) is the thickening of an essentially bordered pb surface, we prove \mathcal{F} is an injective algebra homomorphism, and $\text{Im}\mathcal{F}$ lies in the center of the stated SL_n -skein algebra.

Let α be a (stated) framed oriented arc or a framed oriented knot in (M, \mathcal{N}) . We use $\alpha^{(m)}$ denote the disjoint union of m parallel copies of α (taken in the direction of the framing). We require $\alpha^{(m)}$ lives in a small enough open tubular neighborhood of α . From now on, when we say m parallel copies of an arc or a knot, we always mean taking the disjoint union of m parallel copies in the framing direction (also these m parallel copies live in a small enough open tubular neighborhood). We use $\text{---} \circlearrowleft^m \text{---}$ to denote the m parallel copies of --- .

3.1.1 On m parallel copies of the stated framed oriented arc

In this subsection, we will focus on finding relations for m parallel copies of the stated framed oriented arc.

Lemma 3.1.1 ([LS21; Wan23c]). *In $\mathcal{S}_n(M, \mathcal{N}, v)$, we have*  $= 0$.

Lemma 3.1.2. *In $\mathcal{S}_n(M, \mathcal{N}, v)$, we have*

$$q^{\frac{m}{n}} \text{---} \uparrow \circlearrowleft^m \text{---} - q^{-\frac{m}{n}} \text{---} \uparrow \circlearrowleft^m \text{---} = (q^m - q^{-m}) \text{---} \circlearrowleft^m \text{---} \quad (3.1)$$

where on the left-hand side of the equality $\text{---}\overset{\circ}{m}\text{---}$ is part of m parallel copies of a stated framed oriented arc and the single vertical oriented line is not part of these m parallel copies.

Proof. Using relation (2.4), we get

$$\begin{aligned}
 \text{Eq}_1: & \quad q^{\frac{1}{n}} \text{---}\overset{\circ}{m}\text{---} - q^{-\frac{1}{n}} \text{---}\overset{\circ}{m-1}\text{---} = (q - q^{-1}) \text{---}\overset{\circ}{m-1}\text{---} \\
 \text{Eq}_2: & \quad q^{\frac{1}{n}} \text{---}\overset{\circ}{m-1}\text{---} - q^{-\frac{1}{n}} \text{---}\overset{\circ}{m-2}\text{---} = (q - q^{-1}) \text{---}\overset{\circ}{m-2}\text{---} \\
 & \quad \vdots \\
 \text{Eq}_{m-1}: & \quad q^{\frac{1}{n}} \text{---}\overset{\circ}{m-2}\text{---} - q^{-\frac{1}{n}} \text{---}\overset{\circ}{m-1}\text{---} = (q - q^{-1}) \text{---}\overset{\circ}{m-2}\text{---} \\
 \text{Eq}_m: & \quad q^{\frac{1}{n}} \text{---}\overset{\circ}{m-1}\text{---} - q^{-\frac{1}{n}} \text{---}\overset{\circ}{m}\text{---} = (q - q^{-1}) \text{---}\overset{\circ}{m-1}\text{---}
 \end{aligned}$$

Then $(q^{\frac{2}{n}})^{m-1} \times \text{Eq}_1 + (q^{\frac{2}{n}})^{m-2} \times \text{Eq}_2 + \cdots + q^{\frac{2}{n}} \times \text{Eq}_{m-1} + \text{Eq}_m$, we get

$$\begin{aligned}
 & \quad q^{\frac{2m-1}{n}} \text{---}\overset{\circ}{m}\text{---} - q^{-\frac{1}{n}} \text{---}\overset{\circ}{m}\text{---} = (q - q^{-1}) \left(q^{\frac{2(m-1)}{n}} \text{---}\overset{\circ}{m-1}\text{---} + \right. \\
 & \quad \left. q^{\frac{2(m-2)}{n}} \text{---}\overset{\circ}{m-2}\text{---} + \cdots + q^{\frac{2}{n}} \text{---}\overset{\circ}{m-2}\text{---} + \text{---}\overset{\circ}{m-1}\text{---} \right) \\
 & \quad = (q - q^{-1}) \left(q^{\frac{2(m-1)}{n}} (q^{\frac{n-1}{n}})^{m-1} \text{---}\overset{\circ}{m-1}\text{---} + q^{\frac{2(m-2)}{n}} (q^{\frac{n-1}{n}})^{m-3} \text{---}\overset{\circ}{m-1}\text{---} \right. \\
 & \quad \left. + \cdots + q^{\frac{2}{n}} (q^{\frac{n-1}{n}})^{-(m-3)} \text{---}\overset{\circ}{m-1}\text{---} + (q^{\frac{n-1}{n}})^{-(m-1)} \text{---}\overset{\circ}{m-1}\text{---} \right) \\
 & \quad = q^{\frac{m-1}{n}} (q^m - q^{-m}) \text{---}\overset{\circ}{m-1}\text{---}
 \end{aligned}$$

where the second last equality is because of equation (51) in [LS21] and Lemma 3.1.1. Then we get the equation (3.1). \square

Lemma 3.1.3. *In $\mathcal{S}_n(M, \mathcal{N}, v)$, we have*

$$q^{\frac{m}{n}} \left(\text{---} \downarrow \text{---} \circledast \text{---} \right) - q^{-\frac{m}{n}} \left(\text{---} \downarrow \text{---} \circledast \text{---} \right) = (q^m - q^{-m}) \left(\text{---} \circledast \text{---} \right)$$

where on the left-hand side of the equality $\text{---} \circledast \text{---}$ is part of m parallel copies of a stated framed oriented arc and the single vertical oriented line is not part of these m parallel copies.

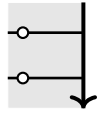
Proof. The proof is the same with the proof for Lemma 3.1.2. \square

Corollary 3.1.4. *If $q^{\frac{m}{n}} = 1$ or -1 , we have*

$$\left(\text{---} \downarrow \text{---} \circledast \text{---} \right) = \left(\text{---} \downarrow \text{---} \circledast \text{---} \right) \tag{3.2}$$

where $\text{---} \circledast \text{---}$ is part of m parallel copies of a stated framed oriented arc and the single vertical line is not part of these m parallel copies. Note that there are four possibilities to give orientations for the n -webs shown in the above local picture.

Proof. Lemmas 3.1.2, 3.1.3. \square

Lemma 3.1.5. *Let α be a framed oriented arc in (M, \mathcal{N}) . Suppose  is a local picture for $\alpha^{(2)}$, and the other two ends that are not shown in the local picture are assigned with the same state. Then we have*

$$\left(\text{---} \downarrow \text{---} \right)_i^j = q \left(\text{---} \downarrow \text{---} \right)_j^i$$

when $j < i$.

Proof. A small tubular open neighborhood of α in M is isomorphic to the thickening of the bigon, with α being identified with the core of the bigon. Then utilizing functoriality we may assume, without loss of generality, that (M, \mathcal{N}) is the thickening of the bigon, α is the core of the bigon. From Theorem 2.1.8 and the definition of $\mathcal{O}_q(SL_n)$, the Lemma is obviously true for the thickening of the bigon.

□

Lemma 3.1.6. *In $\mathcal{S}_n(M, \mathcal{N}, v)$, we have*

$$\begin{array}{l}
 \begin{array}{c} \textcircled{m} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \\ \downarrow \\ j \end{array} = \begin{cases} (q^{-\frac{1}{n} + \delta_{i,j}})^m \begin{array}{c} \textcircled{m} \\ \text{---} \\ j \\ \text{---} \\ i \\ \downarrow \\ j \end{array}, & \text{if } j \geq i, \\
 q^{-\frac{m}{n}}(q - q^{-1})(1 + q^2 + \dots + q^{2(m-1)}) \begin{array}{c} \textcircled{m-1} \\ \text{---} \\ j \\ \text{---} \\ i \\ \text{---} \\ j \\ \downarrow \\ j \end{array} + q^{-\frac{m}{n}} \begin{array}{c} \textcircled{m} \\ \text{---} \\ j \\ \text{---} \\ i \\ \downarrow \\ j \end{array}, & \text{if } j < i, \end{cases}
 \end{array}$$

where all the orientations of the n -webs are the same, that is, they are all pointing towards the marking or pointing out of the marking, on the left-hand side of the equality \textcircled{m} is part of m parallel copies of a stated framed oriented arc and the other single line (the one stated by i) is not part of these m parallel copies.

Proof. When $j \geq i$, it is trivial using relation (2.11).

Suppose $j < i$. We prove this case by using mathematical induction on m . When $m = 1$, it is true because of relation (2.11). Assume it is true for $m - 1$. Then we have

$$\begin{array}{l}
 \begin{array}{c} \textcircled{m} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \\ \downarrow \\ j \end{array} = q^{-\frac{1}{n}}(q - q^{-1}) \begin{array}{c} \textcircled{m-1} \\ \text{---} \\ j \\ \text{---} \\ i \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \\ \downarrow \\ j \end{array} + q^{-\frac{1}{n}} \begin{array}{c} \textcircled{m-1} \\ \text{---} \\ j \\ \text{---} \\ i \\ \text{---} \\ j \\ \downarrow \\ j \end{array} \\
 = q^{-\frac{1}{n}}(q - q^{-1})(q^{\frac{n-1}{n}})^{m-1} \begin{array}{c} \textcircled{m-1} \\ \text{---} \\ j \\ \text{---} \\ i \\ \downarrow \\ j \end{array} + \\
 q^{-\frac{m}{n}}(q - q^{-1})(1 + q^2 + \dots + q^{2(m-2)}) \begin{array}{c} \textcircled{m-2} \\ \text{---} \\ j \\ \text{---} \\ i \\ \text{---} \\ j \\ \downarrow \\ j \end{array} + q^{-\frac{m}{n}} \begin{array}{c} \textcircled{m-1} \\ \text{---} \\ j \\ \text{---} \\ i \\ \downarrow \\ j \end{array} \\
 = q^{-\frac{1}{n}}(q - q^{-1})(q^{\frac{n-1}{n}})^{m-1} q^{m-1} \begin{array}{c} \textcircled{m-1} \\ \text{---} \\ j \\ \text{---} \\ i \\ \text{---} \\ j \\ \downarrow \\ j \end{array} +
 \end{array}$$

Corollary 3.1.8. *If v is a primitive m -th root of unity with m and $2n$ being coprime with each other, we have*

where $\text{---}\textcircled{m}\text{---}$ is part of m parallel copies of a stated framed oriented arc and the other single line (the one stated by i) is not part of these m parallel copies. Note that there are four possibilities for us to give orientations for the stated n -webs shown in the local picture.

Proof. From the assumption, we have $q^m = (q^{\frac{1}{n}})^m = 1$ and q^2 is a primitive m -th root of unity. Then the Corollary 3.1.8 comes from Lemmas 3.1.6, 3.1.7 and Remark 3.1.1. \square

Conventions: In the following of this section, we always assume, unless especially specified, v is a primitive m -th root of unity with m and $2n$ being coprime with each other, all the marked 3-manifolds involved have at least one marking at every component, h is a relative spin structure for (M, \mathcal{N}) .

Recall that the stated skein algebra of the bigon \mathfrak{B} has a Hopf algebra structure, and $\mathcal{S}_n(\mathfrak{B}, v)$ is isomorphic to $\mathcal{O}_q(SL_n)$ as a Hopf algebra. We use Δ, ϵ to denote its coproduct and counit respectively. For any $i, j \in \mathbb{J}$, $b_{i,j} = i \left[\begin{array}{c} \leftarrow \\ \leftarrow \end{array} \right] j$.

Lemma 3.1.9. *In $\mathcal{S}_n(\mathfrak{B}, v)$, we have*

$$\begin{aligned} & \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} (b_{1,\sigma(1)})^{(m)} (b_{2,\sigma(2)})^{(m)} \dots (b_{n,\sigma(n)})^{(m)} \\ &= \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} (b_{\sigma(1),1}^1)^{(m)} (b_{\sigma(2),2})^{(m)} \dots (b_{\sigma(n),n})^{(m)} = 1. \end{aligned}$$

Proof. Theorem 2.1.8, Lemma 2.1.9. \square

Lemma 3.1.10. *In $\mathcal{S}_n(\mathfrak{B}, v)$, we have*

$$\Delta((b_{i,j})^{(m)}) = \sum_{1 \leq k \leq n} (b_{i,k})^{(m)} \otimes (b_{k,j})^{(m)}.$$

Proof. Theorem 2.1.8, Lemma 2.1.9. \square

Definition 3.1.11. Let l be a stated n -web consisting of stated framed oriented arcs. Suppose $l = \cup_{\alpha} \alpha$ where each α is a stated framed oriented arc. Define $l^{(m)} = \cup_{\alpha} \alpha^{(m)}$.

Corollary 3.1.12. Suppose D is a properly embedded disk in a marked 3-manifold (M, \mathcal{N}) with an embedded oriented open interval $\beta \subset D$. Let α be a stated n -web consisting of stated framed oriented arcs. Suppose α is (D, β) -transverse and intersects β in exactly one point. For any state k , let α_k , which is a stated n -web in $Cut_{(D, \beta)}(M, \mathcal{N})$, be the lift of α with both newly created boundary points having the state k . Then $\Theta(\alpha^{(m)}) = \sum_{1 \leq k \leq n} (\alpha_k)^{(m)}$.

Proof. We use the same trick used in Lemma 4.2 in [BL20]. We can assume α has one component. An open small tubular neighborhood of $\alpha \cup D$ in M is isomorphic to the thickening of a bigon such that α is the core of the bigon and D is the thickening of an ideal arc connecting the two ideal points of the bigon. This completes the proof because of functoriality and Lemma 3.1.10. \square

Corollary 3.1.13. Suppose D is a properly embedded disk in a marked 3-manifold (M, \mathcal{N}) with an embedded oriented open interval $\beta \subset D$. Let α be a stated n -web consisting of stated framed oriented arcs. Suppose α is (D, β) -transverse, and $\alpha \cap \beta \neq \emptyset$. For any map $s : \beta \cap \alpha \rightarrow \{1, 2, \dots, n\}$, let α_s , which is a stated n -web in $Cut_{(D, \beta)}(M, \mathcal{N})$, be the lift of α such that for every $P \in \beta \cap \alpha$ the two newly created boundary points corresponding to P both have the state $s(P)$. Then

$$\Theta(\alpha^{(m)}) = \sum_{s: \beta \cap \alpha \rightarrow \{1, 2, \dots, n\}} (\alpha_s)^{(m)}. \quad (3.4)$$

Note that $\Theta(\alpha) = \sum_{s: \beta \cap \alpha \rightarrow \{1, 2, \dots, n\}} \alpha_s$.

Proof. Here we use the technique used in page 24 in [BL20].

Let U be an open small tubular neighborhood of $\alpha \cup D$. Then we have U is the thickening of a pb surface. Because of functoriality, we only need to prove equation (3.4) when $M = U$. Thus we can suppose that α is a simple diagram on a pb surface \mathfrak{S} consisting of stated arcs, c is an interior ideal arc of \mathfrak{S} , and α is transversal to

c. Then equation (3.4) becomes

$$\Theta_c(\alpha^{(m)}) = \sum_{s:c \cap \alpha \rightarrow \{1,2,\dots,n\}} (\alpha(h,s))^{(m)} \quad (3.5)$$

where h be a linear order on the set $\alpha \cap c$ and $\alpha(h,s)$ is defined in subsection 2.1.4.

If $|\alpha \cap c| = 1$, then equation (3.5) follows exactly from Corollary 3.1.12.

Now suppose $|\alpha \cap c| > 1$. Let \mathcal{V} be a finite subset of c and $c \setminus \mathcal{V} = \cup_{i=1}^k c_i$ such that each c_i intersects α at exactly one point. Let $\hat{\mathfrak{S}} = \mathfrak{S} \setminus \mathcal{V}$. Then we are ready to use Lemma 2.1.5. Let \mathfrak{S}' be the result of splitting \mathfrak{S} along c , and $\hat{\mathfrak{S}}'$ be the result of splitting $\hat{\mathfrak{S}}$ along all c_i . The linear order h on $\alpha \cap c$ induces a linear order on $\{c_i \mid 1 \leq i \leq k\}$, which is also denoted as h . Let $\hat{\alpha} \in \mathcal{S}_n(\hat{\mathfrak{S}}, v)$ be the element defined by the same diagram α , but considered as an element in $\mathcal{S}_n(\hat{\mathfrak{S}}, v)$. Then clearly $\iota_*(\hat{\alpha}) = \alpha$, where $\iota_* : \mathcal{S}_n(\hat{\mathfrak{S}}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)$ is the induced map, see Lemma 2.1.5. Since $\hat{\alpha}$ intersects each c_i in exactly one point, from Corollary 3.1.12 we have

$$\Theta_c(\hat{\alpha}^{(m)}) = \sum_{s:c \cap \alpha \rightarrow \{1,2,\dots,n\}} (\hat{\alpha}_s)^{(m)},$$

where $\hat{\alpha}_s$ is the lift of $\hat{\alpha}$ such that for every $P \in c \cap \alpha$ the two newly created boundary points corresponding to P both have the state $s(P)$.

Then we have

$$\begin{aligned} \Theta_c(\alpha) &= \Theta_c(\iota_*(\hat{\alpha})) = (\iota_h)_*(\Theta_c(\hat{\alpha})) \\ &= (\iota_h)_* \left(\sum_{s:c \cap \alpha \rightarrow \{1,2,\dots,n\}} (\hat{\alpha}_s)^{(m)} \right) \\ &= \sum_{s:u \cap \alpha \rightarrow \{1,2,\dots,n\}} (\alpha(h,s))^{(m)} \end{aligned}$$

where $(\iota_h)_*$ is the induced map, see Lemma 2.1.5.

□

Lemma 3.1.14. *We use $P_{1,2}$ to denote the once punctured bigon. Let α (respectively α') be the framed oriented arc in the top left (respectively top right) picture in Figure 3.1. Let $1 \leq i, j \leq n$. Then in $\mathcal{S}_n(P_{1,2}, v)$, we have*

$$(\alpha_{i,j})^{(m)} = (\alpha'_{i,j})^{(m)}.$$

Proof. As in Figure 3.1, we use the red ideal arc to cut $P_{1,2}$. For any state v , we use α_v to denote the lift of $\alpha_{i,j}$ with the newly created two endpoints stated by v , similarly we can define $(\alpha')_v$. From Corollary 3.1.13, we know

$$\Theta((\alpha_{i,j})^{(m)}) = \sum_{1 \leq v \leq n} (\alpha_v)^{(m)}, \quad \Theta((\alpha'_{i,j})^{(m)}) = \sum_{1 \leq v \leq n} ((\alpha')_v)^{(m)}.$$

From Corollary 3.1.4, we know $(\alpha_v)^{(m)} = ((\alpha')_v)^{(m)}$. Thus we have $\Theta(\alpha^{(m)}) = \Theta((\alpha')^{(m)})$. This completes the proof by the injectivity of Θ , see Proposition 2.1.3. \square

Lemma 3.1.15. *We use $P_{1,1}$ to denote the once punctured monogon. Let β (respectively β') be the framed oriented arc in the bottom left (respectively bottom right) picture in Figure 3.1. Let $1 \leq i, j \leq n$. Then in $\mathcal{S}_n(P_{1,1}, v)$, we have*

$$(\beta_{i,j})^{(m)} = (\beta'_{i,j})^{(m)}.$$

Proof. The proof is the same with Lemma 3.1.14. The only difference is that we will use Corollary 3.1.8, instead of Corollary 3.1.4. \square

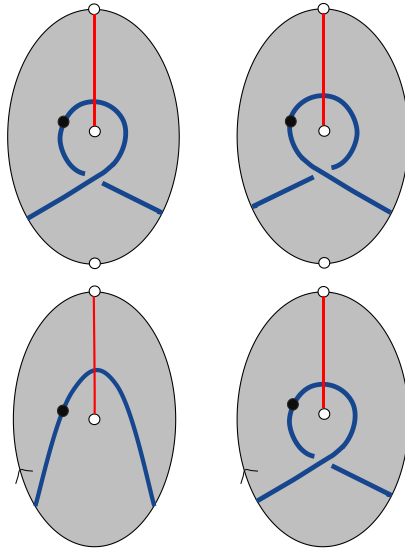


FIGURE 3.1: Blue lines represent framed oriented arcs, where the framing is the one pointing towards readers and the orientation is indicated by the black dot. Red lines are the cutting ideal arcs.

Corollary 3.1.16. *In $\mathcal{S}_n(M, \mathcal{N}, v)$, we have*

$$\begin{array}{c} | \\ \textcircled{m} \\ | \\ \text{---} \textcircled{m} \text{---} \\ | \end{array} = \begin{array}{c} | \\ \textcircled{m} \\ \text{---} \textcircled{m} \text{---} \\ | \end{array} \quad (3.6)$$

where on the left-hand (right-hand) side of the equality the two $\text{---} \textcircled{m} \text{---}$ are parts of the same m parallel copies of a stated framed oriented arc.

Proof. The Corollary can be easily proved by using functoriality and Lemma 3.1.14. \square

Corollary 3.1.17. *In $\mathcal{S}_n(M, \mathcal{N}, v)$, we have*

$$\begin{array}{c} \textcircled{m} \\ \diagdown \\ \textcircled{m} \\ \diagup \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} i \\ \\ j \end{array} = \begin{array}{c} \textcircled{m} \\ \text{---} \\ \textcircled{m} \\ \text{---} \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} j \\ \\ i \end{array}$$

where on the left-hand (right-hand) side of the equality the two $\text{---} \textcircled{m} \text{---}$ are parts of the same m parallel copies of a stated framed oriented arc.

Proof. The Corollary can be easily proved by using functoriality and Lemma 3.1.15. \square

Lemma 3.1.18. *In $\mathcal{S}_n(M, \mathcal{N}, v)$, we have*

$$\text{---} \boxed{\bar{H}} \text{---} \textcircled{m} \downarrow k = \text{---} \textcircled{m} \downarrow k = \text{---} \boxed{H} \text{---} \textcircled{m} \downarrow k$$

where H, \bar{H} are positive half twist and negative half twist respectively, and the parts not shown in the local picture can be arbitrary, that is, $\text{---} \textcircled{m} \text{---}$ may not be part of m parallel copies of some stated framed oriented arc.

Proof. This Lemma can be easily proved by using equation (51) in [LS21] and Lemma 3.1.1. \square

Lemma 3.1.19. *In $\mathcal{S}_n(M, \mathcal{N}, v)$, we have*

$$\sum_{1 \leq k \leq n} (-1)^{k+1} \begin{array}{c} \leftarrow \textcircled{m} \uparrow k \\ \uparrow \\ \downarrow \textcircled{m} \rightarrow \bar{k} \end{array} = \begin{array}{c} \text{---} \textcircled{m} \text{---} \\ \uparrow \end{array} \quad (3.7)$$

where all three $\text{---} \textcircled{m} \text{---}$ are parts of m parallel copies of some stated framed oriented arc.

Proof. Because of functoriality, we can assume besides the components that are shown in equation (3.7), the stated n -webs have no other components. We use β to denote the marking shown in equation (3.7). Then there is a right coaction $\Delta_\beta : \mathcal{S}_n(M, \mathcal{N}, v) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v) \otimes \mathcal{O}_q(SL_n)$, see Subsection 7.1 in [LS21]. The coaction Δ_β is actually defined by cutting out a bigon. From Corollary 3.1.13, we have

$$\Delta_\beta \left(\begin{array}{c} \text{---} \textcircled{m} \text{---} \\ \uparrow \end{array} \right) = \sum_{1 \leq u, v \leq n} \begin{array}{c} \leftarrow \textcircled{m} \uparrow u \\ \uparrow \\ \downarrow \textcircled{m} \rightarrow v \end{array} \otimes \begin{array}{c} u \uparrow \\ \text{---} \textcircled{m} \text{---} \\ v \rightarrow \end{array}.$$

Thus we have

$$\begin{aligned} \begin{array}{c} \text{---} \textcircled{m} \text{---} \\ \uparrow \end{array} &= \sum_{1 \leq u, v \leq n} \varepsilon \left(\begin{array}{c} u \uparrow \\ \text{---} \textcircled{m} \text{---} \\ v \rightarrow \end{array} \right) \begin{array}{c} \leftarrow \textcircled{m} \uparrow u \\ \uparrow \\ \downarrow \textcircled{m} \rightarrow v \end{array} \\ &= \sum_{1 \leq u, v \leq n} \delta_{u, \bar{v}} (c_v)^m \begin{array}{c} \leftarrow \textcircled{m} \uparrow u \\ \uparrow \\ \downarrow \textcircled{m} \rightarrow v \end{array} = \sum_{1 \leq v \leq n} (-1)^{\bar{v}+1} \begin{array}{c} \leftarrow \textcircled{m} \uparrow \bar{v} \\ \uparrow \\ \downarrow \textcircled{m} \rightarrow v \end{array} \end{aligned}$$

where the second equality is because of Lemma 3.1.18 and relation (2.9). \square

Lemma 3.1.20. *In $\mathcal{S}_n(M, \mathcal{N}, v)$, we have*

$$\text{---} \textcircled{m} \text{---} = \text{---} \textcircled{m} \text{---} = d_n \text{---} \textcircled{m} \text{---}$$

where $\text{---} \textcircled{m} \text{---}$ is part of m parallel copies of some stated framed oriented arc.

Proof. Since a positive kink for the m parallel copies of some framed line is isotopic to combining a full positive twist and giving a positive kink for each parallel framed

line, then we have

$$\text{---} \bigcirc \text{---} \textcircled{m} = t^m \text{---} \boxed{F} \text{---} \textcircled{m} = d_n \text{---} \textcircled{m}$$

where the first equality is from relation (2.5) and the second equality is from Lemma 3.1.18.

Similarly we can show $\text{---} \bigcirc \text{---} \textcircled{m} = d_n \text{---} \textcircled{m}$. \square

3.1.2 An action of $\mathcal{S}_n(M, \mathcal{N}, 1)$ on $\mathcal{S}_n(M, \mathcal{N}, v)$

Let (M, \mathcal{N}) be a marked 3-manifold. Recall that for $l = \cup_{\alpha} \alpha$, where each α is a stated framed oriented arc in (M, \mathcal{N}) , we define $l^{(m)}$ to be $\cup_{\alpha} \alpha^{(m)}$.

Lemma 3.1.21. *Let l be a stated n -web consisting of stated framed oriented arcs. Let T_1, T_2 be two isotopic stated n -webs such that $T_1 \cap l = T_2 \cap l = \emptyset$. Then we have*

$$l^{(m)} \cup T_1 = l^{(m)} \cup T_2 \in \mathcal{S}_n(M, \mathcal{N}, v).$$

Proof. Corollaries 3.1.4, 3.1.8. \square

For any stated n -web l consisting of stated framed oriented arcs, we will define a linear map $F_l : \mathcal{S}_n(M, \mathcal{N}, v) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)$. For any stated n -web α , first we isotope α such that $\alpha \cap l = \emptyset$, then define $F_l(\alpha) = l^{(m)} \cup \alpha \in \mathcal{S}_n(M, \mathcal{N}, v)$. From Lemma 3.1.21, we know $F_l(\alpha)$ is independent of how we isotope α . Thus F_l is well-defined on the set of isotopy classes of stated n -webs. Since all the skein relations, used to define stated SL_n -skein modules, are local, F_l preserves all these relations. Thus F_l is a linear map from $\mathcal{S}_n(M, \mathcal{N}, v)$ to $\mathcal{S}_n(M, \mathcal{N}, v)$. We regard empty n -web as a stated n -web consisting of stated framed oriented arcs, then $F_{\emptyset} = Id_{\mathcal{S}_n(M, \mathcal{N}, v)}$.

Recall that we use $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ to denote the set of linear maps from $\mathcal{S}_n(M, \mathcal{N}, v)$ to itself, and $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ has an obvious algebra structure. Then $F_l \in \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ for stated n -web l consisting of stated framed oriented arcs.

Let α, β be any two stated n -webs both consisting of stated framed oriented arcs, then we have $F_{\alpha} F_{\beta} = F_{\beta} F_{\alpha} = F_{\alpha \cup \beta}$ from Corollaries 3.1.4, 3.1.8, where

$\alpha \cup \beta$ is the disjoint union. Let $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ be a subvector space of $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ linearly spanned by F_l for all stated n -webs l consisting of stated framed oriented arcs. The above discussion implies $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ is a commutative subalgebra of $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$.

Lemma 3.1.22. *Let α be any framed oriented arc in (M, \mathcal{N}) . In $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ we have*

$$\begin{aligned} & \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} F_{\alpha_{1,\sigma(1)}} F_{\alpha_{2,\sigma(2)}} \cdots F_{\alpha_{n,\sigma(n)}} \\ &= \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} F_{\alpha_{\sigma(1),1}} F_{\alpha_{\sigma(2),2}} \cdots F_{\alpha_{\sigma(n),n}} = 1. \end{aligned}$$

Proof. Let β be a stated n -web that has no intersection with α . A small tubular open neighborhood U of α in M is isomorphic to the thickening of the bigon, with α being identified with the core of the bigon and $U \cap \beta = \emptyset$. We isotope $\alpha_{i,j}$, $i, j \in \mathbb{J}$, inside U such that there is no intersection among them. Then

$$\begin{aligned} & \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} F_{\alpha_{1,\sigma(1)}} F_{\alpha_{2,\sigma(2)}} \cdots F_{\alpha_{n,\sigma(n)}}(\beta) \\ &= \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} (\alpha_{1,\sigma(1)})^{(m)} \cup (\alpha_{2,\sigma(2)})^{(m)} \cup \cdots \cup (\alpha_{n,\sigma(n)})^{(m)} \cup \beta = \beta. \end{aligned}$$

where the last equality is because of functoriality and Lemma 3.1.9. Thus we have

$$\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} F_{\alpha_{1,\sigma(1)}} F_{\alpha_{2,\sigma(2)}} \cdots F_{\alpha_{n,\sigma(n)}} = 1.$$

Similarly we can prove $\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} F_{\alpha_{\sigma(1),1}} F_{\alpha_{\sigma(2),2}} \cdots F_{\alpha_{\sigma(n),n}} = 1$. \square

Lemma 3.1.23. *For any two stated framed oriented arcs α_1, α_2 , suppose $s(\alpha_1(0)) = s(\alpha_2(0))$, $s(\alpha_1(1)) = s(\alpha_2(1))$, and $[\alpha_1] = [\alpha_2] \in \pi_1^{Mor}(M, \mathcal{N})$.*

(a) *If $h(\widetilde{\alpha}_1) = h(\widetilde{\alpha}_2)$, then $F_{\alpha_1} = F_{\alpha_2} \in \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$.*

(b) *If $h(\widetilde{\alpha}_1) \neq h(\widetilde{\alpha}_2)$, then $F_{\alpha_1} = d_n F_{\alpha_2} \in \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$.*

Proof. Here we use a standard fact that two embeddings of a compact graph in M are homotopic if and only if one can be obtained from the other by crossing changes, height changes, and isotopies [PS00]. Let β be a stated n -web that has no intersection with $\alpha_1 \cup \alpha_2$. From the above standard fact and Corollaries 3.1.4,

3.1.8, 3.1.16, 3.1.17, we know $(\alpha_1)^{(m)} \cup \beta = (\alpha_3)^{(m)} \cup \beta$ where α_3 is exactly α_2 but with different framing and $h(\widetilde{\alpha}_1) = h(\widetilde{\alpha}_3)$. Thus α_3 can be obtained from α_2 by adding a number of kinks, we suppose this number is k . Then we have $(\alpha_1)^{(m)} \cup \beta = (\alpha_3)^{(m)} \cup \beta = d_n^k(\alpha_2)^{(m)} \cup \beta$ from Lemma 3.1.20.

If $h(\widetilde{\alpha}_1) = h(\widetilde{\alpha}_2)$, then $h(\widetilde{\alpha}_2) = h(\widetilde{\alpha}_3)$. Thus k is even, then

$$F_{\alpha_1}(\beta) = (\alpha_1)^{(m)} \cup \beta = d_n^k(\alpha_2)^{(m)} \cup \beta = (\alpha_2)^{(m)} \cup \beta = F_{\alpha_2}(\beta).$$

If $h(\widetilde{\alpha}_1) \neq h(\widetilde{\alpha}_2)$, then $h(\widetilde{\alpha}_2) \neq h(\widetilde{\alpha}_3)$. Thus k is odd, then

$$F_{\alpha_1}(\beta) = (\alpha_1)^{(m)} \cup \beta = d_n^k(\alpha_2)^{(m)} \cup \beta = d_n(\alpha_2)^{(m)} \cup \beta = d_n F_{\alpha_2}(\beta).$$

□

rem A morphism $[\alpha] \in \pi_1^{Mor}(M, \mathcal{N})$ and two integers $1 \leq i, j \leq n$ uniquely determine an element in $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ in the following way: We choose a good representative α such that α is a properly embedded arc in M . Then we give a framing to α respecting \mathcal{N} , that is, the framing at endpoints are given by the velocity vectors of \mathcal{N} . We denote this framed oriented arc as $\hat{\alpha}$. We choose the framing such that $h(\hat{\alpha}) = 0$, then we obtain an element $F_{\hat{\alpha}_{i,j}} \in \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$. Suppose we choose a different good representative α' . We have $[\hat{\alpha}] = [\hat{\alpha}'] \in \pi_1^{Mor}(M, \mathcal{N})$ and $h(\hat{\alpha}') = h(\hat{\alpha}) = 0$. Then $F_{\hat{\alpha}_{i,j}} = F_{\hat{\alpha}'_{i,j}} \in \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ because of Lemma 3.1.23.

We use $S_{i,j}^{[\alpha],m}$ to denote $F_{\hat{\alpha}_{i,j}}$, and use $S^{[\alpha],m}$ to denote an n by n matrix in $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ such that $(S^{[\alpha],m})_{i,j} = S_{i,j}^{[\alpha],m}$, $1 \leq i, j \leq n$.

Then for any stated framed oriented arc $\alpha_{i,j}$, we have

$$F_{\alpha_{i,j}} = d_n^{h(\tilde{\alpha})} S_{i,j}^{[\alpha],m} \in \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}. \quad (3.8)$$

Here we recall the definition for matrix A ,

$$A_{i,j} = (-1)^{i+1} \delta_{i,j}, \quad 1 \leq i, j \leq n.$$

For any complex number $k \in \mathbb{C}$, we will use k to denote $kId_{\mathcal{S}_n(M, \mathcal{N}, v)} \in \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))$ for simplicity.

Proposition 3.1.24. (a) For any two morphisms $[\alpha], [\beta] \in \pi_1^{Mor}(M, \mathcal{N})$, if $[\beta][\alpha]$ makes sense, then $AS^{[\beta*\alpha], m} = AS^{[\beta], m} AS^{[\alpha], m}$.

(b) For any morphism $[\eta] \in \pi_1^{Mor}(M, \mathcal{N})$, we have $\det(S^{[\eta], m}) = 1$. Especially $\det(AS^{[\eta], m}) = 1$.

(c) Suppose $[o] \in \pi_1^{Mor}(M, \mathcal{N})$ is the identity morphism for an object, then $S^{[o], m} = d_n A$. Especially $AS^{[o], m} = I$.

Proof. (a) We have

$$(S^{[\beta], m} AS^{[\alpha], m})_{i,j} = \sum_{1 \leq k \leq n} (-1)^{k+1} S_{i,k}^{[\beta], m} S_{k,j}^{[\alpha], m} = S_{i,j}^{[\beta*\alpha], m} = (S^{[\beta*\alpha], m})_{i,j}$$

where the second equality is from Lemma 3.1.19. Thus we have $AS^{[\beta*\alpha], m} = AS^{[\beta], m} AS^{[\alpha], m}$.

(b) is implied by Lemma 3.1.22.

(c) For $1 \leq i, j \leq n$, we have

$$S_{i,j}^{[o], m} = \left(\begin{array}{c} \uparrow i \\ \leftarrow \\ \downarrow j \end{array} \right)^{(m)} = \delta_{i,j} (c_i^{-1})^m = d_n A_{i,j}.$$

Thus $S^{[o], m} = d_n A$.

□

Suppose M is connected, and the components of \mathcal{N} consist of e_0, e_1, \dots, e_{k-1} where k is a positive integer. If $k \geq 2$, for each $1 \leq t \leq k-1$, let α_t be an oriented path connecting e_0 and e_t with $\alpha_t(0) \in e_0$ and $\alpha_t(1) \in e_t$. We use $[o]$ to denote the identity morphism in $\pi_1^{Mor}(M, e_0)$.

We define an algebra homomorphism $\tilde{\mathcal{F}} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ by defining $\tilde{\mathcal{F}}$ on the above generators, and then check all the relations in (2.19).

Theorem 3.1.25. *There exists an algebra homomorphism*

$$\begin{aligned}\tilde{\mathcal{F}} : \mathcal{S}_n(M, \mathcal{N}, 1) &\rightarrow \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)} \\ S_{i,j}^{[\alpha]} &\mapsto S_{i,j}^{[\alpha],m}\end{aligned}$$

where $[\alpha] \in \pi_1^{\text{Mor}}(M, e_0) \cup \{[\alpha_1], \dots, [\alpha_{k-1}]\}$, $1 \leq i, j \leq n$.

Proof. We know $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ is a commutative algebra. Proposition 3.1.24 shows $\tilde{\mathcal{F}}$ preserves all the relations in (2.19). Thus $\tilde{\mathcal{F}}$ is a well-defined algebra homomorphism. \square

Note that we have $\tilde{\mathcal{F}}(S^{[\alpha]}) = S^{[\alpha],m}$ for all $[\alpha] \in \pi_1^{\text{Mor}}(M, e_0) \cup \{[\alpha_1], \dots, [\alpha_{k-1}]\}$. We will show this is true for any $[\alpha] \in \pi_1^{\text{Mor}}(M, \mathcal{N})$.

The construction for $\tilde{\mathcal{F}}$ in Theorem 3.1.25 depends on the choice of the generators for algebra $\mathcal{S}_n(M, \mathcal{N}, 1)$. We will show $\tilde{\mathcal{F}}$ is independent of the choice of these generators. Actually, we will show a stronger result in the following Theorem.

Theorem 3.1.26. *For any stated n -web l consisting of stated framed oriented arcs, we have $\tilde{\mathcal{F}}(l) = Fl$. Especially $\tilde{\mathcal{F}}$ is independent of the choice of the generators for algebra $\mathcal{S}_n(M, \mathcal{N}, 1)$, and $\tilde{\mathcal{F}}$ is surjective.*

Proof. Note that we will use equations (2.18), (3.8), and Propositions 2.4.3, 3.1.24 in the following proof.

First we show for any $[\alpha] \in \pi_1^{\text{Mor}}(M, \mathcal{N})$ and two integers $1 \leq i, j \leq n$, we have $\tilde{\mathcal{F}}(S_{i,j}^{[\alpha]}) = S_{i,j}^{[\alpha],m}$, that is, to show $\tilde{\mathcal{F}}(S^{[\alpha]}) = S^{[\alpha],m}$. This is clearly true if $k = 1$, since $\pi_1^{\text{Mor}}(M, \mathcal{N}) = \pi_1^{\text{Mor}}(M, e_0)$ when $k = 1$.

Suppose $k \geq 2$. For any $1 \leq t \leq k - 1$, we have

$$\begin{aligned}I &= \tilde{\mathcal{F}}(AS^{[\alpha_t]}AS^{[\alpha_t^{-1}]}) = \tilde{\mathcal{F}}(AS^{[\alpha_t^{-1}]}AS^{[\alpha_t]}) = A\tilde{\mathcal{F}}(S^{[\alpha_t]})A\tilde{\mathcal{F}}(S^{[\alpha_t^{-1}]}) = A\tilde{\mathcal{F}}(S^{[\alpha_t^{-1}]})A\tilde{\mathcal{F}}(S^{[\alpha_t]}) \\ I &= AS^{[\alpha_t],m}AS^{[\alpha_t^{-1}],m} = AS^{[\alpha_t^{-1}],m}AS^{[\alpha_t],m} = A\tilde{\mathcal{F}}(S^{[\alpha_t]})AS^{[\alpha_t^{-1}],m} = AS^{[\alpha_t^{-1}],m}A\tilde{\mathcal{F}}(S^{[\alpha_t]}).\end{aligned}$$

Thus we get $AS^{[\alpha_t^{-1}],m} = A\tilde{\mathcal{F}}(S^{[\alpha_t^{-1}]})$, which implies $S^{[\alpha_t^{-1}],m} = \tilde{\mathcal{F}}(S^{[\alpha_t^{-1}]})$.

For any $[\alpha] \in \pi_1^{\text{Mor}}(M, \mathcal{N})$, suppose $\alpha(0) \in e_u, \alpha(1) \in e_v$. If $u = v = 0$, it is obvious that $\tilde{\mathcal{F}}(S^{[\alpha]}) = S^{[\alpha],m}$. If $u \neq 0, v \neq 0$, then we have $[\alpha_v^{-1} * \alpha * \alpha_u] \in \pi_1^{\text{Mor}}(M, e_0)$.

Then we have

$$\begin{aligned} A\tilde{\mathcal{F}}(S^{[\alpha]}) &= \tilde{\mathcal{F}}(AS^{[\alpha]}) = \tilde{\mathcal{F}}(AS^{[\alpha_v]}AS^{[\alpha_v^{-1}*\alpha*\alpha_u]}AS^{[\alpha_u^{-1}]}) \\ &= A\tilde{\mathcal{F}}(S^{[\alpha_v]})A\tilde{\mathcal{F}}(S^{[\alpha_v^{-1}*\alpha*\alpha_u]})A\tilde{\mathcal{F}}(S^{[\alpha_u^{-1}]}) \\ &= AS^{[\alpha_v],m}AS^{[\alpha_v^{-1}*\alpha*\alpha_u],m}AS^{[\alpha_u^{-1}],m} = AS^{[\alpha],m}. \end{aligned}$$

Thus we have $\hat{\mathcal{F}}(S^{[\alpha]}) = S^{[\alpha],m}$. Similar we can prove $\tilde{\mathcal{F}}(S^{[\alpha]}) = S^{[\alpha],m}$ when $u = 0, v \neq 0$ or $u \neq 0, v = 0$.

For any stated framed oriented arc $\alpha_{i,j}$, we have $\alpha_{i,j} = d_n^{h(\tilde{\alpha})}S_{i,j}^{[\alpha]}$ in $\mathcal{S}_n(M, \mathcal{N}, 1)$.

Then

$$\tilde{\mathcal{F}}(\alpha_{i,j}) = d_n^{h(\tilde{\alpha})}\tilde{\mathcal{F}}(S_{i,j}^{[\alpha]}) = d_n^{h(\tilde{\alpha})}S_{i,j}^{[\alpha],m} = F_{\alpha_{i,j}}.$$

Let $l = \cup_{1 \leq i \leq k} \alpha_i$ where each α_i is a stated framed oriented arc. Then

$$\tilde{\mathcal{F}}(l) = \tilde{\mathcal{F}}(\alpha_1 \cdots \alpha_k) = \tilde{\mathcal{F}}(\alpha_1) \cdots \tilde{\mathcal{F}}(\alpha_k) = F_{\alpha_1} \cdots F_{\alpha_k} = F_{\cup_{1 \leq i \leq k} \alpha_i} = Fl.$$

The surjectivity of $\tilde{\mathcal{F}}$ is obvious. □

Since $\tilde{\mathcal{F}} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ is an algebra homomorphism, this gives an action of $\mathcal{S}_n(M, \mathcal{N}, 1)$ on $\mathcal{S}_n(M, \mathcal{N}, v)$, defined by, for any $\alpha \in \mathcal{S}_n(M, \mathcal{N}, 1), \beta \in \mathcal{S}_n(M, \mathcal{N}, v), \alpha \cdot \beta = \tilde{\mathcal{F}}(\alpha)(\beta)$.

3.1.3 Construction for the Frobenius homomorphism

In this section, we will define the Frobenius homomorphism $\mathcal{F} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)$.

Theorem 3.1.27. *For any marked 3-manifold (M, \mathcal{N}) , there exists a unique linear map*

$$\mathcal{F} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)$$

such that $\mathcal{F}(l) = l^{(m)}$ for any stated n -web l consisting of stated framed oriented arcs.

Proof. Since we assume every component of M contains at least one marking, $\mathcal{S}_n(M, \mathcal{N}, v)$ is a linear span by stated n -webs consisting of stated framed oriented arcs, thus the uniqueness is obvious.

To construct \mathcal{F} , we can suppose M is connected. We use T to denote the linear map from $\text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ to $\mathcal{S}_n(M, \mathcal{N}, v)$, defined by $T(f) = f(\emptyset)$ where $f \in \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ and \emptyset represents the empty stated n -web. Define $\mathcal{F} = T \circ \tilde{\mathcal{F}}$, then \mathcal{F} is a linear map from $\mathcal{S}_n(M, \mathcal{N}, 1)$ to $\mathcal{S}_n(M, \mathcal{N}, v)$ such that $\mathcal{F}(l) = l^{(m)}$ for any stated n -web l consisting of stated framed oriented arcs.

□

rem Let (M, \mathcal{N}) be a marked 3-manifold, and x be an element in $\mathcal{S}_n(M, \mathcal{N}, v)$. Then the proof for Theorem 3.1.27 shows there exists a unique linear map $\mathcal{F}_x : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)$ such that $\mathcal{F}_x(l) = \tilde{\mathcal{F}}(l)(x)$ for any $l \in \mathcal{S}_n(M, \mathcal{N}, 1)$. Then $\mathcal{F} = \mathcal{F}_\emptyset$ where \emptyset is the empty set stated n -web.

For any two elements $\alpha, \beta \in \mathcal{S}_n(M, \mathcal{N}, 1)$, we know $\tilde{\mathcal{F}}(\alpha) \in \text{End}(\mathcal{S}_n(M, \mathcal{N}, v))^{(m)}$ and $\mathcal{F}(\beta) \in \mathcal{S}_n(M, \mathcal{N}, v)$, then $\tilde{\mathcal{F}}(\alpha)(\mathcal{F}(\beta)) \in \mathcal{S}_n(M, \mathcal{N}, v)$. It easy to check we have $\tilde{\mathcal{F}}(\alpha)(\mathcal{F}(\beta)) = \tilde{\mathcal{F}}(\beta)(\mathcal{F}(\alpha)) \in \mathcal{S}_n(M, \mathcal{N}, v)$. Actually if we suppose $\alpha = \sum_{1 \leq i \leq u} k_i \alpha_i, \beta = \sum_{1 \leq j \leq v} t_j \beta_j$ where α_i, β_j consist of stated framed oriented arcs and $\alpha_i \cap \beta_j = \emptyset$, then we have

$$\tilde{\mathcal{F}}(\alpha)(\mathcal{F}(\beta)) = \tilde{\mathcal{F}}(\beta)(\mathcal{F}(\alpha)) = \sum_{1 \leq i \leq u, 1 \leq j \leq v} k_i t_j (\alpha_i)^{(m)} \cup (\beta_j)^{(m)} \in \mathcal{S}_n(M, \mathcal{N}, v). \quad (3.9)$$

We use $\mathcal{S}_n(M, \mathcal{N}, v)^{(m)}$ to denote $\text{Im} \mathcal{F}$, then there is a surjective map

$$\hat{\mathcal{F}} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)^{(m)}$$

induced by \mathcal{F} .

Theorem 3.1.28. *For any marked 3-manifold (M, \mathcal{N}) , there is a commutative algebra structure for $\mathcal{S}_n(M, \mathcal{N}, v)^{(m)}$, which makes $\hat{\mathcal{F}} : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)^{(m)}$ a surjective algebra homomorphism.*

Proof. For any two elements $x, y \in \mathcal{S}_n(M, \mathcal{N}, v)^{(m)}$, suppose $x = \mathcal{F}(\alpha), y = \mathcal{F}(\beta)$ where $\alpha, \beta \in \mathcal{S}_n(M, \mathcal{N}, 1)$, then define $xy = \tilde{\mathcal{F}}(\alpha)(\mathcal{F}(\beta)) \in \mathcal{S}_n(M, \mathcal{N}, v)$. We need

to check this is a well-defined multiplication, that is, to check xy is independent of the choice of α, β . Suppose we also have $x = \mathcal{F}(\alpha_1), y = \mathcal{F}(\beta_1)$ where $\alpha_1, \beta_1 \in \mathcal{S}_n(M, \mathcal{N}, 1)$. Then

$$\tilde{\mathcal{F}}(\alpha)(\mathcal{F}(\beta)) = \tilde{\mathcal{F}}(\alpha)(\mathcal{F}(\beta_1)) = \tilde{\mathcal{F}}(\beta_1)(\mathcal{F}(\alpha)) = \tilde{\mathcal{F}}(\beta_1)(\mathcal{F}(\alpha_1)) = \tilde{\mathcal{F}}(\alpha_1)(\mathcal{F}(\beta_1)),$$

which shows the well-defineness of this multiplication. We also have

$$xy = \tilde{\mathcal{F}}(\alpha)(\mathcal{F}(\beta)) = \tilde{\mathcal{F}}(\beta)(\mathcal{F}(\alpha)) = yx,$$

which shows this multiplication is commutative. From equation (3.9), we can easily show $\hat{\mathcal{F}}$ is a surjective algebra homomorphism. \square

3.1.4 Compatibility between the Frobenius homomorphism and the splitting map

Theorem 3.1.29. *Suppose D is a disk properly embedded into a marked 3-manifold (M, \mathcal{N}) and D contains an embedded oriented open interval β . Let (M', \mathcal{N}') be the result of splitting (M, \mathcal{N}) along (D, β) . Then we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{S}_n(M, \mathcal{N}, 1) & \xrightarrow{\Theta} & \mathcal{S}_n(M', \mathcal{N}', 1) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \mathcal{S}_n(M, \mathcal{N}, v) & \xrightarrow{\Theta} & \mathcal{S}_n(M', \mathcal{N}', v) \end{array} .$$

Proof. We just need to check $\mathcal{F}(\Theta(\alpha)) = \Theta(\mathcal{F}(\alpha))$ for any stated n -web α consisting of stated framed oriented arcs, since $\mathcal{S}_n(M, \mathcal{N}, 1)$ is linearly spanned by all these n -webs. Then Corollary 3.1.13 completes the proof. \square

Theorem 3.1.29 also shows the splitting map restricts to a map

$$\Theta|_{\mathcal{S}_n(M, \mathcal{N}, v)^{(m)}} : \mathcal{S}_n(M, \mathcal{N}, v)^{(m)} \rightarrow \mathcal{S}_n(M', \mathcal{N}', v)^{(m)},$$

and $\Theta|_{\mathcal{S}_n(M, \mathcal{N}, v)^{(m)}} \circ \hat{\mathcal{F}} = \hat{\mathcal{F}} \circ \Theta$. It is also easy to show $\Theta|_{\mathcal{S}_n(M, \mathcal{N}, v)^{(m)}}$ is an algebra homomorphism.

3.1.5 Central elements in $\mathcal{S}_n(\mathfrak{S}, v)$ and the injectivity of the Frobenius homomorphism

When (M, \mathcal{N}) is the thickening of an essentially bordered pb surface \mathfrak{S} , obviously $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ is a subalgebra of $\mathcal{S}_n(\mathfrak{S}, v)$. The algebra structure for $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ inherited from $\mathcal{S}_n(\mathfrak{S}, v)$ is the same with the algebra structure for $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ defined in Theorem 3.1.28. Then $\mathcal{F} : \mathcal{S}_n(\mathfrak{S}, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)$ becomes an algebra homomorphism.

Theorem 3.1.30. *Let \mathfrak{S} be an essentially bordered pb surface, there exists a unique algebra homomorphism $\mathcal{F} : \mathcal{S}_n(\mathfrak{S}, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)$ such that $\text{Im}\mathcal{F}$ lives in the center of $\mathcal{S}_n(\mathfrak{S}, v)$ and $\mathcal{F}(\alpha) = \alpha^{(m)}$ for any stated framed oriented arc α .*

Proof. Lemma 3.1.21, Theorem 3.1.27. □

Then we are trying to show \mathcal{F} in Theorem 3.1.30 is an embedding. We use $\mathcal{O}(SL_n)$ to denote $\mathcal{O}_1(SL_n)$, that is, $\mathcal{O}(SL_n)$ is the coordinate ring of $SL(2, \mathbb{C})$.

Lemma 3.1.31 ([PW91]). *There exists a Hopf algebra homomorphism:*

$$\begin{aligned} F_n : \mathcal{O}(SL_n) &\rightarrow \mathcal{O}_q(SL_n) \\ u_{i,j} &\mapsto (u_{i,j})^m. \end{aligned}$$

Define the monoid

$$\Gamma = \text{Mat}_n(\mathbb{N})/(\text{Id}).$$

Here $\text{Mat}_n(\mathbb{N}) = \mathbb{N}^{n \times n}$ is an additive monoid, and (Id) is the submonoid generated by the identity matrix. Two matrices $p, p' \in \text{Mat}_n(\mathbb{N})$ determine the same element in Γ if and only if $p - p' = k\text{Id}$ for $k \in \mathbb{Z}$. Each $p \in \Gamma$ has a unique lift $\hat{p} \in \text{Mat}_n(\mathbb{N})$ such that $\min_i \hat{p}_{i,i} = 0$.

Proposition 3.1.32 ([Gav07; LY23]). *For any linear order d_{ord} on \mathbb{J}^2 , the set*

$$B^{d_{ord}} := \{b(p) := \prod_{(i,j) \in \mathbb{J}^2} (u_{i,j})^{\hat{p}_{i,j}} \mid p \in \Gamma = \text{Mat}_n(\mathbb{N})/(\text{Id})\}, \quad (3.10)$$

where the product is taken with respect to the order d_{ord} , is a basis of $\mathcal{O}_q(SL_n)$.

Lemma 3.1.33. F_n in Lemma 3.1.31 is an embedding.

Proof. Using Proposition 3.1.32, it is easy to check F_n maps the basis of $\mathcal{O}(SL_n)$ injectively to the basis of $\mathcal{O}_q(SL_n)$. \square

Lemma 3.1.34. $\mathcal{F} : \mathcal{S}_n(\mathfrak{B}, 1) \rightarrow \mathcal{S}_n(\mathfrak{B}, v)$ is injective.

Proof. Here we use the Hopf algebra isomorphism in Theorem 2.1.8.

For any $i, j \in \mathbb{J}$, it is easy to check

$$\mathcal{F}(f_{big}(u_{i,j})) = \mathcal{F}(b_{i,j}) = (b_{i,j})^{(m)}, \quad f_{big}(F_n(u_{i,j})) = f_{big}((u_{i,j})^m) = (b_{i,j})^m = (b_{i,j})^{(m)}.$$

Then we have $\mathcal{F} \circ f_{big} = f_{big} \circ F_n$ since all the maps involved are algebra homomorphisms. Thus \mathcal{F} is injective because f_{big} is an isomorphism and F_n is injective. \square

The proof in the Lemma 3.1.34 actually shows $\mathcal{F} : \mathcal{S}_n(\mathfrak{B}, 1) \rightarrow \mathcal{S}_n(\mathfrak{B}, v)$ is an injective Hopf algebra homomorphism.

We use \mathfrak{T} to denote the standard ideal triangle.

It is well-known that there is a linear isomorphism $QF : \mathcal{S}_n(\mathfrak{B}, v) \otimes \mathcal{S}_n(\mathfrak{B}, v) \rightarrow \mathcal{S}_n(\mathfrak{T}, v)$, see Example 7.9 in [LS21]. Note that QF is not an algebra homomorphism unless $v = 1$.

(3.11)

Lemma 3.1.35. $\mathcal{F} : \mathcal{S}_n(\mathfrak{T}, 1) \rightarrow \mathcal{S}_n(\mathfrak{T}, v)$ is injective.

Proof. Look at the following diagram:

$$\begin{array}{ccc}
\mathcal{S}_n(\mathfrak{B}, 1) \otimes \mathcal{S}_n(\mathfrak{B}, 1) & \xrightarrow{\mathcal{F} \otimes \mathcal{F}} & \mathcal{S}_n(\mathfrak{B}, v) \otimes \mathcal{S}_n(\mathfrak{B}, v) \\
\downarrow QF & & \downarrow QF \\
\mathcal{S}_n(\mathfrak{T}, 1) & \xrightarrow{\mathcal{F}} & \mathcal{S}_n(\mathfrak{T}, v)
\end{array} \quad . \quad (3.12)$$

As a vector space, $\mathcal{S}_n(\mathfrak{B}, 1)$ is generated by parallel stated arcs that are all parallel to the core of \mathfrak{B} (these arcs may have different orientations and states). QF acts on these arcs by extending them to the bottom edge, see equation (3.11). \mathcal{F} acts on these arcs by taking m parallel copies of each arc. Clearly these two actions commute with each other. Thus the diagram in (3.12) is commutative. Then $\mathcal{F} : \mathcal{S}_n(\mathfrak{T}, 1) \rightarrow \mathcal{S}_n(\mathfrak{T}, v)$ is injective since QF is a linear isomorphism and $\mathcal{F} : \mathcal{S}_n(\mathfrak{B}, 1) \rightarrow \mathcal{S}_n(\mathfrak{B}, v)$ is injective, see Lemma 3.1.34. \square

$$\begin{array}{ccc}
\mathcal{S}_n(\mathfrak{S}, 1) \xrightarrow{\Theta} \otimes_{\mathfrak{T} \in \text{tri}(\mathcal{E})} \mathcal{S}_n(\mathfrak{T}, 1) \xrightarrow{T_{\mathcal{E}}} (\otimes_{\mathfrak{T} \in \text{tri}(\mathcal{E})} \mathcal{S}_n(\mathfrak{T}, v)) \otimes (\otimes_{e \in \text{Int}(\mathcal{E})} \mathcal{S}_n(\mathfrak{B}, 1)) \\
\downarrow F_{\mathcal{E}} \qquad \qquad \downarrow \otimes_{\mathfrak{T} \in \text{tri}(\mathcal{E})} \mathcal{F} \qquad \qquad \downarrow (\otimes_{\mathfrak{T} \in \text{tri}(\mathcal{E})} \mathcal{F}) \otimes (\otimes_{e \in \text{Int}(\mathcal{E})} \mathcal{F}) \\
\mathcal{S}_n(\mathfrak{S}, v) \xrightarrow{\Theta} \otimes_{\mathfrak{T} \in \text{tri}(\mathcal{E})} \mathcal{S}_n(\mathfrak{T}, v) \xrightarrow{T_{\mathcal{E}}} (\otimes_{\mathfrak{T} \in \text{tri}(\mathcal{E})} \mathcal{S}_n(\mathfrak{T}, v)) \otimes (\otimes_{e \in \text{Int}(\mathcal{E})} \mathcal{S}_n(\mathfrak{B}, v))
\end{array} \quad (3.13)$$

Please refer to Lemma 2.1.4 for the definition of $T_{\mathcal{E}}$.

Theorem 3.1.36 ($n = 2$ [KQ24], $n = 3$ [Hig23]). *Let \mathfrak{S} be an essentially bordered pb surface, and \mathcal{E} be an ideal triangulation of \mathfrak{S} . Then we have*

(a) *the right square in (3.13) is commutative,*

(b) *there exists a unique algebra homomorphism $F_{\mathcal{E}} : \mathcal{S}_n(\mathfrak{S}, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)$ such that the left square is also commutative,*

(c) *$F_{\mathcal{E}} = \mathcal{F}$, especially we have \mathcal{F} is injective and $F_{\mathcal{E}}$ is independent of the triangulation \mathcal{E} .*

Proof. (a) Theorem 3.1.29.

(b) From Lemma 2.1.4, we know the two rows are exact. Then there exists a unique $F_{\mathcal{E}}$ such that the left square is also commutative because the right square is commutative and the two rows are exact.

(c) From Theorem 3.1.29, we know $\mathcal{F} : \mathcal{S}_n(\mathfrak{S}, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)$ also makes the left square commutative. Then from the uniqueness in (b), we know $F_{\mathcal{E}} = \mathcal{F}$, which implies $F_{\mathcal{E}}$ is independent of \mathcal{E} . Since both Θ and $\otimes_{\mathfrak{T} \in \text{tri}(\mathcal{E})} \mathcal{F}$ are injective (Proposition 2.1.3, Lemma 3.1.35), then $F_{\mathcal{E}} = \mathcal{F}$ is also injective. \square

Corollary 3.1.37. *Let \mathfrak{S} be an essentially bordered pb surface. We have*

$$\hat{\mathcal{F}} : \mathcal{S}_n(\mathfrak{S}, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)^{(m)}$$

is an isomorphism.

Proof. From Theorem 3.1.36, we know $\hat{\mathcal{F}}$ is injective. We also have $\hat{\mathcal{F}}$ is surjective. \square

Let \mathfrak{S} be a pb surface. For each interior puncture p , we have two peripheral loops with vertical framing going around p , which are denoted as a_p, b_p . Clearly both a_p and b_p live in the center of $\mathcal{S}_n(\mathfrak{S}, v)$.

Definition 3.1.38. *Suppose \mathfrak{S} is an essentially bordered pb surface. Let $Z_n(\mathfrak{S})$ be the subalgebra of $\mathcal{S}_n(\mathfrak{S}, v)$ generated by $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ and a_p, b_p for all inner punctures p .*

Corollary 3.1.39. *Suppose \mathfrak{S} is an essentially bordered pb surface. Then $Z_n(\mathfrak{S})$ lives in the center of $\mathcal{S}_n(\mathfrak{S}, v)$, and $Z_n(\mathfrak{S})$ is a finitely generated commutative algebra.*

Proof. Note that $\mathcal{S}_n(\mathfrak{S}, v)^{(m)} = \text{Im} \hat{\mathcal{F}} = \text{Im} \mathcal{F}$. Then from Theorem 3.1.30, we know $Z_n(\mathfrak{S})$ lives in the center of $\mathcal{S}_n(\mathfrak{S}, v)$. Since $\mathcal{S}_n(\mathfrak{S}, 1)$ is finitely generated as an algebra and \mathfrak{S} has finitely many interior punctures, then $Z_n(\mathfrak{S})$ is a finitely generated commutative algebra. \square

3.1.6 Classical shadow

Definition 3.1.40 ([Mil12]). *A commutative algebra is called affine, if it does not contain nonzero nilpotents and it is finitely generated as an algebra.*

Recall that, for a commutative algebra A , we use $\text{MaxSpec}(A)$ to denote the set of maximal ideals of A . Then $\text{MaxSpec}(A)$ is an affine algebraic variety and A is the coordinate ring of $\text{MaxSpec}(A)$, if A is affine [Mil12].

Proposition 3.1.41. *Suppose \mathfrak{S} is an essentially bordered pb surface. Then both $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ and $Z_n(\mathfrak{S})$ are affine algebras.*

Proof. We already know both $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ and $Z_n(\mathfrak{S})$ are finitely generated commutative algebras. Since $\mathcal{S}_n(\mathfrak{S}, v)$ is a domain [LY23], then both $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ and $Z(\mathfrak{S})$ have no nonzero nilpotents.

□

Let A be an algebra, and Z be an affine subalgebra of the center of A . We use Irrep_A to denote the set of finite dimensional irreducible representations considered up to isomorphism (that is, two irreducible representations are considered the same if they are isomorphic). We can omit the subscript for Irrep_A , when there is no confusion with A . Then there is a map $\mathcal{X} : \text{Irrep} \rightarrow \text{MaxSpec}(Z)$ defined as following: Let $\rho : A \rightarrow \text{End}(V)$ be a finite dimensional irreducible representation of A . Since Z is contained in the center of A , for every $x \in Z$ there exists a complex number $r_\rho(x)$ such that $\rho(x) = r_\rho(x)Id_V$. We get an algebra homomorphism $r_\rho : Z \rightarrow \mathbb{C}$. Then the irreducible representation ρ uniquely determines a point $\text{Ker}(r_\rho)$ in $\text{MaxSpec}(Z)$. We define $\mathcal{X}(\rho) = \text{Ker}(r_\rho)$.

From Proposition 3.1.41, we know $Z_n(\mathfrak{S})$ is an affine subalgebra of the center of $\mathcal{S}_n(\mathfrak{S}, v)$, and $\text{MaxSpec}(Z_n(\mathfrak{S}))$ is an affine algebraic variety. Then there is a map

$$\mathcal{X} : \text{Irrep} \rightarrow \text{MaxSpec}(Z_n(\mathfrak{S})).$$

In next section, we will show the center of $\mathcal{S}_n(\mathfrak{S}, v)$ is actually affine, the corresponding map \mathcal{X} is surjective, and \mathcal{X} is injective on the preimage of a Zariski open dense subset.

3.1.7 The Frobenius homomorphism for reduced stated SL_n -skein algebras

The reduced stated SL_n -skein algebra $\overline{\mathcal{S}}_n(\mathfrak{S}, v)$ is defined in subsection 7.1 in [LY23]. For any pb surface \mathfrak{S} ,

$$\overline{\mathcal{S}}_n(\mathfrak{S}, v) = \mathcal{S}_n(\mathfrak{S}, v)/I^{\text{bad}},$$

where I^{bad} is the two sided ideal of $\mathcal{S}_n(\mathfrak{S}, v)$ generated by all bad arcs [LY23]. We use $\overline{\mathcal{S}}_n(\mathfrak{S}, v)^{(m)}$ to denote the image of $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ under the projection from $\mathcal{S}_n(\mathfrak{S}, v)$ to $\overline{\mathcal{S}}_n(\mathfrak{S}, v)$.

For any essentially bordered pb surface \mathfrak{S} , clearly $\mathcal{F} : \mathcal{S}_n(\mathfrak{S}, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)$ sends the bad arc to m parallel copies of the bad arc. Then \mathcal{F} induces an algebra homomorphism $\overline{\mathcal{F}} : \overline{\mathcal{S}}_n(\mathfrak{S}, 1) \rightarrow \overline{\mathcal{S}}_n(\mathfrak{S}, v)$, and $\text{Im}\overline{\mathcal{F}} = \overline{\mathcal{S}}_n(\mathfrak{S}, v)^{(m)}$ lives in the center of $\overline{\mathcal{S}}_n(\mathfrak{S}, v)$.

3.2 The Unicity Theorem for stated SL_n -skein algebras

For any algebra A , we will use $C(A)$ to denote its center.

An algebra A is called **prime** if all $a, b \in A$ satisfy the following condition: if $arb = 0$ for all $r \in A$ then $a = 0$ or $b = 0$. Clearly if A is a domain, then A is prime. An algebra A is **almost Azumaya** if there is a nonzero element $c \in C(A)$ such that A_c is an Azumaya algebra, where A_c is the localization of A by c [FKL19; Kor21].

Definition 3.2.1. *An algebra A is called **affine almost Azumaya** if A satisfies the following conditions [Kor21]:*

- (1) A is finitely generated as an algebra,
- (2) A is prime,
- (3) A is finitely generated as a module over its center.

Theorem 3.2.2 ([FKL19]). *If A is a prime algebra that is finitely generated as a module over its center then A is almost Azumaya. Especially, affine almost Azumaya algebra is almost Azumaya.*

Lemma 3.2.3. *If A is an affine almost Azumaya algebra, then $C(A)$ is an affine algebra.*

Proof. From Lemma 2.1 in [FKL19], we know (1) and (3) in the definition of affine almost Azumaya algebra imply $C(A)$ is a finitely generated algebra. (2) implies $C(A)$ is a domain. \square

Thus $\text{MaxSpec}(C(A))$ is an irreducible affine algebraic variety if A is an affine almost Azumaya algebra.

Suppose A is an algebra, and M is a finitely generated A -module. We define the dimension of M over A to be the minimum number of generators of M over A .

When A is an affine almost Azumaya algebra, we use $\widetilde{C(A)}$ to denote the field of fractions of $C(A)$, and use \widetilde{A} to denote the vector space $A \otimes_{C(A)} \widetilde{C(A)}$ over $\widetilde{C(A)}$. Then we define the rank of A over $C(A)$ to be the dimension of \widetilde{A} over $\widetilde{C(A)}$. Obviously we have the rank of A over $C(A)$ is less than or equal to the dimension of A over $C(A)$.

Theorem 3.2.4 ([FKL19; Kor21]). *If A is an affine almost Azumaya algebra of rank K and of dimension r over its center $C(A)$, then:*

(a) *any irreducible representation of A has dimension at most the square root of K ;*

(b) *The map $\mathcal{X} : \text{Irrep} \rightarrow \text{MaxSpec}(C(A))$, defined in Remark 3.1.6, is surjective, and the cardinality of $\mathcal{X}^{-1}(v)$ is not more than r for any $v \in \text{MaxSpec}(C(A))$;*

(c) *there exists a Zariski open dense subset $U \subset \text{MaxSpec}(C(A))$ such that for any two irreducible representations V_1, V_2 of A with $\mathcal{X}(V_1) = \mathcal{X}(V_2) \in U$, then V_1 and V_2 are isomorphic and have dimension the square root of K . Moreover any representation sending $C(A)$ to scalar operators and whose induced character lies in U is semi-simple.*

Frohman, Kania-Bartoszyńska, and Lê proved $S_2(\mathfrak{S}, v)$ is an affine almost Azumaya algebra when v is a root of unity and $\partial\mathfrak{S} = \emptyset$ [FKL19]. Korinman proved $S_2(\mathfrak{S}, v)$ and $\overline{S}_2(\mathfrak{S}, v)$ are both affine almost Azumaya when v is a root of unity of odd order [Kor21].

3.2.1 More on $\mathcal{S}_n(M, \mathcal{N}, 1)$ -module structure for $\mathcal{S}_n(M, \mathcal{N}, v)$ and the Unicity Theorem for SL_n

In this subsection, we always assume every component of the marked 3-manifold contains at least one marking, the surface is the essentially bordered pb surface, and v is a root of unity of order m with m and $2n$ being coprime with each other. The main goal of this subsection is to show $\mathcal{S}_n(\mathfrak{S}, v)$ is affine almost Azumaya.

From Subsection 3.1.2, we know $\mathcal{S}_n(M, \mathcal{N}, 1)$ acts on $\mathcal{S}_n(M, \mathcal{N}, v)$. This module structure is defined by $l \cdot \alpha = l^{(m)} \cup \alpha$ where $l \in \mathcal{S}_n(M, \mathcal{N}, 1)$ is a stated n -web consisting of stated framed oriented arcs, $\alpha \in \mathcal{S}_n(M, \mathcal{N}, v)$ is a stated n -web in $\mathcal{S}_n(M, \mathcal{N}, v)$, and $l \cap \alpha = \emptyset$.

Let $f : (M_1, \mathcal{N}_1) \rightarrow (M_2, \mathcal{N}_2)$ be an embedding for marked 3-manifolds. We know f induces a linear map $f_* : \mathcal{S}_n(M_1, \mathcal{N}_1, v) \rightarrow \mathcal{S}_n(M_2, \mathcal{N}_2, v)$ and an algebra homomorphism

$$f_* : \mathcal{S}_n(M_1, \mathcal{N}_1, 1) \rightarrow \mathcal{S}_n(M_2, \mathcal{N}_2, 1).$$

Then we have the following Lemma.

Lemma 3.2.5. *The above linear map $f_* : \mathcal{S}_n(M_1, \mathcal{N}_1, v) \rightarrow \mathcal{S}_n(M_2, \mathcal{N}_2, v)$ respects the module structures in a sense that*

$$f_*(x \cdot y) = f_*(x) \cdot f_*(y)$$

where $x \in \mathcal{S}_n(M_1, \mathcal{N}_1, 1), y \in \mathcal{S}_n(M_1, \mathcal{N}_1, v)$.

Proof. Let α be a stated n -web consisting of stated framed oriented arcs, β be any stated n -web in $\mathcal{S}_n(M, \mathcal{N}, v)$, and $\alpha \cap \beta = \emptyset$. Then we have

$$f_*(\alpha \cdot \beta) = f_*(\alpha^{(m)} \cup \beta) = (f_*(\alpha))^{(m)} \cup f_*(\beta) = f_*(\alpha) \cdot f_*(\beta).$$

□

Lemma 3.2.6. *Let $f : (M_1, \mathcal{N}_1) \rightarrow (M_2, \mathcal{N}_2)$ be an embedding for marked 3-manifolds. Suppose the linear map $f_* : \mathcal{S}_n(M_1, \mathcal{N}_1, v) \rightarrow \mathcal{S}_n(M_2, \mathcal{N}_2, v)$ is surjective, and $\mathcal{S}_n(M_1, \mathcal{N}_1, v)$ is finitely generated as an $\mathcal{S}_n(M_1, \mathcal{N}_1, 1)$ -module. Then $\mathcal{S}_n(M_2, \mathcal{N}_2, v)$ is finitely generated as an $\mathcal{S}_n(M_2, \mathcal{N}_2, 1)$ -module, and the dimension of $\mathcal{S}_n(M_2, \mathcal{N}_2, v)$ as an $\mathcal{S}_n(M_2, \mathcal{N}_2, 1)$ -module is not more than the dimension of $\mathcal{S}_n(M_1, \mathcal{N}_1, v)$ as an $\mathcal{S}_n(M_1, \mathcal{N}_1, 1)$ -module.*

Proof. Suppose $\mathcal{S}_n(M_1, \mathcal{N}_1, v)$ is generated by x_1, x_2, \dots, x_t as an $\mathcal{S}_n(M_1, \mathcal{N}_1, 1)$ -module. From Lemma 3.2.5, we get $\mathcal{S}_n(M_2, \mathcal{N}_2, v)$ is generated by

$$f_*(x_1), f_*(x_2), \dots, f_*(x_t)$$

as an $\mathcal{S}_n(M_2, \mathcal{N}_2, 1)$ -module because $f_* : \mathcal{S}_n(M_1, \mathcal{N}_1, v) \rightarrow \mathcal{S}_n(M_2, \mathcal{N}_2, v)$ is surjective. \square

rem For an essentially bordered pb surface \mathfrak{S} , we know $\hat{\mathcal{F}} : \mathcal{S}_n(\mathfrak{S}, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ is an isomorphism, and $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ lives in the center of $\mathcal{S}_n(\mathfrak{S}, v)$. Thus $\mathcal{S}_n(\mathfrak{S}, v)$ has an $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ -module structure given by multiplication. $\mathcal{S}_n(\mathfrak{S}, v)$ also has an $\mathcal{S}_n(\mathfrak{S}, 1)$ -module structure. Then the identity map $Id : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)$ preserves the above two module structures, in a sense that, for any $x \in \mathcal{S}_n(\mathfrak{S}, 1), y \in \mathcal{S}_n(\mathfrak{S}, v)$ we have $x \cdot y = \hat{\mathcal{F}}(x)y$.

Lemma 3.2.7. *$\mathcal{S}_n(\mathcal{B}, v)$ is finitely generated as an $\mathcal{S}_n(\mathcal{B}, 1)$ -module. The dimension of $\mathcal{S}_n(\mathcal{B}, v)$ as an $\mathcal{S}_n(\mathcal{B}, 1)$ -module is at most $m^{n^2} - (m-1)^n m^{n^2-n}$.*

Proof. Let d_{ord} be a linear order on set \mathbb{J}^2 . From Proposition 3.1.32 and Theorem 2.1.8, we know the set

$$\left\{ \prod_{(i,j) \in \mathbb{J}^2} (b_{i,j})^{\hat{p}_{i,j}} \mid p \in \Gamma = \text{Mat}_n(\mathbb{N})/(\text{Id}) \right\}$$

is a basis for $\mathcal{S}_n(\mathcal{B}, v)$, where the product is taken with respect to the order d_{ord} , and

$$\left\{ \prod_{(i,j) \in \mathbb{J}^2} (b_{i,j})^{m\hat{p}_{i,j}} \mid p \in \Gamma = \text{Mat}_n(\mathbb{N})/(\text{Id}) \right\}$$

is a basis for $\mathcal{S}_n(\mathcal{B}, v)^{(m)}$. Clearly $\mathcal{S}_n(\mathcal{B}, v)$ is generated by the set

$$\left\{ \prod_{(i,j) \in \mathbb{J}^2} (b_{i,j})^{p_{i,j}} \mid p_{i,j} \in \mathbb{Z}, 0 \leq p_{i,j} \leq m-1, \exists i \in \mathbb{J} \text{ s.t. } p_{i,i} = 0 \right\} \quad (3.14)$$

as an $\mathcal{S}_n(\mathcal{B}, v)^{(m)}$ -module, where the product is taken with respect to the order d_{ord} . The cardinality of the set in (3.14) is $m^{n^2} - (m-1)^n m^{n^2-n}$.

Then Remark 3.2.1 indicates Lemma 3.2.7. □

Note that when $n > 1$, the set in (3.14) is not independent. We will use $\langle n, m \rangle$ to denote the integer $m^{n^2} - (m-1)^n m^{n^2-n}$.

Lemma 3.2.8. *Suppose \mathfrak{S} is the disjoint union of k bigons, $k \geq 1$. Then $\mathcal{S}_n(\mathfrak{S}, v)$ is finitely generated as an $\mathcal{S}_n(\mathfrak{S}, 1)$ -module. The dimension of $\mathcal{S}_n(\mathfrak{S}, v)$ as an $\mathcal{S}_n(\mathfrak{S}, 1)$ -module is at most $\langle n, m \rangle^k$.*

Proof. Lemma 3.2.7. □

Suppose \mathfrak{S} is an essentially bordered pb surface. Let $B = \{b_1, \dots, b_r\}$ be the collection of properly embedded disjoint compact oriented arcs in \mathfrak{S} . We say B is a **saturated system** if we have (1) after cutting \mathfrak{S} along B , every component of the cutting surface contains exactly one ideal point (2) B is maximal under condition (1).

Let $U(b_1), \dots, U(b_r)$ be a collection of disjoint open tubular neighborhoods of b_1, \dots, b_r , respectively. Each $U(b_i)$ is diffeomorphic with $b_i \times (0, 1)$ (the diffeomorphism is orientation preserving) and we require that $(\partial b_i) \times (0, 1) \subset \partial \mathfrak{S}$. Set $U(B) = \cup_{1 \leq i \leq r} U(b_i)$. Note that each $U(b_i)$ is naturally a bigon, then $U(B)$ is the disjoint union of bigons. From 3.2.9 to 3.2.14, we will always assume \mathfrak{S} is an essentially bordered pb surface.

Theorem 3.2.9 ([LS21]). *Assume $B = \{b_1, \dots, b_r\}$ is a saturated system of arcs of \mathfrak{S} .*

(1) *We have $r = r(\mathfrak{S}) := \sharp(\partial \mathfrak{S}) - E(\mathfrak{S})$, where $\sharp(\partial \mathfrak{S})$ is the number of boundary components of \mathfrak{S} and $E(\mathfrak{S})$ denotes the Euler characteristics of \mathfrak{S} .*

(2) The embedding $U(B) \rightarrow \mathfrak{S}$, with negative c -orders for all boundary components c of \mathfrak{S} , induces a linear isomorphism.

Proposition 3.2.10. *We have*

(a) $\mathcal{S}_n(\mathfrak{S}, v)$ is finitely generated as an $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ -module, and the dimension of $\mathcal{S}_n(\mathfrak{S}, v)$ over $\mathcal{S}_n(\mathfrak{S}, v)^{(m)}$ is at most $(\langle n, m \rangle)^{r(\mathfrak{S})}$.

(b) $\overline{\mathcal{F}}_n(\mathfrak{S}, v)$ is finitely generated as an $\overline{\mathcal{F}}_n(\mathfrak{S}, v)^{(m)}$ -module, and the dimension of $\overline{\mathcal{F}}_n(\mathfrak{S}, v)$ over $\overline{\mathcal{F}}_n(\mathfrak{S}, v)^{(m)}$ is at most $(\langle n, m \rangle)^{r(\mathfrak{S})}$.

Proof. From Lemmas 3.2.6, 3.2.8, and Theorem 3.2.9, and Remark 3.2.1, we can easily prove (a). Trivially (a) implies (b). \square

Lemma 3.2.11. *We have $\mathcal{S}_n(\mathfrak{S}, v)$ (respectively $\overline{\mathcal{F}}_n(\mathfrak{S}, v)$) is finitely generated as a module over the center $C(\mathcal{S}_n(\mathfrak{S}, v))$ (respectively $C(\overline{\mathcal{F}}_n(\mathfrak{S}, v))$). The dimension of $\mathcal{S}_n(\mathfrak{S}, v)$ (respectively $\overline{\mathcal{F}}_n(\mathfrak{S}, v)$) over $C(\mathcal{S}_n(\mathfrak{S}, v))$ (respectively $C(\overline{\mathcal{F}}_n(\mathfrak{S}, v))$) is not more than $(\langle n, m \rangle)^{r(\mathfrak{S})}$.*

Proof. Since $\mathcal{S}_n(\mathfrak{S}, v)^{(m)} \subset C(\mathcal{S}_n(\mathfrak{S}, v))$ and $\overline{\mathcal{F}}_n(\mathfrak{S}, v)^{(m)} \subset C(\overline{\mathcal{F}}_n(\mathfrak{S}, v))$, then Proposition 3.2.10 implies the Lemma. \square

Theorem 3.2.12. *We have $\mathcal{S}_n(\mathfrak{S}, v)$ is affine almost Azumaya. Thus it is also almost Azumaya.*

Proof. Since \mathfrak{S} be an essentially bordered pb surface, then $\mathcal{S}_n(\mathfrak{S}, v)$ is a domain and is a finitely generated algebra, Theorem 6.1 in [LY23]. Combine with Lemma 3.2.11, we get $\mathcal{S}_n(\mathfrak{S}, v)$ is affine almost Azumaya. \square

Corollary 3.2.13. *We have $C(\mathcal{S}_n(\mathfrak{S}, v))$ is affine.*

Proof. Lemma 3.2.3, and Theorem 3.2.12. \square

Then we are ready to state the Unicity Theorem for the stated SL_n -skein algebra when the surface is an essentially bordered pb surface.

Theorem 3.2.14. *Suppose the rank of $\mathcal{S}_n(\mathfrak{S}, v)$ over $C(\mathcal{S}_n(\mathfrak{S}, v))$ is K and the dimension of $\mathcal{S}_n(\mathfrak{S}, v)$ over $C(\mathcal{S}_n(\mathfrak{S}, v))$ is r , then we have*

(a) $K \leq r \leq (\langle n, m \rangle)^{r(\mathfrak{S})}$;

(b) any irreducible representation of $\mathcal{S}_n(\mathfrak{S}, v)$ has dimension at most the square root of K ;

(c) the map $\mathcal{X} : \mathrm{Irrep} \rightarrow \mathrm{MaxSpec}(C(\mathcal{S}_n(\mathfrak{S}, v)))$, defined in Remark 3.1.6, is surjective, and the cardinality of $\mathcal{X}^{-1}(v)$ is not more than r for any $v \in \mathrm{MaxSpec}(C(\mathcal{S}_n(\mathfrak{S}, v)))$;

(d) there exists a Zariski open dense subset $U \subset \mathrm{MaxSpec}(C(\mathcal{S}_n(\mathfrak{S}, v)))$ such that for any two irreducible representations V_1, V_2 of $\mathcal{S}_n(\mathfrak{S}, v)$ with $\mathcal{X}(V_1) = \mathcal{X}(V_2) \in U$, then V_1 and V_2 are isomorphic and have dimension the square root of K . Moreover any representation sending $C(\mathcal{S}_n(\mathfrak{S}, v))$ to scalar operators and whose induced character lies in U is semi-simple.

Proof. Lemma 3.2.11, Theorems 3.2.4 and 3.2.12. □

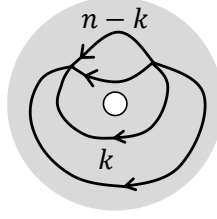
The rank K in Theorem 3.2.14 is very important to understand the representation theory for the stated SL_n -skein algebra. Unfortunately, it is very hard to calculate it. Frohman, Kania-Bartoszynska, Lê precisely calculated this rank K in [FKL21] when $n = 2$ and $\partial\mathfrak{S} = \emptyset$.

We are not clear whether $\overline{\mathcal{S}}_n(\mathfrak{S}, v)$ is a domain (or prime) for general essentially bordered pb surface. In Theorem 8.1 in [LY23], Lê and Yu proved $\overline{\mathcal{S}}_n(\mathfrak{S}, v)$ is a domain when \mathfrak{S} is a polygon. Thus we have the corresponding statements as Theorem 3.2.12, Corollary 3.2.13, and Theorem 3.2.14 for the reduced stated SL_n -skein algebra when \mathfrak{S} is a polygon. For general essentially bordered pb surface, we can get $C(\overline{\mathcal{S}}_n(\mathfrak{S}, v))$ is a finitely generated algebra because $\overline{\mathcal{S}}_n(\mathfrak{S}, v)$ is a finitely generated algebra and finitely generated as a module over its center.

3.3 The center and the PI-degree for the stated SL_n -skein algebra

In this section, we review the results about the center and the rank of the stated SL_n -skein algebra for some special cases in [KW24a; KW24c].

Let \mathfrak{S} be a pb surface containing at least one puncture. For each puncture p and $1 \leq k \leq n - 1$, there is a loop n -web $\alpha_{p,k}$ around p as shown in Figure 3.2. We

FIGURE 3.2: The n -web $\alpha_{p,k}$ around puncture p .

call each $\alpha_{p,k}$ the **peripheral web**. It is easy to see that $\alpha_{p,k}$ is contained in the center of $\mathcal{S}_n(\mathfrak{S}, v)$.

Theorem 3.3.1. [KW24c] *Suppose \mathfrak{S} is a pb surface such that $\partial\mathfrak{S} = \emptyset$ and every component of \mathfrak{S} contains at least one puncture. Suppose v is a root of unity of order m with $\gcd(m, 6) = 1$. We have the following:*

1. *The center of $\mathcal{S}_3(\mathfrak{S}, v)$, as a subalgebra of $\mathcal{S}_3(\mathfrak{S}, v)$, is generated by $\text{Im } \mathcal{F}$ and all the peripheral webs.*
2. *The algebra $\mathcal{S}_3(\mathfrak{S}, v)$ is affine almost Azumaya;*
3. *Suppose \mathfrak{S} is connected with genus g and $p > 0$ punctures, then the rank of $\mathcal{S}_3(\mathfrak{S}, v)$ over its center is*

$$m^{16g-16+6n}.$$

Theorem 3.3.2. [KW24a] *Suppose \mathfrak{S} is a pb surface such that $\partial\mathfrak{S} \neq \emptyset$ and contains no punctures. Suppose v is a root of unity of order m with $\gcd(m, 2n) = 1$. We have the following:*

1. *An element $x \in \mathcal{S}_n(\mathfrak{S}, v)$ is contained in the center of $\mathcal{S}_n(\mathfrak{S}, v)$ if and only if $yx = z$ with $y, z \in \text{Im } \mathcal{F}$.*
2. *Suppose \mathfrak{S} is connected and $\overline{\mathfrak{S}}$ (see Definition 1.1.1) contains t boundary components having even number of punctures, then the rank of $\mathcal{S}_n(\mathfrak{S}, v)$ over its center is*

$$m^{(n^2-1)r(\Sigma)-t(n-1)}.$$

Chapter 4

Stated SL_n -TQFT theory and roots of unity

Suppose that m is a positive integer such that $m \mid n$. Let \mathfrak{S} be a pb surface and $\epsilon \in R$ such that $\epsilon^m = -1$. We will construct an R -linear isomorphism (Thm. 4.2.4)

$$\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}, \epsilon v).$$

As an application, we will establish the injectivity of the splitting map when $v^{2n} = 1$ (Thm. 4.7.4). Furthermore, we will formulate the stated SL_n -TQFT theory (Thm. 4.5.2), which generalizes the results of [CL22b] from SL_2 to general SL_n . This chapter is based on the author's work in [Wan25c].

4.1 Notations for chapter 4

This section introduces notations used in chapter 4.

Let α be an n -web in a marked 3-manifold (M, \mathcal{N}) . The half edge of α containing the starting point (resp. ending point) is called the **starting half edge** (resp. **targeting half edge**).

Recall that our ground ring is a commutative domain R with an invertible element v . In chapter 2, we defined q, c_i, t, a using v . Since this chapter involves different quantum parameters, we introduce new notations for q, c_i, t, a involving v . We set

$q_v^{\frac{1}{2n}} = v$, and define the following constants:

$$c_{i,v} = (-q_v)^{n-i} q_v^{\frac{n-1}{2n}}, \quad t_v = (-1)^{n-1} q_v^{\frac{n^2-1}{n}}, \quad a_v = q_v^{\frac{n+1-2n^2}{4}}.$$

Let \mathfrak{S} be a pb surface. Any stated n -web α in the thickening of \mathfrak{S} can be represented by a **stated n -web diagram** in \mathfrak{S} , which is obtained by projecting α onto \mathfrak{S} . Before performing the projection, we first apply a height-preserving isotopy to α so that its singular points look like the pictures in Figure 4.2 and ensure that the endpoints of α are distinct.

The orientation of $\partial\mathfrak{S}$ induced by the orientation of \mathfrak{S} is called the **positive orientation** of $\partial\mathfrak{S}$. The orientation of $\partial\mathfrak{S}$ opposite to the positive orientation of $\partial\mathfrak{S}$ is called the **negative orientation** of $\partial\mathfrak{S}$. A stated n -web diagram, where the heights of its endpoints are given by the positive (resp. negative) orientation of \mathfrak{S} , is called the **positively ordered stated n -web diagram** (resp. **negatively ordered stated n -web diagram**).

For a stated n -web (resp. a stated n -web diagram) α , we define the following quantities:

- $e(\alpha)$: the number of endpoints of α .
- $t(\alpha)$: the number of endpoints of α that point towards \mathcal{N} (resp. towards the boundary of the pb surface).
- $p(\alpha)$: the number of sinks and sources of α .

4.2 Stated SL_n -skein algebras skewed by roots of unity

For a pb surface \mathfrak{S} , two negatively ordered stated n -web diagrams represent the isotopic stated n -webs in $\mathfrak{S} \times [-1, 1]$ if and only if they are related by a sequence of the ambient isotopies and the five moves in Figure 4.1. See Figure 2 in [FS22] for $n = 3$.

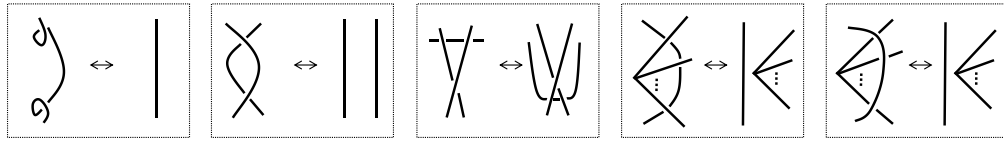


FIGURE 4.1: Five moves for the (stated) n -web diagrams. The orientations for (stated) n -web diagrams are arbitrary.

For a pb surface \mathfrak{S} , we can regard $\mathcal{S}_n(\mathfrak{S}, v)$ as the quotient of the R -module freely generated by the set of ambient isotopy classes of negatively ordered stated n -web diagrams subject to the five moves in Figure 4.1 and relations (2.4)-(2.11). Then the multiplication is defined by stacking the negatively ordered stated n -web diagrams such that the stacking gives the negatively ordered stated n -web diagram.

For a negatively ordered stated n -web diagram α , every crossing point p of α determines a “+” or “-” sign as illustrated in Figure 4.2, and we denote this sign as $w(p)$. Then we define $w(\alpha) = \sum_p w(p)$, where the sum is over all the crossing points of α and we regard the plus sign as 1 and minus sign as -1 . If α has no crossings, we define $w(\alpha) = 0$.



FIGURE 4.2: The sign determined by the left (resp. the right) picture is “+” (resp. “-”).

For any two integers i, j with $j > 0$, suppose $i = kj + r$, where $k, r \in \mathbb{Z}$ and $0 \leq r \leq j - 1$. We define $i(\bmod j)$ to be r .

Let α, β be two negatively ordered stated n -web diagrams. Suppose γ is a negatively ordered stated n -web diagram representing $\alpha\beta$. Then define

$$w(\alpha, \beta) = \left(\sum_p w(p) \right) (\bmod n),$$

where the sum is over all the crossing points between α and β . From the moves in Figure 4.1, we know $w(\alpha, \beta)$ is well-defined and $w(\alpha', \beta') = w(\alpha, \beta)$ if α (resp. β) and α' (resp. β') represent isotopic stated n -webs. Note that $w(\alpha, \beta)$ may not equal $(n - w(\beta, \alpha))(\bmod n)$ in general, but we have $w(\alpha, \beta) = (n - w(\beta, \alpha))(\bmod n)$ if both α and β have no endpoints.

For any integer $0 \leq k \leq n - 1$, we define $\mathcal{S}_n(\mathfrak{S}, v)_k$ as

$$\begin{aligned}
 &R\text{-linear span}\{\text{negatively ordered stated } n\text{-web diagrams } \alpha \mid \\
 &\quad w(\alpha, \alpha') \text{ and } w(\alpha', \alpha) \text{ are multiples of } k, \\
 &\quad \text{for all negatively ordered stated } n\text{-web diagrams } \alpha'\}.
 \end{aligned} \tag{4.1}$$

It is straightforward to verify that $\mathcal{S}_n(\mathfrak{S}, v)_k$ forms a subalgebra of $\mathcal{S}_n(\mathfrak{S}, v)$ whenever $k \mid n$.

4.2.1 On SL_n -skein algebras

Let \mathfrak{S} be a pb surface with an empty boundary. Then the definition of $\mathcal{S}_n(\mathfrak{S}, v)$ is determined by v^2 , that is, $\mathcal{S}_n(\mathfrak{S}, v)$ and $\mathcal{S}_n(\mathfrak{S}, -v)$ are the same (stated) SL_n -skein algebra. Thus we use $\mathcal{S}_n(\mathfrak{S}|v^2)$ to denote $\mathcal{S}_n(\mathfrak{S}, v)$ when $\partial\mathfrak{S} = \emptyset$. Since $\partial\mathfrak{S} = \emptyset$, any (stated) n -web diagram in \mathfrak{S} is a negatively ordered stated n -web diagram.

Theorem 4.2.1. *Let \mathfrak{S} be a pb surface with $\partial\mathfrak{S} = \emptyset$, let m, n be two positive integers with $m \mid n$, and let $u, \epsilon \in R$ be two invertible elements in R such that $u^2 = \epsilon v^2$, $\epsilon^m = 1$. Then there exists an R -linear isomorphism $\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}|v^2) \rightarrow \mathcal{S}_n(\mathfrak{S}|u^2)$, defined by $\varphi_\epsilon(\alpha) = \epsilon^{w(\alpha)}\alpha$ for any n -web diagram α .*

Proof. Suppose α_1 and α_2 are n -web diagrams representing the isotopic n -webs in $\mathfrak{S} \times [-1, 1]$. The five moves in Figure 4.1 imply $w(\alpha_1) \pmod n = w(\alpha_2) \pmod n$. Then we have $\epsilon^{w(\alpha_1)} = \epsilon^{w(\alpha_2)}$ because $\epsilon^n = 1$. Thus φ_ϵ is well-defined on the set of isotopy classes of n -webs in $\mathfrak{S} \times [-1, 1]$.

It suffices to show φ_ϵ preserves relations (2.4)-(2.7). Note that $q_u^{\frac{1}{n}} = \epsilon q_v^{\frac{1}{n}}$, $q_u = q_v$, and $t_u = (-1)^{n-1} q_u^{n-\frac{1}{n}} = \epsilon^{-1} t_v$.

Relation (2.4): We use α_1 to denote the n -web diagram on the righthand side of relation (2.4). Then we have

$$\begin{aligned}
 &\varphi_\epsilon(q_v^{\frac{1}{n}} \begin{array}{|c|} \hline \diagup \diagdown \\ \hline \end{array} - q_v^{-\frac{1}{n}} \begin{array}{|c|} \hline \diagdown \diagup \\ \hline \end{array}) = \epsilon^{w(\alpha_1)} \epsilon q_v^{\frac{1}{n}} \begin{array}{|c|} \hline \diagup \diagdown \\ \hline \end{array} - \epsilon^{w(\alpha_1)} \epsilon^{-1} q_v^{-\frac{1}{n}} \begin{array}{|c|} \hline \diagdown \diagup \\ \hline \end{array} \\
 &= (q_u - q_u^{-1}) \epsilon^{w(\alpha_1)} \begin{array}{|c|} \hline \rightarrow \\ \hline \rightarrow \\ \hline \end{array} = \varphi_\epsilon((q_v - q_v^{-1}) \begin{array}{|c|} \hline \rightarrow \\ \hline \rightarrow \\ \hline \end{array}).
 \end{aligned}$$

Relation (2.5): We use α_2 to denote the n -web diagram on the righthand side of relation (2.5). Then we have

$$\varphi_\epsilon(\text{web with } \alpha_2) = \epsilon^{w(\alpha_2)} \epsilon \text{web with } \alpha_2 = \epsilon^{w(\alpha_2)} \epsilon t_u \text{web with } \alpha_2 = \varphi_\epsilon(t_v \text{web with } \alpha_2).$$

It is trivial that φ_ϵ preserves relation (2.6).

Relation (2.7): We use α_3 to denote the n -web diagram on the lefthand side of relation (2.7). Then we have

$$\begin{aligned} \varphi_\epsilon(\text{web with } \alpha_3) &= \epsilon^{w(\alpha_3)} \text{web with } \alpha_3 \\ &= \epsilon^{w(\alpha_3)} (-q_u)^{\frac{n(n-1)}{2}} \cdot \sum_{\sigma \in S_n} (-q_u^{\frac{1-n}{n}})^{\ell(\sigma)} \text{web with } \sigma_+ \\ &= (-q_v)^{\frac{n(n-1)}{2}} \cdot \sum_{\sigma \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma)} \epsilon^{\ell(\sigma)+w(\alpha_3)} \text{web with } \sigma_+ \\ &= \varphi_\epsilon((-q_v)^{\frac{n(n-1)}{2}} \cdot \sum_{\sigma \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma)} \text{web with } \sigma_+). \end{aligned}$$

Obviously, the inverse of φ_ϵ is $\varphi_{\epsilon^{-1}} : \mathcal{S}_n(\mathfrak{S}|u^2) \rightarrow \mathcal{S}_n(\mathfrak{S}|v^2)$. \square

The map φ_ϵ is not an algebra homomorphism because $\epsilon^{w(\alpha\beta)} \neq \epsilon^{w(\alpha)}\epsilon^{w(\beta)}$ in general. The following Corollary shows it restricts to algebra isomorphism from $\mathcal{S}_n(\mathfrak{S}|v^2)_m$ to $\mathcal{S}_n(\mathfrak{S}|u^2)_m$.

Corollary 4.2.2. *Let \mathfrak{S} be a pb surface with $\partial\mathfrak{S} = \emptyset$, let m, n be two positive integers with $m \mid n$, and let $u, \epsilon \in R$ be two invertible elements in R such that $u^2 = \epsilon v^2$, $\epsilon^m = 1$. Then the linear isomorphism $\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}|v^2) \rightarrow \mathcal{S}_n(\mathfrak{S}|u^2)$ restricts to an algebra isomorphism $\varphi_\epsilon|_{\mathcal{S}_n(\mathfrak{S}|v^2)_m} : \mathcal{S}_n(\mathfrak{S}|v^2)_m \rightarrow \mathcal{S}_n(\mathfrak{S}|u^2)_m$.*

Proof. It is trivial that φ_ϵ restricts to the linear isomorphism $\varphi_\epsilon|_{\mathcal{S}_n(\mathfrak{S}|v^2)_m}$. Let α, α' be two n -web diagrams in \mathfrak{S} that satisfy the conditions in equation (4.1). We know

$w(\alpha, \alpha') = (w(\alpha\alpha') - w(\alpha) - w(\alpha')) \pmod{n}$. Since $m \mid w(\alpha, \alpha')$ and $m \mid n$, we have $m \mid (w(\alpha\alpha') - w(\alpha) - w(\alpha'))$. Thus

$$\varphi_\epsilon(\alpha\alpha') = \epsilon^{w(\alpha\alpha')} \alpha\alpha' = \epsilon^{w(\alpha)+w(\alpha')} \alpha\alpha' = \varphi_\epsilon(\alpha)\varphi_\epsilon(\alpha').$$

□

Corollary 4.2.3. *Suppose \mathfrak{S} is a planar pb surface with an empty boundary, and $2 \mid n$. Then the linear isomorphism $\varphi_{-1} : \mathcal{S}_n(\mathfrak{S}|v^2) \rightarrow \mathcal{S}_n(\mathfrak{S}|-v^2)$ is an algebra isomorphism.*

Proof. Since \mathfrak{S} is a planar pb surface with an empty boundary, we have $\mathcal{S}_n(\mathfrak{S}|v^2)_2 = \mathcal{S}_n(\mathfrak{S}|v^2)$. Then Corollary 4.2.2 completes the proof. □

4.2.2 On stated SL_n -skein algebras

Let \mathfrak{S} be a pb surface, let m, n be two positive integers with $m \mid n$, and let $\epsilon \in R$ with $\epsilon^{2m} = 1$. In this subsection, we will construct an R -linear isomorphism $\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}, \epsilon v)$. We still use the notation φ_ϵ for this R -linear isomorphism because it coincides with the R -linear isomorphism built in Theorem 4.2.1 when $\partial\mathfrak{S} = \emptyset$. We write Theorem 4.2.1 and the following Theorems separately because it is helpful for readers to understand the Theorems and it simplifies the proofs for the following Theorems.

Recall that, for a stated n -web diagram α , we use $e(\alpha)$ to denote the number of endpoints of α , use $t(\alpha)$ to denote the number of endpoints of α that are pointing towards the boundary of the pb surface, and use $p(\alpha)$ to denote the number of sinks and sources of α .

In the following theorem, we construct an R -linear isomorphism from $\mathcal{S}_n(\mathfrak{S}, v)$ to $\mathcal{S}_n(\mathfrak{S}, \epsilon v)$ when $\epsilon^m = -1$ and $m \mid n$. We will show this R -linear isomorphism restricts an algebra isomorphism from $\mathcal{S}_n(\mathfrak{S}, v)_m$ to $\mathcal{S}_n(\mathfrak{S}, \epsilon v)_m$.

Theorem 4.2.4. *Let \mathfrak{S} be a pb surface, let k, m, n be three positive integers with $n = km$, and let $\epsilon \in R$ with $\epsilon^m = -1$. Then there exists an R -linear isomorphism*

$\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}, \epsilon v)$, defined by

$$\varphi_\epsilon(\alpha) = \begin{cases} \epsilon^{2w(\alpha) + \frac{e(\alpha)}{2}} \alpha & k \text{ is even} \\ (-\epsilon)^{2w(\alpha) + \frac{e(\alpha)}{2}} \alpha & k \text{ is odd and } m \text{ is even} \\ (-\epsilon)^{2w(\alpha) + t(\alpha)} \alpha & k \text{ is odd, } m \text{ is odd, and } \frac{k+m}{2} \text{ is odd} \\ (-1)^{p(\alpha)} (-\epsilon)^{2w(\alpha) + t(\alpha)} \alpha & k \text{ is odd, } m \text{ is odd, and } \frac{k+m}{2} \text{ is even,} \end{cases}$$

where $\alpha \in \mathcal{S}_n(\mathfrak{S}, v)$ is a negatively ordered stated n -web diagram. In particular $\varphi_\epsilon(\alpha) = \epsilon^{2(w(\alpha))} \alpha$ if α is a negatively ordered stated n -web diagram without endpoints.

Note that when n is even, the non-negative integer $e(\alpha)$ is always even.

Proof. It is obvious that φ_ϵ preserves the five moves in Figure 4.1. The proof for Theorem 4.2.1 implies φ_ϵ preserves relations (2.4)-(2.7) because $(\epsilon^2)^m = 1$ and $m \mid n$. Trivially $\varphi_{\epsilon^{-1}} : \mathcal{S}_n(\mathfrak{S}, \epsilon v) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)$ is the inverse of φ_ϵ if φ_ϵ is a well-defined R -linear map. Then it suffices to show φ_ϵ preserves relations (2.8)-(2.11).

Let $u = \epsilon v$. We have $q_u = u^{2n} = \epsilon^{2n} v^{2n} = v^{2n} = q_v$, and $c_{i,u} = (-q_u)^{n-i} (q_u^{\frac{1}{2n}})^{n-1} = \epsilon^{n-1} (-q_v)^{n-i} (q_v^{\frac{1}{2n}})^{n-1} = \epsilon^{n-1} c_{i,v}$ for $1 \leq i \leq n$.

Case 1: k is even, and suppose $k = 2\lambda$. We have $\epsilon^n = \epsilon^{mk} = (-1)^k = 1$. Then $c_{i,u} = \epsilon^{n-1} c_{i,v} = \epsilon^{-1} c_{i,v}$ for $1 \leq i \leq n$. From pure calculation, we have $a_u = (-1)^\lambda a_v$.

Relation (2.8): Let U_1 be the open square in relation (2.8), and let $\alpha_1, \alpha_{1,\sigma}$ be two negatively ordered stated n -web diagrams such that α_1 and $\alpha_{1,\sigma}$ are identical outside U_1 and $\alpha_1 \cap U_1$ (resp. $\alpha_{1,\sigma} \cap U_1$) is like the picture on the lefthand (resp. righthand) side of relation (2.8). Then we have $w(\alpha_{1,\sigma}) = w(\alpha_1)$ and $e(\alpha_{1,\sigma}) = e(\alpha_1) + n$. Thus

$$\begin{aligned} \varphi_\epsilon(a_v \sum_{\sigma \in S_n} (-q_v)^{\ell(\sigma)} \alpha_{1,\sigma}) &= (-1)^\lambda a_u \sum_{\sigma \in S_n} (-q_u)^{\ell(\sigma)} \epsilon^{2w(\alpha_1) + \frac{e(\alpha_1) + n}{2}} \alpha_{1,\sigma} \\ &= \epsilon^{2w(\alpha_1) + \frac{e(\alpha_1)}{2}} \alpha_1 = \varphi_\epsilon(\alpha_1). \end{aligned}$$

Relation (2.9): We have

$$\varphi_\epsilon \left(\begin{array}{c} \text{---} i \\ \curvearrowright \\ \downarrow j \end{array} \right) = \epsilon \begin{array}{c} \text{---} i \\ \curvearrowright \\ \downarrow j \end{array} = \delta_{\bar{j},i} \in c_{i,u} \begin{array}{c} \text{---} \\ \downarrow \end{array} = \delta_{\bar{j},i} c_{i,v} \begin{array}{c} \text{---} \\ \downarrow \end{array}.$$

Relation (2.10): Let U_2 be the open square in relation (2.10), and let α_2, α'_2 be two negatively ordered stated n -web diagrams such that α_2 and α'_2 are identical outside U_2 and $\alpha_2 \cap U_2$ (resp. $\alpha'_2 \cap U_2$) is like the picture on the lefthand (resp. righthand) side of relation (2.10). Then we have $w(\alpha'_2) = w(\alpha_2)$ and $e(\alpha'_2) = e(\alpha_2) + 2$. Thus

$$\varphi_\epsilon \left(\sum_{1 \leq i \leq n} c_{\bar{i},v}^{-1} \alpha'_2 \right) = \sum_{1 \leq i \leq n} \epsilon^{-1} c_{\bar{i},u}^{-1} \epsilon^{2w(\alpha_2) + \frac{e(\alpha_2)+2}{2}} \alpha'_2 = \epsilon^{2w(\alpha_2) + \frac{e(\alpha_2)}{2}} \alpha_2 = \varphi_\epsilon(\alpha_2).$$

Relation (2.11): Let U_3 be the open square in relation (2.11), and let $\alpha_3, \alpha'_3, \alpha''_3$ be three negatively ordered stated n -web diagrams such that $\alpha_3, \alpha'_3, \alpha''_3$ are identical

to each other outside U_3 and $\alpha_3 \cap U_3 = \begin{array}{c} \text{---} i \\ \diagdown \\ \text{---} \\ \diagup \\ \downarrow j \end{array}$, $\alpha'_3 \cap U_3 = \begin{array}{c} \text{---} i \\ \text{---} \\ \text{---} \\ \downarrow j \end{array}$, $\alpha''_3 \cap U_3 = \begin{array}{c} \text{---} j \\ \text{---} \\ \text{---} \\ \downarrow i \end{array}$. Then we have $w(\alpha'_3) = w(\alpha''_3) = w(\alpha_3) - 1$ and $e(\alpha'_3) = e(\alpha''_3) = e(\alpha_3)$. Thus

$$\begin{aligned} \varphi_\epsilon \left(q_v^{-\frac{1}{n}} (\delta_{j < i} (q_v - q_v^{-1}) \alpha'_3 + q_v^{\delta_{i,j}} \alpha''_3) \right) &= \epsilon^2 \epsilon^{2(w(\alpha_3)-1) + \frac{e(\alpha_3)}{2}} q_u^{-\frac{1}{n}} (\delta_{j < i} (q_u - q_u^{-1}) \alpha'_3 + q_u^{\delta_{i,j}} \alpha''_3) \\ &= \epsilon^{2w(\alpha_3) + \frac{e(\alpha_3)}{2}} \alpha_3 = \varphi_\epsilon(\alpha_3). \end{aligned}$$

In the following of the proof, we will still use $\alpha_1, \alpha_{1,\mathfrak{S}}, \alpha_2, \alpha'_2, \alpha_3, \alpha'_3, \alpha''_3$ to denote the corresponding negatively ordered stated n -web diagrams as in Case 1.

Case 2: k is odd, m is even, and suppose $m = 2l$. From pure calculation, we have $\epsilon^n = -1$, $c_{i,u} = -\epsilon^{-1} c_{i,v}$, and $a_u = (-\epsilon)^{kl} a_v$.

Relation (2.8): We have

$$\begin{aligned} \varphi_\epsilon \left(a_v \sum_{\sigma \in S_n} (-q_v)^{\ell(\sigma)} \alpha_{1,\sigma} \right) &= (-\epsilon^{-1})^{kl} a_u \sum_{\sigma \in S_n} (-q_u)^{\ell(\sigma)} (-\epsilon)^{2w(\alpha_1) + \frac{e(\alpha_1)+n}{2}} \alpha_{1,\sigma} \\ &= (-\epsilon)^{2w(\alpha_1) + \frac{e(\alpha_1)}{2}} \alpha_1 = \varphi_\epsilon(\alpha_1). \end{aligned}$$

Relation (2.9): We have

$$\varphi_\epsilon \left(\begin{array}{c} i \\ \circlearrowleft \\ j \end{array} \right) = -\epsilon \begin{array}{c} i \\ \circlearrowright \\ j \end{array} = \delta_{\bar{j},i}(-\epsilon) c_{i,u} \begin{array}{c} | \\ | \\ | \end{array} = \delta_{\bar{j},i} c_{i,v} \begin{array}{c} | \\ | \\ | \end{array}.$$

Relation (2.10): We have

$$\varphi_\epsilon \left(\sum_{1 \leq i \leq n} c_{i,v}^{-1} \alpha'_2 \right) = \sum_{1 \leq i \leq n} -\epsilon^{-1} c_{i,u}^{-1} (-\epsilon)^{2w(\alpha_2) + \frac{e(\alpha_2)+2}{2}} \alpha'_2 = (-\epsilon)^{2w(\alpha_2) + \frac{e(\alpha_2)}{2}} \alpha_2 = \varphi_\epsilon(\alpha_2).$$

Relation (2.11): We have

$$\begin{aligned} \varphi_\epsilon \left(q_v^{-\frac{1}{n}} (\delta_{j < i} (q_v - q_v^{-1}) \alpha'_3 + q_v^{\delta_{i,j}} \alpha''_3) \right) &= \epsilon^2 (-\epsilon)^{2(w(\alpha_3)-1) + \frac{e(\alpha_3)}{2}} q_u^{-\frac{1}{n}} (\delta_{j < i} (q_u - q_u^{-1}) \alpha'_3 + q_u^{\delta_{i,j}} \alpha''_3) \\ &= (-\epsilon)^{2w(\alpha_3) + \frac{e(\alpha_3)}{2}} \alpha_3 = \varphi_\epsilon(\alpha_3). \end{aligned}$$

Case 3: k is odd, m is odd, and $\frac{k+m}{2}$ is odd. From pure calculation, we have $\epsilon^n = -1$, $c_{i,u} = -\epsilon^{-1} c_{i,v}$, and $a_u = a_v$. The proof for checking relations (2.9)-(2.11) are similar with the proof for Case 2.

Relation (2.8): We have $w(\alpha_{1,\mathfrak{S}}) = w(\alpha_1)$, and $t(\alpha_{1,\mathfrak{S}}) = t(\alpha_1)$ or $t(\alpha_1) + n$. We always have $(-\epsilon)^{t(\alpha_{1,\mathfrak{S}})} = (-\epsilon)^{t(\alpha_1)}$ because $(-\epsilon)^n = 1$. Then it is a trivial check that φ_ϵ preserves relation (2.8).

Case 4: k is odd, m is odd, and $\frac{k+m}{2}$ is even. From pure calculation, we have $\epsilon^n = -1$, $c_{i,u} = -\epsilon^{-1} c_{i,v}$, and $a_u = -a_v$. The proof is similar with the proof for Case 3.

□

The following theorem is analogous to Theorem 4.2.4. We establish an R -linear isomorphism from $\mathcal{S}_n(\mathfrak{S}, v)$ to $\mathcal{S}_n(\mathfrak{S}, \epsilon v)$ under the conditions $\epsilon^m = 1$ and $m \mid n$. Just like the R -linear isomorphism presented in Theorem 4.2.4, the corresponding isomorphism in the following theorem also confines an algebra isomorphism from $\mathcal{S}_n(\mathfrak{S}, v)_m$ to $\mathcal{S}_n(\mathfrak{S}, \epsilon v)_m$.

Theorem 4.2.5. *Let \mathfrak{S} be a pb surface, let m, n be two positive integers with $m \mid n$, and let $\epsilon \in R$ with $\epsilon^m = 1$. Then there exists an R -linear isomorphism*

$\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}, \epsilon v)$, defined by

$$\varphi_\epsilon(\alpha) = \begin{cases} \epsilon^{2w(\alpha) + \frac{\epsilon(\alpha)}{2}} \alpha & n \text{ is even} \\ \epsilon^{2w(\alpha) + t(\alpha)} \alpha & n \text{ is odd,} \end{cases}$$

where $\alpha \in \mathcal{S}_n(\mathfrak{S}, v)$ is a negatively ordered stated n -web diagram. In particular $\varphi_\epsilon(\alpha) = \epsilon^{2w(\alpha)}$ if α is a negatively ordered stated n -web diagram without endpoints.

Proof. Set $u = \epsilon v$. From pure calculation, we have $\epsilon^n = 1$, $q_u^{\frac{1}{2}} = q_v^{\frac{1}{2}}$, and $c_{i,u} = \epsilon^{-1}c_{i,v}$ for $1 \leq i \leq n$. Note that $a_u = \epsilon^{\frac{n}{2}}a_v$ when n is even and $a_u = a_v$ when n is odd. Then we can prove φ_ϵ is a well-defined linear map using the same technique as Theorem 4.2.4. Trivially $\varphi_{\epsilon^{-1}} : \mathcal{S}_n(\mathfrak{S}, \epsilon v) \rightarrow \mathcal{S}_n(\mathfrak{S}, v)$ is the inverse of φ_ϵ . □

The following Theorem combines Theorems 4.2.4 and 4.2.5.

Theorem 4.2.6. *Let \mathfrak{S} be a pb surface, let k, m, n be three positive integers with $n = km$, and let $\epsilon \in R$ with $\epsilon^{2m} = 1$. Then there exists an R -linear isomorphism $\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}, \epsilon v)$, defined by*

$$\varphi_\epsilon(\alpha) = \begin{cases} \epsilon^{2w(\alpha) + \frac{\epsilon(\alpha)}{2}} \alpha & \epsilon^m = -1, k \text{ is even} \\ (-\epsilon)^{2w(\alpha) + \frac{\epsilon(\alpha)}{2}} \alpha & \epsilon^m = -1, k \text{ is odd, and } m \text{ is even} \\ (-\epsilon)^{2w(\alpha) + t(\alpha)} \alpha & \epsilon^m = -1, k \text{ is odd, } m \text{ is odd, and } \frac{k+m}{2} \text{ is odd} \\ (-1)^{p(\alpha)} (-\epsilon)^{2w(\alpha) + t(\alpha)} \alpha & \epsilon^m = -1, k \text{ is odd, } m \text{ is odd, and } \frac{k+m}{2} \text{ is even} \\ \epsilon^{2w(\alpha) + \frac{\epsilon(\alpha)}{2}} \alpha & \epsilon^m = 1, n \text{ is even} \\ \epsilon^{2w(\alpha) + t(\alpha)} \alpha & \epsilon^m = 1, n \text{ is odd,} \end{cases}$$

where $\alpha \in \mathcal{S}_n(\mathfrak{S}, v)$ is a negatively ordered stated n -web diagram. In particular $\varphi_\epsilon(\alpha) = \epsilon^{2w(\alpha)}$ if α is a negatively ordered stated n -web diagram without endpoints.

Proof. Theorems 4.2.4, 4.2.5. □

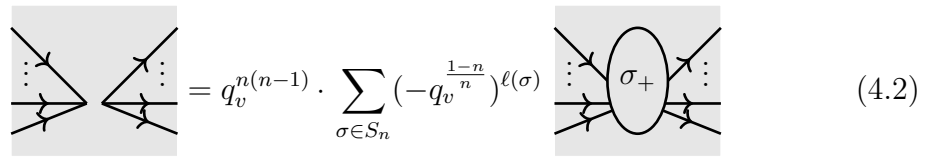
Corollary 4.2.7. *Let \mathfrak{S} be a pb surface, let k, m, n be three positive integers with $m \mid n$, and let $\epsilon \in R$ with $\epsilon^{2m} = 1$. Then the R -linear isomorphism $\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}, \epsilon v)$ restricts to an algebra isomorphism $\varphi_\epsilon|_{\mathcal{S}_n(\mathfrak{S}, v)_m} : \mathcal{S}_n(\mathfrak{S}, v)_m \rightarrow \mathcal{S}_n(\mathfrak{S}, \epsilon v)_m$.*

Proof. The proof is similar with Corollary 4.2.2. □

4.3 Spin structure and the SL_n -skein space

The SU_n -skein module of an oriented 3-manifold is defined in [Sik05]. The SU_n -skein module of the 3-manifold M is built of based n -webs which are defined as the n -webs in M , except that the half-edges incident to any of their n -valent vertices are linearly ordered. The SU_n -skein module, denoted as $\mathcal{S}_n^b(M, v^2)$, is the quotient of the R -module freely generated by the set of isotopy classes of based n -webs subject to relations (2.4)-(2.6) and (4.2). Lê and Sikora proved the equivalence between the SU_n -skein theory and the SL_n -skein theory [LS21]. To simplify the proof, we will work with the SU_n -skein theory in this section (all the discussions can be pulled back to the SL_n -skein theory using the equivalence built by Lê and Sikora).

For a 3-manifold M , Barrett established an isomorphism from $\mathcal{S}_2^b(M, v^2)$ to $\mathcal{S}_2^b(M, -v^2)$ using a spin structure of M [Bar99]. In this section, we are trying to generalize Barrett's work to all n .



$$\text{[web]} = q_v^{n(n-1)} \cdot \sum_{\sigma \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma)} \text{[web with } \sigma_+ \text{]} \quad (4.2)$$

Let α be a based n -web in M , and H be an isotopy of α . We use α^H to denote the based n -web in M obtained from α by doing isotopy H . We use $K(\alpha)$ to denote the number of knot components of α .

rem

Suppose n is a positive even number. Consider a based n -web α in a manifold M with k sinks and sources ($k > 0$), each labeled from 1 to k . Through an isotopy H , we deform α such that, for each $1 \leq i \leq k$, the sink and source labeled by i lie in a small open cube U_i . The intersection $\alpha^H \cap U_i$ resembles the illustration on the left side of relation (4.2).

For $\sigma_1, \sigma_2, \dots, \sigma_k \in S_n$, denote $\alpha_{\sigma_1, \dots, \sigma_k}^H$ as the based n -web obtained from α^H . This web coincides with α^H outside $\cup_{1 \leq i \leq k} U_i$, and within each U_i , the intersection $\alpha_{\sigma_1, \dots, \sigma_k}^H \cap U_i$ follows the pattern on the right side of relation (4.2), with $\sigma_+ = (\sigma_i)_+$.

In $\mathcal{S}_n^b(M, v^2)$, relation (4.2) implies

$$\alpha = q_v^{n(n-1)k} \sum_{\sigma_1, \dots, \sigma_k \in S_n} \left(-q_v^{\frac{1-n}{n}}\right)^{\ell(\sigma_1) + \dots + \ell(\sigma_k)} \alpha_{\sigma_1, \dots, \sigma_k}^H.$$

For each $1 \leq i \leq k$, we know there is a linear order on all the half edges incident to the i -th sink (and source) of α . Then we color the j -th half edge incident to the i -sink (resp. the source) with $(i-1)n+j$ (resp. $(i-1)n+n+1-j$). Then there exists $\tau_\alpha \in S_{kn}$ such that, for each $1 \leq i \leq kn$, the starting half edge colored by i is connected to the targeting half edge colored by $\tau_\alpha(i)$.

Although τ_α is dependent on how we label the sinks and sources of α , the sign $(-1)^{\ell(\tau_\alpha)}$ is irrelevant of the labelings of sinks and sources because n is even. It is easy to show

$$(-1)^{K(\alpha_{\sigma_1, \dots, \sigma_k}^H)} = (-1)^{K(\alpha) + \ell(\tau_\alpha) + \ell(\sigma_1) + \dots + \ell(\sigma_k)}.$$

Suppose n is even, and s is a spin structure of M . For any based n -web α in M , Lê and Sikora defined $s(\alpha) \in \mathbb{Z}_2$ in page 76 in [LS21] with the properties that $s(\text{trivial knot}) = 1$ and $s(\alpha') = s(\alpha) + 1$ if α' has one more positive or negative kink than α .

The following Theorem generalizes Barrett's work [Bar99] to all n .

Theorem 4.3.1. *Let M be a 3-manifold, and let s be a spin structure of M . For each positive integer n , there exists a linear isomorphism $\Phi_n : \mathcal{S}_n^b(M, v^2) \rightarrow \mathcal{S}_n^b(M, -v^2)$, defined by*

$$\Phi_n(\alpha) = \begin{cases} \alpha & n \text{ is odd} \\ (-1)^{s(\alpha) + K(\alpha) + \ell(\tau_\alpha)} & n \text{ is even,} \end{cases}$$

where α is a based n -web in M , the permutation $\tau_\alpha = (1)$ if α has no sinks or sources and τ_α is the permutation defined as in Remark 4.3 if α has sinks and sources. In particular Φ_n is an algebra isomorphism when M is the thickening of an oriented surface.

Proof. When n is odd, relations (2.4)-(2.6) and (4.2) for $\mathcal{S}_n^b(M, v^2)$ are the same with the corresponding relations for $\mathcal{S}_n^b(M, -v^2)$.

Suppose n is even. Obviously, the map Φ_n is well-defined on the set of isotopy classes of based n -webs.

Suppose the based n -web α in M has k sinks (and sources) with $k > 0$. We do the same procedures to α and follow the same notations as in Remark 4.3. In $\mathcal{S}_n^b(M, -v^2)$, we have

$$\begin{aligned}
\Phi_n(\alpha) &= (-1)^{s(\alpha)+K(\alpha)+\ell(\tau_\alpha)} \alpha \\
&= (-1)^{s(\alpha)+K(\alpha)+\ell(\tau_\alpha)} q_v^{n(n-1)k} \sum_{\sigma_1, \dots, \sigma_k \in S_n} \left(q_v^{\frac{1-n}{n}} \right)^{\ell(\sigma_1)+\dots+\ell(\sigma_k)} \alpha_{\sigma_1, \dots, \sigma_k}^H \\
&= (-1)^{s(\alpha)} q_v^{n(n-1)k} \sum_{\sigma_1, \dots, \sigma_k \in S_n} \left(-q_v^{\frac{1-n}{n}} \right)^{\ell(\sigma_1)+\dots+\ell(\sigma_k)} (-1)^{K(\alpha)+\ell(\tau_\alpha)+\ell(\sigma_1)+\dots+\ell(\sigma_k)} \alpha_{\sigma_1, \dots, \sigma_k}^H \\
&= (-1)^{s(\alpha)} q_v^{n(n-1)k} \sum_{\sigma_1, \dots, \sigma_k \in S_n} \left(-q_v^{\frac{1-n}{n}} \right)^{\ell(\sigma_1)+\dots+\ell(\sigma_k)} (-1)^{K(\alpha_{\sigma_1, \dots, \sigma_k}^H)} \alpha_{\sigma_1, \dots, \sigma_k}^H.
\end{aligned} \tag{4.3}$$

Then we are trying to show Φ_n preserves relations (2.4)-(2.6) and (4.2).

Relation (2.4): We use U to denote the small open cube in relation (2.4). Let $\alpha_1, \alpha_2, \alpha_3$ be three based n -webs in M such that they are identical to each other

outside U and $\alpha_1 \cap U = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}$, $\alpha_2 \cap U = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array}$, $\alpha_3 \cap U = \begin{array}{c} \rightarrow \\ \rightarrow \end{array}$. It is easy to show $s(\alpha_1) = s(\alpha_2) = s(\alpha_3)$. If $\alpha_1, \alpha_2, \alpha_3$ contain no sinks or sources, we have $(-1)^{K(\alpha_1)} = (-1)^{K(\alpha_2)} = -(-1)^{K(\alpha_3)}$. Then, in $\mathcal{S}_n^b(M, -v^2)$, we have

$$\begin{aligned}
\Phi_n(q_v^{\frac{1}{n}} \alpha_1 - q_v^{-\frac{1}{n}} \alpha_2) &= q_v^{\frac{1}{n}} (-1)^{s(\alpha_1)+K(\alpha_1)} \alpha_1 - q_v^{-\frac{1}{n}} (-1)^{s(\alpha_2)+K(\alpha_2)} \alpha_2 \\
&= (-1)^{s(\alpha_3)+K(\alpha_3)} (q_v^{-\frac{1}{n}} \alpha_2 - q_v^{\frac{1}{n}} \alpha_1) = (-1)^{s(\alpha_3)+K(\alpha_3)} (q_v - q_v^{-1}) \alpha_3 = \Phi_n((q_v - q_v^{-1}) \alpha_3).
\end{aligned}$$

If $\alpha_1, \alpha_2, \alpha_3$ contain sinks and sources, we label the sinks (and sources) in the same way for $\alpha_1, \alpha_2, \alpha_3$. We can use the same isotopy H to pair the sinks and sources for $\alpha_1, \alpha_2, \alpha_3$ such that the support of H is contained in $M \setminus U$. Let $\sigma_1, \dots, \sigma_k \in S_n$. We have $(\alpha_1)_{\sigma_1, \dots, \sigma_k}^H, (\alpha_2)_{\sigma_1, \dots, \sigma_k}^H, (\alpha_3)_{\sigma_1, \dots, \sigma_k}^H$ are identical to each other outside U , and

$$(\alpha_1)_{\sigma_1, \dots, \sigma_k}^H \cap U = \alpha_1 \cap U, \quad (\alpha_2)_{\sigma_1, \dots, \sigma_k}^H \cap U = \alpha_2 \cap U, \quad (\alpha_3)_{\sigma_1, \dots, \sigma_k}^H \cap U = \alpha_3 \cap U.$$

Futhermore, $(-1)^{K((\alpha_1)_{\sigma_1, \dots, \sigma_k}^H)} = (-1)^{K((\alpha_2)_{\sigma_1, \dots, \sigma_k}^H)} = -(-1)^{K((\alpha_3)_{\sigma_1, \dots, \sigma_k}^H)}$. Note that, in $\mathcal{S}_n^b(M, -v^2)$, we have

$$q_v^{-\frac{1}{n}}(\alpha_2)_{\sigma_1, \dots, \sigma_k}^H - q_v^{\frac{1}{n}}(\alpha_1)_{\sigma_1, \dots, \sigma_k}^H = (q_v - q_v^{-1})(\alpha_3)_{\sigma_1, \dots, \sigma_k}^H.$$

From equation (4.3), we have

$$\begin{aligned} & \Phi_n(q_v^{\frac{1}{n}}\alpha_1 - q_v^{-\frac{1}{n}}\alpha_2) \\ = & (-1)^{s(\alpha_1)}q_v^{n(n-1)k} \sum_{\sigma_1, \dots, \sigma_k \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma_1)+\dots+\ell(\sigma_k)} (-1)^{K((\alpha_1)_{\sigma_1, \dots, \alpha_k}^H)} q_v^{\frac{1}{n}}(\alpha_1)_{\sigma_1, \dots, \alpha_k}^H \\ & - (-1)^{s(\alpha_2)}q_v^{n(n-1)k} \sum_{\sigma_1, \dots, \sigma_k \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma_1)+\dots+\ell(\sigma_k)} (-1)^{K((\alpha_2)_{\sigma_1, \dots, \alpha_k}^H)} q_v^{-\frac{1}{n}}(\alpha_2)_{\sigma_1, \dots, \alpha_k}^H \\ = & (-1)^{s(\alpha_3)}q_v^{n(n-1)k} \\ & \sum_{\sigma_1, \dots, \sigma_k \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma_1)+\dots+\ell(\sigma_k)} (-1)^{K((\alpha_3)_{\sigma_1, \dots, \alpha_k}^H)} (q_v^{-\frac{1}{n}}(\alpha_2)_{\sigma_1, \dots, \alpha_k}^H - q_v^{\frac{1}{n}}(\alpha_1)_{\sigma_1, \dots, \alpha_k}^H) \\ = & (q_v - q_v^{-1})(-1)^{s(\alpha_3)}q_v^{n(n-1)k} \sum_{\sigma_1, \dots, \sigma_k \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma_1)+\dots+\ell(\sigma_k)} (-1)^{K((\alpha_3)_{\sigma_1, \dots, \alpha_k}^H)} (\alpha_3)_{\sigma_1, \dots, \alpha_k}^H \\ = & \Phi_n((q_v - q_v^{-1})\alpha_3). \end{aligned}$$

It is trivial that Φ_n preserves relations (2.5) and (2.6).

Relation (4.2): In relation (4.2), the open cube is denoted by V . Consider two based n -webs in M , namely β and β_σ , with the property that they are identical outside V . The intersections $\beta \cap V$ and $\beta_\sigma \cap V$ correspond to pictures on the lefthand and righthand sides of equation (4.2), respectively.

If β_σ has no sinks or sources, equation (4.3) implies

$$\begin{aligned} \Phi_n(\beta) &= (-1)^{s(\beta)}q_v^{n(n-1)} \sum_{\sigma \in S_n} (-1)^{K(\beta_\sigma)} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma)} \beta_\sigma \\ &= q_v^{n(n-1)} \sum_{\sigma \in S_n} (-1)^{K(\beta_\sigma)+s(\beta_\sigma)} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma)} \beta_\sigma \\ &= \Phi_n(q_v^{n(n-1)} \sum_{\sigma \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma)} \beta_\sigma). \end{aligned}$$

Suppose β_σ has k sinks (and sources) with $k > 0$. We label all the sinks (and sources) of β_σ from 1 to k . We label the sink (resp. source) of β shown in the

relation (4.2) by 1, and label the rest of the sinks (resp. sources) of β from 2 to $k+1$ such that, for each $1 \leq i \leq k$, the sink (resp. source) of β labeled by $i+1$ is the sink (resp. source) of β_σ labeled by i . Since the sink and the source of β labeled by 1 are already paired together, there exists an isotopy F of β such that the support of F is contained in $M \setminus V$ and, for each $2 \leq i \leq k+1$, the sink and the source of β^F labeled by i are like the picture on the lefthand side of relation (4.2). Obviously, we can regard F as an isotopy of β_σ , and, for each $1 \leq i \leq k$, the sink and the source of $(\beta_\sigma)^F$ labeled by i are like the picture on the lefthand side of relation (4.2). Then we have

$$\begin{aligned}
& \Phi_n(q_v^{n(n-1)} \sum_{\sigma \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma)} \beta_\sigma) \\
&= q_v^{n(n-1)} \sum_{\sigma \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma)} (-1)^{s(\beta_\sigma)} q_v^{n(n-1)k} \\
&\quad \sum_{\sigma_1, \dots, \sigma_k \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma_1) + \dots + \ell(\sigma_k)} (-1)^{K((\beta_\sigma)_{\sigma_1, \dots, \alpha_k}^F)} (\beta_\sigma)_{\sigma_1, \dots, \alpha_k}^F \\
&= q_v^{n(n-1)(k+1)} (-1)^{s(\beta)} \sum_{\sigma, \sigma_1, \dots, \sigma_k \in S_n} (-q_v^{\frac{1-n}{n}})^{\ell(\sigma) + \ell(\sigma_1) + \dots + \ell(\sigma_k)} (-1)^{K(\beta_{\sigma, \sigma_1, \dots, \alpha_k}^F)} \beta_{\sigma, \sigma_1, \dots, \alpha_k}^F \\
&= \Phi_n(\beta).
\end{aligned}$$

□

4.4 The R -linear isomorphism between $\mathcal{S}_n(M, \mathcal{N}, v)$ and $\mathcal{S}_n(M, \mathcal{N}, -v)$

For any marked 3-manifold (M, \mathcal{N}) and any positive integer n , we will show there exists an R -linear isomorphism $\Psi_n : \mathcal{S}_n(M, \mathcal{N}, v) \rightarrow \mathcal{S}_n(M, \mathcal{N}, -v)$ such that it is an algebra isomorphism when (M, \mathcal{N}) is the thickening of a pb surface and $\Psi_n(\alpha) = \alpha$ for any n -web α in (M, \mathcal{N}) without endpoints.

Let $u = -v$. We have $q_u^{\frac{1}{n}} = u^2 = v^2 = q_v^{\frac{1}{n}}$, $q_u = u^{2n} = v^{2n} = q_v$. Then relations (2.4)-(2.7) for $\mathcal{S}_n(M, \mathcal{N}, u)$ are exactly the same with these relations for $\mathcal{S}_n(M, \mathcal{N}, v)$.

Theorem 4.4.1. *Let (M, \mathcal{N}) be a marked 3-manifold, and let n be any positive integer. There exists an R -linear isomorphism $\Psi_n : \mathcal{S}_n(M, \mathcal{N}, v) \rightarrow \mathcal{S}_n(M, \mathcal{N}, -v)$*

defined by

$$\Psi_n(\alpha) = \begin{cases} (-1)^{\frac{(n-1)e(\alpha)}{2}} \alpha & n \text{ is odd} \\ (-1)^{\frac{e(\alpha)}{2}} \alpha & n \text{ is even,} \end{cases}$$

where $\alpha \in \mathcal{S}_n(M, \mathcal{N}, v)$ is a stated n -web in (M, \mathcal{N}) . In particular, Ψ_n is an algebra isomorphism when (M, \mathcal{N}) is the thickening of a pb surface and $\Psi_n(\alpha) = \alpha$ for any stated n -web α in (M, \mathcal{N}) without endpoints.

Proof. It is obvious that Ψ_n is well-defined on the set of isotopy classes of stated n -webs in (M, \mathcal{N}) . Then it suffices to show Ψ_n preserves relations (2.4)-(2.11).

Let $u = -v$. From the previous discussion, we know $q_u^{\frac{1}{2}} = q_v^{\frac{1}{2}}, q_u = q_v$. Then, for each $1 \leq i \leq n$, we have

$$c_{i,u} = (-q_u)^{n-i} (q_u^{\frac{1}{2}})^{n-1} = (-1)^{n-1} (-q_v^{n-i}) q_v^{\frac{n-1}{2}} = (-1)^{n-1} c_{i,v}. \quad (4.4)$$

When n is odd, we suppose $n = 2k + 1$. Equation (4.4) implies that $c_{i,u} = c_{i,v}$ for $1 \leq i \leq n$. We have $q_u^{\frac{1}{2}} = u^n = (-v)^n = -v^n = -q_v^{\frac{1}{2}}$, and $a_u = q_u^{\frac{n+1}{4}} q_u^{-\frac{n^2}{2}} = q_u^{\frac{k+1}{2}} q_u^{-\frac{n^2}{2}} = (-1)^k q_v^{\frac{k+1}{2}} q_v^{-\frac{n^2}{2}} = (-1)^k a_v$. Trivially Ψ_n preserves relations (2.4)-(2.7) and (2.9)-(2.11). Let α (resp. α_σ) be the stated n -web on the lefthand (resp. righthand) side of relation (2.8), then we have $e(\alpha_\sigma) = e(\alpha) + n$. We have

$$\Psi_n(a_v \sum_{\sigma \in S_n} (-q_v)^{\ell(\sigma)} \alpha_\sigma) = (-1)^k a_u \sum_{\sigma \in S_n} (-q_u)^{\ell(\sigma)} (-1)^{k(e(\alpha)+n)} \alpha_\sigma = (-1)^{k e(\alpha)} \alpha = \Psi_n(\alpha).$$

When n is even, suppose $n = 2\lambda$. Equation (4.4) implies that $c_{i,u} = -c_{i,v}$ for $1 \leq i \leq n$. We have $q_u^{\frac{1}{2}} = u^n = (-v)^n = v^n = q_v^{\frac{1}{2}}$, $q_u^{\frac{1}{4}} = u^\lambda = (-1)^\lambda v^\lambda = (-1)^\lambda q_v^{\frac{1}{4}}$, and $a_u = q_u^{\frac{n+1-2n^2}{4}} = (-1)^{\lambda(n+1-2n^2)} q_v^{\frac{n+1-2n^2}{4}} = (-1)^\lambda a_v$. Trivially Ψ_n preserves relations (2.4)-(2.7).

Relation (2.8): We use β_1 (resp. $\beta_{1,\sigma}$) to denote the stated n -web on the lefthand (resp. righthand) side of relation (2.8). Then $e(\beta_{1,\sigma}) = e(\beta_1) + n$. We have

$$\begin{aligned} \Psi_n(a_v \sum_{\sigma \in S_n} (-q_v)^{\ell(\sigma)} \beta_{1,\sigma}) &= (-1)^\lambda a_u \sum_{\sigma \in S_n} (-q_u)^{\ell(\sigma)} (-1)^{\frac{1}{2}(e(\beta_1)+n)} \beta_{1,\sigma} \\ &= (-1)^{\frac{e(\beta_1)}{2}} \beta_1 = \Psi_n(\beta_1). \end{aligned}$$

Relation (2.9): We have

$$\Psi_n \left(\begin{array}{c} i \\ \circlearrowleft \\ j \end{array} \right) = - \begin{array}{c} i \\ \circlearrowright \\ j \end{array} = -\delta_{j,i} c_{i,u} \left| \begin{array}{c} \\ \\ \end{array} \right| = \delta_{j,i} c_{i,v} \left| \begin{array}{c} \\ \\ \end{array} \right|.$$

Relation (2.10): We use β_2 (resp. β'_2) to denote the stated n -web on the lefthand (resp. righthand) side of relation (2.10). Then $e(\beta'_2) = e(\beta_2) + 2$. We have

$$\Psi_n \left(\sum_{1 \leq i \leq n} c_{i,v}^{-1} \beta'_2 \right) = \sum_{1 \leq i \leq n} -c_{i,u}^{-1} (-1)^{\frac{1}{2}(e(\beta_2)+2)} \beta'_2 = (-1)^{\frac{1}{2}e(\beta_2)} \beta_2 = \Psi_n(\beta_2).$$

It is a trivial check that Ψ_n preserves relation (2.11). \square

rem Theorem 4.4.1 is also true when (M, \mathcal{N}) is the generalized marked 3-manifold, please refer to the next section for the definition of the generalized marked 3-manifold.

Let (M, \mathcal{N}) be a marked 3-manifold, and e be an oriented open interval in ∂M such that there is no intersection between the closure of \mathcal{N} and the closure of e . Then $(M, \mathcal{N} \cup e)$ is also a marked 3-manifold, and we say $(M, \mathcal{N} \cup e)$ is obtained from (M, \mathcal{N}) by adding one extra marking e . We use l_e to denote the R -linear map from $\mathcal{S}_n(M, \mathcal{N}, v)$ to $\mathcal{S}_n(M, \mathcal{N} \cup e, v)$ induced by the embedding from (M, \mathcal{N}) to $(M, \mathcal{N} \cup e)$. We will call the R -linear map l_e the **adding marking map**. We have the following obvious Corollary.

Corollary 4.4.2. *For each positive integer n , the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{S}_n(M, \mathcal{N}, v) & \xrightarrow{l_e} & \mathcal{S}_n(M, \mathcal{N} \cup e, v) \\ \downarrow \Psi_n & & \downarrow \Psi_n \\ \mathcal{S}_n(M, \mathcal{N}, -v) & \xrightarrow{l_e} & \mathcal{S}_n(M, \mathcal{N} \cup e, -v) \end{array}.$$

4.5 Stated SL_n -TQFT

The idea of this section is motivated by Costantino and Lê's work [CL22b]. They formulated the stated TQFT theory for SL_2 . In this section, we will generalize their results to SL_n .

A **generalized marked 3-manifold** a pair (M, \mathcal{N}) , where M is an oriented 3-manifold, and $\mathcal{N} \subset \partial M$ is a one dimensional submanifold consisting of oriented circles and oriented open intervals such that there is no intersection between the closure of any two components. We use \mathcal{N}_1 to denote the subset of \mathcal{N} consisting of all the oriented open intervals.

The definition of the stated SL_n -skein module of a generalized marked 3-manifold is the same with the definition of the stated SL_n -skein module of a marked 3-manifold. For a generalized marked 3-manifold (M, \mathcal{N}) , we also use $\mathcal{S}_n(M, \mathcal{N}, v)$ to denote the stated SL_n -skein module of (M, \mathcal{N}) . The classical limit of the stated SL_n -skein module of the generalized marked 3-manifold is well-studied in [Wan23c].

A **marked surface** is a pair $(\mathfrak{S}, \mathcal{P})$, where \mathfrak{S} is an oriented compact surface and \mathcal{P} is a set of finite points in $\partial\mathfrak{S}$, called marked points. We assume that every point of \mathcal{P} is signed by " - " or " + ", and every component of \mathfrak{S} contains at least one marked point.

For a marked surface $(\mathfrak{S}, \mathcal{P})$, we can define a marked 3-manifold (M, \mathcal{N}) , where $M = \mathfrak{S} \times [-1, 1]$, $\mathcal{N} = \mathcal{P} \times (-1, 1)$. For any $p \in \mathcal{P}$, the orientation of $\{p\} \times (-1, 1)$ is the positive (resp. negative) orientation of $(-1, 1)$ if the sign of p is positive (resp. negative). We call (M, \mathcal{N}) the thickening of $(\mathfrak{S}, \mathcal{P})$, and define $\mathcal{S}_n(\mathfrak{S}, \mathcal{P}, v)$ to be $\mathcal{S}_n(M, \mathcal{N}, v)$. Then $\mathcal{S}_n(\mathfrak{S}, \mathcal{P}, v)$ has an algebra structure given by stacking the stated n -webs, that is, for any two stated n -webs α, β , the product $\alpha\beta$ is defined to be stacking α above β .

Let (M, \mathcal{N}) be a generalized marked 3-manifold, and $(\mathfrak{S}, \mathcal{P})$ be a marked surface. Suppose $\phi : \mathfrak{S} \rightarrow \partial M$ is an embedding such that $\phi(\mathfrak{S}) \cap \overline{\mathcal{N}} = \phi(\mathfrak{S}) \cap \overline{\mathcal{N}}_1 = \phi(\mathcal{P})$. For any $p \in \mathcal{P}$, we assume the sign of p is + (resp. -) if ϕ is orientation preserving and the component of $\overline{\mathcal{N}}$ connecting to p points towards (resp. away from) p , and the sign of p is - (resp. +) if ϕ is orientation reversing and the component of $\overline{\mathcal{N}}$ connecting to p points towards (resp. away from) p .

Then a closed regular neighborhood $U(\mathfrak{S})$ of $\phi(\mathfrak{S})$ is isomorphic to the thickening of $(\mathfrak{S}, \mathcal{P})$, we use ψ to denote this isomorphism from $(\mathfrak{S}, \mathcal{P}) \times [-1, 1]$ to $U(\mathfrak{S})$. Then $\mathcal{S}_n(\mathfrak{S}, \mathcal{P}, v)$ has a left (resp. right) action on $\mathcal{S}_n(M, \mathcal{N}, v)$ if ϕ is orientation preserving (resp. reversing). For any stated n -webs α in $(\mathfrak{S}, \mathcal{P}) \times [-1, 1]$ and $\beta \in (M, \mathcal{N})$, we first isotope β such that β is away from $U(\mathfrak{S})$, the action of α on β is given by $\psi(\alpha) \cup \beta$. Costantino and Lê defined the above actions for $n = 2$ in subsection 4.5 in [CL22b]. Here we recalled their construction and generalized it to all n in an obvious way.

For a generalized marked 3-manifold (M, \mathcal{N}) , Costantino and Lê defined the so called strict subsurface in [CL22b]. Here we recall their definition. A **strict subsurface** \mathfrak{S} of (M, \mathcal{N}) is a proper embedding $\mathfrak{S} \rightarrow M$ of a compact surface such that \mathfrak{S} is transversal to \mathcal{N} and every connected component of \mathfrak{S} intersects \mathcal{N} . Define $\text{Sl}_{\mathfrak{S}}(M, \mathcal{N})$ to be (M', \mathcal{N}') , where $M' = M \setminus \mathfrak{S}$ and $\mathcal{N}' = \mathcal{N} \setminus \mathfrak{S}$. Define $\mathcal{P} = \mathfrak{S} \cap \mathcal{N}$. For a point $p \in \mathcal{P}$, define its sign to be $+$ or $-$ according as the orientation of M is equal the orientation of \mathfrak{S} followed by the orientation of the tangent to \mathcal{N} at p or not. Then $(\mathfrak{S}, \mathcal{P})$ is a marked surface and there is a right and a left action of $\mathcal{S}_n(\mathfrak{S}, \mathcal{P}, v)$ on $\mathcal{S}_n(\text{Sl}_{\mathfrak{S}}(M, \mathcal{N}), v)$.

The obvious embedding from $\text{Sl}_{\mathfrak{S}}(M, \mathcal{N})$ to (M, \mathcal{N}) induces an R -linear map

$$\varphi_{\mathfrak{S}} : \mathcal{S}_n(\text{Sl}_{\mathfrak{S}}(M, \mathcal{N}), v) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v).$$

Theorem 4.5.1. *Assume \mathfrak{S} is a strict subsurface of a generalized marked 3-manifold (M, \mathcal{N}) . Then the linear map $\varphi_{\mathfrak{S}} : \mathcal{S}_n(\text{Sl}_{\mathfrak{S}}(M, \mathcal{N}), v) \rightarrow \mathcal{S}_n(M, \mathcal{N}, v)$ is surjective and its kernel is the R -span of $\{a \cdot x - x \cdot a \mid a \in \mathcal{S}_n(\mathfrak{S}, \mathcal{P}, v), x \in \mathcal{S}_n(\text{Sl}_{\mathfrak{S}}(M, \mathcal{N}), v)\}$.*

Note that

$$\mathcal{S}_n(\text{Sl}_{\mathfrak{S}}(M, \mathcal{N}), v) / R\text{-span}\{a \cdot x - x \cdot a \mid a \in \mathcal{S}_n(\mathfrak{S}, \mathcal{P}, v), x \in \mathcal{S}_n(\text{Sl}_{\mathfrak{S}}(M, \mathcal{N}), v)\},$$

denoted as $\text{HH}_0(\mathcal{S}_n(\text{Sl}_{\mathfrak{S}}(M, \mathcal{N}), v))$, is the 0-th Hochschild homology of the $\mathcal{S}_n(\mathfrak{S}, \mathcal{P}, v)$ -bimodule $\mathcal{S}_n(\text{Sl}_{\mathfrak{S}}(M, \mathcal{N}), v)$.

Proof. The proof of Theorem 5.1 in [CL22b] almost works here. We only need to check the following equations and the parallel equations, obtained from the following equations by reversing the arrows of all the stated n -webs, in $HH_0(\mathcal{S}_n(\mathrm{Sl}_{\mathfrak{S}}(M, \mathcal{N}), v))$:

$$\begin{array}{c}
 \begin{array}{|c} \hline \text{Web with } n \text{ strands merging to } 1 \\ \hline \end{array} \\
 = \sum_{1 \leq i_1, \dots, i_n \leq n} c_{i_1, v}^{-1} \cdots c_{i_n, v}^{-1} \\
 \begin{array}{|c} \hline \text{Web with } n \text{ strands merging to } 1 \text{ and then splitting into } i_1, \dots, i_n \\ \hline \end{array} \\
 \end{array} \tag{4.5}$$

$$\begin{array}{c}
 \begin{array}{|c} \hline \text{Web with strands } i \text{ and } j \text{ crossing} \\ \hline \end{array} \\
 = \sum_{1 \leq i, j \leq n} c_{i, v}^{-1} c_{j, v}^{-1} \\
 \begin{array}{|c} \hline \text{Web with strands } \bar{i} \text{ and } \bar{j} \text{ crossing} \\ \hline \end{array} \\
 \end{array} \tag{4.6}$$

$$\begin{array}{c}
 \begin{array}{|c} \hline \text{Web with strand } j \text{ crossing strands } i \text{ and } \bar{i} \\ \hline \end{array} \\
 = \sum_{1 \leq i \leq n} c_{i, v}^{-1} \\
 \begin{array}{|c} \hline \text{Web with strands } i \text{ and } \bar{i} \text{ crossing strand } j \\ \hline \end{array} \\
 \end{array} \tag{4.7}$$

Equation (4.5) and its parallel equation come from equation (53) in [LS21] and relation (2.8). Equation (4.6) and its parallel equation come from relations (2.9) and (2.10). Equation (4.7) and its parallel equation come from relation (2.9).

Using equations (4.5)-(4.7) and the techniques used in the proof of Theorem 5.1 in [CL22b], we can easily prove Theorem 4.5.1.

□

A **decorated manifold** is 5-tuple $\mathbb{M} = (M, \partial^+ M, \partial^- M, \partial^s M, \mathcal{N})$ with the following properties: (1) (M, \mathcal{N}) is a generalized marked 3-manifold; (2) $\partial^+ M, \partial^- M, \partial^s M \subset \partial M$ are compact surfaces with boundaries with disjoint interiors and oriented as induced by the orientation of M , such that $\partial^+ M \cup \partial^- M \cup \partial^s M = \partial M$ and $\partial^+ M \cap \partial^- M = \emptyset$; (3) $(\partial^{\pm 1} M, \overline{\mathcal{N}} \cap \partial^{\pm 1} M)$ are marked surfaces, where the signs of the marked points are determined by the orientation of \mathcal{N} and $\partial^{\pm} M$. Please refer to Definition 6.1 in [CL22b] for the detailed definition of decorated manifolds.

In Definition 6.2 in [CL22b], Costantino and Lê defined **the category of decorated cobordisms**, denoted as DeCob. Here we briefly recall the definition of DeCob. The objects of DeCob are marked surfaces. A morphism from \mathfrak{S}_- to \mathfrak{S}_+ is the diffeomorphism class of a decorated manifold \mathbb{M} endowed with diffeomorphisms $\phi_{\pm 1} : \partial^{\pm} M \rightarrow \mathfrak{S}_{\pm 1}$ with ϕ_+ orientation preserving and ϕ_- orientation reversing. The composition of morphisms are obtained by gluing the decorated manifolds along the common marked surface. The identity morphism is the thickening of the corresponding marked surface. Then DeCob is a symmetric monoidal with \otimes given by disjoint union.

Let Mor be the ‘‘Morita category’’ whose objects are R -algebras and morphisms are isomorphism classes of bimodules in the category of R -modules. The composition is given by the tensor product over the mid algebra. The identity of an algebra A is the isomorphism class of A as left and right bimodule over itself via left and right multiplication. It is a symmetric monoidal category with the tensor product \otimes_R .

For any object $(\mathfrak{S}, \mathcal{P}) \in \text{DeCob}$, we define $\mathcal{S}_n((\mathfrak{S}, \mathcal{P}))$ to be $\mathcal{S}_n(\mathfrak{S}, \mathcal{P}, v)$, which is an object in Mor. For any morphism $\mathbb{M} = (M, \partial^+ M, \partial^- M, \partial^s M, \mathcal{N}) : \mathfrak{S}_- \rightarrow \mathfrak{S}_+$, we define $\mathcal{S}_n(\mathbb{M})$ to be $\mathcal{S}_n(M, \mathcal{N}, v)$, which is a right module over $\mathcal{S}_n(\mathfrak{S}_-)$ and a left module over $\mathcal{S}_n(\mathfrak{S}_+)$. Then $\mathcal{S}_n(\mathbb{M})$ is a morphism from $\mathcal{S}_n(\mathfrak{S}_-)$ to $\mathcal{S}_n(\mathfrak{S}_+)$ in Mor.

The following Theorem is the main result of this section, which generalizes Costantino and Lê's result, Theorem 6.5 in [CL22b]. They proved the following Theorem for $n = 2$.

Theorem 4.5.2. *For each positive integer n , $\mathcal{S}_n : DeCob \rightarrow Mor$ is a symmetric monoidal functor.*

Proof. Costantino and Lê's proof for Theorem 6.5 in [CL22b] works here. We briefly recall their proof.

It is obvious that \mathcal{S}_n preserves the symmetric monoidal structure and maps the identity morphism to the identity morphism. It suffices to show \mathcal{S}_n is compatible with the composition of the morphisms. This follows from Theorem 4.5.1. \square

4.6 Stated SL_n -skein modules at 4-th roots of unity

In this section, we will focus on the stated SL_n -skein modules at $\epsilon \in R$ with $\epsilon^4 = 1$. For example $R = \mathbb{C}$ and $\epsilon = \pm 1, \pm i$, where $i \in \mathbb{C}$ with $i^2 = -1$. Let (M, \mathcal{N}) be a connected marked 3-manifold with $\mathcal{N} \neq \emptyset$, and (M, \mathcal{N}') be a marked 3-manifold obtained from (M, \mathcal{N}) by adding one extra marking. We will prove $\mathcal{S}_n(M, \mathcal{N}', \epsilon) = \mathcal{S}_n(M, \mathcal{N}, \epsilon) \otimes_R \mathcal{O}_{q_\epsilon}(SL_n)$, and the R -linear map from $\mathcal{S}_n(M, \mathcal{N}, \epsilon)$ to $\mathcal{S}_n(M, \mathcal{N}', \epsilon)$ induced by the obvious embedding from (M, \mathcal{N}) to (M, \mathcal{N}') is injective.

In this section, we will always assume $\epsilon \in R$ with $\epsilon^4 = 1$. Then $q_\epsilon^{\frac{1}{n}} = \pm 1, q_\epsilon = \pm 1$.

Let (M, \mathcal{N}) be a marked 3-manifold. In $\mathcal{S}_n(M, \mathcal{N}, \epsilon)$, we have

$$\begin{array}{c}
 \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}, \quad \begin{array}{c} \circ \diagdown \diagup \\ \circ \diagup \diagdown \end{array} \begin{array}{c} i \\ j \end{array} = q_\epsilon^{-\frac{1}{n} + \delta_{i,j}} \begin{array}{c} \circ \diagdown \diagup \\ \circ \diagup \diagdown \end{array} \begin{array}{c} j \\ i \end{array}, \quad \begin{array}{c} \circ \diagdown \diagup \\ \bullet \diagup \diagdown \end{array} \begin{array}{c} i \\ j \end{array} = q_\epsilon^{\frac{1}{n} - \delta_{i,\bar{j}}} \begin{array}{c} \circ \diagdown \diagup \\ \bullet \diagup \diagdown \end{array} \begin{array}{c} j \\ i \end{array}. \\
 (4.8)
 \end{array}$$

From left to right in equation (4.8), the first equality comes from relation (2.4), the second equality comes from relation (2.11), and the third equality comes from Proposition 3.2 in [Wan23c]. Note that $q_\epsilon^{-\frac{1}{n} + \delta_{i,j}}, q_\epsilon^{\frac{1}{n} - \delta_{i,\bar{j}}} = \pm 1$. If we set $a_{i,j} =$

$q_\epsilon^{-\frac{1}{n} + \delta_{i,j}}$, $b_{i,j} = q_\epsilon^{\frac{1}{n} - \delta_{i,\bar{j}}}$, then we have

$$a_{i,j} b_{i,\bar{j}} = 1. \quad (4.9)$$

Let (M, \mathcal{N}) be a connected marked 3-manifold. The following Theorem shows the R -linear structure for $\mathcal{S}_n(M, \mathcal{N}, \epsilon)$ is only dependent on M and the number of components of \mathcal{N} and is independent of where these components lie in ∂M .

Recall that D denotes the 2-dimensional closed disk.

Theorem 4.6.1. *Let (M, \mathcal{N}) be a connected marked 3-manifold. Assume (M, \mathcal{N}_1) (resp. (M, \mathcal{N}_2)) is obtained from (M, \mathcal{N}) by adding one extra marking e_1 (resp. e_2), and $l_1 : \mathcal{S}_n(M, \mathcal{N}, \epsilon) \rightarrow \mathcal{S}_n(M, \mathcal{N}_1, \epsilon)$ (resp. $l_2 : \mathcal{S}_n(M, \mathcal{N}, \epsilon) \rightarrow \mathcal{S}_n(M, \mathcal{N}_2, \epsilon)$) is the R -linear map induced by the embedding from (M, \mathcal{N}) to (M, \mathcal{N}_1) (resp. (M, \mathcal{N}_2)). Then there exists an R -linear isomorphism $F : \mathcal{S}_n(M, \mathcal{N}_1, \epsilon) \rightarrow \mathcal{S}_n(M, \mathcal{N}_2, \epsilon)$ such that $F \circ l_1 = l_2$.*

Proof. For each $i = 1, 2$, the oriented closed interval \bar{e}_i is an embedding from $[0, 1]$ to ∂M . Let $c : [0, 1] \rightarrow M$ be a smooth proper embedding such that $c(0) = \bar{e}_1(1)$ and $c(1) = \bar{e}_2(0)$. Let $U(c)$ be a closed regular neighborhood of c , that is $U(c)$ is diffeomorphic to $c \times D$, such that $U(c) \cap \partial M = D \times \{c(0), c(1)\}$. We also require $U(c) \cap \bar{\mathcal{N}}_1 = U(c) \cap \bar{e}_1 \neq \bar{e}_1$ and $U(c) \cap \bar{\mathcal{N}}_2 = U(c) \cap \bar{e}_2 \neq \bar{e}_2$ are closed intervals. Then $c \times \partial D$ is a strict subsurface for both (M, \mathcal{N}_1) and (M, \mathcal{N}_2) .

For each $i = 1, 2$, define $e'_i = e_i \setminus U(c)$, and the orientation of e'_i is inherited from e_i . Let $M' = M \setminus (c \times \text{int}(D))$, $\mathcal{N}'_1 = \mathcal{N}_1 \setminus U(c) = (\mathcal{N}_1 \setminus e_1) \cup e'_1$, $\mathcal{N}'_2 = \mathcal{N}_2 \setminus U(c) = (\mathcal{N}_2 \setminus e_2) \cup e'_2$, and $h_1 : \mathcal{S}_n(M', \mathcal{N}'_1, \epsilon) \rightarrow \mathcal{S}_n(M, \mathcal{N}_1, \epsilon)$ (resp. $h_2 : \mathcal{S}_n(M', \mathcal{N}'_2, \epsilon) \rightarrow \mathcal{S}_n(M, \mathcal{N}_2, \epsilon)$) be the R -linear map induced by the embedding from (M', \mathcal{N}'_1) to (M, \mathcal{N}_1) (resp. from (M', \mathcal{N}'_2) to (M, \mathcal{N}_2)). Theorem 4.5.1 implies h_1 (resp. h_2) is surjective and $\text{Ker}(h_1)$ (resp. $\text{Ker}(h_2)$) is generated by the relation depicted in the left (resp. right) picture in Figure 4.3.

There is a copy of c in $c \times \partial D$, denoted as c' , such that $c'(0) = \bar{e}'_1(1)$ and $c'(1) = \bar{e}'_2(0)$. Let $V(c) \in M'$ be a small open regular neighborhood of $e'_1 \cup e'_2 \cup c'$ (here the ambient 3-manifold is M') such that $V(c)$ retracts to $e'_1 \cup e'_2 \cup c'$ and $V(c) \cap \bar{\mathcal{N}} = \emptyset$. Let f be the isomorphism from (M', \mathcal{N}'_1) to (M', \mathcal{N}'_2) that drags e'_1 to e'_2 along c' such that f is identity on $M' \setminus V(c)$. The induced R -linear map $f_* : \mathcal{S}_n(M', \mathcal{N}'_1, \epsilon) \rightarrow \mathcal{S}_n(M', \mathcal{N}'_2, \epsilon)$ is illustrated in Figure 4.4.

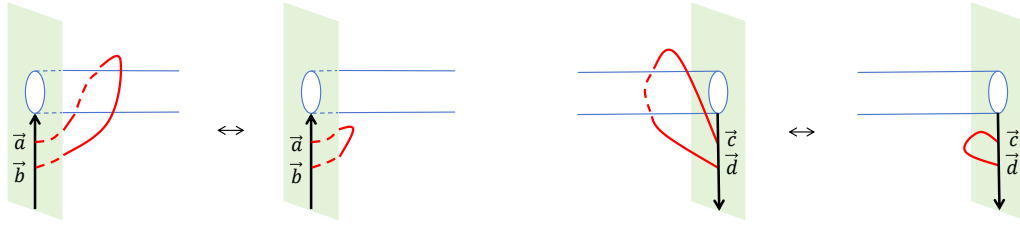


FIGURE 4.3: The faint green portions belong to $\partial M'$, and the blue sections pertain to $c \times \partial D$. The red curves have arbitrary orientations and represent parallel copies of stated arcs. The states for these parallel copies of arcs are illustrated by vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ whose entries are integers between 1 and n . In the same relation, the lefthand side and the righthand side are compatible with each other regarding the orientations of the red curves. The black arrow in the left (resp. right) picture is a part of e'_1 (resp. e'_2). The left (resp. right) picture is intended for relations that generate $\text{Ker}(h_1)$ (resp. $\text{Ker}(h_2)$).

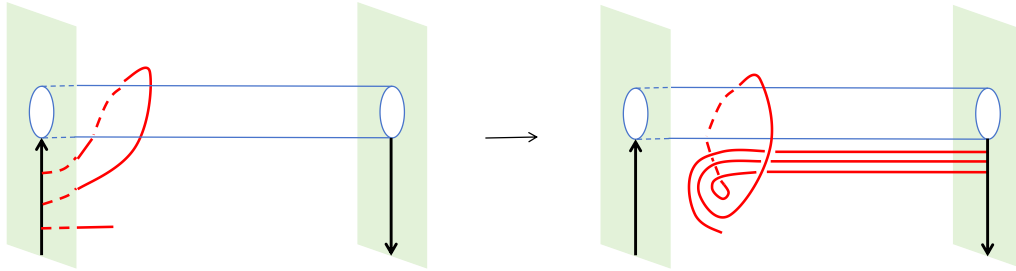


FIGURE 4.4: The picture illustrates the R -linear map f_* .

Using relations in (4.8) and equation (4.9), we can easily show $f_*(\text{Ker}(h_1)) = \text{Ker}(h_2)$. Similarly, we have $(f^{-1})_*(\text{Ker}(h_2)) = \text{Ker}(h_1)$. Then f_* (resp. $(f^{-1})_*$) induces an R -linear map $F : \mathcal{S}_n(M, \mathcal{N}_1, \epsilon) \rightarrow \mathcal{S}_n(M, \mathcal{N}_2, \epsilon)$ (resp. $G : \mathcal{S}_n(M, \mathcal{N}_2, \epsilon) \rightarrow \mathcal{S}_n(M, \mathcal{N}_1, \epsilon)$). Obviously, the R -linear maps F and G are inverse to each other.

From the definition of F , it is easy to show $F \circ l_1 = l_2$.

□

rem The author proved Theorem 4.6.1 for general quantum parameter (that is, ϵ is replaced with general invertible element $v \in R$) when e_1, e_2 lie in the same component of ∂M , please refer to Lemma 6.4 in [Wan23c].

Note that Theorem 4.6.1 is not true for general quantum parameter when e_1, e_2 lie in different components of ∂M , please refer to [CL22b] for counterexamples.

The following Theorem is the main result of this section.

Theorem 4.6.2. *Let (M, \mathcal{N}) be a connected marked 3-manifold with $\mathcal{N} \neq \emptyset$, and let (M, \mathcal{N}') be another marked 3-manifold obtained from (M, \mathcal{N}) by adding one extra marking e' . We use $l_{e'}$ to denote the R -linear map from $\mathcal{S}_n(M, \mathcal{N}, \epsilon) \rightarrow \mathcal{S}_n(M, \mathcal{N}', \epsilon)$ induced by the embedding from (M, \mathcal{N}) to (M, \mathcal{N}') . Then we have*

(1) *The R -linear map $l_{e'}$ is injective.*

(2) $\mathcal{S}_n(M, \mathcal{N}', \epsilon) \simeq \mathcal{S}_n(M, \mathcal{N}, \epsilon) \otimes_R \mathcal{O}_{q_\epsilon}(\mathrm{SL}_n)$.

Proof. Since \mathcal{N} is not empty, we suppose e is a component of \mathcal{N} .

Case 1: e and e' lie in a same component of ∂M . Theorem 6.10 in [Wan23c] implies this case.

Case 2: e and e' lie in different components of ∂M . Assume (M, \mathcal{N}'') is obtained from (M, \mathcal{N}) by adding one extra marking e'' with e'' and e lying in a same component of ∂M . We use $l_{e''}$ to denote the R -linear map from $\mathcal{S}_n(M, \mathcal{N}, \epsilon) \rightarrow \mathcal{S}_n(M, \mathcal{N}'', \epsilon)$ induced by the embedding from (M, \mathcal{N}) to (M, \mathcal{N}'') . From Case 1, we know $l_{e''}$ is injective and $\mathcal{S}_n(M, \mathcal{N}'', \epsilon) \simeq \mathcal{S}_n(M, \mathcal{N}, \epsilon) \otimes_R \mathcal{O}_{q_\epsilon}(\mathrm{SL}_n)$. Then Theorem 4.6.1 implies $\mathrm{Ker}(l_{e'}) = \mathrm{Ker}(l_{e''}) = 0$ and

$$\mathcal{S}_n(M, \mathcal{N}', \epsilon) \simeq \mathcal{S}_n(M, \mathcal{N}'', \epsilon) \simeq \mathcal{S}_n(M, \mathcal{N}, \epsilon) \otimes_R \mathcal{O}_{q_\epsilon}(\mathrm{SL}_n).$$

□

4.7 Applications: Injectivity of the splitting map in special cases

Lê and Sikora formulated the following Conjecture regarding the injectivity of the splitting map (see subsection 2.1.4).

Conjecture 4.7.1 (Conjecture 7.12 in [LS21]). *For any pb surface \mathfrak{S} and any ideal arc c , the splitting map $\Theta_c : \mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathrm{Cut}_c(\mathfrak{S}), v)$ is injective.*

In this section, we will use and recall some results from [Wan23c]. Although the author worked with \mathbb{C} in [Wan23c], instead of general commutative domain, it is easy to check that all the results, referenced from [Wan23c], are true with general

commutative domain R (the proving techniques in [Wan23c] can be applied to the commutative domain R).

Lê proved Conjecture 4.7.1 when $n = 2$ [Lê18], Higgins proved the case when $n = 3$ [Hig23], Lê and Sikora proved Conjecture 4.7.1 when \mathfrak{S} is connected and has non-empty boundary. The author proved the splitting map is injective for all marked 3-manifolds when the quantum parameter $v = 1$ [Wan23c]. Let (M, \mathcal{N}) be marked 3-manifold, let D be a properly embedded disk in M , and let e be an open oriented interval in D . The author proved $\Theta_{(D,e)}$ is injective if the component of ∂M containing ∂D contains at least one marking, Corollary 6.11 in [Wan23c].

Using the results in previous sections, we will prove Conjecture 4.7.1 when $v^{2m} = 1$ and $m \mid n$, and prove any splitting map for the marked 3-manifold (M, \mathcal{N}) is injective when every component of M contains at least one marking and $v^4 = 1$. We will also show the splitting map for any marked 3-manifold and any splitting disk is injective when $v = -1$.

We call the pb surface, obtained from D by removing one ideal point in ∂D , as **monogon**, denoted as D_1 . Then it is easy to show $\mathcal{S}_n(D_1, v) \simeq R$ as R -algebras.

Let \mathfrak{S} be a pb surface, and p be an ideal point of \mathfrak{S} . Suppose c_p is a trivial ideal arc at p , that is, the two endpoints of c_p are both p and c_p bounds an embedded monogon. Then $\text{Cut}_{c_p}(\mathfrak{S})$ is the disjoint union of a monogon and \mathfrak{S}_p . We use Θ_p to denote the following composition:

$$\mathcal{S}_n(\mathfrak{S}, v) \rightarrow \mathcal{S}_n(\mathfrak{S}_p, v) \otimes_R \mathcal{S}_n(D_1, v) \simeq \mathcal{S}_n(\mathfrak{S}_p, v),$$

where the map from $\mathcal{S}_n(\mathfrak{S}, v)$ to $\mathcal{S}_n(\mathfrak{S}_p, v) \otimes_R \mathcal{S}_n(D_1, v)$ is the splitting map Θ_{c_p} . For any stated n -web diagram α in \mathfrak{S} , we can isotope α such that there is no intersection between α and the monogon bounded by c_p . Then we have $\Theta_p(\alpha) = \alpha$. Actually this is another way to define Θ_p .

Lemma 4.7.2 (Corollary 8.2 in [LS21]). *Let \mathfrak{S} be a pb surface with an ideal point p and an ideal arc c . We have $\text{Ker}(\Theta_p) = \text{Ker}(\Theta_c)$.*

rem If we try to understand Θ_p in the thickening of pb surfaces, it is easy to see Θ_p is actually adding marking map. Then Lemma 4.7.2 actually states the kernel of the adding marking map equals to the kernel of the splitting map. The author

proved this for general marked 3-manifolds. The kernel of the adding marking map equals the kernel of the splitting map if the adding marking and the boundary of the splitting disk belong to the same component of ∂M (for the thickening of pb surfaces, this is always true), Theorem 6.7 in [Wan23c].

Lê and Sikora also proved $\text{Ker}(\Theta_p) = \text{Ker}(\Theta_{p'})$ for any two ideal points p, p' of a connected pb surface, Theorem 8.1 in [LS21]. If we interpret Θ_p and $\Theta_{p'}$ as adding marking maps, the reason why $\text{Ker}(\Theta_p) = \text{Ker}(\Theta_{p'})$ becomes very clear, please refer to Lemma 6.4 in [Wan23c]. In this Lemma, the author proved the kernels of two adding marking maps are same if the two adding markings lie in a same boundary component of the marked 3-manifold.

Lemma 4.7.3. *Let \mathfrak{S} be a pb surface with an ideal point p , let m, n be two positive integers with $m \mid n$, and let $\epsilon \in R$ with $\epsilon^{2m} = 1$. Then $\Theta_p : \mathcal{S}_n(\mathfrak{S}, \epsilon) \rightarrow \mathcal{S}_n(\mathfrak{S}_p, \epsilon)$ is injective.*

Proof. From Theorem 4.2.6, we know there exist linear isomorphisms $\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}, \epsilon)$ and $\varphi_\epsilon : \mathcal{S}_n(\mathfrak{S}_p, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}_p, \epsilon)$. The definition of φ_ϵ shows the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}_n(\mathfrak{S}, 1) & \xrightarrow{\varphi_\epsilon} & \mathcal{S}_n(\mathfrak{S}, \epsilon) \\ \downarrow \Theta_p & & \downarrow \Theta_p \\ \mathcal{S}_n(\mathfrak{S}_p, 1) & \xrightarrow{\varphi_\epsilon} & \mathcal{S}_n(\mathfrak{S}_p, \epsilon) \end{array} . \quad (4.10)$$

Corollary 6.8 in [Wan23c] and Lemma 4.7.2 imply $\Theta_p : \mathcal{S}_n(\mathfrak{S}, 1) \rightarrow \mathcal{S}_n(\mathfrak{S}_p, 1)$ is injective. Then $\Theta_p : \mathcal{S}_n(\mathfrak{S}, \epsilon) \rightarrow \mathcal{S}_n(\mathfrak{S}_p, \epsilon)$ is also injective because the two φ_ϵ in diagram (4.10) are R -linear isomorphisms. \square

The following Theorem provides affirmative examples for Conjecture 4.7.1.

Theorem 4.7.4. *Let \mathfrak{S} be a pb surface with an ideal arc c , let m, n be two positive integers with $m \mid n$, and let $\epsilon \in R$ with $\epsilon^{2m} = 1$. Then the splitting map $\Theta_c : \mathcal{S}_n(\mathfrak{S}, \epsilon) \rightarrow \mathcal{S}_n(\text{Cut}_c(\mathfrak{S}), \epsilon)$ is injective.*

Proof. Lemmas 4.7.2 and 4.7.3. \square

Theorem 4.7.5. *Let (M, \mathcal{N}) be a connected marked 3-manifold with $\mathcal{N} \neq \emptyset$, let D be a properly embedded disk in M , and let e be an open oriented interval in D . Suppose $\epsilon \in R$ such that $\epsilon^4 = 1$. We have $\Theta_{(D, \epsilon)} : \mathcal{S}_n(M, \mathcal{N}, \epsilon) \rightarrow \mathcal{S}_n(\text{Cut}_{(D, \epsilon)}(M, \mathcal{N}), \epsilon)$ is injective.*

Proof. Theorem 4.6.2 and Theorem 6.7 in [Wan23c]. \square

Lemma 4.7.6. *Let (M, \mathcal{N}) be a marked 3-manifold, and let (M, \mathcal{N}') be another marked 3-manifold obtained from (M, \mathcal{N}) by adding one extra marking e . Then the adding marking map $l_e : \mathcal{S}_n(M, \mathcal{N}, -1) \rightarrow \mathcal{S}_n(M, \mathcal{N}', -1)$ is injective.*

Proof. From Corollary 4.4.2, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_n(M, \mathcal{N}, 1) & \xrightarrow{l_e} & \mathcal{S}_n(M, \mathcal{N} \cup e, 1) \\ \downarrow \Psi_n & & \downarrow \Psi_n \\ \mathcal{S}_n(M, \mathcal{N}, -1) & \xrightarrow{l_e} & \mathcal{S}_n(M, \mathcal{N} \cup e, -1) \end{array},$$

where Ψ_n is the linear isomorphism constructed in Theorem 4.4.1. Then

$$l_e : \mathcal{S}_n(M, \mathcal{N}, -1) \rightarrow \mathcal{S}_n(M, \mathcal{N}', -1)$$

is injective because $l_e : \mathcal{S}_n(M, \mathcal{N}, 1) \rightarrow \mathcal{S}_n(M, \mathcal{N}', 1)$ is injective, Corollary 6.6 in [Wan23c]. \square

Theorem 4.7.7. *Let (M, \mathcal{N}) be marked 3-manifold, let D be a properly embedded disk in M , and let e be an open oriented interval in D . We have $\Theta_{(D, \epsilon)} : \mathcal{S}_n(M, \mathcal{N}, -1) \rightarrow \mathcal{S}_n(\text{Cut}_{(D, \epsilon)}(M, \mathcal{N}), -1)$ is injective.*

Proof. Lemma 4.7.6 and Theorem 6.7 in [Wan23c]. \square

Conjecture 4.7.8. *Let (M, \mathcal{N}) be marked 3-manifold, let D be a properly embedded disk in M , and let e be an open oriented interval in D . Suppose $\epsilon \in R$ such that $\epsilon^4 = 1$. We have $\Theta_{(D, \epsilon)} : \mathcal{S}_n(M, \mathcal{N}, \epsilon) \rightarrow \mathcal{S}_n(\text{Cut}_{(D, \epsilon)}(M, \mathcal{N}), \epsilon)$ is injective.*

We have the following confirmative examples for Conjecture 4.7.8: (1) $\epsilon = \pm 1$ (2) ϵ is a primitive 4-th root of unity and every component of M contains at least marking. So to prove Conjecture 4.7.8, it suffices to show the adding marking map

is injective when we add one extra marking to the 3-manifold without markings and ϵ is a primitive 4-th root of unity.

Part II

Kauffman bracket skein theory

Chapter 5

Finiteness and dimension of the stated skein modules over the Frobenius

In chapters 5-7, we focus on the (stated) skein theory introduced in section 1.1. Let R be a commutative domain with an invertible element $q^{\frac{1}{2}}$. We denote by $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ the stated skein module of the marked 3-manifold (M, \mathcal{N}) .

In chapters 5-6, we assume that $R = \mathbb{C}$ and that $q^{\frac{1}{2}}$ is a nonzero complex number. This chapter is based on the author's work in [Wan23a]. When $q^{\frac{1}{2}}$ is a root of unity of odd order, there exists a \mathbb{C} -linear map

$$\mathcal{F}: \mathcal{S}_1(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N}),$$

called the **Frobenius map** for the stated skein module, see subsection 5.1.2. The Frobenius map induces an $\mathcal{S}_1(M, \mathcal{N})$ -module structure on $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$. We will show that $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ is a finitely generated $\mathcal{S}_1(M, \mathcal{N})$ -module when M is compact (Thm. 5.3.1). Furthermore, we will establish an upper bound for the rank of $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ over $\mathcal{S}_1(M, \mathcal{N})$ (Thm. 5.5.36).

5.1 Preliminaries

Let A be an algebra, we use $C(A)$ to denote the center of A . Suppose B is a finitely generated A -module. We use $\dim_A B$ to denote the lower bound of the number of generators of B over A . We call an algebra A a domain, if $xy = 0$ for two elements $x, y \in A$, we have $x = 0$ or $y = 0$.

Lemma 5.1.1. *Suppose A is a commutative domain, and B is a module over A freely generated by n elements. Then $\dim_A B = n$.*

Proof. Clearly we have $\dim_A B \leq n$. It suffices to show $\dim_A B \geq n$. Let $S = A - \{0\}$.

We use \tilde{A} to denote the localization of A over the multiplicative set S , that is, \tilde{A} is the field of fractions of A . Then $B \otimes_A \tilde{A}$ is a vector space over \tilde{A} . From the assumption, we have $B \simeq A^{\oplus n}$ as A -modules. Then we have

$$B \otimes_A \tilde{A} \simeq (A^{\oplus n}) \otimes_A \tilde{A} = (A \otimes_A \tilde{A})^{\oplus n} \simeq \tilde{A}^{\oplus n} \quad (5.1)$$

where all the isomorphisms in equation (5.1) are isomorphisms between \tilde{A} -vector spaces. Thus $\dim_{\tilde{A}}(B \otimes_A \tilde{A}) = n$. Let $\{b_1, \dots, b_k\}$ be any finite generating set for B over A . Then we have $\{b_1 \otimes_A 1, \dots, b_k \otimes_A 1\}$ linearly spans $B \otimes_A \tilde{A}$, thus we have $\dim_{\tilde{A}}(B \otimes_A \tilde{A}) = n \leq k$. Then we get $n \leq \dim_A B$. \square

5.1.1 Stated skein modules and algebra

Recall that we use $\mathcal{S}_{q^{1.2}}(M, \mathcal{N})$ to denote the stated skein module of a marked 3-manifold (M, \mathcal{N}) (see section 1.1). When $\mathcal{N} = \emptyset$, the stated skein module $\mathcal{S}_{q^{1.2}}(M, \emptyset)$ is just the skein module of M (see section 1.1). Since relations (1.2) and (1.3) only involve q . We use $\mathcal{S}_q(M)$ to denote $\mathcal{S}_{q^{1.2}}(M, \emptyset)$.

Let $(M_1, \mathcal{N}_1), (M_2, \mathcal{N}_2)$ be two marked 3-manifolds, and $f : M_1 \rightarrow M_2$ be an orientation preversing embedding. If furthermore $f(\mathcal{N}_1) \subset \mathcal{N}_2$ and preserves the orientations of the marking, we say f is an embedding from (M_1, \mathcal{N}_1) to (M_2, \mathcal{N}_2) . Obviously f induces a linear map $f_{\sharp} : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$. We say f is an isomorphism if there exists another embedding $g : (M_2, \mathcal{N}_2) \rightarrow (M_1, \mathcal{N}_1)$

such that g and f are inverse to each other. We also say (M_1, \mathcal{N}_1) is isomorphic to (M_2, \mathcal{N}_2) .

Recall that, for a pb surface \mathfrak{S} (Def. 1.1.1), we use $\mathcal{S}_{q^{1/2}}(\mathfrak{S})$ to denote the stated skein algebra of \mathfrak{S} (see section 1.1).

Definition 5.1.2. *If a pb surface has an empty boundary, we call it a **closed pb surface**. A **totally closed pb surface** is a closed pb surface without punctures.*

When the pb surface \mathfrak{S} is closed (i.e. $\partial\mathfrak{S} = \emptyset$), the stated skein algebra $\mathcal{S}_{q^{1/2}}(\mathfrak{S})$ is the (Kauffman bracket) skein algebra. For the same reason as the skein module, we denote the skein algebra as $\mathcal{S}_q(\mathfrak{S})$.

Here we recall the definition of the marked surface. A **marked surface** is a pair (Σ, \mathcal{P}) , where Σ is a compact surface and \mathcal{P} is a finite subset of $\partial\Sigma$. A boundary component, which has no intersection with \mathcal{P} , is called the **unmarked boundary component**.

Recall that for a marked surface (Σ, \mathcal{P}) , we can define a marked 3-manifold (M, \mathcal{N}) where $M = \Sigma \times [0, 1]$ and $\mathcal{N} = \cup_{x \in \mathcal{P}} (\{x\} \times (0, 1))$ and the orientation of \mathcal{N} is given by the positive direction of $(0, 1)$. Then (M, \mathcal{N}) is called the thickening of (Σ, \mathcal{P}) , and we define $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P}) = \mathcal{S}_{q^{1/2}}(M, \mathcal{N})$. Similarly, as the pb surface, $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})$ has an algebra structure given by stacking the stated tangles. When $\mathcal{P} = \emptyset$, we use $\mathcal{S}_q(\Sigma)$ to denote $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})$.

For an oriented surface Σ , the orientation of $\partial\Sigma$ induced from the orientation of Σ is called the positive orientation of $\partial\Sigma$. For any marked surface (Σ, \mathcal{P}) , we can define a corresponding pb surface \mathfrak{S} by the following procedures:

1. Let \mathcal{Q} be obtained from \mathcal{P} by slightly moving \mathcal{P} along $\partial\Sigma$ in the positive direction of $\partial\Sigma$.
2. Let Σ' be obtained from Σ by replacing all unmarked boundary components with interior punctures.
3. Define $\mathfrak{S} = \Sigma' - \mathcal{Q}$.

Conversely, we can recover (Σ, \mathcal{P}) from \mathfrak{S} . It is easy to see that (Σ, \mathcal{P}) and \mathfrak{S} have the same skein theory.

The notation of the pb surface is convenient for defining the ideal triangulation. The notation of the marked surface also has its own convenience because the thickening of the marked surface is a compact marked 3-manifold.

Remark 5.1.3. *For two (marked) 3-manifolds, we say they have the same skein theory if their (stated) skein modules are isomorphic as \mathbb{C} -vector spaces in a natural way.*

For two surfaces, we say they have the same skein theory if their (stated) skein algebras are isomorphic as \mathbb{C} -algebras in a natural way.

5.1.2 Chebyshev polynomial and Frobenius map

Chebyshev polynomials are defined by the following recurrence relation:

$$Q_n(x) = xQ_{n-1}(x) - Q_{n-2}(x).$$

The Chebyshev polynomial of the first kind, denoted as $T_n(x)$, is defined by setting $T_0(x) = 2, T_1(x) = x$. The Chebyshev polynomial of the second kind, denoted as $S_n(x)$, is defined by setting $S_0(x) = 1, S_1(x) = x$. For $n \geq 2$, we have $T_n(x) = S_n(x) - S_{n-2}(x)$.

Let (M, \mathcal{N}) be a marked 3-manifold. Suppose β is a framed knot or stated framed arc in (M, \mathcal{N}) , and k is a positive integer. We define $\beta^{(k)}$ to be obtained from β by taking k parallel copies of β in the framing direction and define $\beta^{(0)}$ to be the empty stated tangle in (M, \mathcal{N}) . Suppose $P(x) = \sum_{0 \leq k \leq m} a_k x^k \in \mathbb{C}[x]$, then we define

$$\alpha^{[P]} = \sum_{0 \leq k \leq m} a_k \alpha^{(k)} \in \mathcal{S}_{q^{1/2}}(M, \mathcal{N}).$$

Suppose $q^{\frac{1}{2}}$ is a root of unity of order N with N odd. Then there is a linear map, called the Frobenius map, $\mathcal{F} : \mathcal{S}_1(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ [BL20; BW16a; Wan23c]. Let l be any stated (M, \mathcal{N}) -tangle, suppose $l = \alpha_1 \cup \cdots \cup \alpha_k \cup \beta_1 \cup \cdots \cup \beta_m$ where $\alpha_i, 1 \leq i \leq k$, are stated framed arcs and $\beta_j, 1 \leq j \leq m$, are framed knots. Then

$$\mathcal{F}(l) = \alpha_1^{(N)} \cup \cdots \cup \alpha_k^{(N)} \cup \beta_1^{[TN]} \cup \cdots \cup \beta_m^{[TN]}. \quad (5.2)$$

Let α be any stated (M, \mathcal{N}) -tangle, then $\mathcal{F}(\alpha)$ is transparent [BL20; Wan23c], in a sense that we have the following two relations in $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$:

$$\begin{array}{c} \text{X} \end{array} \mathcal{F}(\alpha) = \begin{array}{c} \text{X} \end{array} \mathcal{F}(\alpha), \quad \begin{array}{c} \text{V} \\ \bullet \\ i \quad j \end{array} \mathcal{F}(\alpha) = \begin{array}{c} \text{V} \\ \bullet \\ i \quad j \end{array} \mathcal{F}(\alpha) \quad (5.3)$$

where $i, j = -, +$.

Then we define an action of $\mathcal{S}_1(M, \mathcal{N})$ on $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$. For any elements $\alpha = c_1\alpha_1 + \dots + c_m\alpha_m \in \mathcal{S}_1(M, \mathcal{N})$ and $l = k_1l_1 + \dots + k_nl_n \in \mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ where $c_j, k_i \in \mathbb{C}$, α_j and l_i are stated tangles in (M, \mathcal{N}) and $\alpha_j \cap l_i = \emptyset$, we define

$$\alpha \cdot l = \mathcal{F}(\alpha) \cup l = \sum_{1 \leq j \leq m, 1 \leq i \leq n} c_j k_i \mathcal{F}(\alpha_j) \cup l_i. \quad (5.4)$$

From equation (5.3) and the fact that \mathcal{F} preserves relations (1.2)-(1.6), the above action is well-defined.

We use $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})^{(N)}$ to denote $\text{Im} \mathcal{F}$.

Theorem 5.1.4. $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})^{(N)}$ has a commutative algebra structure, which makes

$$\mathcal{F} : \mathcal{S}_1(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})^{(N)}$$

a surjective algebra homomorphism.

Proof. For any two elements $\alpha, l \in \mathcal{S}_1(M, \mathcal{N})$, first we prove $\alpha \cdot \mathcal{F}(l) = l \cdot \mathcal{F}(\alpha)$. Suppose $\alpha = c_1\alpha_1 + \dots + c_m\alpha_m \in \mathcal{S}_1(M, \mathcal{N})$ and $l = k_1l_1 + \dots + k_nl_n \in \mathcal{S}_1(M, \mathcal{N})$ where $c_j, k_i \in \mathbb{C}$, α_j and l_i are stated tangles in (M, \mathcal{N}) and $\alpha_j \cap l_i = \emptyset$. Then from equation (5.4), we know

$$\alpha \cdot \mathcal{F}(l) = \sum_{1 \leq j \leq m, 1 \leq i \leq n} c_j k_i \alpha_j \cdot \mathcal{F}(l_i) = \sum_{1 \leq j \leq m, 1 \leq i \leq n} c_j k_i \mathcal{F}(\alpha_j) \cup \mathcal{F}(l_i) = l \cdot \mathcal{F}(\alpha). \quad (5.5)$$

For any two elements $x, y \in \mathcal{S}_{q^{1/2}}(M, \mathcal{N})^{(N)}$, suppose $x = \mathcal{F}(u)$ where $u \in \mathcal{S}_1(M, \mathcal{N})$. Then we define $xy = u \cdot y$. To show this multiplication is well-defined, we have to show xy is independent of the choice of u . Suppose we also have

$x = \mathcal{F}(u')$ where $u' \in \mathcal{S}_1(M, \mathcal{N})$. We also suppose $y = \mathcal{F}(v)$ where $v \in \mathcal{S}_1(M, \mathcal{N})$. Then

$$u \cdot y = u \cdot \mathcal{F}(v) = v \cdot \mathcal{F}(u) = v \cdot \mathcal{F}(u') = u' \cdot \mathcal{F}(v) = u' \cdot y.$$

We also have

$$xy = u \cdot y = u \cdot \mathcal{F}(v) = v \cdot \mathcal{F}(u) = v \cdot x = yx.$$

Thus the above defined algebra structure is commutative.

Equation (5.5) shows $\mathcal{F} : \mathcal{S}_1(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})^{(N)}$ is an algebra homomorphism. \square

For a pb surface \mathfrak{S} , we have $\mathcal{F} : \mathcal{S}_1(\mathfrak{S}) \rightarrow \mathcal{S}_{q^{1/2}}(\mathfrak{S})$ is an algebra homomorphism. We also use $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}$ to denote $\text{Im}\mathcal{F}$. Then the multiplication structure of $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}$ inherited from $\mathcal{S}_{q^{1/2}}(\mathfrak{S})$ coincides with the one defined in Theorem 5.1.4. Then we have $\mathcal{F} : \mathcal{S}_1(\mathfrak{S}) \rightarrow \mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}$ is an isomorphism and $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)} \subset C(\mathcal{S}_{q^{1/2}}(\mathfrak{S}))$ [BL20; KQ24; Wan23c]. Obviously $\mathcal{S}_{q^{1/2}}(\mathfrak{S})$ has an $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}$ -module structure defined by multiplication. We know $\mathcal{S}_{q^{1/2}}(\mathfrak{S})$ also has an $\mathcal{S}_1(\mathfrak{S})$ -module structure. Then these two module structures are equivalent via $\text{Id}_{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}$ and \mathcal{F} . That is, for any $x \in \mathcal{S}_1(\mathfrak{S}), y \in \mathcal{S}_{q^{1/2}}(\mathfrak{S})$, we have $x \cdot y = \mathcal{F}(x)y$.

Roots of Unity Assumptions:

Throughout this chapter, we follow the conventions below when making roots of unity assumptions:

- (R1) When working with the stated skein module or algebra, we assume that $q^{1/2}$ is a root of unity of odd order N .
- (R2) When working with the skein module or algebra (i.e., the non-stated case), we assume that q^2 is a root of unity of odd order N . In this case, we set $\zeta = q^N = \pm 1$.

5.2 On skein modules

The skein module $\mathcal{S}_q(M)$ for a 3-manifold M does not involve boundary and is only defined by relations (1.2) and (1.3). Suppose q^2 is a primitive N -th root of

unity with N odd. Then there exists a linear map

$$\mathcal{F} : \mathcal{S}_\zeta(M) \rightarrow \mathcal{S}_q(M),$$

where $\zeta = q^N = \pm 1$ [BL20; BW16a]. The definition here is the same with equation (5.2), except l does not contain arcs.

We still use $\mathcal{S}_q(M)^{(N)}$ to denote $\text{Im}\mathcal{F}$. From [BL20], we know elements in $\mathcal{S}_q(M)^{(N)}$ are transparent. Similarly, as in subsection 5.1.2, the skein module $\mathcal{S}_q(M)$ has an $\mathcal{S}_\zeta(M)$ -module structure, and $\mathcal{S}_q(M)^{(N)}$ has a commutative algebra structure, which makes $\mathcal{F} : \mathcal{S}_\zeta(M) \rightarrow \mathcal{S}_q(M)^{(N)}$ a surjective algebra homomorphism. For any $\eta \in \mathcal{S}_\zeta(M)$ and $\gamma \in \mathcal{S}_q(M)$, we use $\eta \cdot \gamma$ to denote the action of $\mathcal{S}_\zeta(M)$ on $\mathcal{S}_q(M)$ (the same notation as in subsection 5.1.2).

For a closed pb surface \mathfrak{S} , we have $\mathcal{F} : \mathcal{S}_\zeta(\mathfrak{S}) \rightarrow \mathcal{S}_q(\mathfrak{S})$ is an algebra embedding [BL20; BW16a]. We use $\mathcal{S}_q(\mathfrak{S})^{(N)}$ to denote $\text{Im}\mathcal{F}$, then $\mathcal{S}_q(\mathfrak{S})^{(N)}$ also acts on $\mathcal{S}_q(\mathfrak{S})$ by the multiplication. For all $x \in \mathcal{S}_\zeta(\mathfrak{S}), y \in \mathcal{S}_q(\mathfrak{S})$, we have $x \cdot y = \mathcal{F}(x)y$.

In this section, we will always have the (R2) assumption.

Theorem 5.2.1. ([FKL22]) *Suppose we have the (R2) assumption. For any compact 3-manifold M , we have $\mathcal{S}_q(M)$ is finitely generated over $\mathcal{S}_\zeta(M)$.*

In this section, we use the connection between Theorem 5.2.1 and the finiteness of the skein module of the 3-manifold M over $\mathcal{S}_q(\partial M)$ to give sufficient conditions for $\mathcal{S}_q(M)$ being finitely generated over $\mathcal{S}_q(\partial M)$. We also provide a way to estimate an upper bound for $\dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M)$ when M is compact, which will be used in subsection 5.5.1 to give a precise upper bound for $\dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M)$.

5.2.1 Some functorialities

Let $f : M_1 \rightarrow M_2$ be an embedding between two 3-manifolds. We know f induces a linear map $f_\# : \mathcal{S}_q(M_1) \rightarrow \mathcal{S}_q(M_2)$ and an algebra homomorphism $f_\# : \mathcal{S}_\zeta(M_1) \rightarrow \mathcal{S}_\zeta(M_2)$. Actually we have the following lemma.

Lemma 5.2.2. *Let $f : M_1 \rightarrow M_2$ be an embedding between two 3-manifolds. Then $f_\#$ restricts to an algebra homomorphism $f_\# : \mathcal{S}_q(M_1)^{(N)} \rightarrow \mathcal{S}_q(M_2)^{(N)}$, and we*

have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_\zeta(M_1) & \xrightarrow{f_\#} & \mathcal{S}_\zeta(M_2) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \mathcal{S}_q(M_1) & \xrightarrow{f_\#} & \mathcal{S}_q(M_2) \end{array} .$$

What's more, $f_\#$ respects the module structures in a sense that, for any

$$x \in \mathcal{S}_\zeta(M_1), y \in \mathcal{S}_q(M_1),$$

we have $f_\#(x \cdot y) = f_\#(x) \cdot f_\#(y)$.

Lemma 5.2.3. *Let $f : M_1 \rightarrow M_2$ be an embedding between two 3-manifolds. Suppose $f_\# : \mathcal{S}_q(M_1) \rightarrow \mathcal{S}_q(M_2)$ is surjective and $\mathcal{S}_q(M_1)$ is finitely generated over $\mathcal{S}_\zeta(M_1)$. Then we have $\mathcal{S}_q(M_2)$ is finitely generated over $\mathcal{S}_\zeta(M_2)$ and*

$$\dim_{\mathcal{S}_\zeta(M_2)} \mathcal{S}_q(M_2) \leq \dim_{\mathcal{S}_\zeta(M_1)} \mathcal{S}_q(M_1).$$

Proof. Suppose $\dim_{\mathcal{S}_\zeta(M_1)} \mathcal{S}_q(M_1) = n$ and $\mathcal{S}_q(M_1)$ is generated by x_1, \dots, x_n over $\mathcal{S}_\zeta(M_1)$. Then $\mathcal{S}_q(M_2)$ is finitely generated by $f_\#(x_1), \dots, f_\#(x_n)$ over $\mathcal{S}_\zeta(M_2)$ because of the surjectivity of $f_\#$ and Lemma 5.2.2. \square

5.2.2 Boundary action on skein modules

Let M be a 3-manifold. Then the skein algebra $\mathcal{S}_q(\partial M)$ has an action on $\mathcal{S}_q(M)$. We can identify $\partial M \times [0, 1]$ with a regular closed tubular neighborhood $U(\partial M)$ of ∂M such that ∂M is $\partial M \times \{1\}$. We use L to denote the embedding from $U(\partial M)$ to M . Then for any skein $l \in \mathcal{S}_q(\partial M)$, we have $L_\#(l)$ is the skein l in $\mathcal{S}_q(M)$. Then for any skein α in $\mathcal{S}_q(\partial M)$ and $\beta \in \mathcal{S}_q(M)$, we first isotope β such that $\beta \in M - U(\partial M)$, then define $\alpha * \beta := L_\#(\alpha) \cup \beta$.

We know that $\mathcal{S}_\zeta(M)$ also acts on $\mathcal{S}_q(M)$. Then it is easy to show these two actions commute with each other, that is, for any elements $\alpha \in \mathcal{S}_q(\partial M), \beta \in \mathcal{S}_\zeta(M), \eta \in \mathcal{S}_q(M)$ we have $\alpha * (\beta \cdot \eta) = \beta \cdot (\alpha * \eta)$.

For elements $l_1 \in \mathcal{S}_\zeta(\partial M), l_2 \in \mathcal{S}_q(\partial M), l_3 \in \mathcal{S}_q(M)$, we have

$$(l_1 \cdot l_2) * l_3 = (\mathcal{F}(l_1)l_2) * l_3 = \mathcal{F}(l_1) * (l_2 * l_3) = L_\#(l_1) \cdot (l_2 * l_3). \quad (5.6)$$

5.2.3 Relations between $\mathcal{S}_q(M)$ over $\mathcal{S}_\zeta(M)$ and $\mathcal{S}_q(M)$ over $\mathcal{S}_q(\partial M)$

When the ground ring is $\mathbb{Q}(q)$, Detcherry conjectured that the skein module of a compact 3-manifold M is finitely generated as a module over the skein algebra $\mathcal{S}_q(\partial M)$ [Det21]. When the ground ring is \mathbb{C} and q^2 is a root of unity of odd order, this conjecture fails. A simple counterexample is $S^1 \times S^2$. But Detcherry's conjecture still holds for some 3-manifolds under this section's context, that is when the ground ring is \mathbb{C} and q^2 is a root of unity of odd order, for example the lens spaces and the complement of the two bridge knot (or link) [BF05; HP93; LT14; Lê06]. In this subsection, we will give sufficient conditions for $\mathcal{S}_q(M)$ being finitely generated over $\mathcal{S}_q(\partial M)$.

A surface \mathfrak{S} is of finite type if there exists a compact surface $\overline{\mathfrak{S}}$ and a finite subset $A \subset \overline{\mathfrak{S}}$ such that \mathfrak{S} is homeomorphic to $\overline{\mathfrak{S}} \setminus A$.

Theorem 5.2.4. (Theorem 3.11 in [AF17]) *Suppose we have the (R2) assumption and \mathfrak{S} is a finite type surface. Then $\mathcal{S}_q(\mathfrak{S})$ is finitely generated over $\mathcal{S}_\zeta(\mathfrak{S})$ (or $\mathcal{S}_q(\mathfrak{S})^{(N)}$).*

We state a convention that $\mathcal{S}_q(\emptyset)$ is \mathbb{C} .

Proposition 5.2.5. *Let M be a compact 3-manifold with ∂M being a finite type surface. Suppose $\mathcal{S}_q(M)$ is finitely generated over $\mathcal{S}_q(\partial M)$. Then $\mathcal{S}_q(M)$ is finitely generated over $\mathcal{S}_\zeta(M)$, and we have*

$$\dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M) \leq (\dim_{\mathcal{S}_q(\partial M)} \mathcal{S}_q(M)) (\dim_{\mathcal{S}_\zeta(\partial M)} \mathcal{S}_q(\partial M)).$$

Proof. When $\partial M = \emptyset$, obviously we have

$$\dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M) \leq \dim_{\mathbb{C}} \mathcal{S}_q(M).$$

Then we look at the case when $\partial M \neq \emptyset$. Suppose $\dim_{\mathcal{S}_q(\partial M)} \mathcal{S}_q(M) = r$, and $\mathcal{S}_q(M)$ is generated by A_1, \dots, A_r as an $\mathcal{S}_q(\partial M)$ -module. From Theorem 5.2.4, we can suppose $\dim_{\mathcal{S}_\zeta(\partial M)} \mathcal{S}_q(\partial M) = s$, and $\mathcal{S}_q(\partial M)$ is generated by B_1, \dots, B_s as an $\mathcal{S}_\zeta(\partial M)$ -module. It suffices to show $\mathcal{S}_q(M)$ is generated by $B_i * A_j, 1 \leq i \leq s, 1 \leq j \leq r$ as an $\mathcal{S}_\zeta(M)$ -module.

Let C be any element in $\mathcal{S}_q(M)$. Then $C = C_1 * A_1 + \dots + C_r * A_r$ where $C_1, \dots, C_r \in \mathcal{S}_q(\partial M)$. For each $1 \leq j \leq r$, we can suppose

$$C_j = \sum_{1 \leq i \leq s} D_{j,i} \cdot B_i$$

where $D_{j,i} \in \mathcal{S}_\zeta(\partial M)$. Then we have

$$\begin{aligned} C &= \sum_{1 \leq j \leq r} C_j * A_j = \sum_{1 \leq j \leq r} \left(\sum_{1 \leq i \leq s} D_{j,i} \cdot B_i \right) * A_j \\ &= \sum_{1 \leq j \leq r, 1 \leq i \leq s} (D_{j,i} \cdot B_i) * A_j = \sum_{1 \leq j \leq r, 1 \leq i \leq s} L_{\#}(D_{j,i}) \cdot (B_i * A_j). \end{aligned}$$

where the last equality is because of equation (5.6), and $L_{\#}(D_{j,i}) \in \mathcal{S}_\zeta(M)$, for $1 \leq j \leq r, 1 \leq i \leq s$. \square

Proposition 5.2.6. *Let M be a compact 3-manifold. If $\mathcal{S}_q(M)$ is finitely generated over $\mathcal{S}_\zeta(M)$ and $\mathcal{S}_\zeta(M)$ is finitely generated over $\mathcal{S}_\zeta(\partial M)$, then $\mathcal{S}_q(M)$ is finitely generated over $\mathcal{S}_q(\partial M)$ and $\dim_{\mathcal{S}_q(\partial M)} \mathcal{S}_q(M) \leq (\dim_{\mathcal{S}_\zeta(\partial M)} \mathcal{S}_\zeta(M))(\dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M))$.*

Proof. Assume $\dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M) = n$, and $\mathcal{S}_q(M)$ is generated by b_1, \dots, b_n over $\mathcal{S}_\zeta(M)$. Assume $\dim_{\mathcal{S}_\zeta(\partial M)} \mathcal{S}_\zeta(M) = m$, and $\mathcal{S}_\zeta(M)$ is generated by c_1, \dots, c_m over $\mathcal{S}_\zeta(\partial M)$. It suffices to show $\mathcal{S}_q(M)$ is generated by $c_i \cdot b_j, 1 \leq i \leq m, 1 \leq j \leq n$ over $\mathcal{S}_q(\partial M)$.

For any $a \in \mathcal{S}_q(M)$, we can suppose $a = \sum_{1 \leq i \leq n} a_i \cdot b_i$ where $a_i \in \mathcal{S}_\zeta(M), 1 \leq i \leq n$. For each $1 \leq i \leq n$, we suppose $a_i = \sum_{1 \leq j \leq m} d_{i,j} * c_j$ where $d_{i,j} \in \mathcal{S}_\zeta(\partial M), 1 \leq i \leq n, 1 \leq j \leq m$. Then we have

$$a = \sum_{1 \leq i \leq n} a_i \cdot b_i = \sum_{1 \leq i \leq n, 1 \leq j \leq m} (d_{i,j} * c_j) \cdot b_i = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \mathcal{F}(d_{i,j}) * (c_j \cdot b_i)$$

where $\mathcal{F}(d_{i,j}) \in \mathcal{S}_q(\partial M), 1 \leq i \leq n, 1 \leq j \leq m$. \square

Theorem 5.2.7. *Suppose we have the (R2) assumption. Let M be a compact 3-manifold. If $\mathcal{S}_\zeta(M)$ is finitely generated over $\mathcal{S}_\zeta(\partial M)$, then $\mathcal{S}_q(M)$ is finitely generated over $\mathcal{S}_q(\partial M)$ and*

$$\dim_{\mathcal{S}_q(\partial M)} \mathcal{S}_q(M) \leq (\dim_{\mathcal{S}_\zeta(\partial M)} \mathcal{S}_\zeta(M)) (\dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M)).$$

Proof. Theorem 5.2.1 and Proposition 5.2.6. □

5.2.4 Handlebody and compression body

A 3-manifold H is called a **compression body** if there exists a connected totally closed surface \mathfrak{S} such that H is obtained from $\mathfrak{S} \times [0, 1]$ by attaching 2-handles along mutually disjoint loops in $\mathfrak{S} \times \{1\}$ and filling in some resulting 2-sphere boundary components with 3-dimensional solid balls. We denote $\mathfrak{S} \times \{0\}$ by $\partial^+ H$ and $\partial H - \partial^+ H$ by $\partial^- H$. We define the genus of H to be the genus of \mathfrak{S} . A compression body H is called a **handlebody** if $\partial^- H = \emptyset$ [SSS05].

Theorem 5.2.8. ([SSS05]) *Let M be a connected compact 3-manifold. Then $M = H \cup H'$ where H is a handlebody and H' is a compression body such that $\partial^- H' = \partial M$ and $\partial^+ H' = \partial H$.*

The decomposition in Theorem 5.2.8 is called the Heegaard splitting of M . We define the genus (or Heegaard genus) of M to be the minimum genus of the handlebody among all the Heegaard splittings.

Let (M, \mathcal{N}) be a compact marked 3-manifold, we define the genus of (M, \mathcal{N}) to be the genus of M . When \mathcal{N} is empty we could just call the genus of (M, \emptyset) as the genus of M .

For any nonnegative integer k , we P_k to denote the surface obtained from S^2 by removing k points. Let H be a handlebody of genus g , then H has the same skein theory with $P_{g+1} \times [0, 1]$.

5.2.5 An upper bound for $\dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M)$

Let M be a compact 3-manifold. We can assume M is a connected compact 3-manifold with genus g . Then suppose the Heegaard splitting of M is $M = H \cup H'$

where H is a handlebody of genus g and H' is a compression body. Then we can isotope any skein α in M into H . Thus the embedding $f : H \rightarrow M$ induces a surjective linear map $f_{\sharp} : \mathcal{S}_q(H) \rightarrow \mathcal{S}_q(M)$. From Lemma 5.2.3, we can get

$$\dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M) \leq \dim_{\mathcal{S}_\zeta(H)} \mathcal{S}_q(H) = \dim_{\mathcal{S}_\zeta(P_{g+1})} \mathcal{S}_q(P_{g+1}) = \dim_{\mathcal{S}_q(P_{g+1})^{(N)}} \mathcal{S}_q(P_{g+1}). \quad (5.7)$$

From Theorem 5.2.4, we know $\dim_{\mathcal{S}_q(P_{g+1})^{(N)}} \mathcal{S}_q(P_{g+1})$ is finite. The above discussion also offers a proof for Theorem 5.2.1.

5.2.6 Finiteness of the character-reduced skein module

Let M be a 3-manifold. Recall that we use $\mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(M)$ to denote the SL_2 character variety of M (Equation (1.1)). Each $\rho \in \mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(M)$ induces an algebra homomorphism $f_\rho : \mathcal{S}_\zeta(M) \rightarrow \mathbb{C}$, furthermore this correspondence is a bijection from $\mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(M)$ to the set of algebra homomorphisms from $\mathcal{S}_\zeta(M)$ to \mathbb{C} [PS00].

Let I be an ideal of $\mathcal{S}_\zeta(M)$, we

$$I\mathcal{S}_q(M) = \{x_1 \cdot y_1 + \cdots + x_k \cdot y_k \mid k \in \mathbb{N}, x_i \in \mathcal{S}_\zeta(M), y_i \in \mathcal{S}_q(M), 1 \leq i \leq k\}$$

(note that $x_1 \cdot y_1 + \cdots + x_k \cdot y_k$ is defined to be 0 when $k = 0$). Then $I\mathcal{S}_q(M)$ is an $\mathcal{S}_\zeta(M)$ -submodule of $\mathcal{S}_q(M)$.

For any element $\rho \in \mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(M)$, the **character-reduced skein module** $\mathcal{S}_q(M)_\rho$ of M with respect to ρ is defined to be

$$\mathcal{S}_q(M)/I_\rho \quad (5.8)$$

where $I_\rho = (\mathrm{Ker} f_\rho)\mathcal{S}_q(M)$ [DKS23; FKL22].

Remark 5.2.9. For any maximal ideal I of $\mathcal{S}_\zeta(M)$, which is point in $\mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(M)$, the character-reduced skein module is still an $\mathcal{S}_\zeta(M)$ -module since $I\mathcal{S}_q(M)$ is an $\mathcal{S}_\zeta(M)$ -submodule. Actually $\mathcal{S}_\zeta(M)$ -module structure for $\mathcal{S}_q(M)_I$ coincides with the linear structure over \mathbb{C} by the identity map. The maximal ideal I of $\mathcal{S}_\zeta(M)$ corresponds to an algebra homomorphism $f_I : \mathcal{S}_\zeta(M) \rightarrow \mathbb{C}$, then for any element

$\alpha \in \mathcal{S}_\zeta(M), \beta \in \mathcal{S}_q(M)$, we have

$$\alpha \cdot (\beta + I\mathcal{S}_q(M)) = \alpha \cdot \beta + I\mathcal{S}_q(M) = f_I(\alpha)(\beta + I\mathcal{S}_q(M)).$$

From Remark 5.2.9, we know the character-reduced skein module is obtained by sending elements in $\mathcal{S}_\zeta(M)$ to complex numbers, we can easily get the following result by Theorem 5.2.1.

Theorem 5.2.10. *Suppose we have the (R2) assumption. Let M be a compact 3-manifold, and $\rho \in \mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(M)$. Then $\mathcal{S}_q(M)_\rho$ is finite-dimensional over \mathbb{C} , and $\dim_{\mathbb{C}} \mathcal{S}_q(M)_\rho \leq \dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M)$.*

5.3 On stated skein modules

In this section, we always have the (R1) assumption. The main goal of this section is to prove the following Theorem.

Theorem 5.3.1. *Suppose we have the (R1) assumption. Then $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ is finitely generated over $\mathcal{S}_1(M, \mathcal{N})$ for any compact marked 3-manifold (M, \mathcal{N}) .*

5.3.1 Some functorialities

Let $f : (M_1, \mathcal{N}_1) \rightarrow (M_2, \mathcal{N}_2)$ be an embedding between two marked 3-manifolds. Then f induces a linear map $f_\# : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$, which restricts to an algebra homomorphism $f_\# : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1)^{(N)} \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)^{(N)}$. Similarly as in Lemma 5.2.2, we have f induces an algebra homomorphism $f_\# : \mathcal{S}_1(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_1(M_2, \mathcal{N}_2)$, and $f_\#$ respects the module structures, in a sense that, for any $x \in \mathcal{S}_1(M_1, \mathcal{N}_1), y \in \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1)$ we have $f_\#(x \cdot y) = f_\#(x) \cdot f_\#(y)$.

Lemma 5.3.2. *Let $f : (M_1, \mathcal{N}_1) \rightarrow (M_2, \mathcal{N}_2)$ be an embedding between two marked 3-manifolds. Suppose $f_\# : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$ is surjective and Theorem 5.3.1 holds for (M_1, \mathcal{N}_1) . Then Theorem 5.3.1 also holds for (M_2, \mathcal{N}_2) and*

$$\dim_{\mathcal{S}_1(M_2, \mathcal{N}_2)} \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2) \leq \dim_{\mathcal{S}_1(M_1, \mathcal{N}_1)} \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1).$$

Proof. The proof is similar to Lemma 5.2.3. □

5.3.2 Good strict subsurface

Let (M, \mathcal{N}) be a compact marked 3-manifold, and \mathfrak{S} be a properly embedded surface in M such that every component of \mathfrak{S} intersects \mathcal{N} . Let $V(\mathfrak{S})$ be a closed regular neighborhood of \mathfrak{S} , that is, $V(\mathfrak{S})$ is isomorphic to $\mathfrak{S} \times [0, 1]$ by an orientation preserving diffeomorphism such that \mathfrak{S} is identified with $\mathfrak{S} \times \{1/2\}$ and $\partial\mathfrak{S} \times [0, 1] \subset \partial M$. Then define $Cut_{\mathfrak{S}}(M, \mathcal{N}) = (M', \mathcal{N}')$ where $M' = M - \mathfrak{S} \times (0, 1)$ and $\mathcal{N}' = \mathcal{N} - \mathfrak{S} \times [0, 1]$ (here we identify $\mathfrak{S} \times [0, 1]$ with $V(\mathfrak{S})$). \mathfrak{S} is called a **good strict subsurface** of (M, \mathcal{N}) , if $Cut_{\mathfrak{S}}(M, \mathcal{N}) = (M', \mathcal{N}')$ is isomorphic to the thickening of a marked surface.

Theorem 5.3.3. *Let (Σ, \mathcal{P}) be any marked surface, then $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})$ is finitely generated over $\mathcal{S}_1(\Sigma, \mathcal{P})$ (or $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})^{(N)}$).*

Proof. We can assume Σ is connected. When \mathcal{P} is empty, the stated skein algebra is just the skein algebra. This case was proved in Theorem 5.2.1. When \mathcal{P} is not empty, Theorem 5.3.3 was proved in [Kor21; Wan23c]. \square

Proposition 5.3.4. *(Proposition 8.14 [Wan23c]) Assume (M, \mathcal{N}) is a marked 3-manifold which contains a good strict subsurface \mathfrak{S} . Then we have $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ is a finitely generated $\mathcal{S}_1(M, \mathcal{N})$ -module, and*

$$\dim_{\mathcal{S}_1(M, \mathcal{N})} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \leq \dim_{\mathcal{S}_{q^{1/2}}(Cut_{\mathfrak{S}}(M, \mathcal{N}))^{(N)}} \mathcal{S}_{q^{1/2}}(Cut_{\mathfrak{S}}(M, \mathcal{N})).$$

Proof. Here we briefly review the proof in [Wan23c]. We use (M', \mathcal{N}') to denote $Cut_{\mathfrak{S}}(M, \mathcal{N})$. Since \mathfrak{S} is a good strict subsurface, then (M', \mathcal{N}') has the same skein theory with the thickening of a marked surface. Then Theorem 5.3.3 shows $\mathcal{S}_{q^{1/2}}(M', \mathcal{N}')$ is finitely generated over $\mathcal{S}_1(M', \mathcal{N}')$.

Meanwhile the obvious embedding $f : (M', \mathcal{N}') \rightarrow (M, \mathcal{N})$ induces a surjective linear map $f_{\#} : \mathcal{S}_{q^{1/2}}(M', \mathcal{N}') \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})$. Then Lemma 5.3.2 completes the proof. Also, we conclude

$$\dim_{\mathcal{S}_1(M, \mathcal{N})} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \leq \dim_{\mathcal{S}_1(M', \mathcal{N}')} \mathcal{S}_{q^{1/2}}(M', \mathcal{N}').$$

Since (M', \mathcal{N}') is isomorphic to the thickening of a marked surface, we have

$$\dim_{\mathcal{S}_1(M', \mathcal{N}')} \mathcal{S}_{q^{1/2}}(M', \mathcal{N}') = \dim_{\mathcal{S}_{q^{1/2}}(Cut_{\mathfrak{S}}(M, \mathcal{N}))^{(N)}} \mathcal{S}_{q^{1/2}}(Cut_{\mathfrak{S}}(M, \mathcal{N})).$$

□

For any two nonnegative integers g, p , we use $\mathfrak{S}_{g,p}$ to denote the compact oriented surface with p boundary components and genus g . For any two nonnegative integers a, b with $b > 0$, we use ${}_{a,b}\mathfrak{S}$ to denote the pb surface obtained from $\mathfrak{S}_{a,1}$ by removing b points on the unique boundary component of $\mathfrak{S}_{a,1}$, we use ${}_{a,b}\Sigma$ to denote the marked surface $(\mathfrak{S}_{a,1}, \mathcal{P})$ where \mathcal{P} is the subset of the unique boundary component of $\mathfrak{S}_{a,1}$ consisting of k points. For the positive integer k , we use D_k to denote the pb surface obtained from the disk by removing k points on the boundary, we use Σ_k to denote the marked surface (D, K) where D is the disk and $K \subset \partial D$ consisting of k points. We call D_2 the bigon, and use $\lambda_{q^{1/2}}$ to denote $\dim_{\mathcal{S}_{q^{1/2}}(D_2)^{(N)}} \mathcal{S}_{q^{1/2}}(D_2)$.

Let k, g be two nonnegative integers, we use $M_{g,k}$ to denote the marked 3-manifold (M, \mathcal{N}) , where $M = \mathfrak{S}_{g,0} \times [0, 1]$ and $\mathcal{N} \subset \mathfrak{S}_{g,0} \times \{1\}$ such that \mathcal{N} has k components. Note that $M_{g,k}$ is defined up to diffeomorphism.

Lemma 5.3.5. *For any two nonnegative integers g, k , we have $\mathcal{S}_{q^{1/2}}(M_{g,k})$ is finitely generated over $\mathcal{S}_1(M_{g,k})$. When $k > 0$, we have*

$$\dim_{\mathcal{S}_1(M_{g,k})} \mathcal{S}_{q^{1/2}}(M_{g,k}) \leq \dim_{\mathcal{S}_{q^{1/2}}(g,k\mathfrak{S})^{(N)}} \mathcal{S}_{q^{1/2}}(g,k\mathfrak{S}).$$

Proof. When $k = 0$, the marking set of $M_{g,k}$ is empty, and the stated skein module is just the skein module. Then Theorem 5.2.1 shows this case.

Then we look at the case when k is positive. From the definition, we know $M_{g,k}$ is just $\mathfrak{S}_{g,0} \times [0, 1]$ with k markings on $\mathfrak{S}_{g,0} \times \{1\}$. We use \mathcal{N} to denote the union of all the markings of $M_{g,k}$. Let c be a closed curve in $\mathfrak{S}_{g,0}$ such that c bounds an embedded disk in $\mathfrak{S}_{g,0}$ and $c \times \{1\}$ intersects \mathcal{N} transversely in exactly one point. Then $c \times [0, 1]$ is a strict subsurface of $M_{g,k}$. We also have $\text{Cut}_{c \times [0,1]}(M_{g,k})$ is isomorphic with the thickening of ${}_{g,k}\Sigma \cup \Sigma_1$. Thus $c \times [0, 1]$ is a good strict subsurface of $M_{g,k}$. Then Theorem 5.3.3 and Proposition 5.3.4 shows $\mathcal{S}_{q^{1/2}}(M_{g,k})$ is finitely generated over $\mathcal{S}_1(M_{g,k})$, and

$$\dim_{\mathcal{S}_1(M_{g,k})} \mathcal{S}_{q^{1/2}}(M_{g,k}) \leq \dim_{\mathcal{S}_{q^{1/2}}(g,k\Sigma)^{(N)}} \mathcal{S}_{q^{1/2}}(g,k\Sigma) = \dim_{\mathcal{S}_{q^{1/2}}(g,k\mathfrak{S})^{(N)}} \mathcal{S}_{q^{1/2}}(g,k\mathfrak{S}).$$

□

5.3.3 Attaching 2-handles and cutting out open 3-balls for marked 3-manifolds

Let (M, \mathcal{N}) be a marked 3-manifold, and c_1, \dots, c_k be a collection of closed disjoint curves on ∂M such that for each $1 \leq i \leq k$ there is no intersection between c_i and \mathcal{N} . Then we can define a new marked 3-manifold (M', \mathcal{N}') where M' is obtained from M by attaching 2-handles along c_1, \dots, c_k and $\mathcal{N}' = \mathcal{N}$. We say (M', \mathcal{N}') is obtained from (M, \mathcal{N}) by attaching 2-handles along c_1, \dots, c_k . We use $f : (M, \mathcal{N}) \rightarrow (M', \mathcal{N}')$ to denote the obvious embedding. Then we have the following obvious Lemma.

Lemma 5.3.6. $f : (M, \mathcal{N}) \rightarrow (M', \mathcal{N}')$ induces a surjective linear map

$$f_{\#} : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M', \mathcal{N}').$$

Lemma 5.3.7. Let (M, \mathcal{N}) be a marked 3-manifold such that $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ is finitely generated over $\mathcal{S}_1(M, \mathcal{N})$. Suppose (M', \mathcal{N}') is obtained from (M, \mathcal{N}) by attaching 2-handles. Then $\mathcal{S}_{q^{1/2}}(M', \mathcal{N}')$ is also finitely generated over $\mathcal{S}_1(M', \mathcal{N}')$, and

$$\dim_{\mathcal{S}_1(M', \mathcal{N}')} \mathcal{S}_{q^{1/2}}(M', \mathcal{N}') \leq \dim_{\mathcal{S}_{q^{1/2}}(M, \mathcal{N})^{(N)}} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}).$$

Proof. This follows from Lemmas 5.3.2 and 5.3.6. □

Remark 5.3.8. Let (M, \mathcal{N}) be a marked 3-manifold, and B be an embedded 3-dimensional solid ball contained in the interior of M . Let M' be obtained from M by cutting out the interior of B , conversely M is obtained from M' by filling ∂B with a solid 3-dimensional ball. Clearly, (M, \mathcal{N}) and (M', \mathcal{N}) have the same skein theory.

Lemma 5.3.9. Let (M, \mathcal{N}) be a marked 3-manifold such that M is a compression body H and \mathcal{N} is a nonempty subset of $\partial^- H$. Then $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ is finitely generated over $\mathcal{S}_1(M, \mathcal{N})$, and

$$\dim_{\mathcal{S}_1(M, \mathcal{N})} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \leq \dim_{\mathcal{S}_{q^{1/2}}(g, k \mathfrak{S})^{(N)}} \mathcal{S}_{q^{1/2}}(g, k \mathfrak{S})$$

where g is the genus of H and k is the number of components of \mathcal{N} .

Proof. From the definition of the compression body, we know (M, \mathcal{N}) is obtained from $M_{g, k}$ by attaching 2-handles along mutually disjoint loops in $\mathfrak{S} \times \{1\}$ and

filling in some resulting 2-sphere boundary components with 3-dimensional solid balls. Then Lemmas 5.3.5, 5.3.7 and Remark 5.3.8 complete the proof. \square

5.3.4 Proof for Theorem 5.3.1

Proof of Theorem 5.3.1. Let (M, \mathcal{N}) be a compact marked 3-manifold. We can assume (M, \mathcal{N}) is a connected compact marked 3-manifold. When $\mathcal{N} = \emptyset$, Theorem 5.2.1 shows this case. Then suppose $\mathcal{N} \neq \emptyset$. From Theorem 5.2.8, we have $M = H \cup H'$ where H is a handlebody and H' is a compression body such that $\partial^- H' = \partial M$ and $\partial^+ H' = \partial H$. Then $\mathcal{N} \subset \partial M = \partial^- H'$. Let f be the obvious embedding from (H', \mathcal{N}) to (M, \mathcal{N}) . Then f induces a surjective linear map $f_{\#} : \mathcal{S}_{q^{1/2}}(H', \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ since we can isotope all stated (M, \mathcal{N}) -tangles away from H . Then Lemmas 5.3.2 and 5.3.9 show $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ is finitely generated over $\mathcal{S}_1(M, \mathcal{N})$, and

$$\dim_{\mathcal{S}_1(M, \mathcal{N})} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \leq \dim_{\mathcal{S}_{q^{1/2}}(g', k) \mathfrak{S}^{(N)}} \mathcal{S}_{q^{1/2}}(g', k) \mathfrak{S} \quad (5.9)$$

where g' is the genus of H' and k is the number of components of \mathcal{N} . \square

Remark 5.3.10. *Let (M, \mathcal{N}) be a connected compact marked 3-manifold with $\mathcal{N} \neq \emptyset$. Suppose the genus of (M, \mathcal{N}) is g and \mathcal{N} has k components. Then equation (5.9) shows*

$$\dim_{\mathcal{S}_1(M, \mathcal{N})} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \leq \dim_{\mathcal{S}_{q^{1/2}}(g, k) \mathfrak{S}^{(N)}} \mathcal{S}_{q^{1/2}}(g, k) \mathfrak{S}.$$

5.3.5 Representation-reduced stated skein modules

Let (M, \mathcal{N}) be a marked 3-manifold. For any ideal I of $\mathcal{S}_1(M, \mathcal{N})$, we define

$$I \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) = \{x_1 \cdot \alpha_1 + \cdots + x_k \cdot \alpha_k \mid k \in \mathbb{N}, x_i \in I, \alpha_i \in \mathcal{S}_{q^{1/2}}(M, \mathcal{N}), 1 \leq i \leq k\}$$

(when $k = 0$, we define $x_1 \cdot \alpha_1 + \cdots + x_k \cdot \alpha_k = 0$).

Assume (M, \mathcal{N}) is a connected marked 3-manifold. Let $\pi_1(M, \mathcal{N})$ be the fundamental groupoid of (M, \mathcal{N}) when $\mathcal{N} \neq \emptyset$, see Definition 2.2.5. Then we define $\mathcal{X}(M, \mathcal{N})$ to be the set of homomorphisms from $\pi_1(M, \mathcal{N})$ to $SL_2(\mathbb{C})$ when $\mathcal{N} \neq \emptyset$, and define $\mathcal{X}(M, \mathcal{N})$ to be $\mathfrak{X}_{SL_2(\mathbb{C})}(M)$ when $\mathcal{N} = \emptyset$. Then $\mathcal{X}(M, \mathcal{N})$ is an algebraic set. For any $\rho \in \mathcal{X}(M, \mathcal{N})$, there is an algebra homomorphism $f_{\rho} : \mathcal{S}_1(M, \mathcal{N}) \rightarrow \mathbb{C}$.

This offers a one to one correspondence between $\mathcal{X}(M, \mathcal{N})$ and the set of algebra homomorphisms from $\mathcal{S}_1(M, \mathcal{N})$ to \mathbb{C} , see Theorem 2.2.9. Although we assume M to be connected, the above discussion can be easily and naturally generalized to general marked 3-manifolds.

For any $\rho \in \mathcal{X}(M, \mathcal{N})$, we define the **representation-reduced stated skein module** $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})_\rho$ of (M, \mathcal{N}) with respect to ρ to be

$$\mathcal{S}_{q^{1/2}}(M, \mathcal{N})/I_\rho$$

where $I_\rho = (\text{Ker } f_\rho)\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$. When \mathcal{N} is empty, $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})_\rho$ coincides with the character-reduced skein module in equation (5.8).

Then we have the parallel statements for the representation-reduced stated skein module as in Remark 5.2.9.

Then Theorem 5.3.1 clearly shows the following Theorem.

Theorem 5.3.11. *Suppose we have the (R1) assumption. Let (M, \mathcal{N}) be a compact marked 3-manifold, and $\rho \in \mathcal{X}(M, \mathcal{N})$. Then $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})_\rho$ is finite-dimensional over \mathbb{C} , and*

$$\dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_\rho \leq \dim_{\mathcal{S}_1(M, \mathcal{N})} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}).$$

5.4 Dimension of $C(\mathcal{S}_q(\mathfrak{S}))$ over $\mathcal{S}_q(\mathfrak{S})^{(N)}$ when \mathfrak{S} is a closed pb surface

In this section, we always assume \mathfrak{S} is a connected closed pb surface, and q^2 is a primitive N -th root of unity with N odd. The following is the main result of this section.

Theorem 5.4.1. *Suppose we have the (R2) assumption. Let \mathfrak{S} be a connected closed pb surface with p punctures and with negative Euler characteristic. Then $C(\mathcal{S}_q(\mathfrak{S}))$ is freely generated over $\mathcal{S}_q(\mathfrak{S})^{(N)}$ by N^p elements. Especially we have*

$$\dim_{\mathcal{S}_q(\mathfrak{S})^{(N)}} C(\mathcal{S}_q(\mathfrak{S})) = N^p.$$

Recall that, for any nonnegative integer k , we use P_k to denote the surface obtained from S^2 by removing k points.

Corollary 5.4.2. *We have $\mathcal{S}_q(P_3)$ is freely generated over $\mathcal{S}_q(P_3)^{(N)}$ by N^3 elements. Especially*

$$\dim_{\mathcal{S}_q(P_3)^{(N)}} \mathcal{S}_q(P_3) = N^3.$$

Proof. We have $C(\mathcal{S}_q(P_3)) = \mathcal{S}_q(P_3)$ [Lê06; Prz91]. Then Corollary 5.4.2 follows from Theorem 5.4.1. \square

Remark 5.4.3. *The basis for $C(\mathcal{S}_q(\mathfrak{S}))$ in Theorem 3.5 in [FKL21] can show Theorem 5.4.1. In Remark 5.4 in [FKL19], Charles Frohman, Joanna Kania-Bartoszyńska, and Thang Lê also claimed the statement in Theorem 5.4.1.*

In the remainder of this section, we provide a proof of Theorem 5.4.1 using the quantum trace map.

5.4.1 Chekhov-Fock algebra and Quantum trace map

Let \mathfrak{S} be a closed pb surface with genus g and p punctures, where $p > 0$. For an ideal triangulation τ and a nonzero complex number μ , we have the associated Chekhov-Fock algebra \mathcal{T}_τ^μ see [BW11; BW16a; BW17; Liu09] for more details. Suppose the set of edges of τ is $\{e_1, \dots, e_n\}$ where $n = 6g + 3p - 6$. Set b_{ij} to be the number of times an end of the edge e_j immediately succeeds an end of e_i when going counterclockwise around a puncture of \mathfrak{S} , and define $\sigma_{ij} = b_{ij} - b_{ji} \in \{-2, -1, 0, 1, 2\}$. As an algebra \mathcal{T}_τ^μ is generated by $Y_1^{\pm 1}, Y_2^{\pm 1}, \dots, Y_n^{\pm 1}$ and subject to the relations:

$$Y_i Y_i^{-1} = Y_i^{-1} Y_i = 1, Y_i Y_j = \mu^{2\sigma_{ij}} Y_j Y_i,$$

where each Y_i is associated to the edge e_i . For any $Y_{i_1}, Y_{i_2}, \dots, Y_{i_k}$, we use $[Y_{i_1} Y_{i_2} \dots Y_{i_k}]$ to denote

$$\mu^{-\sum_{1 \leq j < l \leq k} \sigma_{jl}} Y_{i_1} Y_{i_2} \dots Y_{i_k}.$$

For any $\vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^{\oplus n}$, we use $Y^{\vec{k}}$ to denote $[Y_1^{k_1} Y_2^{k_2} \dots Y_n^{k_n}]$. Then the set $\{Y^{\vec{k}} \mid \vec{k} \in \mathbb{Z}^{\oplus n}\}$ is a basis of \mathcal{T}_τ^μ .

For any $\vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^{\oplus n}$, we say \vec{k} satisfies the balanced condition if for every ideal triangle T in τ with three edges labeled by i_1, i_2, i_3 we have $k_{i_1} + k_{i_2} + k_{i_3}$

is even. We use \mathcal{Z}_τ^μ , called the balanced Chekhov-Fock algebra, to denote the subalgebra of \mathcal{T}_τ^μ generated by all $Y^{\vec{k}}$ with \vec{k} satisfying the balanced condition.

Suppose v is a puncture of \mathfrak{S} , we use H_v to denote $[Y_{i_1} Y_{i_2} \dots Y_{i_{k_v}}] \in \mathcal{T}_\tau^\mu$ where $e_{i_1}, e_{i_2}, \dots, e_{i_{k_v}}$ are all the edges that connect to the vertex v (note that there maybe some repetitions among $e_{i_1}, e_{i_2}, \dots, e_{i_{k_v}}$). It is easy to show $H_v \in C(\mathcal{T}_\tau^\mu)$.

Set $\mu = q^{1/2}$. Then there is an algebra embedding [BW11]

$$\text{Tr}_\tau^q : \mathcal{S}_q(\mathfrak{S}) \rightarrow \mathcal{T}_\tau^\mu.$$

We set $\nu = \mu^{N^2}$, then there is an algebra embedding $F : \mathcal{T}_\tau^\nu \rightarrow \mathcal{T}_\tau^\mu$ defined by $F(Y^{\vec{k}}) = Y^{N\vec{k}}$. We also have $\nu^2 = \mu^{2N^2} = q^{N^2} = q^N = \zeta$. Then we have the following commutative diagram [BW16a]:

$$\begin{array}{ccc} \mathcal{S}_\zeta(\mathfrak{S}) & \xrightarrow{\text{Tr}_\tau^\zeta} & \mathcal{T}_\tau^\nu \\ \downarrow \mathcal{F} & & \downarrow F \\ \mathcal{S}_q(\mathfrak{S}) & \xrightarrow{\text{Tr}_\tau^q} & \mathcal{T}_\tau^\mu \end{array} \quad (5.10)$$

The image of the quantum trace map lies in the balanced Chekhov-Fock algebra [BW11]. Obviously, F restricts to an algebra embedding $F : \mathcal{Z}_\tau^\nu \rightarrow \mathcal{Z}_\tau^\mu$. Thus we can replace \mathcal{T}_τ^ν (respectively \mathcal{T}_τ^μ) with \mathcal{Z}_τ^ν (respectively \mathcal{Z}_τ^μ) in the above diagram, and the new diagram still commutes.

5.4.2 On the balanced Chekhov-Fock algebra

We assume the punctures of \mathfrak{S} are denoted v_1, \dots, v_p . For each $1 \leq i \leq p$, we use H_i to denote H_{v_i} .

Lemma 5.4.4. *There exist elements $Z_1, \dots, Z_n \in \mathcal{Z}_\tau^\mu$ such that (1) for $1 \leq i \leq p$ we have $Z_i = H_i$, (2) for any pair $1 \leq i < j \leq n$ we have $Z_i Z_j = \mu^{2b_{ij}} Z_j Z_i$ where $b_{ij} \in \mathbb{Z}$, (3) $\{Z^{\vec{k}} \mid \vec{k} \in \mathbb{Z}^{\oplus n}\}$ is a basis of \mathcal{Z}_τ^μ where $Z^{\vec{k}}$ is defined in the same way as $Y^{\vec{k}}$.*

Proof. It is an immediate consequence from Lemma 12 in [BW17]. □

For any two distinct $\vec{k}, \vec{t} \in \mathbb{Z}^{\oplus n}$, we have $Z^{\vec{k}} \neq Z^{\vec{t}}$ because they are basis elements in Lemma 5.4.4 (3). For any $Z^{\vec{k}}$ with $\vec{k} = (k_1, \dots, k_n)$, we define $\mathfrak{S}(Z^{\vec{k}}) = (k_1, \dots, k_p) \in \mathbb{Z}^{\oplus p}$. For any element $x \in \mathbb{Z}^{\oplus p}$, let D_x be the subvector space of \mathcal{Z}_τ^μ spanned by all $Z^{\vec{k}}$ with $\mathfrak{S}(Z^{\vec{k}}) = x$. Then we have $\mathcal{Z}_\tau^\mu = \bigoplus_{x \in \mathbb{Z}^{\oplus p}} D_x$. Note that this grading is compatible with the algebra structure for \mathcal{Z}_τ^μ , that is, $D_x D_y \subset D_{x+y}$.

For any two different elements $a = (a_1, \dots, a_p), b = (b_1, \dots, b_p) \in \mathbb{Z}^{\oplus p}$, there exists $1 \leq i \leq p$ such that $a_t = b_t$ when $1 \leq t \leq i-1$ and $a_i \neq b_i$, then we define $a < b$ (respectively $b < a$) if $a_i < b_i$ (respectively $b_i < a_i$). Simply speaking, the linear order " \leq " on $\mathbb{Z}^{\oplus p}$ is lexicographic order.

For any nonzero element $l \in \mathcal{Z}_\tau^\mu$, we suppose $l = l_{a_1} + \dots + l_{a_k}$ where $a_i, 1 \leq i \leq k$, are k distinct elements in $\mathbb{Z}^{\oplus p}$ and l_{a_i} is a nonzero element in $D_{a_i}, 1 \leq i \leq k$. Then we define $\deg(l) = \max\{a_1, \dots, a_k\}$. For any two nonzero elements $l_1, l_2 \in \mathcal{Z}_\tau^\mu$, clearly we have $\deg(l_1 l_2) = \deg(l_1) + \deg(l_2)$.

For \mathcal{Z}_τ^ν , we define the same $Z_1, \dots, Z_n \in \mathcal{Z}_\tau^\nu$ as Z_τ^μ in Lemma 5.4.4. That is, for each $1 \leq i \leq n$, suppose $Z_i = Y^{\vec{k}_i} \in \mathcal{Z}_\tau^\mu$, then similarly we define $Z_i = Y^{\vec{k}_i} \in \mathcal{Z}_\tau^\nu$. Recall that we have an algebra embedding $F : \mathcal{Z}_\tau^\nu \rightarrow \mathcal{Z}_\tau^\mu$, which is a restriction of $F : \mathcal{T}_\tau^\nu \rightarrow \mathcal{T}_\tau^\mu$. Then for each $1 \leq i \leq n$, we have $F(Z_i) = F(Y^{\vec{k}_i}) = Y^{N\vec{k}_i} = (Y^{\vec{k}_i})^N = (Z_i)^N$.

5.4.3 Proof of Theorem 5.4.1

Proof of Theorem 5.4.1. Recall that T_N is the N -th Chebyshev polynomial of the first kind, see subsection 5.1.2. Let $T_N = \sum_{0 \leq t \leq N} \lambda_t x^t$. Note that $\lambda_N = 1$.

When $p = 0$, we have $C(\mathcal{S}_q(\mathfrak{S})) = \mathcal{S}_q(\mathfrak{S})^{(N)}$ [FKL19], then the statement of Theorem 5.4.1 is clearly true for this case.

Then we look at the case when $p > 0$. Here we use all the notations in subsection 5.4.2. For each puncture v_i , we use d_i to denote the loop going around v_i with vertical framing. From [FKL19], we know $C(\mathcal{S}_q(\mathfrak{S}))$ is generated by $\mathcal{S}_q(\mathfrak{S})^{(N)}$ and $\{d_1, \dots, d_p\}$ as a subalgebra of $\mathcal{S}_q(\mathfrak{S})$. For any $\vec{k} = (k_1, \dots, k_p) \in \mathbb{N}^{\oplus p}$, we use $d^{\vec{k}}$ to denote $(d_1)^{k_1} \dots (d_p)^{k_p}$. Let $\mathcal{C} = \{(k_1, \dots, k_p) \in \mathbb{N}^{\oplus p} \mid 0 \leq k_i \leq N-1, 1 \leq i \leq p\}$. Then we want to show $C(\mathcal{S}_q(\mathfrak{S}))$ is freely generated by $\{d^{\vec{k}} \mid \vec{k} \in \mathcal{C}\}$ over $\mathcal{S}_q(\mathfrak{S})^{(N)}$.

First we want to show $C(\mathcal{S}_q(\mathfrak{S}))$ is generated by $\{d^{\vec{k}} \mid \vec{k} \in \mathcal{C}\}$ over $\mathcal{S}_q(\mathfrak{S})^{(N)}$. Let V be the $\mathcal{S}_q(\mathfrak{S})^{(N)}$ -submodule of $C(\mathcal{S}_q(\mathfrak{S}))$ generated by $\{d^{\vec{k}} \mid \vec{k} \in \mathcal{C}\}$. Then it suffices to show $d^{\vec{k}} \in V$ for all $\vec{k} \in \mathbb{N}^{\oplus p}$. For any $\vec{k} \in \mathbb{N}^{\oplus p-1}$, $1 \leq t \leq p, a \in \mathbb{N}$, we define $\vec{k}_a^t = (k_1, \dots, k_{t-1}, a, k_t, \dots, k_{p-1}) \in \mathbb{N}^{\oplus p}$. For any $\vec{k} \in \mathbb{N}^{\oplus p-1}$ with $0 \leq k_i \leq N-1$ for all $1 \leq i \leq p-1$ and $a \in \mathbb{N}$, we want to show $d^{\vec{k}_a^1} \in V$. We already have $d^{\vec{k}_a^1} \in V$ when $0 \leq a \leq N-1$. Then suppose we have $d^{\vec{k}_a^1} \in V$ when $a \leq m$ where $m \geq N-1$. From the assumption, we have $d^{\vec{k}_{m+1-N}^1} \in V$, then we have

$$T_N(d_1)d^{\vec{k}_{m+1-N}^1} = \sum_{0 \leq t \leq N} \lambda_t d_1^t d^{\vec{k}_{m+1-N}^1} = \sum_{0 \leq t \leq N} \lambda_t d^{\vec{k}_{t+m+1-N}^1} \in V.$$

From the assumption, we have $d^{\vec{k}_{t+m+1-N}^1} \in V$ when $0 \leq t \leq N-1$, then $d^{\vec{k}_{m+1}^1} \in V$. From mathematical induction, we get $d^{\vec{k}_a^1} \in V$ when $\vec{k} \in \mathbb{N}^{\oplus p-1}$ with $0 \leq k_i \leq N-1$ for all $1 \leq i \leq p-1$ and $a \in \mathbb{N}$. Using the same trick as above, we can prove $d^{\vec{k}_a^2} \in V$ when $\vec{k} \in \mathbb{N}^{\oplus p-1}$ with $0 \leq k_i \leq N-1$ for all $2 \leq i \leq p-1$ and $a \in \mathbb{N}$. Repeat the above process, eventually we can prove $d^{\vec{k}_a^p} \in V$ when $\vec{k} \in \mathbb{N}^{\oplus p-1}$ and $a \in \mathbb{N}$. This actually shows $d^{\vec{k}} \in V$ for all $\vec{k} \in \mathbb{N}^{\oplus p}$.

Then we try to show the independence of $\{d^{\vec{k}} \mid \vec{k} \in \mathcal{C}\}$. Suppose

$$\sum_{\vec{k} \in \mathcal{C}} \mathcal{F}(l_{\vec{k}})d^{\vec{k}} = 0$$

where $l_{\vec{k}} \in \mathcal{S}_{\zeta}(\mathfrak{S})$. Then we want to show $l_{\vec{k}} = 0$ for all $\vec{k} \in \mathcal{C}$. Suppose on the contrary, then $\mathcal{C}_0 = \{\vec{k} \in \mathcal{C} \mid l_{\vec{k}} \neq 0\} \neq \emptyset$. We have

$$\begin{aligned} 0 &= Tr_{\tau}^q\left(\sum_{\vec{k} \in \mathcal{C}} \mathcal{F}(l_{\vec{k}})d^{\vec{k}}\right) = \sum_{\vec{k} \in \mathcal{C}_0} Tr_{\tau}^q \mathcal{F}(l_{\vec{k}})Tr_{\tau}^q(d^{\vec{k}}) \\ &= \sum_{\vec{k} \in \mathcal{C}_0} FTr_{\tau}^{\zeta}(l_{\vec{k}})(Z_1 + Z_1^{-1})^{k_1} \cdots (Z_p + Z_p^{-1})^{k_p} \text{ where } \vec{k} = (k_1, \dots, k_p). \end{aligned}$$

Since, for each $\vec{k} \in \mathcal{C}_0$, $l_{\vec{k}} \neq 0$, then $\deg(FTr_{\tau}^{\zeta}(l_{\vec{k}}))$ is well-defined. We have $\deg(FTr_{\tau}^{\zeta}(l_{\vec{k}})) = Nx_{\vec{k}}$ for some $x_{\vec{k}} \in \mathbb{Z}^{\oplus p}$ because of the definition of F . Then

$$\deg(FTr_{\tau}^{\zeta}(l_{\vec{k}})(Z_1 + Z_1^{-1})^{k_1} \cdots (Z_p + Z_p^{-1})^{k_p}) = Nx_{\vec{k}} + \vec{k}.$$

For any $\vec{a} \neq \vec{b} \in \mathcal{C}_0$, we have $Nx_{\vec{a}} + \vec{a} \neq Nx_{\vec{b}} + \vec{b}$. Otherwise we have $\vec{a} - \vec{b} = N(x_{\vec{b}} - x_{\vec{a}})$. But $0 \leq a_i, b_i \leq N-1$, thus we get $\vec{a} - \vec{b} = N(x_{\vec{b}} - x_{\vec{a}}) = 0$, which contradicts

with $\vec{a} \neq \vec{b}$. Thus we have $\deg(\text{Tr}_\tau^q(\sum_{\vec{k} \in \mathcal{C}} \mathcal{F}(l_{\vec{k}})d^{\vec{k}})) = \max\{Nx_{\vec{k}} + \vec{k} \mid \vec{k} \in \mathcal{C}_0\}$, which contradicts with $0 = \text{Tr}_\tau^q(\sum_{\vec{k} \in \mathcal{C}} \mathcal{F}(l_{\vec{k}})d^{\vec{k}})$. Then $l_{\vec{k}} = 0$ for all $\vec{k} \in \mathcal{C}$. \square

5.5 Dimension of stated skein modules over Frobenius

We showed the finiteness of stated skein modules over Frobenius. In this section, we focus on estimating the dimension of stated skein modules over Frobenius. For simplicity, we will assume all the 3-manifolds and pb surfaces, mentioned in this section, are connected.

Let \mathfrak{S} be a pb surface. Recall that $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)} = \text{Im}\mathcal{F}$ is a commutative domain when $q^{1/2}$ is a root of unity of order N with N odd, where $\mathcal{F} : \mathcal{S}_1(\mathfrak{S}) \rightarrow \mathcal{S}_{q^{1/2}}(\mathfrak{S})$ is the Frobenius map. We use $\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}^{(N)}$ to denote the field of fractions of $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}$ and use $\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}$ to denote $\mathcal{S}_{q^{1/2}}(\mathfrak{S}) \otimes_{\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}} \widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}^{(N)}$.

Define

$$K_{\mathfrak{S}, q^{1/2}} := \dim_{\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}^{(N)}} \widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}, \quad (5.11)$$

and define

$$\lambda_{\mathfrak{S}, q^{1/2}} := \dim_{\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}} \mathcal{S}_{q^{1/2}}(\mathfrak{S}), \quad (5.12)$$

where $\dim_{\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}} \mathcal{S}_{q^{1/2}}(\mathfrak{S})$ is the lower bound of the number of generators of $\mathcal{S}_{q^{1/2}}(\mathfrak{S})$ over $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}$. Then clearly we have $K_{\mathfrak{S}, q^{1/2}} \leq \lambda_{\mathfrak{S}, q^{1/2}}$. In this section, we will give an explicit calculation for $K_{\mathfrak{S}, q^{1/2}}$, and give a lower bound and an upper bound for $\lambda_{\mathfrak{S}, q^{1/2}}$.

5.5.1 When the marking set is empty

In this subsection, we always have the (R2) assumption. From Theorem 5.2.1, we know, for any compact 3-manifold M , $\mathcal{S}_q(M)$ is finitely generated over $\mathcal{S}_\zeta(M)$ where $\zeta = q^N = \pm 1$. We will give an upper bound for $\dim_{\mathcal{S}_\zeta(M)} \mathcal{S}_q(M)$ in this subsection.

For a closed pb surface \mathfrak{S} , we know $\mathcal{S}_q(\mathfrak{S})$ is a domain. Then $\mathcal{S}_q(\mathfrak{S})^{(N)} = \text{Im}\mathcal{F}$ is a commutative domain. We use $\widetilde{\mathcal{S}_q(\mathfrak{S})}^{(N)}$ to denote the field of fractions of $\mathcal{S}_q(\mathfrak{S})^{(N)}$, and use $\widetilde{\mathcal{S}_q(\mathfrak{S})}$ to denote $\mathcal{S}_q(\mathfrak{S}) \otimes_{\mathcal{S}_q(\mathfrak{S})^{(N)}} \widetilde{\mathcal{S}_q(\mathfrak{S})}^{(N)}$. Then $\widetilde{\mathcal{S}_q(\mathfrak{S})}$ is a vector space over the field $\widetilde{\mathcal{S}_q(\mathfrak{S})}^{(N)}$. Clearly we have $\dim_{\mathcal{S}_q(\mathfrak{S})^{(N)}} \mathcal{S}_q(\mathfrak{S}) \geq \dim_{\widetilde{\mathcal{S}_q(\mathfrak{S})}^{(N)}} \widetilde{\mathcal{S}_q(\mathfrak{S})}$.

Lemma 5.5.1. *Suppose \mathfrak{S} is a closed pb surface with genus g and p punctures and the Euler characteristic $\chi(\mathfrak{S}) < 0$. Then there exist $N^{6g-6+3p}$ elements in $\mathcal{S}_q(\mathfrak{S})$ which are linearly independent over $\mathcal{S}_q(\mathfrak{S})^{(N)}$.*

Proof. Corollary 5.2 in [FKL21] and Theorem 5.4.1. □

Lemma 5.5.2. *Suppose \mathfrak{S} is a closed pb surface with genus g and p punctures and the Euler characteristic $\chi(\mathfrak{S}) < 0$. Then $\dim_{\mathcal{S}_q(\mathfrak{S})^{(N)}} \mathcal{S}_q(\mathfrak{S}) \geq \dim_{\widetilde{\mathcal{S}_q(\mathfrak{S})}^{(N)}} \widetilde{\mathcal{S}_q(\mathfrak{S})} \geq N^{6g-6+3p}$.*

Proof. Lemma 5.5.1. □

Theorem 5.5.3. (Corollary 3.10 in [FK18]) *Suppose \mathfrak{S} is a closed pb surface with genus g and p punctures and the Euler characteristic $\chi(\mathfrak{S}) < 0$. Then $\dim_{\widetilde{\mathcal{S}_q(\mathfrak{S})}^{(N)}} \widetilde{\mathcal{S}_q(\mathfrak{S})} = N^{6g-6+3p} = N^{-3\chi(\mathfrak{S})}$.*

Thang Lê informed us that Theorem 6.1 in [FKL21] shows the above Theorem.

Lemma 5.5.4. *Let T be the solid torus. Then we have $\mathcal{S}_q(T)$ is a free $\mathcal{S}_q(T)^{(N)}$ -module generated by N elements. Especially, we have*

$$\dim_{\widetilde{\mathcal{S}_q(T)}^{(N)}} \widetilde{\mathcal{S}_q(T)} = \dim_{\mathcal{S}_q(T)^{(N)}} \mathcal{S}_q(T) = \dim_{\mathcal{S}_q(T)} \mathcal{S}_q(T) = N.$$

Proof. It is well-known that $\mathcal{S}_q(T) = \mathbb{C}[x]$ and $\mathcal{S}_q(T)^{(N)} = \mathbb{C}[T_N(x)]$. Clearly $\mathcal{S}_q(T)$ is freely generated by $1, x, \dots, x^{N-1}$ over $\mathcal{S}_q(T)^{(N)}$.

□

Remark 5.5.5. *There are only four closed pb surfaces with non-negative Euler characteristic: closed torus T, P_0, P_1, P_2 . Proposition 5.5 in [FA16] shows $\dim_{\widetilde{\mathcal{S}_q(T)}^{(N)}} \widetilde{\mathcal{S}_q(T)} = N^2$. Obviously we have $\dim_{\widetilde{\mathcal{S}_q(P_i)}^{(N)}} \widetilde{\mathcal{S}_q(P_i)} = 1$ when $i = 0, 1$. Lemma 5.5.4 shows $\dim_{\widetilde{\mathcal{S}_q(P_2)}^{(N)}} \widetilde{\mathcal{S}_q(P_2)} = N$. In [FA16], Frohman and Abdiel calculated $\dim_{\widetilde{\mathcal{S}_q(\mathfrak{S})}^{(N)}} \widetilde{\mathcal{S}_q(\mathfrak{S})}$ when \mathfrak{S} is the closed torus, once punctured torus, P_2, P_3 , which coincide with Theorem 5.5.3 and Lemma 5.5.4.*

Let \mathcal{A} be a \mathbb{C} -algebra and $a_1, \dots, a_m \in \mathcal{A}$. The algebra \mathcal{A} is orderly finitely generated by a_1, \dots, a_m if \mathcal{A} is \mathbb{C} -linearly spanned by $\{a_1^{k_1} \cdots a_m^{k_m} \mid k_i \in \mathbb{N} \text{ for } 1 \leq i \leq m\}$.

Theorem 5.5.6. (First paragraph in the proof of Theorem 6.2 in [PS00]) Suppose \mathfrak{S} is a closed pb surface with genus g and p punctures, where $p \geq 1$. Then there exist closed curves $\alpha_1, \dots, \alpha_k$ such that $\mathcal{S}_q(\mathfrak{S})$ is orderly finitely generated by $\alpha_1, \dots, \alpha_k$, where $k = 2^{2g+p-1} - 1$.

Theorem 5.5.7. Suppose \mathfrak{S} is a closed pb surface with genus g and p punctures, where $p \geq 1$, and the Euler characteristic $\chi(\mathfrak{S}) < 0$. Then

$$N^{6g-6+3p} \leq \dim_{\mathcal{S}_q(\mathfrak{S})(N)} \mathcal{S}_q(\mathfrak{S}) \leq N^{2^{2g+p-1}-1}.$$

Proof. From Theorem 5.5.6, we know there exist $k = 2^{2g+p-1} - 1$ disjoint solid tori embedded in $\mathfrak{S} \times [0, 1]$ such that the embedding f from the union of these k solid tori, which is denoted as M' , to $\mathfrak{S} \times [0, 1]$ induces a surjective linear map $f_{\sharp} : \mathcal{S}_q(M') \rightarrow \mathcal{S}_q(\mathfrak{S} \times [0, 1])$. Then from Lemma 5.2.3, we have

$$\dim_{\mathcal{S}_q(\mathfrak{S})(N)} \mathcal{S}_q(\mathfrak{S}) = \dim_{\mathcal{S}_q(\mathfrak{S} \times [0, 1])(N)} \mathcal{S}_q(\mathfrak{S} \times [0, 1]) \leq \dim_{\mathcal{S}_q(M')(N)} \mathcal{S}_q(M') = N^k$$

where the last equality comes from Lemma 5.5.4. Then Lemma 5.5.2 completes the proof. \square

Theorem 5.5.8. Suppose \mathfrak{S} is a totally closed pb surface with genus g and the Euler characteristic $\chi(\mathfrak{S}) < 0$. Then

$$N^{6g-6} \leq \dim_{\mathcal{S}_q(\mathfrak{S})(N)} \mathcal{S}_q(\mathfrak{S}) \leq N^{2^{2g}-1}.$$

Proof. Let \mathfrak{S}' be a pb surface obtained from \mathfrak{S} by removing one point in \mathfrak{S} . Then the embedding $l : \mathfrak{S}' \rightarrow \mathfrak{S}$ induces a surjective algebra homomorphism $l_{\sharp} : \mathcal{S}_q(\mathfrak{S}') \rightarrow \mathcal{S}_q(\mathfrak{S})$. Then Lemma 5.2.3 and Theorem 5.5.7 show

$$\dim_{\mathcal{S}_q(\mathfrak{S})(N)} \mathcal{S}_q(\mathfrak{S}) \leq \dim_{\mathcal{S}_q(\mathfrak{S}')(N)} \mathcal{S}_q(\mathfrak{S}') \leq N^{2^{2g}-1}.$$

Then Lemma 5.5.2 completes the proof. \square

Theorem 5.5.9. Let M be a compact 3-manifold with genus g , we have

$$\dim_{\mathcal{S}_q(M)} \mathcal{S}_q(M) \leq N^{2^g-1}.$$

Epecially the dimension of any character-reduced skein module of M is not more than N^{2g-1} .

Proof. Equation (5.7), Lemma 5.5.4 and Theorems 5.2.10, 5.5.7. □

Corollary 5.5.10. *Let M be a compact 3-manifold with genus g . If $\mathcal{S}_\zeta(M)$ is finitely generated over $\mathcal{S}_\zeta(\partial M)$ where $\zeta = q^N = \pm 1$, then*

$$\dim_{\mathcal{S}_q(\partial M)} \mathcal{S}_q(M) \leq N^{2g-1} (\dim_{\mathcal{S}_\zeta(\partial M)} \mathcal{S}_\zeta(M)).$$

Proof. Theorems 5.2.7 and 5.5.9. □

In the remainder of this subsection, we will focus on proving $\mathcal{F} : \mathcal{S}_\zeta(S^1 \times S^2) \rightarrow \mathcal{S}_q(S^1 \times S^2)$ is an isomorphism, which shows $\mathcal{S}_q(S^1 \times S^2)$ is freely generated over $\mathcal{S}_\zeta(S^1 \times S^2)$ by the empty skein.

Recall that $T_n(x)$ and $S_n(x)$ are Chebyshev polynomials of the first kind and of the second kind respectively. Now we define a new sequence of polynomials $A_n(x)$ by setting $A_n(x) = S_n(x)$, $n = 1, 2$, and $A_n(x) = S_n(x) + A_{n-2}(x)$ when $n > 2$. Suppose D is an embedded disk in S^2 , then $S^1 \times D$ is an embedded solid torus in $S^1 \times S^2$. We use P to denote the origin of the disk D , and use x to denote the skein in $S^1 \times D$ represented by the closed line $\{P\} \times S^1$ with vertical framing. It is well-known that $\mathcal{S}_q(S^1 \times D)$ is actually $\mathbb{C}[x]$.

For any polynomial $f(x) \in \mathbb{C}[x]$, we can regard $f(x)$ as an element in $\mathcal{S}_q(S^1 \times S^2)$ by the embedding from $S^1 \times D$ to $S^1 \times S^2$. Then it is obvious that $1, x, x^2, \dots$ span $\mathcal{S}_q(S^1 \times S^2)$, and $1, T_N(x), T_N(x)^2, \dots$ span $\mathcal{S}_q(S^1 \times S^2)^{(N)}$.

Theorem 5.5.11. $\mathcal{F} : \mathcal{S}_\zeta(S^1 \times S^2) \rightarrow \mathcal{S}_q(S^1 \times S^2)$ is an isomorphism. Especially, we have $\mathcal{S}_q(S^1 \times S^2)$ is freely generated over $\mathcal{S}_\zeta(S^1 \times S^2)$ by the empty skein, and $\mathcal{S}_q(S^1 \times S^2)$ has a commutative algebra structure.

Proof. If N is 1, it is obvious.

Suppose $N \geq 3$. From [HP95], we know $\mathcal{S}_\zeta(S^1 \times S^2)$ has a basis

$$\{1, A_1(x), \dots, A_k(x), \dots\}.$$

Thus $\{1, T_1(x), \dots, T_k(x), \dots\}$ is also a basis for $\mathcal{S}_q(S^1 \times S^2)$. For any $k \in \mathbb{N}^*$, we have $\mathcal{F}(T_k(x)) = T_{kN}(x)$. Thus it suffices to show $\{1, T_N(x), \dots, T_{kN}(x), \dots\}$ is a basis for $\mathcal{S}_q(S^1 \times S^2)$.

For each positive integer i , we set e_i to be $A_i(x) \in \mathcal{S}_q(S^1 \times S^2)$. From [HP95], we know

$$\mathcal{S}_q(S^1 \times S^2) = \mathbb{C}\emptyset \oplus_{1 \leq i < +\infty, N \nmid i+2} \mathbb{C}e_i,$$

and $e_i = 0$ for $1 \leq i < +\infty, N \nmid i+2$. Then

$$\{1\} \cup \{e_i \mid \exists k \in \mathbb{N}^* \text{ such that } i+2 = kN\} \quad (5.13)$$

is a basis for $\mathcal{S}_q(S^1 \times S^2)$.

Suppose $i+2 = kN$ for some positive integer k . We have

$$\begin{aligned} T_{i+2}(x) &= S_{i+2}(x) - S_i(x) = A_{i+2}(x) - A_i(x) - (A_i(x) - A_{i-2}(x)) \\ &= A_{i+2}(x) + A_{i-2}(x) - 2A_i(x) = e_{i+2} + e_{i-2} - 2e_i. \end{aligned}$$

Since $N \mid i+2$, N is odd and N is not 1, we have $N \nmid i+4$ and $N \nmid i$. Then $e_{i+2} = e_{i-2} = 0$. Thus we have $T_{i+2}(x) = -2e_i$, then $e_i = -\frac{1}{2}T_{kN}(x) \in \mathcal{S}_q(S^1 \times S^2)$. Note that there is a special case where $N = 3, k = 1$, but using the same technique as above we can still get $e_i = -\frac{1}{2}T_{kN}(x) \in \mathcal{S}_q(S^1 \times S^2)$. Then basis in equation (5.13) shows $\{1, T_N(x), \dots, T_{kN}(x), \dots\}$ is a basis for $\mathcal{S}_q(S^1 \times S^2)$. \square

Corollary 5.5.12. *The character-reduced skein module for $S^1 \times S^2$ always has dimension one.*

Remark 5.5.13. *Charles Frohman, Joanna Kania-Bartoszyńska, and Thang Lê proved the character-reduced skein module of any closed 3-manifold with respect to any non-central character is 1-dimensional [FKL23; FKL22].*

Corollary 5.5.12 shows the character-reduced skein module for $S^1 \times S^2$ with respect to any central character is still 1-dimensional.

5.5.2 On $\dim_{\mathcal{S}_{q^{1/2}}(D_2)^{(N)}} \mathcal{S}_{q^{1/2}}(D_2)$

In the remainder of this section, we always have the (R1) assumption. Recall that D_2 is the bigon, and $\lambda_{q^{1/2}}$ denotes $\dim_{\mathcal{S}_{q^{1/2}}(D_2)^{(N)}} \mathcal{S}_{q^{1/2}}(D_2)$. From [CL22a],

we know $\mathcal{S}_{q^{1/2}}(D_2)$ is just $\mathcal{O}_q(SL_2)$, where $\mathcal{O}_q(SL_2)$ is generated by a, b, c, d and subject to the following relations:

$$\begin{aligned} ca &= q^2 ac, db = q^2 bd, ba = q^2 ab, dc = q^2 cd, \\ bc &= cb, ad - q^{-2}bc = 1, da - q^2cb = 1. \end{aligned}$$

Let A_q be the the subalgebra of $\mathcal{O}_q(SL_2)$ generated by a^N, b^N, c^N, d^N . With the identification between $\mathcal{S}_{q^{1/2}}(D_2)$ and $\mathcal{O}_q(SL_2)$, $\mathcal{S}_{q^{1/2}}(D_2)^{(N)}$ is just A_q . Thus we have $\lambda_{q^{1/2}} = \dim_{A_q} \mathcal{O}_q(SL_2)$. Since $\mathcal{O}_q(SL_2)$ is a domain, then A_q is a commutative domain. We use \widetilde{A}_q to denote the field of fractions of A_q , and use $\widetilde{\mathcal{O}_q(SL_2)}$ to denote $\mathcal{O}_q(SL_2) \otimes_{A_q} \widetilde{A}_q$. We use $K_{q^{1/2}}$ to denote

$$\dim_{\widetilde{A}_q} \widetilde{\mathcal{O}_q(SL_2)} = \dim_{\mathcal{S}_{q^{1/2}}(D_2)^{(N)}} \widetilde{\mathcal{S}_{q^{1/2}}(D_2)}.$$

Clearly we have $K_{q^{1/2}} \leq \lambda_{q^{1/2}}$.

Remark 5.5.14. $K_{q^{1/2}}$ is actually the rank of $\mathcal{O}_q(SL_2)$ over $\mathcal{O}(SL_2)$. In [BG12] chapter 3, there is a general calculation for this rank for all semisimple Lie groups (our case is SL_2). In this subsection, we will give an elementary way to calculate $K_{q^{1/2}}$, and give a clear basis for $\widetilde{\mathcal{O}_q(SL_2)}$ over \widetilde{A}_q .

Since the map from $\mathcal{O}_q(SL_2)$ to $\widetilde{\mathcal{O}_q(SL_2)}$, given by $x \mapsto x \otimes 1$, for $x \in \mathcal{O}_q(SL_2)$, is injective, we can regard any element in $\mathcal{O}_q(SL_2)$ as an element in $\widetilde{\mathcal{O}_q(SL_2)}$ by this embedding.

Lemma 5.5.15. Suppose $\alpha_1, \dots, \alpha_n \in \mathcal{O}_q(SL_2)$ are linearly independent over A_q . When we regard all α_i as elements in $\widetilde{\mathcal{O}_q(SL_2)}$, $\alpha_1, \dots, \alpha_n$ are linearly independent over \widetilde{A}_q . Especially we have $\lambda_{q^{1/2}} \geq K_{q^{1/2}} \geq n$.

We define $\Lambda = \{(k_1, k_2, k_3, k_4) \mid k_1, k_2, k_3, k_4 \in \mathbb{N}, k_1 k_2 = 0\}$. For any

$$\vec{k} = (k_1, k_2, k_3, k_4) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N},$$

we define $O_{\vec{k}}$ to be $a^{k_1} d^{k_2} b^{k_3} c^{k_4} \in \mathcal{O}_q(SL_2)$.

Lemma 5.5.16. ([Gav07; Wan23c]) The set $\{O_{\vec{k}} \mid \vec{k} \in \Lambda\}$ is a basis for $\mathcal{O}_q(SL_2)$, and the set $\{O_{N\vec{k}} \mid \vec{k} \in \Lambda\}$ is a basis for A_q .

For any positive integer t , define a subset of $\mathcal{O}_q(SL_2)$ as

$$E_t = \{k_t b^t c^t + \cdots + k_1 bc + 1 \mid k_t = q^s \text{ for some integer } s\}.$$

We define $E_0 = \{1\}$. Then for any $l, t \in \mathbb{N}$, we have $E_l E_t \subset E_{l+t}$.

Lemma 5.5.17. *For any $t \in \mathbb{N}$, define $dE_t = \{df \mid f \in E_t\}$, $E_t d = \{fd \mid f \in E_t\}$. Then we have $dE_t = E_t d$. Similarly, we also have $aE_t = E_t a$.*

Proof. Note that $d(bc) = q^4(bc)d$, which obviously shows the Lemma. □

Lemma 5.5.18. *For any $t \in \mathbb{N}$, we have $a^t d^t \in E_t$.*

Proof. We prove this Lemma by using mathematical induction on t . Note that $ad = q^{-2}bc + 1$, then the Lemma is obviously true when $t = 0, 1$. We suppose $a^t d^t = f \in E_t$, then we want to show $a^{t+1} d^{t+1} \in E_{t+1}$. We have

$$a^{t+1} d^{t+1} = afd = adg = (q^{-2}bc + 1)g \in E_{t+1}$$

where $g \in E_t$. □

We use \leq_O to denote the lexicographic order on $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Then for any nonzero element $x \in \mathcal{O}_q(SL_2)$, we have $x = \sum_{\vec{k} \in \Lambda_x} l_{\vec{k}} O_{\vec{k}}$ where Λ_x is a finite subset of Λ and $l_{\vec{k}} \in \mathbb{C}^*$ for all $\vec{k} \in \Lambda_x$. Then we define $\deg(x) = \max(\Lambda_x)$ where $\max(\Lambda_x)$ is the maximal element in Λ_x under the linear order \leq_O . Then we have the following two Lemmas.

Lemma 5.5.19. *Let x_1, \dots, x_k be k ($k > 0$) nonzero elements in $\mathcal{O}_q(SL_2)$. Suppose $\deg(x_1), \dots, \deg(x_k)$ are k distinct elements in Λ , then $x_1 + \cdots + x_k \neq 0$.*

Proof. It is obvious. □

We define a function $\varphi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \Lambda$ by

$$\varphi(\vec{k}) = \begin{cases} (0, 0, k_3 + k_1, k_4 + k_2) & k_1 = k_2 \\ (k_1 - k_2, 0, k_3 + k_2, k_4 + k_2) & k_1 > k_2 \\ (0, k_2 - k_1, k_3 + k_1, k_4 + k_1) & k_1 < k_2 \end{cases}$$

where $\vec{k} = (k_1, k_2, k_3, k_4) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

Lemma 5.5.20. For any $\vec{k} \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, we have $\deg(O_{\vec{k}}) = \varphi(\vec{k})$.

Proof. Let $\vec{k} = (k_1, k_2, k_3, k_4) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

(1) Suppose $k_1 = k_2$. From Lemma 5.5.18 we have

$$O_{\vec{k}} = a^{k_1} d^{k_1} b^{k_3} c^{k_4} = f b^{k_3} c^{k_4}$$

where $f \in E_{k_1}$. Thus $\deg(O_{\vec{k}}) = \deg(f b^{k_3} c^{k_4}) = (0, 0, k_3 + k_1, k_4 + k_1) = \varphi(\vec{k})$.

(2) Suppose $k_1 > k_2$. From Lemma 5.5.18 we have

$$O_{\vec{k}} = a^{k_1} d^{k_2} b^{k_3} c^{k_4} = a^{k_1 - k_2} h b^{k_3} c^{k_4}$$

where $h \in E_{k_2}$. Thus $\deg(O_{\vec{k}}) = \deg(a^{k_1 - k_2} h b^{k_3} c^{k_4}) = (k_1 - k_2, 0, k_3 + k_2, k_4 + k_2) = \varphi(\vec{k})$.

(3) Suppose $k_1 < k_2$. From Lemma 5.5.18 we have

$$O_{\vec{k}} = a^{k_1} d^{k_2} b^{k_3} c^{k_4} = x d^{k_2 - k_1} b^{k_3} c^{k_4}$$

where $x \in E_{k_1}$. From Lemma 5.5.17, we know there exists $y \in E_{k_1}$ such that $x d^{k_2 - k_1} = d^{k_2 - k_1} y$. Then $\deg(O_{\vec{k}}) = \deg(d^{k_2 - k_1} y b^{k_3} c^{k_4}) = (0, k_2 - k_1, k_3 + k_1, k_4 + k_1) = \varphi(\vec{k})$. \square

There is an obvious partition for Λ , defined by $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$, where

$$\begin{aligned} \Lambda_0 &= \{(0, 0, k_3, k_4) \mid k_3, k_4 \in \mathbb{N}\} \\ \Lambda_1 &= \{(k_1, 0, k_3, k_4) \mid k_1 \in \mathbb{N}^*, k_3, k_4 \in \mathbb{N}\} \\ \Lambda_2 &= \{(0, k_2, k_3, k_4) \mid k_2 \in \mathbb{N}^*, k_3, k_4 \in \mathbb{N}\}. \end{aligned}$$

Then for any $\vec{k} \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, we have $\varphi(\vec{k}) \in \Lambda_0$ if and only if $k_1 = k_2$, $\varphi(\vec{k}) \in \Lambda_1$ if and only if $k_1 > k_2$, $\varphi(\vec{k}) \in \Lambda_2$ if and only if $k_1 < k_2$.

Lemma 5.5.21. For any $\vec{u}, \vec{v} \in \Lambda$, if $u_1 - u_2 = v_1 - v_2$, then we have $u_1 = v_1$ and $u_2 = v_2$.

Proof. Suppose $\vec{u} \in \Lambda_i, \vec{v} \in \Lambda_j$ where $i, j = 0, 1, 2$. Note that if $i \neq j$, we cannot have $u_1 - u_2 = v_1 - v_2$, thus $i = j$. Then trivially, $u_1 - u_2 = v_1 - v_2$ shows $u_1 = v_1$ and $u_2 = v_2$. \square

Define a subset \mathcal{D} of Λ by

$$\mathcal{D} = \{(0, k_2, k_3, k_4) \in \Lambda \mid 0 \leq k_2, k_3, k_4 \leq N - 1\}.$$

Define a map $\psi : \Lambda \times \mathcal{D} \rightarrow \Lambda$ by $\psi(\vec{u}, \vec{v}) = \varphi(N\vec{u} + \vec{v})$ where $\vec{u} \in \Lambda, \vec{v} \in \mathcal{D}$. Then we have the following Lemma.

Lemma 5.5.22. *The above map ψ is injective.*

Proof. Assume $\psi(\vec{u}, \vec{v}) = \psi(\vec{m}, \vec{n})$, then we want to show $\vec{u} = \vec{m}$ and $\vec{v} = \vec{n}$.

(1) Suppose $\psi(\vec{u}, \vec{v}) = \psi(\vec{m}, \vec{n}) \in \Lambda_0$. Then we have $Nu_1 + v_1 = Nu_2 + v_2$ and $Nm_1 + n_1 = Nm_2 + n_2$. Since $\vec{v} \in \mathcal{D}$, we have $v_1 = 0$, then $N(u_1 - u_2) = v_2$, that is $N \mid v_2$. We also have $0 \leq v_2 \leq N - 1$, then $v_2 = 0$. Then we get $u_1 = u_2$. Since $\vec{u} \in \Lambda$, we have $u_1 = u_2 = 0$. Similar, we can get $m_1 = m_2 = n_1 = n_2 = 0$.

Then we have

$$\psi(\vec{u}, \vec{v}) = (0, 0, Nu_3 + v_3, Nu_4 + v_4) = \psi(\vec{m}, \vec{n}) = (0, 0, Nm_3 + n_3, Nm_4 + n_4).$$

For each $i = 3, 4$, we have $Nu_i + v_i = Nm_i + n_i$. Since $0 \leq v_i, n_i \leq N - 1$, we have $u_i = m_i, v_i = n_i$. Thus we get $\vec{u} = \vec{m}$ and $\vec{v} = \vec{n}$.

(2) Suppose $\psi(\vec{u}, \vec{v}) = \psi(\vec{m}, \vec{n}) \in \Lambda_1$. Then we have $Nu_1 + v_1 = Nu_1 > Nu_2 + v_2$ and $Nm_1 + n_1 = Nm_1 > Nm_2 + n_2$. From the definition of ψ , we know

$$\begin{aligned} \psi(\vec{u}, \vec{v}) &= (N(u_1 - u_2) - v_2, 0, Nu_3 + v_3 + Nu_2 + v_2, \\ &\quad Nu_4 + v_4 + Nu_2 + v_2) \\ \psi(\vec{m}, \vec{n}) &= (N(m_1 - m_2) - n_2, 0, Nm_3 + n_3 + Nm_2 + n_2, \\ &\quad Nm_4 + n_4 + Nm_2 + n_2). \end{aligned}$$

From $\psi(\vec{u}, \vec{v}) = \psi(\vec{m}, \vec{n})$, we get $N(u_1 - u_2) - v_2 = N(m_1 - m_2) - n_2$, then $v_2 = n_2$ and $u_1 - u_2 = m_1 - m_2$ since $0 \leq v_2, n_2 \leq N - 1$. Then Lemma 5.5.21 shows $u_1 = m_1$ and $u_2 = m_2$. From $\psi(\vec{u}, \vec{v}) = \psi(\vec{m}, \vec{n})$, we can also get $Nu_i + v_i = Nm_i + n_i$ for

each $i = 3, 4$. Similarly as above we can have $u_i = m_i, v_i = n_i$ for each $i = 3, 4$. Thus we get $\vec{u} = \vec{m}$ and $\vec{v} = \vec{n}$.

(3) Suppose $\psi(\vec{u}, \vec{v}) = \psi(\vec{m}, \vec{n}) \in \Lambda_2$. The proving technique is similar with case (2).

□

Lemma 5.5.23. *We have $\{O_{\vec{k}} \mid \vec{k} \in \mathcal{D}\}$ is linearly independent over A_q .*

Proof. It surfices to show

$$\sum_{\vec{k} \in \mathcal{D}'} \alpha_{\vec{k}} O_{\vec{k}} \neq 0$$

where \mathcal{D}' is a nonempty finite subset of \mathcal{D} and $\alpha_{\vec{k}}$ is a nonzero element in A_q for each $\vec{k} \in \mathcal{D}'$.

For each $\vec{k} \in \mathcal{D}'$, from Lemma 5.5.16 we can suppose $\alpha_{\vec{k}} = \sum_{\vec{v} \in \Lambda_{\vec{k}}} l_{\vec{k}, \vec{v}} O_{N\vec{v}}$ where $\Lambda_{\vec{k}}$ is a nonempty finite subset of Λ and $l_{\vec{k}, \vec{v}} \neq 0$ for each $\vec{v} \in \Lambda_{\vec{k}}$. Then we have

$$\sum_{\vec{k} \in \mathcal{D}'} \alpha_{\vec{k}} O_{\vec{k}} = \sum_{\vec{k} \in \mathcal{D}'} \sum_{\vec{v} \in \Lambda_{\vec{k}}} l_{\vec{k}, \vec{v}} O_{N\vec{v}} O_{\vec{k}} = \sum_{\vec{k} \in \mathcal{D}'} \sum_{\vec{v} \in \Lambda_{\vec{k}}} l_{\vec{k}, \vec{v}} O_{N\vec{v} + \vec{k}}.$$

From Lemma 5.5.20, we know $\deg(O_{N\vec{v} + \vec{k}}) = \varphi(N\vec{v} + \vec{k}) = \psi(\vec{v}, \vec{k})$. From Lemma 5.5.22, we know $\deg(O_{N\vec{v} + \vec{k}}), \vec{k} \in \mathcal{D}', \vec{v} \in \Lambda_{\vec{k}}$, are distinct. Then from Lemma 5.5.19, we get

$$\sum_{\vec{k} \in \mathcal{D}'} \alpha_{\vec{k}} O_{\vec{k}} = \sum_{\vec{k} \in \mathcal{D}'} \sum_{\vec{v} \in \Lambda_{\vec{k}}} l_{\vec{k}, \vec{v}} O_{N\vec{v} + \vec{k}} \neq 0.$$

□

Corollary 5.5.24. *We have $\lambda_{q^{1/2}} \geq K_{q^{1/2}} \geq N^3$.*

Proof. Lemmas 5.5.15, 5.5.23 and the fact that $|\mathcal{D}| = N^3$.

□

Theorem 5.5.25. *When we regard $\{O_{\vec{k}} \mid \vec{k} \in \mathcal{D}\}$ as a subset of $\widetilde{\mathcal{O}}_q(SL_2)$, it is a basis of $\widetilde{\mathcal{O}}_q(SL_2)$ over \widetilde{A}_q . Especially we have $K_{q^{1/2}} = N^3$.*

Proof. Lemmas 5.5.15 and 5.5.23 show $\{O_{\vec{k}} \mid \vec{k} \in \mathcal{D}\}$ are linearly independent over \widetilde{A}_q . Let V be the linear span of $\{O_{\vec{k}} \mid \vec{k} \in \mathcal{D}\}$ over \widetilde{A}_q . Then it suffices to show $V = \widetilde{\mathcal{O}}_q(SL_2)$.

For any $k_1, k_2, k_3 \in \mathbb{N}$, clearly we have $d^{k_1} b^{k_2} c^{k_3} \in V$. Then we want to show for any $k_1, k_2, k_3, k_4 \in \mathbb{N}$ we have $a^{k_1} d^{k_2} b^{k_3} c^{k_4} \in V$. Suppose $k_1 = uN + v$ where $u, v \in \mathbb{N}$ and $0 \leq v \leq N - 1$. Then we have

$$\begin{aligned} d^N(a^{k_1} d^{k_2} b^{k_3} c^{k_4}) &= a^{uN} a^v d^N d^{k_2} b^{k_3} c^{k_4} = a^{uN} a^v d^v d^{N-v} d^{k_2} b^{k_3} c^{k_4} \\ &= a^{uN} f d^{N+k_2-v} b^{k_3} c^{k_4} = a^{uN} d^{N+k_2-v} h b^{k_3} c^{k_4} \in V. \end{aligned}$$

where $f, h \in E_v$. Since d^N is invertible, we have $a^{k_1} d^{k_2} b^{k_3} c^{k_4} \in V$. \square

Remark 5.5.26. During the proof of Theorem 5.5.25, we only used the invertibility of d^N . The same technique can show $\{a^{k_1} b^{k_2} c^{k_3} \mid 0 \leq k_1, k_2, k_3 \leq N - 1\}$ span $\widetilde{\mathcal{O}_q(SL_2)}$ by using the invertibility of a^N . The representation-reduced stated skein module of D_2 is defined by a matrix $\begin{pmatrix} x_a & x_b \\ x_c & x_d \end{pmatrix} \in SL_2(\mathbb{C})$. We regard $\begin{pmatrix} x_a & x_b \\ x_c & x_d \end{pmatrix} \in$

$SL_2(\mathbb{C})$ as an algebra homomorphism from A_q to \mathbb{C} such that it maps $\begin{pmatrix} a^N & b^N \\ c^N & d^N \end{pmatrix}$ to

$\begin{pmatrix} x_a & x_b \\ x_c & x_d \end{pmatrix}$. Then the above discussion shows $\{a^{k_1} b^{k_2} c^{k_3} \mid 0 \leq k_1, k_2, k_3 \leq N - 1\}$ (respectively $\{d^{k_1} b^{k_2} c^{k_3} \mid 0 \leq k_1, k_2, k_3 \leq N - 1\}$) spans the corresponding representation-reduced stated skein module of D_2 when $x_a \neq 0$ (respectively $x_d \neq 0$).

We will try to calculate the dimension for representation-reduced stated skein module in future work.

Define

$$\mathcal{B} = \{(N - k_1, 0, k_2, k_3) \mid 1 \leq k_1 \leq N - 1, 0 \leq k_2, k_3 \leq N - 1, k_2 < k_1 \text{ or } k_3 < k_1\} \subset \Lambda$$

Lemma 5.5.27. We have $\mathcal{O}_q(SL_2)$ is linearly spanned by $\{O_{\vec{k}} \mid \vec{k} \in \mathcal{D} \cup \mathcal{B}\}$ over A_q . Especially $\lambda_{q^{1/2}} \leq |\mathcal{D} \cup \mathcal{B}| = 2N^3 - \frac{N(N+1)(2N+1)}{6}$.

Proof. Let U be the linear span of $\{O_{\vec{k}} \mid \vec{k} \in \mathcal{D} \cup \mathcal{B}\}$ over A_q . Then it suffices to show $a^{N-k_1} b^{k_2} c^{k_3} \in U$ for any $1 \leq k_1 \leq N - 1, 0 \leq k_2, k_3 \leq N - 1$. From the definition of \mathcal{B} , we know $a^{N-k_1} b^{k_2} c^{k_3} \in U$ if $k_2 < k_1$ or $k_3 < k_1$. Then we suppose $k_2 \geq k_1$ and $k_3 \geq k_1$, and we use k to denote $k_2 - k_1 + k_3 - k_1$. Then we will prove this case by using mathematical induction on k . From Lemma 5.5.18, we have

$$a^N b^{k_1} = a^{N-k_1} (l_{k_1} b^{k_1} c^{k_1} + l_{k_1-1} b^{k_1-1} c^{k_1-1} + \dots + 1)$$

where l_{k_1} is a nonzero complex number. Then we have

$$a^{N-k_1}b^{k_1}c^{k_1} = s_{k_1}a^N b^{k_1} + s_{k_1-1}a^{N-k_1}b^{k_1-1}c^{k_1-1} + \dots + s_0a^{N-k_1} \in U. \quad (5.14)$$

Equation (5.14) shows $a^{N-k_1}b^{k_2}c^{k_3} \in U$ when $k = 0$. Then suppose $a^{N-k_1}b^{k_2}c^{k_3} \in U$ when $k = k_2 - k_1 + k_3 - k_1 \leq m$ ($m \in \mathbb{N}$), then we look at the case when $k = m + 1$. We left multiply equation (5.14) on both sides by $b^{k_2-k_1}c^{k_3-k_1}$, we get

$$a^{N-k_1}b^{k_2}c^{k_3} = s_{k_1}a^N b^{k_2}c^{k_3-k_1} + s_{k_1-1}a^{N-k_1}b^{k_2-1}c^{k_3-1} + \dots + s_0a^{N-k_1}b^{k_2-k_1}c^{k_3-k_1} \in U.$$

□

5.5.3 When (M, \mathcal{N}) is the thickening of a pb surface

Let \mathfrak{S} be a pb surface. Recall that $r(\mathfrak{S}) = -\chi(\mathfrak{S}) + \sharp\partial\mathfrak{S}$, where $\chi(\mathfrak{S})$ is the Euler characteristic of \mathfrak{S} and $\sharp\partial\mathfrak{S}$ is the number of boundary components of \mathfrak{S} . Note that when $\partial\mathfrak{S} = \emptyset$, we have $r(\mathfrak{S}) = -\chi(\mathfrak{S}) = 2g - 2 + p$ where g is the genus and p is the number of punctures. Theorem 7.13 in [LS21] shows the following Lemma.

Lemma 5.5.28. *Let \mathfrak{S} be a pb surface with nonempty boundary. Then we have a linear isomorphism*

$$F : \mathcal{S}_{q^{1/2}}(D_2)^{\otimes r(\mathfrak{S})} \rightarrow \mathcal{S}_{q^{1/2}}(\mathfrak{S})$$

such that F restricts to an algebra isomorphism

$$F : (\mathcal{S}_{q^{1/2}}(D_2)^{(N)})^{\otimes r(\mathfrak{S})} \rightarrow (\mathcal{S}_{q^{1/2}}(\mathfrak{S}))^{(N)},$$

and F preserves module structures, that is

$$F(\alpha \cdot \beta) = F(\alpha) \cdot F(\beta)$$

for any $\alpha \in (\mathcal{S}_{q^{1/2}}(D_2)^{(N)})^{\otimes r(\mathfrak{S})}$, $\beta \in \mathcal{S}_{q^{1/2}}(D_2)^{\otimes r(\mathfrak{S})}$. Especially we have

$$\dim_{\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}} \mathcal{S}_{q^{1/2}}(\mathfrak{S}) = \dim_{(\mathcal{S}_{q^{1/2}}(D_2)^{(N)})^{\otimes r(\mathfrak{S})}} \mathcal{S}_{q^{1/2}}(D_2)^{\otimes r(\mathfrak{S})} = \dim_{(A_q)^{\otimes r(\mathfrak{S})}} \mathcal{O}_q(SL_2)^{\otimes r(\mathfrak{S})}.$$

Lemma 5.5.29. *For any positive integer k , $\mathcal{O}_q(SL_2)^{\otimes k}$ contains N^{3k} elements which are linearly independent over $(A_q)^{\otimes k}$.*

Proof. From Lemma 5.5.23, we know there exist N^3 elements $x_1, \dots, x_{N^3} \in \mathcal{O}_q(SL_2)$ which are linearly independent over A_q . Then we have $\bigoplus_{1 \leq i \leq N^3} A_q x_i \subset \mathcal{O}_q(SL_2)$. Thus we have

$$\begin{aligned} & (\bigoplus_{1 \leq i \leq N^3} A_q x_i) \otimes \cdots \otimes (\bigoplus_{1 \leq i \leq N^3} A_q x_i) = \bigoplus_{1 \leq i_1, \dots, i_k \leq N^3} (A_q x_{i_1} \otimes \cdots \otimes A_q x_{i_k}) \\ & = \bigoplus_{1 \leq i_1, \dots, i_k \leq N^3} ((A_q)^{\otimes k} x_{i_1} \otimes \cdots \otimes x_{i_k}) \subset \mathcal{O}_q(SL_2)^{\otimes k}. \end{aligned}$$

This shows that $\{x_{i_1} \otimes \cdots \otimes x_{i_k} \mid 1 \leq i_1, \dots, i_k \leq N^3\}$ are linearly independent over $(A_q)^{\otimes k}$. \square

Recall that $\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}^{(N)}$ is the field of fractions of $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}$,

$$\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})} = \mathcal{S}_{q^{1/2}}(\mathfrak{S}) \otimes_{\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}} \widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}^{(N)},$$

$K_{\mathfrak{S}, q^{1/2}} = \dim_{\widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}^{(N)}} \widetilde{\mathcal{S}_{q^{1/2}}(\mathfrak{S})}$, and $\lambda_{\mathfrak{S}, q^{1/2}} = \dim_{\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}} \mathcal{S}_{q^{1/2}}(\mathfrak{S})$.

Lemma 5.5.30. *Let \mathfrak{S} be a pb surface, we require $\chi(\mathfrak{S})$ is negative if $\partial\mathfrak{S} = \emptyset$. Then there exist $N^{3r(\mathfrak{S})}$ elements in $\mathcal{S}_{q^{1/2}}(\mathfrak{S})$, which are linearly independent over $\mathcal{S}_{q^{1/2}}(\mathfrak{S})^{(N)}$. Especially we have*

$$N^{3r(\mathfrak{S})} \leq K_{\mathfrak{S}, q^{1/2}} \leq \lambda_{\mathfrak{S}, q^{1/2}}.$$

Proof. Lemma 5.5.1 shows Lemma 5.5.30 when $\partial\mathfrak{S} = \emptyset$. When $\partial\mathfrak{S} \neq \emptyset$, Lemmas 5.5.28 and 5.5.29 show Lemma 5.5.30. \square

Theorem 5.5.31. *Suppose we have the (R1) assumption. Let \mathfrak{S} be a pb surface with nonempty boundary. Then we have*

$$N^{3r(\mathfrak{S})} \leq \lambda_{\mathfrak{S}, q^{1/2}} \leq \left(2N^3 - \frac{N(N+1)(2N+1)}{6}\right)^{r(\mathfrak{S})},$$

where $\lambda_{\mathfrak{S}, q^{1/2}}$ is defined in (5.12).

Proof. Lemmas 5.5.27, 5.5.28 and 5.5.30. \square

Corollary 5.5.32. *Let \mathfrak{S} be a pb surface with nonempty boundary. Then we have*

$$\lim_{N \text{ odd} \rightarrow +\infty} \frac{\log \lambda_{\mathfrak{S}, q^{1/2}}}{\log N} = 3r(\mathfrak{S}).$$

Theorem 5.5.33. *Suppose we have the (R1) assumption. Let \mathfrak{S} be a pb surface, we require $\chi(\mathfrak{S})$ is negative if $\partial\mathfrak{S} = \emptyset$. Then we have $K_{\mathfrak{S},q^{1/2}} = N^{3r(\mathfrak{S})}$, where $K_{\mathfrak{S},q^{1/2}}$ is defined in (5.11).*

Proof. When $\partial = \emptyset$, Theorem 5.5.33 was proved in Theorem 5.5.3.

Then assume $\partial\mathfrak{S} \neq \emptyset$. From Lemma 5.5.30, it suffices to show $K_{\mathfrak{S},q^{1/2}} \leq N^{3r(\mathfrak{S})}$. We use $\widetilde{(A_q)^{\otimes r(\mathfrak{S})}}$ to denote the field of fractions of the commutative domain $(A_q)^{\otimes r(\mathfrak{S})}$, and use $\widetilde{\mathcal{O}_q(SL_2)^{\otimes r(\mathfrak{S})}}$ to denote $\mathcal{O}_q(SL_2)^{\otimes r(\mathfrak{S})} \otimes_{(A_q)^{\otimes r(\mathfrak{S})}} \widetilde{(A_q)^{\otimes r(\mathfrak{S})}}$. Then from Lemma 5.5.28, it suffices to show $\dim_{\widetilde{(A_q)^{\otimes r(\mathfrak{S})}}} \widetilde{\mathcal{O}_q(SL_2)^{\otimes r(\mathfrak{S})}} \leq N^{3r(\mathfrak{S})}$. To simplify notation, we will regard elements in $\mathcal{O}_q(SL_2)^{\otimes r(\mathfrak{S})}$ as elements in $\widetilde{\mathcal{O}_q(SL_2)^{\otimes r(\mathfrak{S})}}$ via the obvious embedding from $\mathcal{O}_q(SL_2)^{\otimes r(\mathfrak{S})}$ to $\widetilde{\mathcal{O}_q(SL_2)^{\otimes r(\mathfrak{S})}}$.

We use x_1, \dots, x_{N^3} to denote the basis elements in Theorem 5.5.25, then x_1, \dots, x_{N^3} are actually elements in $\mathcal{O}_q(SL_2)$. Let V be the sub-vector space of $\widetilde{\mathcal{O}_q(SL_2)^{\otimes r(\mathfrak{S})}}$ linearly spanned by $\{x_{i_1} \otimes \dots \otimes x_{i_{r(\mathfrak{S})}} \mid 1 \leq i_1, \dots, i_{r(\mathfrak{S})} \leq N^3\}$. Let $y_1, \dots, y_{r(\mathfrak{S})} \in \mathcal{O}_q(SL_2)$ be $r(\mathfrak{S})$ elements in $\mathcal{O}_q(SL_2)$. For each $1 \leq i \leq r(\mathfrak{S})$, from Theorem 5.5.25 there exist $u_i \in A_q - \{0\}$ and $v_{i,1}, \dots, v_{i,N^3} \in A_q$ such that $u_i y_i = \sum_{1 \leq j \leq N^3} v_{i,j} x_j$. Then we have

$$\begin{aligned} & (u_1 \otimes \dots \otimes u_{r(\mathfrak{S})})(y_1 \otimes \dots \otimes y_{r(\mathfrak{S})}) = u_1 y_1 \otimes \dots \otimes u_{r(\mathfrak{S})} y_{r(\mathfrak{S})} \\ &= \sum_{1 \leq j_1, \dots, j_{r(\mathfrak{S})} \leq N^3} v_{1,j_1} x_{j_1} \otimes \dots \otimes v_{r(\mathfrak{S}),j_{r(\mathfrak{S})}} x_{j_{r(\mathfrak{S})}} \\ &= \sum_{1 \leq j_1, \dots, j_{r(\mathfrak{S})} \leq N^3} (v_{1,j_1} \otimes \dots \otimes v_{r(\mathfrak{S}),j_{r(\mathfrak{S})}})(x_{j_1} \otimes \dots \otimes x_{j_{r(\mathfrak{S})}}) \end{aligned}$$

where $u_1 \otimes \dots \otimes u_{r(\mathfrak{S})} \in (A_q)^{\otimes r(\mathfrak{S})} - \{0\}$ and $v_{1,j_1} \otimes \dots \otimes v_{r(\mathfrak{S}),j_{r(\mathfrak{S})}} \in (A_q)^{\otimes r(\mathfrak{S})}$. Thus $y_1 \otimes \dots \otimes y_{r(\mathfrak{S})} \in V$. Then clearly we have $V = \widetilde{\mathcal{O}_q(SL_2)^{\otimes r(\mathfrak{S})}}$. This completes the proof since the cardinality of $\{x_{i_1} \otimes \dots \otimes x_{i_{r(\mathfrak{S})}} \mid 1 \leq i_1, \dots, i_{r(\mathfrak{S})} \leq N^3\}$ is $N^{3r(\mathfrak{S})}$. \square

Remark 5.5.34. *When $\partial\mathfrak{S} \neq \emptyset$, obviously we have*

$$\{x_{i_1} \otimes \dots \otimes x_{i_{r(\mathfrak{S})}} \mid 1 \leq i_1, \dots, i_{r(\mathfrak{S})} \leq N^3\}$$

in the proof for Theorem 5.5.33 is actually a basis.

Remark 5.5.35. *For general stated SL_n -skein algebra, the Frobenius map was built when $\partial\mathfrak{S} \neq \emptyset$ [Wan23c]. From Theorem 7.3 in chapter 3 in [BG12] for SL_n ,*

and a general statement for Lemma 6.28 for SL_n , using the same techniques we can generalize Theorem 5.5.33 to SL_n , and the corresponding dimension is $N^{(n^2-1)r(\mathfrak{S})}$.

5.5.4 When the marking set is nonempty

For a compact marked 3-manifold (M, \mathcal{N}) , we will give an upper bound for

$$\dim_{\mathcal{S}_1(M, \mathcal{N})} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}),$$

in this subsection. From Theorem 5.3.11, this offers an upper bound for the dimension of the representation-reduced stated skein module.

Theorem 5.5.36. *Suppose we have the (R1) assumption. Let (M, \mathcal{N}) be a compact marked 3-manifold with $\mathcal{N} \neq \emptyset$. Suppose the genus of (M, \mathcal{N}) is g , and \mathcal{N} has k components. Then we have*

$$\dim_{\mathcal{S}_1(M, \mathcal{N})} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \leq \left(2N^3 - \frac{N(N+1)(2N+1)}{6}\right)^{2g+k-1}.$$

Proof. Remark 5.3.10, Theorem 5.5.31. □

Theorem 5.5.37. *Suppose we have the (R1) assumption. Let (M, \mathcal{N}) be a compact marked 3-manifold with $\mathcal{N} \neq \emptyset$. Suppose the genus of (M, \mathcal{N}) is g , and \mathcal{N} has k components. Then we have the dimension of any representation-reduced stated skein module of (M, \mathcal{N}) is not more than*

$$\left(2N^3 - \frac{N(N+1)(2N+1)}{6}\right)^{2g+k-1}.$$

Chapter 6

Representation-reduced stated skein modules and algebras

This chapter is based on the author's work in [Wan25b]. We introduce the splitting map for the representation-reduced stated skein module, see section 6.2, and prove its injectivity in certain special cases (Thm. 6.2.8). Furthermore, we show that the representation-reduced stated skein module of a handlebody is the irreducible representation of the stated skein algebra of its boundary (Thm. 6.3.1). Finally, we establish that the dimension of the representation-reduced skein module for closed 3-manifolds is one (Thm. 6.3.3).

Let $(\Sigma_1, \mathcal{P}_1)$ and $(\Sigma_2, \mathcal{P}_2)$ be two marked surfaces. An embedding $f : \Sigma_1 \rightarrow \Sigma_2$ is called an embedding from $(\Sigma_1, \mathcal{P}_1)$ to $(\Sigma_2, \mathcal{P}_2)$ if $f(\mathcal{P}_1) \subset \mathcal{P}_2$. Clearly f induces an algebra homomorphism $f_* : \mathcal{S}_{q^{1/2}}(\Sigma_1, \mathcal{P}_1) \rightarrow \mathcal{S}_{q^{1/2}}(\Sigma_2, \mathcal{P}_2)$. Similarly as the marked 3-manifold, we define two marked surfaces are isomorphic to each other, if there exist two embeddings respectively from $(\Sigma_1, \mathcal{P}_1)$ to $(\Sigma_2, \mathcal{P}_2)$ and from $(\Sigma_2, \mathcal{P}_2)$ to $(\Sigma_1, \mathcal{P}_1)$ such that they are inverse to each other.

In the remainder of this chapter, we will always assume $q^{\frac{1}{2}}$ is a root of unity of odd order N .

6.1 Representation-reduced stated skein modules and algebras

In this section, we will recall the representation-reduced stated skein module introduced in subsection 5.3.5. Then we will introduce the representation-reduced stated skein algebra of the pb surface. We will discuss the functorality of the representation-reduced stated skein module.

Let (M, \mathcal{N}) be a marked 3-manifold. Recall that we use $\text{MaxSpec}(\mathcal{S}_1(M, \mathcal{N}))$ to denote the set of all maximal ideals of $\mathcal{S}_1(M, \mathcal{N})$. We also can regard $\text{MaxSpec}(\mathcal{S}_1(M, \mathcal{N}))$ as the set of algebra homomorphisms from $\mathcal{S}_1(M, \mathcal{N})$ to \mathbb{C} .

For any element $\rho \in \text{MaxSpec}(\mathcal{S}_1(M, \mathcal{N}))$, the commutative algebra $\mathcal{S}_1(M, \mathcal{N})$ has an action on \mathbb{C} induced by ρ , that is, for $x \in \mathcal{S}_1(M, \mathcal{N}), k \in \mathbb{C}, x \cdot k = \rho(x)k$. Recall that $\mathcal{S}_1(M, \mathcal{N})$ also has an action on $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$. Then we define

$$\mathcal{S}_{q^{1/2}}(M, \mathcal{N})_\rho = \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \otimes_{\mathcal{S}_1(M, \mathcal{N})}^\rho \mathbb{C} \simeq \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) / \text{Ker}(\rho) \cdot \mathcal{S}_{q^{1/2}}(M, \mathcal{N}),$$

where the superscript for $\otimes_{\mathcal{S}_1(M, \mathcal{N})}^\rho$ is to imply the action of $\mathcal{S}_1(M, \mathcal{N})$ on \mathbb{C} is induced by ρ . We call $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})_\rho$ the representation-reduced stated skein module of (M, \mathcal{N}) related to ρ , or just, the representation-reduced stated skein module when (M, \mathcal{N}) and ρ are clear.

When the marked 3-manifold (M, \mathcal{N}) is the thickening of a marked surface (Σ, \mathcal{P}) and $\rho \in \text{MaxSpec}(\mathcal{S}_1(\Sigma, \mathcal{P}))$, we use $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})_\rho$ to denote $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})_\rho$. Then $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})_\rho$ is a quotient algebra of $\mathcal{S}_{q^{1/2}}(\Sigma, \mathcal{P})$.

Definition 6.1.1. *Let (M_1, \mathcal{N}_1) and (M_2, \mathcal{N}_2) be two marked 3-manifolds. Recall that $q^{\frac{1}{2}}$ is a root of unity of odd order N . Suppose for each such a $q^{\frac{1}{2}}$, there exists a linear map $\varphi_{q^{1/2}} : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$. We can omit the subscript for $\varphi_{q^{1/2}}$ when there is no confusion. We say $\varphi : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$ behaves well with respect to the Frobenius map if it satisfies the following conditions:*

(1) *If $q^{1/2} = 1$, then $\varphi : \mathcal{S}_1(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_1(M_2, \mathcal{N}_2)$ is an algebra homomorphism.*

(2) For each $q^{\frac{1}{2}}$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_1(M_1, \mathcal{N}_1) & \xrightarrow{\varphi_1} & \mathcal{S}_1(M_2, \mathcal{N}_2) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) & \xrightarrow{\varphi_{q^{1/2}}} & \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2) \end{array} .$$

(3) For each $q^{\frac{1}{2}}$, we have $\varphi_{q^{1/2}} : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$ preserves module structures, in a sense that, $\varphi_{q^{1/2}}(\alpha \cdot \beta) = \varphi_1(\alpha) \cdot \varphi_{q^{1/2}}(\beta)$ for any $\alpha \in \mathcal{S}_1(M_1, \mathcal{N}_1)$, $\beta \in \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1)$.

Lemma 6.1.2. Suppose $\varphi : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$ behaves well with respect to the Frobenius map. We know the algebra homomorphism $\varphi_1 : \mathcal{S}_1(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_1(M_2, \mathcal{N}_2)$ induces a map: $\varphi^* : \text{MaxSpec}(\mathcal{S}_1(M_2, \mathcal{N}_2)) \rightarrow \text{MaxSpec}(\mathcal{S}_1(M_1, \mathcal{N}_1))$. Then, for each $q^{\frac{1}{2}}$ and $\rho \in \text{MaxSpec}(\mathcal{S}_1(M_2, \mathcal{N}_2))$, the linear map

$$\varphi_{q^{1/2}} : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$$

induces a linear map

$$\varphi_\rho : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1)_{\varphi^*(\rho)} \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)_\rho$$

defined by sending $x \otimes k$ to $\varphi(x) \otimes k$, where $x \in \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1)$ and $k \in \mathbb{C}$.

Proof. For each $q^{\frac{1}{2}}$ and $\rho \in \text{MaxSpec}(\mathcal{S}_1(M_2, \mathcal{N}_2))$, define

$$\overline{\varphi}_\rho : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \times \mathbb{C} \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)_\rho, (x, k) \rightarrow \varphi_{q^{1/2}}(x) \otimes k.$$

Clearly $\overline{\varphi}_\rho$ is bilinear. For any $x \in \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1)$, $y \in \mathcal{S}_1(M_1, \mathcal{N}_1)$ and $k \in \mathbb{C}$, we have

$$\begin{aligned} \overline{\varphi}_\rho(x \cdot y, k) &= \varphi_{q^{1/2}}(x \cdot y) \otimes k = \varphi_{q^{1/2}}(x) \cdot \varphi_1(y) \otimes k \\ &= \varphi_{q^{1/2}}(x) \otimes \varphi_1(y) \cdot k = \varphi_{q^{1/2}}(x) \otimes \rho(\varphi_1(y))k = \overline{\varphi}_\rho(x, y \cdot k). \end{aligned}$$

Thus $\overline{\varphi}_\rho$ induces the linear map $\varphi_\rho : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1)_{\varphi^*(\rho)} \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)_\rho$. \square

We will show the embedding between two marked 3-manifolds induces a linear map between the corresponding representation-reduced stated skein modules, and the representation-reduced stated skein module of two disjoint marked 3-manifolds is isomorphic to the tensor product of their representation-reduced stated skein modules.

Lemma 6.1.3. *Let $f : (M_1, \mathcal{N}_1) \rightarrow (M_2, \mathcal{N}_2)$ be an embedding between two marked 3-manifolds. Then we have $f_* : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$ behaves well with respect to the Frobenius map. Here f_* is the \mathbb{C} -linear map induced by f .*

Proof. Condition (1) is trivial.

Condition (2): For any framed knot or stated framed arc α in (M_1, \mathcal{N}_1) and any non-negative integer k , obviously, we have $f_*(\alpha^{(k)}) = (f_*(\alpha))^{(k)}$. Let $P(x) = \sum_{1 \leq k \leq n} \lambda_k x^k \in \mathbb{C}[x]$, then we have

$$f_*(\alpha^{[P]}) = \sum_{1 \leq k \leq n} \lambda_k f_*(\alpha^{(k)}) = \sum_{1 \leq k \leq n} \lambda_k (f_*(\alpha))^{(k)} = f_*(\alpha)^{[P]}.$$

This shows f_* and \mathcal{F} commute with each other from the definition of \mathcal{F} .

Condition (3): For any disjoint stated (M_1, \mathcal{N}_1) -tangles α, β , we have

$$f_*(\alpha \cdot \beta) = f_*(\mathcal{F}(\alpha) \cup \beta) = f_*(\mathcal{F}(\alpha)) \cup f_*(\beta) = \mathcal{F}(f_*(\alpha)) \cup f_*(\beta) = f_*(\alpha) \cdot f_*(\beta)$$

(here we regard α as an element in $\mathcal{S}_1(M_1, \mathcal{N}_1)$ and regard β as an element in $\mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1)$).

□

For any $\rho \in \text{MaxSpec}(\mathcal{S}_1(M_2, \mathcal{N}_2))$, Lemma 6.1.2 implies that

$$f_* : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$$

induces a linear map

$$f_\rho : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1)_{f^*(\rho)} \rightarrow \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)_\rho.$$

Let (M_1, \mathcal{N}_1) and (M_2, \mathcal{N}_2) be two marked 3-manifolds. Then for each $q^{\frac{1}{2}}$, there exists a linear isomorphism

$$\kappa : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \otimes \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2) \rightarrow \mathcal{S}_{q^{1/2}}((M_1, \mathcal{N}_1) \cup (M_2, \mathcal{N}_2)),$$

defined by $\kappa(\alpha \otimes \beta) = \alpha \cup \beta$, where α is a stated (M_1, \mathcal{N}_1) -tangle and β is a stated (M_2, \mathcal{N}_2) -tangle.

Lemma 6.1.4. *We have*

$$\kappa : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \otimes \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2) \rightarrow \mathcal{S}_{q^{1/2}}((M_1, \mathcal{N}_1) \cup (M_2, \mathcal{N}_2))$$

behaves well with respect to the Frobenius map (the Frobenius map for $\mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \otimes \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)$ is defined by sending $x \otimes y$ to $\mathcal{F}(x) \otimes \mathcal{F}(y)$, where $x \in \mathcal{S}_1(M_1, \mathcal{N}_1)$ and $y \in \mathcal{S}_1(M_2, \mathcal{N}_2)$).

Proof. The proof is trivial. □

It is easy to show that $\text{MaxSpec}(\mathcal{S}_1(M_1, \mathcal{N}_1) \otimes \mathcal{S}_1(M_2, \mathcal{N}_2)) = \text{MaxSpec}(\mathcal{S}_1(M_1, \mathcal{N}_1)) \times \text{MaxSpec}(\mathcal{S}_1(M_2, \mathcal{N}_2))$, that is, any $\rho \in \text{MaxSpec}(\mathcal{S}_1(M_1, \mathcal{N}_1) \otimes \mathcal{S}_1(M_2, \mathcal{N}_2))$ is a pair (ρ_1, ρ_2) , where $\rho_i \in \text{MaxSpec}(\mathcal{S}_1(M_i, \mathcal{N}_i))$ for $i = 1, 2$ (here we regard (ρ_1, ρ_2) as an algebra homomorphism from $\mathcal{S}_1(M_1, \mathcal{N}_1) \otimes \mathcal{S}_1(M_2, \mathcal{N}_2)$ to \mathbb{C} , define by $(\rho_1, \rho_2)(x_1 \otimes x_2) = \rho_1(x_1)\rho_2(x_2)$, where $x_i \in \mathcal{S}_1(M_i, \mathcal{N}_i)$ for $i = 1, 2$).

For any $\rho \in \text{MaxSpec}(\mathcal{S}_1((M_1, \mathcal{N}_1) \cup (M_2, \mathcal{N}_2)))$ and $i = 1, 2$, we use ρ_i to denote the composition $\mathcal{S}_1(M_i, \mathcal{N}_i) \rightarrow \mathcal{S}_1((M_1, \mathcal{N}_1) \cup (M_2, \mathcal{N}_2)) \rightarrow \mathbb{C}$, where the first map is induced by the embedding from (M_i, \mathcal{N}_i) to $(M_1, \mathcal{N}_1) \cup (M_2, \mathcal{N}_2)$ and the second map is ρ . Then $\kappa^*(\rho) = (\rho_1, \rho_2)$.

For any $\rho \in \text{MaxSpec}(\mathcal{S}_1((M_1, \mathcal{N}_1) \cup (M_2, \mathcal{N}_2)))$ with $\kappa^*(\rho) = (\rho_1, \rho_2)$ and any $q^{\frac{1}{2}}$, the linear isomorphism,

$$\kappa : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1) \otimes \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2) \rightarrow \mathcal{S}_{q^{1/2}}((M_1, \mathcal{N}_1) \cup (M_2, \mathcal{N}_2))$$

induces the following linear isomorphism

$$\kappa_\rho : \mathcal{S}_{q^{1/2}}(M_1, \mathcal{N}_1)_{\rho_1} \otimes \mathcal{S}_{q^{1/2}}(M_2, \mathcal{N}_2)_{\rho_2} \rightarrow \mathcal{S}_{q^{1/2}}((M_1, \mathcal{N}_1) \cup (M_2, \mathcal{N}_2))_\rho, \quad (6.1)$$

defined by $\kappa_\rho((x_1 \otimes k_1) \otimes (x_2 \otimes k_2)) = \kappa(x_1 \otimes x_2) \otimes k_1 k_2$, where $k_1, k_2 \in \mathbb{C}$ and $x_i \in \mathcal{S}_{q^{1/2}}(M_i, \mathcal{N}_i)$ for $i = 1, 2$.

6.2 The splitting map for representation-reduced stated skein modules

The existence of the splitting map is a very important property for stated skein modules. Thang Lê constructed the quantum trace map using the splitting map [Lê18] (the quantum trace map was originally constructed in [BW11]). The splitting map also gives the stated skein algebra of the bigon a comultiplication structure (it actually has a Hopf algebra structure) [CL22a].

In this section, we prove the splitting map for stated skein modules induces the splitting map for the representation-reduced stated skein modules. We also show the splitting map for the representation-reduced stated skein modules is injective if there exists at least one component of \mathcal{N} such that this component and the boundary of the splitting disk belong to the same component of ∂M .

6.2.1 The splitting map

Let (M, \mathcal{N}) be a marked 3-manifold, and let (D, u) be a pair, where D , called the splitting disk, is a properly embedded disk in M and u is an embedded oriented open interval in D . Suppose $U(D)$ is an open regular neighborhood of D such that $U(D)$ is isomorphic to $D \times (0, 1)$ and $\partial U(D) = \partial D \times (0, 1)$. Let $M' = M \setminus U(D)$. Then there exists a projection $\text{pr} : M' \rightarrow M$. Suppose $\text{pr}^{-1}(u) = u_1 \cup u_2$, where both u_1 and u_2 are oriented open intervals in $\partial M'$. Define $\text{Cut}_{(D,u)}(M, \mathcal{N}) = (M', \mathcal{N}')$, where $\mathcal{N}' = \mathcal{N} \cup u_1 \cup u_2$.

For any stated (M, \mathcal{N}) -tangle α , we isotope α such that $\alpha \cap D = \alpha \cap u$ and at each point in $\alpha \cap u$ the framing of α is given by the the velocity vector of u . Let s be a map from $\alpha \cap u$ to $\{-, +\}$. We define a stated (M', \mathcal{N}') -tangle $\alpha(s)$ in the following way: the (M', \mathcal{N}') -tangle is $\text{pr}^{-1}(\alpha)$; the states for $\text{pr}^{-1}(\alpha) \cap \mathcal{N}'$ are inherited from α ; for each point $u \in \alpha \cap u$, we state the two endpoints $\text{pr}^{-1}(u)$ with $s(u)$. Then there exists a linear map $\Theta_{(D,u)} : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M', \mathcal{N}')$ such

that $\Theta_{(D,u)}(\alpha) = \sum_{s:\alpha \cap u \rightarrow \{-,+\}} \alpha(s)$ [BL20]. When there is no confusion, we can omit the subscript for $\Theta_{(D,u)}$.

Lemma 6.2.1. *We have $\Theta : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M', \mathcal{N}')$ behaves well with respect to the Frobenius map.*

Proof. Condition (1) is trivial from the definition of Θ .

Condition (2) is proved in [BL20].

Condition (3): For any disjoint stated (M, \mathcal{N}) -tangles α and β , we have

$$\Theta(\alpha \cdot \beta) = \Theta(\mathcal{F}(\alpha) \cup \beta) = \Theta(\mathcal{F}(\alpha)) \cup \Theta(\beta) = \mathcal{F}(\Theta(\alpha)) \cup \Theta(\beta) = \Theta(\alpha) \cdot \Theta(\beta),$$

where we regard α as an element in $\mathcal{S}_1(M, \mathcal{N})$ and regard β as an element in $\mathcal{S}_{q^{1/2}}(M, \mathcal{N})$.

□

Remark 6.2.2. *There is a surjective algebra homomorphism $\Phi : \mathcal{S}_1(M, \mathcal{N}) \rightarrow R_2(M, \mathcal{N})$, where $R_2(M, \mathcal{N})$ is the coordinate ring of some algebraic set (Thm. 2.2.9). We have $\text{Ker} \Phi$ consists of all nilpotents of $\mathcal{S}_1(M, \mathcal{N})$ (Thm. 2.4.19). Then Φ induces a bijection $\Phi^* : \text{MaxSpec}(R_2(M, \mathcal{N})) \rightarrow \text{MaxSpec}(\mathcal{S}_1(M, \mathcal{N}))$. Proposition 2.2.11 implies there is a surjective map*

$$\nu^* : \text{MaxSpec}(R_2(M', \mathcal{N}')) \rightarrow \text{MaxSpec}(R_2(M, \mathcal{N}))$$

induced by an algebra homomorphism $\nu : R_2(M, \mathcal{N}) \rightarrow R_2(M', \mathcal{N}')$. Theorem 2.2.12 shows $\Phi \circ \Theta = \nu \circ \Phi$. Thus $\Theta^ \circ \Phi^* = \Phi^* \circ \nu^*$. Then we have Θ^* is surjective since Φ^* is a bijection and ν^* is surjective.*

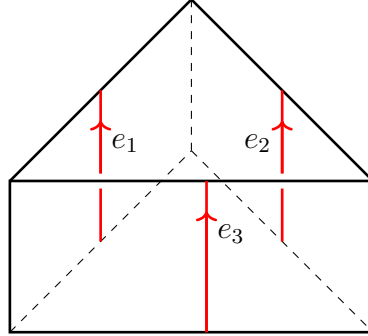
For any $q^{\frac{1}{2}}$ and any $\rho \in \text{MaxSpec}(\mathcal{S}_1(M', \mathcal{N}'))$, Lemma 6.1.2 implies that $\Theta : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M', \mathcal{N}')$ induces a linear map

$$\Theta_\rho : \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_{\Theta^*(\rho)} \rightarrow \mathcal{S}_{q^{1/2}}(M', \mathcal{N}')_\rho.$$

We will call Θ_ρ the splitting map for the representation-reduced stated skein module. Suppose ∂D , where D is the splitting disk, is contained in the boundary component V . In subsection 6.2.4, we will prove Θ_ρ is injective when $V \cap \mathcal{N} \neq \emptyset$.

6.2.2 Gluing the thickening of the marked triangle

Let \mathbb{T} denote the marked 3-manifold in the following picture:



Then \mathbb{T} is the thickening of \mathfrak{T} .

For each $i = 1, 2, 3$, let D_i be an embedded disk in $\partial\mathbb{T}$ such that the closure of e_i is contained in the interior of D_i and there is no intersection among these three disks. From now on, when we draw \mathbb{T} , we may omit all the black lines, that is, we only draw three red arrows. We also only draw involved markings and stated tangles when we try to draw stated tangles in marked 3-manifolds.

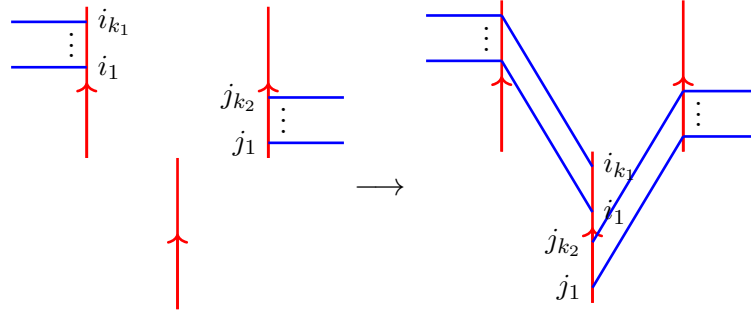
Let (M, \mathcal{N}) be any marked 3-manifold with $\sharp\mathcal{N} \geq 2$. Suppose e'_1, e'_2 are two components of \mathcal{N} . For each $i = 1, 2$, let D'_i be an embedded disk on the boundary of M such that the intersection between the closure of \mathcal{N} and D_i is the closure of e_i and the closure of e_i is contained in the interior of D_i and $D_1 \cap D_2 = \emptyset$. For each $i = 1, 2$, let $\phi_i : D'_i \rightarrow D_i$ be a diffeomorphism such that $\phi_i(e'_i) = e_i$ and ϕ_i preserves the orientations of e_i and e'_i . We set

$$M_{e'_1\mathbb{T}e'_2} = (M \cup \mathbb{T}) / (\phi_i(x) = x, x \in D'_i, i = 1, 2), \quad \mathcal{N}_{e'_1\mathbb{T}e'_2} = (\mathcal{N} - (e'_1 \cup e'_2)) \cup e_3.$$

Then $(M_{e'_1\mathbb{T}e'_2}, \mathcal{N}_{e'_1\mathbb{T}e'_2})$ is a marked 3-manifold. We use $(M, \mathcal{N})_{e'_1\mathbb{T}e'_2}$ to denote this marked 3-manifold.

Then there is a linear isomorphism $QF_{e'_1, e'_2} : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M_{e'_1\mathbb{T}e'_2}, \mathcal{N}_{e'_1\mathbb{T}e'_2})$ [CL22a; Wan23c]. We use ι to denote the obvious embedding from M to $M_{e'_1\mathbb{T}e'_2}$. For any stated (M, \mathcal{N}) -tangles, we extend the ends of $\iota(\alpha)$ on each $e_i, i = 1, 2$, to e_3 such that the framing of extended parts contained in \mathbb{T} is given by the positive direction of $[0, 1]$ and all the ends on e_3 extended from e_1 are higher than all the

ends extended from e_2 . To be precise, see the following picture:



where the blue lines are parts of stated tangles and the framing in the picture is given by the red arrows. To simplify notation, we normally omit the subscript for $QF_{e'_1, e'_2}$ when there is no confusion.

Lemma 6.2.3. *We have $QF : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M_{e'_1 \mathbb{T} e'_2}, \mathcal{N}_{e'_1 \mathbb{T} e'_2})$ behaves well with respect to the Frobenius map.*

Proof. Condition (1) is trivial from the definition of the map QF .

Condition (2): We know $\mathcal{F}(\alpha) = \alpha^{(N)}$ for any stated arc α . Then the operation of taking N parallel copies and the operation of QF commute with each other. This completes the proof for condition (2).

Condition (3): The proof is similar with Lemma 6.2.1.

□

For any $q^{\frac{1}{2}}$ and $\rho \in \text{MaxSpec}(\mathcal{S}_1(M_{e'_1 \mathbb{T} e'_2}, \mathcal{N}_{e'_1 \mathbb{T} e'_2}))$, Lemma 6.1.2 implies $QF : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M_{e'_1 \mathbb{T} e'_2}, \mathcal{N}_{e'_1 \mathbb{T} e'_2})$ induces the linear isomorphism

$$QF_\rho : \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_{QF^*(\rho)} \rightarrow \mathcal{S}_{q^{1/2}}(M_{e'_1 \mathbb{T} e'_2}, \mathcal{N}_{e'_1 \mathbb{T} e'_2})_\rho.$$

6.2.3 Adding an extra marking

Let (M, \mathcal{N}) be a marked 3-manifold, and let e be an embedded oriented open interval in ∂M such that there is no intersection between the closure of e and the closure of \mathcal{N} . Define a new marked 3-manifold (M, \mathcal{N}') , where $\mathcal{N}' = \mathcal{N} \cup e$. We

say (M, \mathcal{N}') is obtained from (M, \mathcal{N}) by adding one extra marking e . We use $l : (M, \mathcal{N}) \rightarrow (M, \mathcal{N}')$ to denote the obvious embedding.

Let \mathbb{B} be the three dimensional solid ball with two markings on its boundary, that is, \mathbb{B} is the thickening of the bigon. We label one marking of \mathbb{B} as b .

Lemma 6.2.4. *Let (M, \mathcal{N}) be a marked 3-manifold, and let (M, \mathcal{N}') be obtained from (M, \mathcal{N}) by adding one extra marking e . Suppose e is contained in the component U of ∂M and $U \cap \mathcal{N} \neq \emptyset$. Then we have the following results:*

(a) *For any $\rho \in \text{MaxSpec}(\mathcal{S}_1(M, \mathcal{N}'))$, we have*

$$l_\rho : \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_{l^*(\rho)} \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N}')_\rho$$

is injective.

(b) *For any $\rho \in \text{MaxSpec}(\mathcal{S}_1(M, \mathcal{N}'))$, there exists $\rho' \in \text{MaxSpec}(\mathcal{S}_1(\mathbb{B}))$ such that*

$$\mathcal{S}_{q^{1/2}}(M, \mathcal{N}')_\rho \simeq \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_{l^*(\rho)} \otimes \mathcal{S}_{q^{1/2}}(\mathbb{B})_{\rho'}.$$

Especially, we have

$$\dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}')_\rho = N^3 \dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_{l^*(\rho)}.$$

Proof. This Lemma is implied by Theorem 6.10 in [Wan23c] since all the maps in Theorem 6.10 in [Wan23c] behave well with respect to the Frobenius map as shown in Lemmas 6.1.3, 6.2.3.

From Theorem 4.10 in [Wan23b], we know $\dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(\mathbb{B})_{\rho'} = N^3$. So we have

$$\dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(M, \mathcal{N}')_\rho = N^3 \dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_{l^*(\rho)}.$$

□

6.2.4 Injectivity for the splitting map

In this subsection we will show the splitting map for the representation-reduced stated skein modules is injective when the boundary component of the 3-manifold, containing the boundary of the splitting disk, contains at least one marking.

We define $c_+(q^{\frac{1}{2}}) = q^{\frac{1}{2}}$ and $c_-(q^{\frac{1}{2}}) = -(q^{\frac{1}{2}})^5$. We define a map $\bar{\cdot} : \{-, +\} \rightarrow \{-, +\}$ such that $\overline{+} = -$ and $\overline{-} = +$. Note that $c_+(q^{\frac{1}{2}})^N = c_+(1)$ and $c_-(q^{\frac{1}{2}})^N = c_-(1)$.

We use \bar{H} to denote the negative half-twist for parallel strands, please see Figure 6.1.

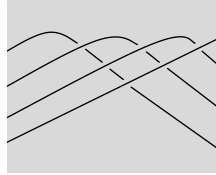


FIGURE 6.1: The negative half-twist for four strands.

Lemma 6.2.5. ([LS21]) *Let (M, \mathcal{N}) be a marked 3-manifold, and e be a component of \mathcal{N} . Then there is a linear isomorphism $h_e : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ given by*

$$h_e \left(\begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \\ \text{---} \\ \downarrow \\ i_k \\ \vdots \\ i_2 \\ \vdots \\ i_1 \end{array} \right) = \left(\frac{1}{\prod_{j=1}^k c_{i_j}(q^{\frac{1}{2}})} \right) \cdot \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \\ \text{---} \\ \downarrow \\ \bar{i}_1 \\ \vdots \\ \bar{i}_2 \\ \vdots \\ \bar{i}_k \end{array}$$

where $i_j \in \{-, +\}$, for $1 \leq j \leq k$, and the thick line with an arrow is a part of the marking e .

Lemma 6.2.6. *Let (M, \mathcal{N}) be a marked 3-manifold, and e be a component of \mathcal{N} . We have $h_e : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})$ behaves well with respect the Frobenius map.*

Proof. Condition (1) is trivial.

Condition (2): We use $\text{---} \textcircled{N} \text{---} \downarrow i$ to denote N parallel copies of the corresponding stated arc (they have the same state). Thus $\mathcal{F}(\text{---} \downarrow i) = \text{---} \textcircled{N} \text{---} \downarrow i$. We

use $\begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \\ \text{---} \\ \downarrow \\ \bar{H} \\ \vdots \\ \text{---} \\ \text{---} \\ \downarrow \\ \text{---} \end{array}$ to denote the picture obtained from parallel strands by first

taking the negative half-twist and then taking N parallel copies of each strand. We use $\text{---} \textcircled{N} \text{---} \bar{H}$ to denote the picture obtained from N parallel strands by taking the negative half-twist.

We have

$$\begin{aligned}
 h_e \circ \mathcal{F} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \\ \text{---} \\ \downarrow \\ i_1 \end{array} \begin{array}{c} i_k \\ \vdots \\ i_2 \\ i_1 \end{array} \right) &= h_e \left(\begin{array}{c} \text{---} \circlearrowleft N \\ \text{---} \\ \vdots \\ \text{---} \circlearrowleft N \\ \text{---} \circlearrowleft N \\ \downarrow \\ i_1 \end{array} \begin{array}{c} i_k \\ \vdots \\ i_2 \\ i_1 \end{array} \right) \\
 &= \left(\frac{1}{\prod_{j=1}^k c_{i_j} (q^{\frac{1}{2}})^N} \right) \cdot \begin{array}{c} \text{---} \\ \text{---} \circlearrowleft N \text{---} \overline{H} \\ \vdots \\ \text{---} \circlearrowleft N \text{---} \overline{H} \\ \text{---} \circlearrowleft N \text{---} \overline{H} \\ \downarrow \\ \overline{i_1} \\ \overline{i_2} \\ \vdots \\ \overline{i_k} \end{array} \\
 &= \left(\frac{1}{\prod_{j=1}^k c_{i_j} (1)} \right) \cdot \begin{array}{c} \text{---} \\ \text{---} \circlearrowleft N \\ \vdots \\ \text{---} \circlearrowleft N \\ \text{---} \circlearrowleft N \\ \downarrow \\ \overline{i_1} \\ \overline{i_2} \\ \vdots \\ \overline{i_k} \end{array},
 \end{aligned}$$

where the last equality is because of relations (1.2) and (1.5).

$$\begin{aligned}
 \mathcal{F} \circ h_e \left(\begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \\ \text{---} \\ \downarrow \\ i_1 \end{array} \begin{array}{c} i_k \\ \vdots \\ i_2 \\ i_1 \end{array} \right) &= \left(\frac{1}{\prod_{j=1}^k c_{i_j} (1)} \right) \cdot \mathcal{F} \left(\begin{array}{c} \text{---} \\ \text{---} \circlearrowleft N \\ \vdots \\ \text{---} \circlearrowleft N \\ \text{---} \circlearrowleft N \\ \downarrow \\ \overline{i_1} \\ \overline{i_2} \\ \vdots \\ \overline{i_k} \end{array} \right) \\
 &= \left(\frac{1}{\prod_{j=1}^k c_{i_j} (1)} \right) \cdot \begin{array}{c} \text{---} \\ \text{---} \circlearrowleft N \\ \vdots \\ \text{---} \circlearrowleft N \\ \text{---} \circlearrowleft N \\ \downarrow \\ \overline{i_1} \\ \overline{i_2} \\ \vdots \\ \overline{i_k} \end{array}.
 \end{aligned}$$

Condition (3): The proof is similar with the proof for Condition (2). □

For any $q^{\frac{1}{2}}$ and $\rho \in \text{MaxSpec}(\mathcal{S}_1(M, \mathcal{N}))$, Lemma 6.1.2 implies

$$h_e : \mathcal{S}_{q^{1/2}}(M, \mathcal{N}) \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})$$

induces the linear isomorphism

$$(h_e)_\rho : \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_{(h_e)^*(\rho)} \rightarrow \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_\rho.$$

Let (M, \mathcal{N}) be a marked 3-manifold, let D be a properly embedded disk in M , and let u be an embedded oriented open interval in D . Then $\text{Cut}_{(D,u)}(M, \mathcal{N})$ has two copies of u , denoted them as u_1, u_2 . Suppose \mathcal{N}' is obtained from \mathcal{N} by adding one extra marking e such that e and ∂D belong to the same component of ∂M . We use $l : (M, \mathcal{N}) \rightarrow (M, \mathcal{N}')$ to denote the obvious embedding. We have (M, \mathcal{N}') is isomorphic to $(\text{Cut}_{(D,u)}(M, \mathcal{N}))_{u_1 \mathbb{T} u_2}$.

Lemma 6.2.7. *There exists an isomorphism $\varphi : (M, \mathcal{N}') \rightarrow (\text{Cut}_{(D,u)}(M, \mathcal{N}))_{u_1 \mathbb{T} u_2}$ such that*

$$QF_{u_1, u_2} \circ h_{u_2} \circ \Theta_{(D,u)} = \varphi_* \circ l_*.$$

Proof. Proposition 6.3 in [Wan23c]. □

Theorem 6.2.8. *Let (M, \mathcal{N}) be a marked 3-manifold, let D be a properly embedded disk in M , and let u be an embedded oriented open interval in D . Suppose the component V of ∂M contains ∂D and $V \cap \mathcal{N} \neq \emptyset$. Then, for any $\rho \in \text{MaxSpec}(\mathcal{S}_1(\text{Cut}_{(D,u)}(M, \mathcal{N})))$, we have*

$$\Theta_\rho : \mathcal{S}_{q^{1/2}}(M, \mathcal{N})_{\Theta^*(\rho)} \rightarrow \mathcal{S}_{q^{1/2}}(\text{Cut}_{(D,u)}(M, \mathcal{N}))_\rho$$

is injective.

Proof. Here we use the notations in Lemma 6.2.7. Suppose

$$\rho' \in \text{MaxSpec}(\mathcal{S}_1((\text{Cut}_{(D,u)}(M, \mathcal{N}))_{u_1 \mathbb{T} u_2})).$$

Lemmas 6.1.3, 6.2.1, 6.2.3 and 6.2.7 imply

$$(QF_{u_1, u_2})_{\rho'} \circ (h_{u_2})_{QF^*(\rho')} \circ (\Theta_{(D,u)})_{(h_{u_2})^*(QF^*(\rho'))} = \varphi_{\rho'} \circ l_{\varphi^*(\rho)}.$$

Since $(QF_{u_1, u_2})_{\rho'}$, $(h_{u_2})_{QF^*(\rho')}$, $\varphi_{\rho'}$ are all linear isomorphisms, we have

$$\text{Ker}((\Theta_{(D,u)})_{(h_{u_2})^*(QF^*(\rho'))}) = \text{Ker}(l_{\varphi^*(\rho)}).$$

Lemma 6.2.4 implies $\text{Ker}((\Theta_{(D,u)})_{(h_{u_2})^*(QF^*(\rho'))}) = \text{Ker}(l_{\varphi^*(\rho)}) = 0$. This completes the proof because both $(h_{u_2})^*$ and QF^* are bijections. □

Remark 6.2.9. *When the marking set is empty, we expect the splitting map for the representation-reduced stated module is injective if M is the thickening of an oriented surface.*

6.3 Geometric stated skein

Let M be an oriented connected closed 3-manifold. Recall that, for any positive integer k , we use M_k to denote the marked 3-manifold obtained from M by removing k open three dimensional balls and adding one marking to each newly created sphere boundary component. Then M_k is defined up to isomorphism. In this section, we will show $\dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(M_k)_\rho = 1$ for any $\rho \in \text{MaxSpec}(\mathcal{S}_1(M_k))$.

We use H_g to denote the genus g handlebody. For any positive integer k , we use $H_{g,k}$ to denote the marked 3-manifold obtained from H_g by adding k markings on ∂H_g . Then $H_{g,k}$ is defined up to isomorphism. We use \mathcal{N}_k to denote the union of all the markings in $H_{g,k}$, and use $\overline{\mathcal{N}_k}$ to denote the closure of \mathcal{N}_k . To simplify the notation, we will use H to denote $H_{g,k}$ in this section.

For each marking e of H , we can regard its closure \bar{e} as an embedding from $[0, 1]$ to ∂H_g . We choose an embedded disk D_e (respectively D'_e) in ∂H_g such that $\bar{e} \subset D_e$ (respectively $\bar{e} \subset D'_e$) and $\bar{e} \cap \partial D_e = \bar{e}(0)$ (respectively $\bar{e} \cap \partial D'_e = \bar{e}(1)$). We also require there is no intersection among D_e (or D'_e). Then $\partial_-(H)$ (respectively $\partial_+(H)$) is obtained from ∂H_g by removing the interior of all D_e (respectively D'_e), please refer to Figures 6.2 and 6.3. $\partial_-(H)$ (respectively $\partial_+(H)$) is a marked surface with marked points $\partial_-(H) \cap \overline{\mathcal{N}_k}$ (respectively $\partial_+(H) \cap \overline{\mathcal{N}_k}$).

Then $\mathcal{S}_{q^{1/2}}(\partial_-(H))$ has a right action on $\mathcal{S}_{q^{1/2}}(H)$, and $\mathcal{S}_{q^{1/2}}(\partial_+(H))$ has a left action on $\mathcal{S}_{q^{1/2}}(H)$.

We use Σ_g to denote the closed surface of genus g . For any positive integer k , we use $\Sigma_{g,k}$ to denote the marked surface obtained from Σ_g by removing k open disks and equipping each newly created boundary component with one marked point. To simplify the notation, we will use Σ to denote $\Sigma_{g,k}$ in this section. We have $\Sigma \simeq \partial_+(H) \simeq \partial_-(H)$. Then any isomorphism $f : \Sigma \rightarrow \partial_+(H)$ induces a left action of $\mathcal{S}_{q^{1/2}}(\Sigma)$ on $\mathcal{S}_{q^{1/2}}(H)$. Similarly, any isomorphism $g : \Sigma \rightarrow \partial_-(H)$ induces a right action of $\mathcal{S}_{q^{1/2}}(\Sigma)$ on $\mathcal{S}_{q^{1/2}}(H)$.

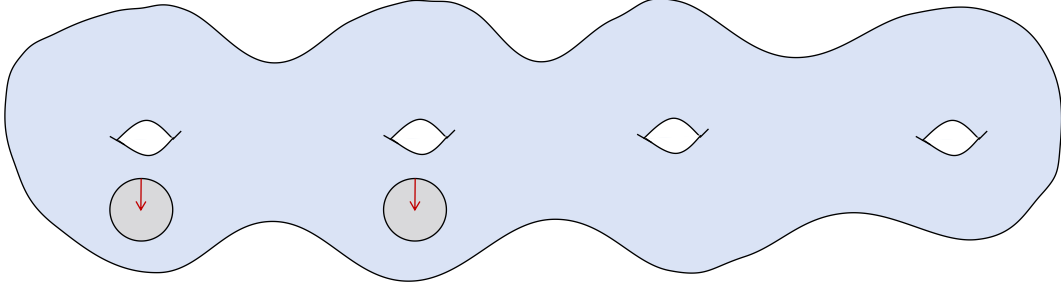


FIGURE 6.2: The picture is for H , where $g = 4$ and $k = 2$. The gray disks are embedded disks in the boundary of the handlebody. The red arrows contained in the gray disks are markings. The sub-surface of ∂H_g colored by blue is denoted as $\partial_-(H)$ (as a surface, $\partial_-(H)$ is obtained from ∂H_g by removing k open disks). It is a marked surface with marked points $\partial_-(H) \cap \overline{\mathcal{N}}_k$.

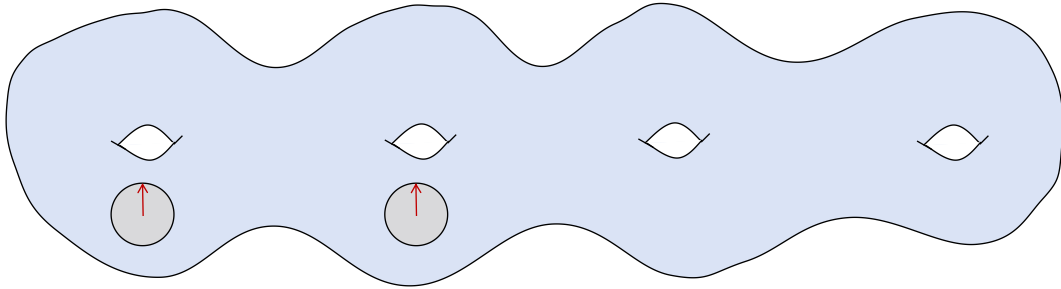


FIGURE 6.3: The picture is for H , where $g = 4$ and $k = 2$. The gray disks are embedded disks in the boundary of the handlebody. The red arrows contained in the gray disks are markings. The sub-surface of ∂H_g colored by blue is denoted as $\partial_+(H)$ (as a surface, $\partial_+(H)$ is obtained from ∂H_g by removing k open disks). It is a marked surface with marked points $\partial_+(H) \cap \overline{\mathcal{N}}_k$.

We know there is an injective algebra homomorphism $\mathcal{F} : \mathcal{S}_1(\Sigma) \rightarrow \mathcal{S}_{q^{1/2}}(\Sigma)$. From [Yu23], we know $\text{Im}\mathcal{F}$ is the center of $\mathcal{S}_{q^{1/2}}(\Sigma)$.

Let $\theta : \mathcal{S}_{q^{1/2}}(\Sigma) \rightarrow \text{End}(V)$ be an irreducible representation of $\mathcal{S}_{q^{1/2}}(\Sigma)$. Then θ induces an algebra homomorphism $\varepsilon_\theta : \mathcal{S}_1(\Sigma) \rightarrow \mathbb{C}$ such that $\varepsilon_\theta(x)Id_V = \theta(\mathcal{F}(x))$ for any $x \in \mathcal{S}_1(\Sigma)$. We will call ε_θ the **classical shadow** of θ . Then the Azumaya locus $\mathcal{A}(\mathcal{S}_{q^{1/2}}(\Sigma))$ of $\mathcal{S}_{q^{1/2}}(\Sigma)$ is defined to be

$$\mathcal{A}(\mathcal{S}_{q^{1/2}}(\Sigma)) = \{\rho \in \text{MaxSpec}(\mathcal{S}_1(\Sigma)) \mid \text{there exists a unique irreducible representation } \theta \text{ of } \mathcal{S}_{q^{1/2}}(\Sigma) \text{ such that } \varepsilon_\theta = \rho\}.$$

An irreducible representation of $\mathcal{S}_{q^{1/2}}(\Sigma)$ is called an **Azumaya representation** if its classical shadow lives in the Azumaya locus of $\mathcal{S}_{q^{1/2}}(\Sigma)$.

Since $\text{Im}\mathcal{F}$ is the center of $\mathcal{S}_{q^{1/2}}(\Sigma)$, the PI-dimension of $\mathcal{S}_{q^{1/2}}(\Sigma)$ is equal to the square root of the rank of $\mathcal{S}_{q^{1/2}}(\Sigma)$ over $\mathcal{S}_1(\Sigma)$. From [Wan23a; Yu23], we know the PI-dimension of $\mathcal{S}_{q^{1/2}}(\Sigma)$ is $N^{3(g+k-1)}$. For any positive integer n , we use $\text{Mat}_n(\mathbb{C})$ to denote the algebra of all n by n complex matrices. Then

$$\mathcal{A}(\mathcal{S}_{q^{1/2}}(\Sigma)) = \{\rho \in \text{MaxSpec}(\mathcal{S}_1(\Sigma)) \mid \mathcal{S}_{q^{1/2}}(\Sigma)_\rho \simeq \text{Mat}_n(\mathbb{C}), \text{ where } n = N^{3(g+k-1)}\}.$$

Suppose $\rho \in \text{MaxSpec}(\mathcal{S}_1(H))$ and f is an isomorphism from Σ to $\partial_+(H)$. It is easy to show that the left action of $\mathcal{S}_{q^{1/2}}(\Sigma)$ (induced by f) on $\mathcal{S}_{q^{1/2}}(H)$ reduces to a left action of $\mathcal{S}_{q^{1/2}}(\Sigma)$ on $\mathcal{S}_{q^{1/2}}(H)_\rho$.

We can regard $\partial_+(H) \times [0, 1]$ as a closed regular neighborhood of $\partial_+(H)$. We use L to denote the embedding from $\partial_+(H) \times [0, 1]$ to H . Then L induces an algebra homomorphism $L_* : \mathcal{S}_1(\partial_+(H)) \rightarrow \mathcal{S}_1(H)$. Then L_* induces a map $L^* : \text{MaxSpec}(\mathcal{S}_1(H)) \rightarrow \text{MaxSpec}(\mathcal{S}_1(\partial_+(H)))$.

Any isomorphism $f : \Sigma \rightarrow \partial_+(H)$ induces an isomorphism from $\Sigma \times [0, 1]$ to $\partial_+(H) \times [0, 1]$, which is still denoted as f .

In the following theorem, we prove $\mathcal{S}_{q^{1/2}}(H)_\rho$ is an Azumaya representation for $\mathcal{S}_{q^{1/2}}(\Sigma)$. This generalizes Theorem 12.1 in [FKL23] (they prove a parallel result for the non-stated case) to the stated case.

For any marked point p of Σ , we define a curve $\alpha(p)$ as in Figure 6.4.

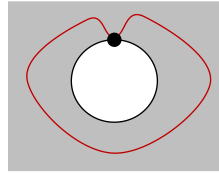


FIGURE 6.4: The circle is one of the boundary components of Σ , the black dot is the marked point p on this boundary component. The red curve is $\alpha(p)$.

Theorem 6.3.1. *Let k be a positive integer, let $\rho \in \text{MaxSpec}(\mathcal{S}_1(H))$, and let f be an isomorphism from Σ to $\partial_+(H)$. Then we have the followings:*

- (a) $L^*(\rho) \in \mathcal{A}(\mathcal{S}_{q^{1/2}}(\partial_+(H)))$.

(b) $\mathcal{S}_{q^{1/2}}(H)_\rho$ is an irreducible representation of $\mathcal{S}_{q^{1/2}}(\Sigma)$ whose classical shadow is $f^*(L^*(\rho))$. Meanwhile, this irreducible representation is an Azumaya representation.

Proof. (a) For each marked point p of Σ , we have $L(f(\alpha(p)))$ is trivial in H_g (that is, it bounds an embedded disk in H_g). Then, for any $\rho \in \text{MaxSpec}(\mathcal{S}_1(H))$, Theorem 8.1 in [KK23] shows

$$L^*(\rho) \in \mathcal{A}(\mathcal{S}_{q^{1/2}}(\partial_+(H))).$$

(b) From the above discussion, we know $\mathcal{S}_{q^{1/2}}(\Sigma)$ has a left action on $\mathcal{S}_{q^{1/2}}(H)_\rho$ (induced by f). For any stated tangle α in $\Sigma \times [0, 1]$ and any stated tangle β in H , we have

$$\begin{aligned} \mathcal{F}(\alpha) \cdot (\beta \otimes 1) &= (L_*(f_*(\mathcal{F}(\alpha))) \cup \beta) \otimes 1 = (\mathcal{F}(L_*(f_*(\alpha))) \cup \beta) \otimes 1 \\ &= (L_*(f_*(\alpha)) \cdot \beta) \otimes 1 = \beta \otimes L_*(f_*(\alpha)) \cdot 1 = \rho(L_*(f_*(\alpha)))\beta \otimes 1, \end{aligned}$$

where we regard α as an element in $\mathcal{S}_1(\Sigma)$ and regard β as an element in $\mathcal{S}_{q^{1/2}}(H)$. Then the classical shadow of the representation $\mathcal{S}_{q^{1/2}}(H)_\rho$ (as a representation of $\mathcal{S}_{q^{1/2}}(\Sigma)$) is $f^*(L^*(\rho))$. Thus this representation reduces to a representation of $\mathcal{S}_{q^{1/2}}(\Sigma)_{f^*(L^*(\rho))}$. We have $\mathcal{S}_{q^{1/2}}(\Sigma)_{f^*(L^*(\rho))} \simeq \text{Mat}_{N^{3(g+k-1)}}(\mathbb{C})$ because $L^*(\rho) \in \mathcal{A}(\mathcal{S}_{q^{1/2}}(\partial_+(H)))$ and f is an isomorphism. From Theorem 4.10 in [Wan23b], we know $\dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(H)_\rho = N^{3(g+k-1)}$. Then $\mathcal{S}_{q^{1/2}}(H)_\rho$ is an irreducible representation of $\mathcal{S}_{q^{1/2}}(\Sigma)_{f^*(L^*(\rho))}$. This completes the proof. \square

To distinguish two copies of H . We use H^+ and H^- to denote them (that is $H^+ = H^- = H$, but we denote these two copies with different notations). We also use ∂H^+ to denote $\partial_+(H)$ and use ∂H^- to denote $\partial_-(H)$.

We suppose H has a fixed orientation, and the orientations of $\partial_+(H)$ and $\partial_-(H)$ are inherited from H . Let f be an isomorphism from Σ to $\partial_+(H)$, and let g be an isomorphism from Σ to $\partial_-(H)$. Then $f \circ g^{-1}$ is an isomorphism from ∂H^- to ∂H^+ (here we require $f \circ g^{-1}$ is an orientation reversing isomorphism).

We use $H^- \cup_{f \circ g^{-1}} H^+$ to denote $(H^- \cup H^+)/ (x = f(g^{-1}(x)), x \in \partial H^-)$. Then $H^- \cup_{f \circ g^{-1}} H^+$ is a marked 3-manifold. Actually, there exists an oriented closed

3-manifold M such that M_k is isomorphic to $H^- \cup_{f \circ g^{-1}} H^+$. Meanwhile, for any oriented connected closed 3-manifold M and any positive integer k , there exist a non-negative integer g and isomorphisms $f : \Sigma \rightarrow \partial_+(H) = \partial H^+$, $g : \Sigma \rightarrow \partial_-(H) = \partial H^-$ such that $H^- \cup_{f \circ g^{-1}} H^+$ is isomorphic to M_k .

We use L_+ (respectively L_-) to denote the obvious embedding from H^+ (respectively H^-) to $H^- \cup_{f \circ g^{-1}} H^+$. Then we have $L_+ \circ f = L_- \circ g$, which is an embedding from Σ to $H^- \cup_{f \circ g^{-1}} H^+$. Then $L_+ \circ f$ induces an algebra homomorphism

$$(L_+ \circ f)_* = (L_+)_* \circ f_* : \mathcal{S}_1(\Sigma) \rightarrow \mathcal{S}_1(H^- \cup_{f \circ g^{-1}} H^+).$$

It further induces a map

$$(L_+ \circ f)^* = f^* \circ (L_+)^* : \text{MaxSpec}(\mathcal{S}_1(H^- \cup_{f \circ g^{-1}} H^+)) \rightarrow \text{MaxSpec}(\mathcal{S}_1(\Sigma)).$$

Suppose R is a commutative unital ring, U and V are two R -modules, and I is an ideal of R . We state the following results for classical module theory without giving proofs.

$$\text{We have } U \otimes_R R/I \simeq \frac{U}{I \cdot U} \text{ as } R\text{-modules.} \quad (6.2)$$

$$\text{We have } U \otimes_R V \simeq \frac{U}{I \cdot U} \otimes_{R/I} V \text{ as } R\text{-modules if } I \cdot V = 0. \quad (6.3)$$

Theorem 6.3.2. *Let k be a positive integer, let f (respectively g) be an isomorphism from Σ to ∂H^+ (respectively ∂H^-) such that $f \circ g^{-1}$ is orientation reversing isomorphism, and let $\rho \in \text{MaxSpec}(\mathcal{S}_1(H^- \cup_{f \circ g^{-1}} H^+))$. Then we have the followings:*

$$(a) (L_+ \circ f)^*(\rho) \in \mathcal{A}(\mathcal{S}_{q^{1/2}}(\Sigma))$$

$$(b) \dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(H^- \cup_{f \circ g^{-1}} H^+)_{\rho} = 1.$$

Proof. (a) The proof is similar with the proof for (a) in Theorem 6.3.1. Theorem 8.1 in [KK23] shows

$$(L_+ \circ f)^*(\rho) \in \mathcal{A}(\mathcal{S}_{q^{1/2}}(\Sigma)),$$

for any $\rho \in \text{MaxSpec}(\mathcal{S}_1(H^- \cup_{f \circ g^{-1}} H^+))$.

(b) We will also regard f (respectively g) as an embedding from Σ to H^+ (respectively H^-). Here we use a technique used in page 48 in [FKL23]. We know $\mathcal{S}_{q^{1/2}}(\Sigma)$ has a left (respectively right) action on $\mathcal{S}_{q^{1/2}}(H^+)$ (respectively $\mathcal{S}_{q^{1/2}}(H^-)$). From Theorem 6.5 in [CL22a], we have

$$\mathcal{S}_{q^{1/2}}(H^-) \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \mathcal{S}_{q^{1/2}}(H^+) \simeq \mathcal{S}_{q^{1/2}}(H^- \cup_{f \circ g^{-1}} H^+). \quad (6.4)$$

This isomorphism is given by sending $\alpha \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \beta$ to $L_-(\alpha) \cup L_+(\beta)$, where α (respectively β) is a stated tangle in H^- (respectively H^+).

The surjective algebra homomorphism $f_* : \mathcal{S}_1(\Sigma) \rightarrow \mathcal{S}_1(H^+)$ induces an action of $\mathcal{S}_1(\Sigma)$ on $\mathcal{S}_{q^{1/2}}(H^+)$. For any $\rho' \in \text{MaxSpec}(\mathcal{S}_1(H^+))$, we have

$$\text{Ker}(\rho') \cdot \mathcal{S}_{q^{1/2}}(H^+) = \text{Ker}(\rho' \circ f_*) \cdot \mathcal{S}_{q^{1/2}}(H^+) \quad (6.5)$$

because $f_* : \mathcal{S}_1(\Sigma) \rightarrow \mathcal{S}_1(H^+)$ is surjective. We have the similar discussion for H^- and $H^- \cup_{f \circ g^{-1}} H^+$. From the isomorphism between $\mathcal{S}_{q^{1/2}}(H^-) \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \mathcal{S}_{q^{1/2}}(H^+)$ and $\mathcal{S}_{q^{1/2}}(H^- \cup_{f \circ g^{-1}} H^+)$, we have $\mathcal{S}_1(\Sigma)$ also acts on $\mathcal{S}_{q^{1/2}}(H^-) \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \mathcal{S}_{q^{1/2}}(H^+)$. The action is given by

$$\alpha \cdot (\beta_1 \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \beta_2) = \alpha \cdot \beta_1 \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \beta_2 = \beta_1 \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \alpha \cdot \beta_2, \quad (6.6)$$

where $\alpha \in \mathcal{S}_1(\Sigma)$, $\beta_1 \in \mathcal{S}_{q^{1/2}}(H^-)$ and $\beta_2 \in \mathcal{S}_{q^{1/2}}(H^+)$.

Then we have

$$\begin{aligned} \mathcal{S}_{q^{1/2}}(H^- \cup_{f \circ g^{-1}} H^+)_{\rho} &\simeq \frac{\mathcal{S}_{q^{1/2}}(H^- \cup_{f \circ g^{-1}} H^+)}{\text{Ker}(\rho) \cdot \mathcal{S}_{q^{1/2}}(H^- \cup_{f \circ g^{-1}} H^+)} \\ &\simeq \frac{\mathcal{S}_{q^{1/2}}(H^- \cup_{f \circ g^{-1}} H^+)}{\text{Ker}((L_+ \circ f)^*(\rho)) \cdot \mathcal{S}_{q^{1/2}}(H^+ \cup_{f \circ g^{-1}} H^-)} \\ &\simeq \frac{\mathcal{S}_{q^{1/2}}(H^-) \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \mathcal{S}_{q^{1/2}}(H^+)}{\text{Ker}((L_+ \circ f)^*(\rho)) \cdot (\mathcal{S}_{q^{1/2}}(H^-) \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \mathcal{S}_{q^{1/2}}(H^+))} \\ &\simeq (\mathcal{S}_{q^{1/2}}(H^-) \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \mathcal{S}_{q^{1/2}}(H^+)) \otimes_{\mathcal{S}_1(\Sigma)} \frac{\mathcal{S}_1(\Sigma)}{\text{Ker}((L_+ \circ f)^*(\rho))} \\ &\simeq \mathcal{S}_{q^{1/2}}(H^-) \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \left(\mathcal{S}_{q^{1/2}}(H^+) \otimes_{\mathcal{S}_1(\Sigma)} \frac{\mathcal{S}_1(\Sigma)}{\text{Ker}((L_+ \circ f)^*(\rho))} \right) \\ &\simeq \mathcal{S}_{q^{1/2}}(H^-) \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \left(\frac{\mathcal{S}_{q^{1/2}}(H^+)}{\text{Ker}((L_+ \circ f)^*(\rho)) \cdot \mathcal{S}_{q^{1/2}}(H^+)} \right) \end{aligned}$$

$$\begin{aligned}
&\simeq \mathcal{S}_{q^{1/2}}(H^-) \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \left(\frac{\mathcal{S}_{q^{1/2}}(H^+)}{\text{Ker}((L_+)^*(\rho)) \cdot \mathcal{S}_{q^{1/2}}(H^+)} \right) \\
&\simeq \mathcal{S}_{q^{1/2}}(H^-) \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)} \mathcal{S}_{q^{1/2}}(H^+)_{(L_+)^*(\rho)} \\
&\simeq \mathcal{S}_{q^{1/2}}(H^-)_{(L_-)^*(\rho)} \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)_{(L_+ \circ f)^*(\rho)}} \mathcal{S}_{q^{1/2}}(H^+)_{(L_+)^*(\rho)},
\end{aligned}$$

where the first and the eighth isomorphisms come from definitions, the second and the seventh isomorphisms come from equation (6.5) (or a similar discussion), the third isomorphism comes from (6.4), the fourth and the sixth isomorphisms come from equation (6.2), the fifth isomorphism comes from equation (6.6), the last isomorphism comes from equation (6.3). From (a), we know $(L_+ \circ f)^*(\rho) \in \mathcal{A}(\mathcal{S}_{q^{1/2}}(\Sigma))$. Then Theorem 6.3.1 implies that

$$\mathcal{S}_{q^{1/2}}(H^-)_{(L_-)^*(\rho)} \otimes_{\mathcal{S}_{q^{1/2}}(\Sigma)_{(L_+ \circ f)^*(\rho)}} \mathcal{S}_{q^{1/2}}(H^+)_{(L_+)^*(\rho)} \simeq \mathbb{C}^d \otimes_{\text{Mat}_d(\mathbb{C})} \mathbb{C}^d \simeq \mathbb{C},$$

where $d = 3(g + k - 1)$. □

The following Theorem generalizes Theorem 12.2 in [FKL23] (they proved a parallel result for the non-stated case) to the stated case. But, for the non-stated case, we have restrictions for ρ .

Theorem 6.3.3. *Let M be any oriented connected closed 3-manifold, and let k be any positive integer. For any $\rho \in \text{MaxSpec}(\mathcal{S}_1(M_k))$, we have $\dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(M_k)_\rho = 1$.*

Proof. Since M_k is isomorphic to some $H^- \cup_{f \circ g^{-1}} H^+$, Theorem 6.3.2 implies the Theorem. □

Corollary 6.3.4. *Suppose n is a positive integer. For each $1 \leq i \leq n$, let $M(i)$ be an oriented connected closed 3-manifold, and let k_i be a positive integer. For any $\rho \in \text{MaxSpec}(\mathcal{S}_1(\cup_{1 \leq i \leq n} M(i)_{k_i}))$, we have $\dim_{\mathbb{C}} \mathcal{S}_{q^{1/2}}(\cup_{1 \leq i \leq n} M(i)_{k_i})_\rho = 1$.*

Proof. Equation (6.1) and Theorem 6.3.3. □

Chapter 7

The Volume Conjecture from the skein algebra perspective

This chapter is based on the author's work in [Wan25a]. We will discuss the Volume Conjecture of Bonahon, Wong, and Yang [BWY21; BWY22], which they formulated using surface diffeomorphisms and intertwiners between irreducible representations of the skein algebra. Specifically, we will compute these intertwiners for the case of a closed torus and examine their traces (Lem. 7.3.4, Thm. 7.3.5, and 7.3.11). We will also consider cases where the surface diffeomorphism is periodic (Conj. 7.4.4, Thm. 7.4.5, and 7.4.12).

To align with the terminology in [Wan25a], we introduce new notations for the Kauffman bracket skein algebra and the SL_2 -character variety in this chapter.

7.1 Preliminaries

7.1.1 The $\mathrm{SL}_2(\mathbb{C})$ character variety and the Kauffman bracket skein algebra

Let S be an oriented surface of finite type. The corresponding character variety

$$\mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(S) = \mathrm{Hom}(\pi_1(S), \mathrm{SL}_2(\mathbb{C})) // \mathrm{SL}_2(\mathbb{C})$$

is the set of the group homomorphisms from the fundamental group of S to $SL_2(\mathbb{C})$ with the equivalence relation that two homomorphisms are equivalent if and only if they have the same character (see equation (1.1)).

The Kauffman bracket skein algebra $SK_{q^{1/2}}(S)$ of a surface S , as a vector space over the complex field \mathbb{C} , is generated by all isotopic framed links in $S \times [0, 1]$, subject to the skein relation:

$$K_1 = q^{-1/2}K_\infty + q^{1/2}K_0,$$

where K_1, K_∞, K_0 are three links that differ in a small neighborhood as shown in Figure 7.1, and the trivial knot relation: $K \amalg \bigcirc = -(q + q^{-1})K$ where \bigcirc is a simple knot bounding a disk that has no intersection with K . For any two links $[L_1], [L_2]$, the multiplication $[L_1][L_2]$ is defined by stacking L_2 above L_1 . Here $q^{1/2}$ is a nonzero complex number. The skein algebra $SK_{q^{1/2}}(S)$ is a quantization for the regular ring of the character variety $\mathfrak{X}_{SL_2(\mathbb{C})}(S)$ [Bul97].

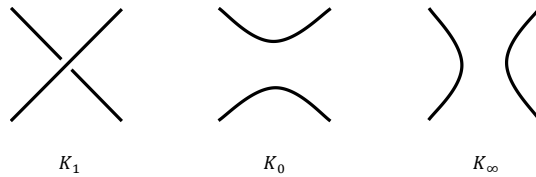


FIGURE 7.1: The Kauffman bracket skein relation.

7.1.2 Classical shadow and Unicity theorem

We recall some notations and constructions for the classical shadow [BW16a]. When q is a primitive n -root of unity with n odd and $(q^{1/2})^n = -1$, Bonahon and Wong found a fascinating algebra homomorphism $T^{q^{1/2}}$ from $SK_{-1}(S)$ to $SK_{q^{1/2}}(S)$, called the Chebyshev homomorphism. Bonahon and Wong proved that $\text{Im}(T^{q^{1/2}})$ is contained in the center of $SK_{q^{1/2}}(S)$. If K is a simple knot with vertical framing, then $T^{q^{1/2}}([K]) = T_n([K])$ where T_n is the n -th Chebyshev polynomial of the first type.

Let $\rho : SK_{q^{1/2}}(S) \rightarrow \text{End}(V)$ be an irreducible representation of $SK_{q^{1/2}}(S)$. Then there exists an algebra homomorphism κ_ρ from $SK_{-1}(S)$ to \mathbb{C} such that $\rho \circ T^{q^{1/2}}(X) = \kappa_\rho(X)Id_V$ for any X in $SK_{-1}(S)$. According to [Bul97], there exists a unique character $[\gamma] \in \mathfrak{X}_{SL_2(\mathbb{C})}(S)$ such that $Tr^\gamma = \kappa_\rho$. Recall that Tr^γ is an

algebra homomorphism from $SK_{-1}(S)$ to \mathbb{C} defined by $Tr^\gamma([K]) = -\text{Trace}\gamma(K)$ where $[K]$ is a simple knot. For every puncture v , we use P_v to denote the loop going around this puncture. There is a complex number p_v such that $\rho([P_v]) = p_v Id_V$. Then an irreducible representation of $SK_{q^{1/2}}(S)$ gives a character $[\gamma]$, called the classical shadow of this irreducible representation, and puncture weights $\{p_v\}_v$, with the relation that $-\text{Trace}\gamma(\alpha_v) = T_n(p_v)$ where α_v denotes the element in the fundamental group of S going around the puncture v .

Theorem 7.1.1 ([BW16a; BW17; BW19; BWY21]). *Let q be a primitive n -root of unity with n odd, and $(q^{1/2})^n = -1$. Then an irreducible representation $\rho : SK_{q^{1/2}} \rightarrow \text{End}(V)$ uniquely determines:*

(1) a character $[\gamma] \in \mathfrak{X}_{\text{SL}_2(\mathbb{C})}(S)$, represented by a group homomorphism $\gamma : \pi_1(S) \rightarrow \text{SL}_2(\mathbb{C})$;

(2) a weight p_v associated to each puncture v of S such that $T_n(p_v) = -\text{Trace}\gamma(\alpha_v)$.

Conversely, every data of a character $\gamma \in \mathfrak{X}_{\text{SL}_2(\mathbb{C})}(S)$ and of puncture weights $p_v \in \mathbb{C}$ satisfying the above condition is realized by an irreducible representation $\rho : SK_{q^{1/2}}(S) \rightarrow \text{End}(V)$.

It turns out that every character in an open dense subset of $\mathfrak{X}_{\text{SL}_2(\mathbb{C})}(S)$ corresponds to a unique irreducible representation of the skein algebra.

Theorem 7.1.2 ([BWY21; FKL19; GJS24]). *Suppose that $[\gamma]$ is in the smooth part of $\mathfrak{X}_{\text{SL}_2(\mathbb{C})}(S)$ or, equivalently, that it is realized by an irreducible homomorphism $\gamma : \pi_1(S) \rightarrow \text{SL}_2(\mathbb{C})$. Then the irreducible representation $\rho : SK_{q^{1/2}}(S) \rightarrow \text{End}(V)$ in Theorem 7.1.1 is unique up to isomorphism of representations. This representation has dimension $\dim V = n^{3g+p-3}$ if S has genus g and p punctures.*

7.1.3 Volume conjecture for surface diffeomorphisms

Bonahon-Wong-Yang constructed the so called Kauffman bracket intertwiners [BWY21; BWY22]. They used these intertwiners to formulate the volume conjecture for surface diffeomorphisms. Here we recall their construction for Kauffman bracket intertwiners.

For a surface S , let φ be a diffeomorphism of S . Obviously φ induces an isomorphism φ_* from $\pi_1(S)$ to $\pi_1(S)$. Then φ_* induces an action on $\mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(S)$ defined by $\varphi^*([\gamma]) = [\gamma\varphi_*]$ where γ is a representative for $[\gamma]$. Although φ_* is only defined up to conjugation, we have φ^* is well-defined. Actually the mapping class group $\mathrm{Mod}(S)$ acts on $\mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(S)$. We say an element $[\gamma] \in \mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is invariant under a diffeomorphism φ , or the element it represents in $\mathrm{Mod}(S)$, if $\varphi^*([\gamma]) = [\gamma]$.

The algebra isomorphism induced by φ from $SK_{q^{1/2}}(S)$ to itself is defined by $\varphi_{\sharp}([K]) = [\varphi \times \mathrm{Id}_{[0,1]}(K)]$ where K is a framed link in $S \times [0, 1]$. Actually the mapping class group $\mathrm{Mod}(S)$ acts on $SK_{q^{1/2}}(S)$.

Let φ be any diffeomorphism for surface S , and let $[\gamma] \in \mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(S)$ be a φ -invariant smooth character. For each puncture v , select a complex number θ_v such that $\mathrm{Trace}\gamma(\alpha_v) = -e^{\theta_v} - e^{-\theta_v}$. Since $[\gamma]$ is φ -invariant, we can choose θ_v to be φ -invariant, that is, $\theta_v = \theta_{\varphi(v)}$. Then set $p_v = e^{\frac{\theta_v}{n}} + e^{-\frac{\theta_v}{n}}$, we have $T_n(p_v) = -\mathrm{Trace}(\alpha_v)$ and $\{p_v\}_v$ are invariant under the action of φ . Suppose ρ is an irreducible representation associated to $[\gamma]$ and puncture weights p_v . Then $\rho \circ \varphi_{\sharp}$ is also an irreducible representation associated to $[\gamma]$ and puncture weights p_v . By the unicity theorem, we know there exists an intertwiner $\Lambda_{\varphi, \gamma}^q$ such that

$$\rho \circ \varphi_{\sharp}(X) = \Lambda_{\varphi, \gamma}^q \circ \rho(X) \circ (\Lambda_{\varphi, \gamma}^q)^{-1}$$

for every $X \in SK_{q^{1/2}}(S)$. We normalize the intertwiner such that $|\det(\Lambda_{\varphi, \gamma}^q)| = 1$.

Conjecture 7.1.3 ([BWY21; BWY22]). *Let the pseudo-Anosov surface diffeomorphism $\varphi : S \rightarrow S$, the φ -invariant smooth character $[\gamma] \in \mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and the φ -invariant puncture weights p_v as above be given. For every odd n , consider the primitive n -root of unity $q_n = e^{2\pi i/n}$ and choose $(q_n)^{1/2} = e^{\pi i/n}$. Then*

$$\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\mathrm{Trace} \Lambda_{\varphi, \gamma}^{q_n}| = \frac{1}{4\pi} \mathrm{vol}_{\mathrm{hyp}}(M_{\varphi}),$$

where $\mathrm{vol}_{\mathrm{hyp}}(M_{\varphi})$ is the volume of the complete hyperbolic metric of the mapping torus M_{φ} .

7.1.4 Ideal triangulation and intertwiners obtained from Chekhov-Fock algebras

Let S be an oriented surface with punctures, and let $\tau = \{e_1, \dots, e_m\}$ be an ideal triangulation for S , where e_1, \dots, e_m are non-isotopic disjoint embedded arcs in S connecting punctures such that all these arcs cut S into triangles. We call e_1, \dots, e_m the edges of τ . An edge weight system for τ is an m -tuple, $a = (a_1, \dots, a_m)$, where a_i is a nonzero complex number for each $1 \leq i \leq m$. The pair (τ, a) determines a character $[\bar{\gamma}]$ in $\mathcal{X}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, please refer to section 8 in [BL07] or section 3 in [BWY21] for more details.

For each ideal triangulation τ , there is a Chekhov-Fock algebra \mathcal{T}_τ^q corresponding to τ , where q is a nonzero complex number. As an algebra over \mathbb{C} , the Chekhov-Fock algebra \mathcal{T}_τ^q is generated by $X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_m^{\pm 1}$ subject to the relations:

$$X_i X_i^{-1} = X_i^{-1} X_i = 1, X_i X_j = q^{2\sigma_{ij}} X_j X_i.$$

Each X_i is associated to the i -th edge in the ideal triangulation τ , and σ_{ij} is an integer determined by τ , see [BL07; BW11; Liu09] for more details. If we replace q with $q^{\frac{1}{4}}$, we get the so called Chekhov-Fock square root algebra $\mathcal{T}_\tau^{q^{\frac{1}{4}}}$. It is well known that \mathcal{T}_τ^q is an Ore domain. We will use $\hat{\mathcal{T}}_\tau^q$ to denote the ring of fractions of \mathcal{T}_τ^q (that is the localization over all nonzero elements).

Let τ, τ' be any two ideal triangulations for S . Then there is an algebra isomorphism $\Phi_{\tau\tau'}^q : \hat{\mathcal{T}}_{\tau'}^q \rightarrow \hat{\mathcal{T}}_\tau^q$, called be the Chekhov-Fock coordinate change isomorphism [Liu09].

For an ideal triangulation τ , there are two operations. (1) Reindexing: obtain a new ideal triangulation τ' by reindexing all the edges in τ . (2) Diagonal exchange: for any $1 \leq i \leq m$, define a new ideal triangulation $\tau' = \{e'_1, \dots, e'_m\}$, where $e'_j = e_j$ for every $j \neq i$ and e'_i is the other diagonal of the square formed by the two faces of τ that are adjacent to e_i .

Let τ be an ideal triangulation, let $a = (a_1, \dots, a_m)$ be an edge weight system for τ . Suppose $\tau' = \{e'_1, \dots, e'_m\}$ is obtained from τ by reindexing such that $e'_i = e_{\sigma(i)}$ for $1 \leq i \leq m$, where σ is a permutation for $\{1, \dots, m\}$. Then we define an edge weight system $a' = (a'_1, \dots, a'_m)$ for τ' by setting $a'_i = a_{\sigma(i)}$ for $1 \leq i \leq m$. If τ' is obtained from τ by the diagonal exchange, we define an edge weight system a' for τ'

using formulas in Proposition 3 in [Liu09]. We will say a' is an edge weight system for τ' derived from the pair (τ, a) . Then (τ', a') determines the same character in $\mathcal{X}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ as (τ, a) [BL07; BWY21].

A sequence of ideal triangulations $\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)}$ is called an ideal triangulation sweep if, for each $1 \leq i \leq k-1$, we have $\tau^{(i+1)}$ is obtained from $\tau^{(i)}$ by reindexing or the diagonal exchange. A sequence of edge weight systems $a^{(0)}, a^{(1)}, \dots, a^{(k)}$ is called an edge weight system sweep for the ideal triangulation sweep $\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)}$, if the edge weight system $a^{(i+1)}$ for $\tau^{(i+1)}$ is derived from $(\tau^{(i)}, a^{(i)})$ for each $0 \leq i \leq k-1$. Note that the sequence $a^{(0)}, a^{(1)}, \dots, a^{(k)}$ is completely determined by $a^{(0)}$. If in addition $a^{(0)} = a^{(k)}$, we call the sequence $a^{(0)}, a^{(1)}, \dots, a^{(k)}$ a periodic edge weight system for the ideal triangulation sweep $\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)}$.

Suppose q is a primitive n -root of unity with n odd. Let φ be an orientation preserving diffeomorphism for surface S , and let $\tau = \tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)} = \varphi(\tau)$ be an ideal triangulation sweep. Suppose $a = a^{(0)}, a^{(1)}, \dots, a^{(k)} = a$ is a periodic edge weight system for $\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)}$ (the existence of the periodic edge weight system is guaranteed by Lemma 11 in [BWY21]), which defines a φ -invariant character $[\bar{\gamma}] \in \mathcal{X}_{\mathrm{PSL}_2(\mathbb{C})}(S)$. Then, for each puncture v , we can choose a nonzero complex number h_v such that $h_v = h_{\varphi(v)}$ for every puncture v and $(h_v)^n = a_{i_1} a_{i_2} \dots a_{i_j}$ for every puncture v adjacent to the edges $e_{i_1}, e_{i_2}, \dots, e_{i_j}$. From Proposition 13 in [BWY21], we know a and puncture weights h_v uniquely determine an irreducible representation $\bar{\rho} : \mathcal{T}_\tau^q \rightarrow \mathrm{End}(V)$ for the Chekhov-Fock algebra \mathcal{T}_τ^q such that $\bar{\rho}(X_i^n) = a_i$ for $1 \leq i \leq m$ and $\bar{\rho}(H_v) = h_v$ for each puncture v , where H_v is a central element in \mathcal{T}_τ^q associated to each puncture v . Let $\Phi_{\tau\varphi(\tau)}^q : \hat{\mathcal{T}}_{\varphi(\tau)}^q \rightarrow \hat{\mathcal{T}}_\tau^q$ be the Chekhov-Fock coordinate change isomorphism, and let $\Psi_{\varphi(\tau)\tau}^q : \hat{\mathcal{T}}_\tau^q \rightarrow \hat{\mathcal{T}}_{\varphi(\tau)}^q$ be the algebra isomorphism induced by φ . Then $\bar{\rho} \simeq \bar{\rho} \circ \Phi_{\tau\varphi(\tau)}^q \circ \Psi_{\varphi(\tau)\tau}^q$, so there exists an intertwiner $\bar{\Lambda}_{\varphi, \bar{\gamma}}^q$ such that

$$\bar{\rho} \circ \Phi_{\tau\varphi(\tau)}^q \circ \Psi_{\varphi(\tau)\tau}^q(X) = \bar{\Lambda}_{\varphi, \bar{\gamma}}^q \circ \bar{\rho}(X) \circ (\bar{\Lambda}_{\varphi, \bar{\gamma}}^q)^{-1}$$

for every $X \in \mathcal{T}_\tau^q$.

Under certain conditions, the trace of intertwiners obtained from Chekhov-Fock algebras equals the trace of intertwiners obtained from skein algebras, see Theorem 16 in [BWY21]. We will use this equality to calculate the trace of intertwiners obtained from skein algebras for the once punctured torus in section 7.4.

From now on, we will always assume $q^{1/2}$ is a primitive n -root of -1 with n odd.

7.2 Irreducible representation construction for $SK_{q^{1/2}}(T^2)$ and $SK_{q^{1/2}}(S_{1,1})$

In order to get Kauffman bracket intertwiners, we want to find the explicit irreducible representations associated to given characters and puncture weights. Here we construct irreducible representations for skein algebras of the closed torus T^2 and the once punctured torus $S_{1,1}$. In section 7.3, we will use these irreducible representations to calculate intertwiners for the closed torus.

7.2.1 An algebraic embedding for $SK_{q^{1/2}}(T^2)$

Let $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$ be the algebra generated by X, X^{-1}, Y, Y^{-1} , subject to the relations $XY = qYX, XX^{-1} = X^{-1}X = 1, YY^{-1} = Y^{-1}Y = 1$. Frohman-Gelca built an algebraic embedding [FG00]:

$$G_{q^{1/2}} : SK_{q^{1/2}}(T^2) \rightarrow \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$$

$$(a, b)_T \mapsto \theta_{(a,b)} + \theta_{(-a,-b)},$$

where $(a, b)_T$ is the simple link associated to two integers a, b , and $\theta_{(a,b)} = q^{-ab/2} X^a Y^b$.

If $\gcd(a, b) = 1$ (with the convention that $\gcd(\pm 1, 0) = \gcd(0, \pm 1) = 1$), then $(a, b)_T$ is represented by the simple knot (a, b) in $\mathbb{R}^2/\mathbb{Z}^2$ with vertical framing. If $\gcd(a, b) = k$ and $a = a'k, b = b'k$, then $(a, b)_T = T_k((a', b'))$ where T_k is the k -th

Chebyshev polynomial of the first type. We have $\theta_{(a,b)}\theta_{(c,d)} = q^{1/2} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \theta_{(a+c, b+d)}$, and $(\theta_{(a,b)})^{-1} = \theta_{(-a,-b)}$. Since $\theta_{(a,b)} + \theta_{(-a,-b)} = q^{-ab/2}(X^a Y^b + X^{-a} Y^{-b})$, then $\text{Im}G_{q^{1/2}} = \text{span} \langle X^a Y^b + X^{-a} Y^{-b} | (a, b) \in \mathbb{Z} \times \mathbb{Z} \rangle$.

Let $T_{q^{1/2}}$ be the Chebyshev homomorphism from the skein algebra $SK_{-1}(T^2)$ to $SK_{q^{1/2}}(T^2)$ defined in [BW16a], and let $F_{q^{1/2}} : \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{-1} \rightarrow \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$ defined by $X \mapsto X^n, Y \mapsto Y^n$. It is easy to check that we have $F_{q^{1/2}}G_{-1} = G_{q^{1/2}}T_{q^{1/2}}$.

7.2.2 Irreducible representations for $SK_{q^{1/2}}(T^2)$

Bonahon-Liu described the irreducible representations of $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$ [BL07]. Let V denote the n -dimensional vector space over the complex field with basis e_0, e_1, \dots, e_{n-1} , and let u, v be any two nonzero complex numbers. Set $\rho_{u,v}(X)e_i = uq^i e_i$, $\rho_{u,v}(Y)e_i = ve_{i+1}$, where the indices are considered modulo n , then $\rho_{u,v}$ is an irreducible representation. Any irreducible representation of $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$ is isomorphic to a representation $\rho_{u,v}$, and $\rho_{u,v} \simeq \rho_{u',v'}$ if and only if $u^n = (u')^n, v^n = (v')^n$.

It is well-known that $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ where $\alpha = (1, 0)$ and $\beta = (0, 1)$. For any $[\gamma] \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(T^2)$, we have $[\gamma]$ has a representative γ such that $\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and $\gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$ because $\pi_1(T^2)$ is commutative.

For any given character $[\gamma] \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(T^2)$, the following Theorem offers a representation of $SK_{q^{1/2}}(T^2)$ whose classical shadow is $[\gamma]$. For this Theorem, we use the fact that $ab + a + b \equiv \text{gcd}(a, b) \pmod{2}$ for any two integers a, b (recall that $\text{gcd}(\pm 1, 0) = \text{gcd}(0, \pm 1) = 1$).

Theorem 7.2.1. *Choose $u, v \in \mathbb{C}$ such that $u^n = -\lambda_1, v^n = -\lambda_2$ or $u^n = -\lambda_1^{-1}, v^n = -\lambda_2^{-1}$, then the classical shadow of $\rho_{u,v}G_{q^{1/2}}$ is $[\gamma]$.*

Proof. To show the classical shadow of $\rho_{u,v}G_{q^{1/2}}$ is $[\gamma]$, it suffices to show $\rho_{u,v}G_{q^{1/2}}(T_{q^{1/2}}((a, b)_T)) = \text{Tr}^\gamma((a, b)_T)Id_V$ for all $(a, b)_T \in SK_{-1}(T^2)$. First we have

$$\begin{aligned} \rho_{u,v}G_{q^{1/2}}(T_{q^{1/2}}((a, b)_T)) &= \rho_{u,v}(F_{q^{1/2}}G_{-1}((a, b)_T)) \\ &= \rho_{u,v}(F_{q^{1/2}}(\theta_{(a,b)} + \theta_{(-a,-b)})) = \rho_{u,v}(\theta_{(na,nb)} + \theta_{(-na,-nb)}) \\ &= \rho_{u,v}((-1)^{ab}X^{na}Y^{nb} + (-1)^{ab}X^{-na}Y^{-nb}) \\ &= (-1)^{ab}[(\rho_{u,v}(X))^{na}(\rho_{u,v}(Y))^{nb} + (\rho_{u,v}(X))^{-na}(\rho_{u,v}(Y))^{-nb}] \\ &= (-1)^{ab}[(u^n)^a(v^n)^b + (u^n)^{-a}(v^n)^{-b}]Id_V \\ &= (-1)^{ab+a+b}[\lambda_1^a\lambda_2^b + \lambda_1^{-a}\lambda_2^{-b}]Id_V. \end{aligned}$$

Suppose $\text{gcd}(a, b) = d$ and $a = a'd, b = b'd$, then we have

$$\begin{aligned} \text{Tr}^\gamma((a, b)_T) &= \text{Tr}^\gamma(T_d((a', b'))) = T_d(\text{Tr}^\gamma((a', b'))) \\ &= T_d(-\text{Trace}(\gamma((a', b')))) = T_d(-\text{Trace}(\gamma(a'\alpha + b'\beta))) \end{aligned}$$

$$\begin{aligned}
&= T_d(-\text{Trace}((\gamma(\alpha))^{a'}(\gamma(\beta))^{b'})) = T_d((- \lambda_1^{a'} \lambda_2^{b'}) + (- \lambda_1^{-a'} \lambda_2^{-b'})) \\
&= (- \lambda_1^{a'} \lambda_2^{b'})^d + (- \lambda_1^{-a'} \lambda_2^{-b'})^d = (-1)^d [\lambda_1^{da'} \lambda_2^{BL1997rings'} + \lambda_1^{-da'} \lambda_2^{-BL1997rings'}] \\
&= (-1)^{ab+a+b} [\lambda_1^a \lambda_2^b + \lambda_1^{-a} \lambda_2^{-b}].
\end{aligned}$$

□

We can easily get the following Theorem by using the representation theory.

Theorem 7.2.2. *Under the same assumption as in Theorem 7.2.1, we have the following conclusions:*

(a) *if $\lambda_1 \neq \pm 1$ or $\lambda_2 \neq \pm 1$, the representation $\rho_{u,v} G_{q^{1/2}}$ is irreducible.*

(b) *if $\lambda_1 = \pm 1$ and $\lambda_2 = \pm 1$, we have V has only two irreducible subrepresentations V_1, V_2 , and $V = V_1 \oplus V_2$, and $\dim(V_1) = (n+1)/2$, $\dim(V_2) = (n-1)/2$, especially $V_1 = \text{span}\langle e_0, e_1 + e_{n-1}, e_2 + e_{n-2}, \dots, e_{(n-1)/2} + e_{(n+1)/2} \rangle$ and $V_2 = \text{span}\langle e_1 - e_{n-1}, e_2 - e_{n-2}, \dots, e_{(n-1)/2} - e_{(n+1)/2} \rangle$ if $u = \pm 1, v = \pm 1$.*

rem

The Azumaya locus of $SK_{q^{1/2}}(T^2)$ is a subset of $\mathcal{X}_{\text{SL}_2(\mathbb{C})}(T^2)$. An element in $\mathcal{X}_{\text{SL}_2(\mathbb{C})}(T^2)$ lives in the Azumaya locus if it corresponds to a unique irreducible representation of $SK_{q^{1/2}}(T^2)$ (the correspondence is the one in Theorem 7.1.1). We know the PI-dimension of $SK_{q^{1/2}}(T^2)$ is n . Then a character $[\gamma] \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(T^2)$ lives in the Azumaya locus if and only if there exists an irreducible representation of $SK_{q^{1/2}}(T^2)$ of dimension n whose classical shadow is $[\gamma]$. So from Theorem 7.2.2, we get $[\gamma]$ lives in the Azumaya locus if and only if $\lambda_1 \neq \pm 1$ or $\lambda_2 \neq \pm 1$, where $\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and $\gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$. In [KK22], the authors proved that the character lives in the Azumaya locus of the skein algebra of a closed surface if and only if the character is noncentral.

In [BW16b], Bonahon-Wong proved the Witten-Reshetikhin-Turaev representation of the Kauffman bracket skein algebra is irreducible and whose classical shadow is the trivial character. For the closed torus T^2 , we use V_{T^2} to denote the Witten-Reshetikhin-Turaev representation of $SK_{q^{1/2}}(T^2)$. We know $\dim V_{T^2} = \frac{n-1}{2}$ with basis $v_1, v_2, \dots, v_{\frac{n-1}{2}}$ where v_k is the skein in the solid torus represented by $2k - 2$

nontrivial parallel closed curves, which are parallel to the core of the solid torus, with the $(2k - 2)$ -th Jones-Wenzel idempotent inserted. In Theorem 7.2.2 with $\lambda_1 = \lambda_2 = 1$, we have V_2 is isomorphic to V_{T^2} as representations for $SK_{q^{1/2}}(T^2)$, and the isomorphism is given by

$$e_{2k-1} - e_{n-2k+1} \mapsto v_k, \forall 1 \leq k \leq \frac{n-1}{2}.$$

7.2.3 Irreducible representations for $SK_{q^{1/2}}(S_{1,1})$

We want to find the explicit irreducible representations of $SK_{q^{1/2}}(S_{1,1})$ corresponding to given characters and puncture weights. Let $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$ be the algebra generated by X_1, X_2, X_3 subject to the relations:

$$X_1X_2 = qX_2X_1, X_2X_3 = qX_3X_2, X_3X_1 = qX_1X_2, X_iX_i^{-1} = X_i^{-1}X_i = 1.$$

We have $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}] = \mathcal{T}_\tau^{q^{1/4}}(S_{1,1})$ where τ is the ideal triangulation in Figure 7.2.

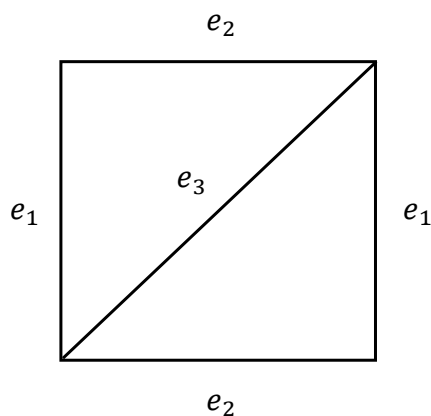
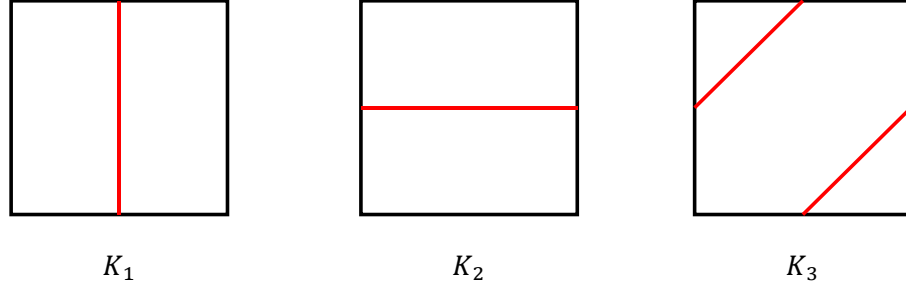


FIGURE 7.2: The ideal triangulation τ .

We define the skeins K_1, K_2, K_3 in the skein algebra $SK_{q^{1/2}}(S_{1,1})$ using Figure 7.3.

FIGURE 7.3: Skeins K_1, K_2, K_3 .

According to [BP00], the algebra $SK_{q^{1/2}}(S_{1,1})$ is generated by K_1, K_2, K_3 subject to relations:

$$\begin{aligned} q^{-1/2}K_1K_2 - q^{1/2}K_2K_1 &= (q^{-1} - q)K_3, \\ q^{-1/2}K_2K_3 - q^{1/2}K_3K_2 &= (q^{-1} - q)K_1, \\ q^{-1/2}K_3K_1 - q^{1/2}K_1K_3 &= (q^{-1} - q)K_2. \end{aligned}$$

Let P be the loop around the puncture in $S_{1,1}$, then

$$P = q^{-1/2}K_1K_2K_3 - q^{-1}K_1^2 - qK_2^2 - q^{-1}K_3^2 + q + q^{-1}.$$

Lemma 7.2.3. *There is an algebraic embedding $F : S_{q^{1/2}}(S_{1,1}) \rightarrow \mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$ such that*

$$\begin{aligned} F(K_1) &= [X_2X_3] + [X_2^{-1}X_3^{-1}] + [X_2X_3^{-1}], \\ F(K_2) &= [X_3X_1] + [X_3^{-1}X_1^{-1}] + [X_3X_1^{-1}], \\ F(K_3) &= [X_1X_2] + [X_1^{-1}X_2^{-1}] + [X_1X_2^{-1}], \\ F(P) &= [X_1^2X_2^2X_3^2] + [X_1^{-2}X_2^{-2}X_3^{-2}]. \end{aligned} \tag{7.1}$$

Proof. Actually F is just the quantum trace map constructed in Theorem 11 in [BW11] if we regard $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$ as the Chekhov-Fock square root algebra associated to the ideal triangulation in Figure 7.2 where X_i corresponds to e_i for $i = 1, 2, 3$. \square

Let V be an n dimensional vector space over the complex field with basis w_0, w_1, \dots, w_{n-1} .

We can define a representation $\rho_{r_1, r_2, r_3} : \mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}] \rightarrow \text{End}(V)$ such that

$$\begin{aligned} \rho_{r_1, r_2, r_3}(X_1)w_i &= r_1q^i w_i, \\ \rho_{r_1, r_2, r_3}(X_2)w_i &= r_2q^{-i} w_{i+1}, \\ \rho_{r_1, r_2, r_3}(X_3)w_i &= r_3w_{i-1}, \end{aligned} \tag{7.2}$$

where r_1, r_2, r_3 are nonzero complex numbers. We can get $\rho_{r_1, r_2, r_3}([X_1X_2X_3]) = r_1r_2r_3q^{1/2}Id_V$.

Lemma 7.2.4. *For any three nonzero complex numbers r_1, r_2, r_3 , we have ρ_{r_1, r_2, r_3} is an irreducible representation of $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$. Furthermore every irreducible representation of $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$ is isomorphic to a representation ρ_{r_1, r_2, r_3} , and $\rho_{r_1, r_2, r_3} \simeq \rho_{s_1, s_2, s_3}$ if and only if*

$$r_1^n = s_1^n, r_2^n = s_2^n, r_3^n = s_3^n, r_1r_2r_3 = s_1s_2s_3.$$

For any $\gamma \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(S_{1,1})$ and a nonzero complex number p such that $T_n(p) = -\text{Trace}\gamma(P)$ where P is the loop going around the only puncture in $S_{1,1}$, let $t_i = -\text{Trace}\gamma(K_i)$ for $i = 1, 2, 3$. According to [Tak15], we have

$$T_n(p) = -t_1t_2t_3 - t_1^2 - t_2^2 - t_3^2 + 2.$$

Lemma 7.2.5. *Let x, y be two indeterminates such that $xy = q^{-2}yx$, then $T_n(x + x^{-1} + y) = x^{-n} + x^n + y^n$ for $n \geq 1$.*

For any given character $[\gamma] \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(S_{1,1})$, the following Theorem offers a representation of $SK_{q^{1/2}}(S_{1,1})$ whose classical shadow is $[\gamma]$.

Theorem 7.2.6. *With the above notation, we have $\rho_{r_1, r_2, r_3}F$ is a representation of $SK_{q^{1/2}}(S_{1,1})$. The classical shadow of $\rho_{r_1, r_2, r_3}F$ is γ and $\rho_{r_1, r_2, r_3}F(P) = pId_V$ if and only if we have the following equations:*

$$\begin{aligned} r_2^n r_3^n + r_2^{-n} r_3^{-n} + r_2^n r_3^{-n} &= -t_1, \\ r_3^n r_1^n + r_3^{-n} r_1^{-n} + r_3^n r_1^{-n} &= -t_2, \\ r_1^n r_2^n + r_1^{-n} r_2^{-n} + r_1^n r_2^{-n} &= -t_3, \\ r_1^2 r_2^2 r_3^2 q + r_1^{-2} r_2^{-2} r_3^{-2} q^{-1} &= p. \end{aligned} \tag{7.3}$$

Proof. It is easy to see

$$\begin{aligned} [X_2X_3][X_2X_3^{-1}] &= q^{-2}[X_2X_3^{-1}][X_2X_3], \\ [X_3X_1][X_3X_1^{-1}] &= q^{-2}[X_3X_1^{-1}][X_3X_1], \\ [X_1X_2][X_1X_2^{-1}] &= q^{-2}[X_1X_2^{-1}][X_1X_2]. \end{aligned}$$

From Lemma 7.2.5, we get

$$\begin{aligned} T_n(\rho_{r_1, r_2, r_3} F(K_1)) &= T_n(\rho_{r_1, r_2, r_3} ([X_2X_3] + [X_2^{-1}X_3^{-1}] + [X_2X_3^{-1}])) \\ &= \rho_{r_1, r_2, r_3} (T_n([X_2X_3] + [X_2^{-1}X_3^{-1}] + [X_2X_3^{-1}])) \\ &= \rho_{r_1, r_2, r_3} ([X_2^n X_3^n] + [X_2^{-n} X_3^{-n}] + [X_2^n X_3^{-n}]) \\ &= \rho_{r_1, r_2, r_3} (-(X_2^n X_3^n + X_2^{-n} X_3^{-n} + X_2^n X_3^{-n})) \\ &= -(r_2^n r_3^n + r_2^{-n} r_3^{-n} + r_2^n r_3^{-n}) Id_V. \end{aligned} \tag{7.4}$$

Similarly we can get

$$\begin{aligned} T_n(\rho_{r_1, r_2, r_3} F(K_2)) &= -(r_3^n r_1^n + r_3^{-n} r_1^{-n} + r_3^n r_1^{-n}) Id_V, \\ T_n(\rho_{r_1, r_2, r_3} F(K_3)) &= -(r_1^n r_2^n + r_1^{-n} r_2^{-n} + r_1^n r_2^{-n}) Id_V. \end{aligned} \tag{7.5}$$

And

$$\begin{aligned} \rho_{r_1, r_2, r_3} F(P) &= \rho_{r_1, r_2, r_3} ([X_1^2 X_2^2 X_1^2] + [X_1^{-2} X_2^{-2} X_1^{-2}]) \\ &= \rho_{r_1, r_2, r_3} ([X_1^2 X_2^2 X_1^2]) + \rho_{r_1, r_2, r_3} ([X_1^{-2} X_2^{-2} X_1^{-2}]) \\ &= (r_1^2 r_2^2 r_3^2 q + r_1^{-2} r_2^{-2} r_3^{-2} q^{-1}) Id_V. \end{aligned} \tag{7.6}$$

From equations 7.4, 7.5, 7.6 and the fact that K_1, K_2, K_3 generate the algebra $SK_{-1}(S_{1,1})$, we can get the conclusions in Theorem 7.2.6.

□

rem At first glance, it seems like, in equation 7.3, we may not be able to get solutions, but actually the fourth one is a consequence of first three equations because we have the relation $T_n(p) = -t_1 t_2 t_3 - t_1^2 - t_2^2 - t_3^2 + 2$. In fact, to get solutions, we

only need to solve equations:

$$\begin{aligned} yz + y^{-1}z^{-1} + yz^{-1} &= -t_1, \\ zx + z^{-1}x^{-1} + zx^{-1} &= -t_2, \\ xy + x^{-1}y^{-1} + xy^{-1} &= -t_3. \end{aligned}$$

Let $Y_i = X_i^2$ for $i = 1, 2, 3$, then

$$Y_1Y_2 = q^4Y_2Y_1, Y_2Y_3 = q^4Y_3Y_2, Y_3Y_1 = q^4Y_1Y_3, Y_iY_i^{-1} = Y_i^{-1}Y_i = 1.$$

The subalgebra of $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$ generated by $Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}$ is $\mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}]$.

Here we recall a lemma from [BWY21] for irreducible representations for $\mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}]$.

Let V be an n dimensional vector space over \mathbb{C} with basis w_0, w_1, \dots, w_{n-1} . For any three nonzero complex numbers y_1, y_2, y_3 , define $\rho_{y_1, y_2, y_3} : \mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}] \rightarrow \text{End}(V)$ such that

$$\begin{aligned} \rho_{y_1, y_2, y_3}(Y_1)(w_i) &= y_1q^{4i}w_i, \\ \rho_{y_1, y_2, y_3}(Y_2)(w_i) &= y_2q^{-2i}w_{i+1}, \\ \rho_{y_1, y_2, y_3}(Y_3)(w_i) &= y_3q^{-2i}w_{i-1}. \end{aligned} \tag{7.7}$$

Lemma 7.2.7 ([BWY21]). (1) For any three nonzero complex numbers y_1, y_2, y_3 , the representation ρ_{y_1, y_2, y_3} is irreducible.

(2) Every irreducible representation of $\mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}]$ is isomorphic to a representation ρ_{y_1, y_2, y_3} .

(3) For ρ_{y_1, y_2, y_3} and $\rho_{y'_1, y'_2, y'_3}$, they are isomorphic if and only if $y_1^n = (y'_1)^n, y_2^n = (y'_2)^n, y_3^n = (y'_3)^n$ and $y_1y_2y_3 = y'_1y'_2y'_3$.

7.3 Calculation of intertwiners for the closed torus

7.3.1 Construction of intertwiners for the closed torus

The mapping class group of the closed torus is $SL(2, \mathbb{Z})$ [FM11]. For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, we hope to find invariant characters under A . For a $[\gamma] \in$

$\mathcal{X}_{SL_2(\mathbb{C})}(T^2)$, we choose a representative with $\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and $\gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$

where α and β denote loops $(1, 0)$ and $(0, 1)$ in $\mathbb{R}^2/\mathbb{Z}^2$ respectively. We have $[\gamma]$ is invariant under A if and only if $\text{Trace}(\gamma(A_*(z))) = \text{Trace}(\gamma(z))$ for all $z \in \pi_1(T^2)$.

For any $(k_1, k_2) \in \mathbb{Z} \oplus \mathbb{Z}$, we have

$$A_*(k_1\alpha + k_2\beta) = (k_1, k_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (k_1a + k_2c, k_1b + k_2d) = (k_1a + k_2c)\alpha + (k_1b + k_2d)\beta,$$

$$\begin{aligned} \gamma(A_*(k_1\alpha + k_2\beta)) &= \gamma[(k_1a + k_2c)\alpha + (k_1b + k_2d)\beta] = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}^{k_1a+k_2c} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}^{k_1b+k_2d} \\ &= \begin{pmatrix} \lambda_1^{k_1a+k_2c} \lambda_2^{k_1b+k_2d} & 0 \\ 0 & \lambda_1^{-(k_1a+k_2c)} \lambda_2^{-(k_1b+k_2d)} \end{pmatrix}, \end{aligned}$$

$$\gamma(k_1\alpha + k_2\beta) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}^{k_1} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}^{k_2} = \begin{pmatrix} \lambda_1^{k_1} \lambda_2^{k_2} & 0 \\ 0 & \lambda_1^{-k_1} \lambda_2^{-k_2} \end{pmatrix}.$$

Then it is easy to show that $[\gamma]$ is A -invariant if and only if $\lambda_1 = \lambda_1^a \lambda_2^b$, $\lambda_2 = \lambda_1^c \lambda_2^d$ or $\lambda_1 = \lambda_1^{-a} \lambda_2^{-b}$, $\lambda_2 = \lambda_1^{-c} \lambda_2^{-d}$.

And A also induces two algebra isomorphisms $F_{A,+}$ and $F_{A,-}$ from $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$ to itself defined by

$$F_{A,+}(\theta_{(i,j)}) = \theta_{(i,j)A}, F_{A,-}(\theta_{(i,j)}) = \theta_{(-i,-j)A}.$$

Then $F_{A,+}$ and $F_{A,-}$ are well-defined because

$$\theta_{(i,j)A} \theta_{(k,l)A} = q^{1/2} \begin{bmatrix} (i,j)A \\ (k,l)A \end{bmatrix} \theta_{(i+k,j+l)A} = q^{1/2} \begin{bmatrix} i & j \\ k & l \end{bmatrix}_{[A]} \theta_{(i+k,j+l)A} = q^{1/2} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \theta_{(i+k,j+l)A},$$

and similarly

$$\theta_{(-i,-j)A}\theta_{(-k,-l)A} = q^{1/2} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \theta_{(-i-k,-j-l)A}.$$

In section 7.2, we know there is an embedding

$$G_{q^{1/2}} : SK_{q^{1/2}}(T^2) \rightarrow \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}.$$

For the following discussion we will omit the subscript for $G_{q^{1/2}}$ when there is no confusion.

Lemma 7.3.1. *The following diagram is commutative:*

$$\begin{array}{ccc} SK_{q^{1/2}}(T^2) & \xrightarrow{G} & \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}} \\ \downarrow A_{\#} & & \downarrow F_A \\ SK_{q^{1/2}}(T^2) & \xrightarrow{G} & \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}} \end{array}$$

where F_A is $F_{A,+}$ or $F_{A,-}$.

Proof. We only prove the case when $F_A = F_{A,+}$.

First we show $A_{\#}((k, l)_T) = ((k, l)A)_T$. Assume $\gcd(k, l) = j$ and $k = k'j, l = l'j$, then we have

$$A_{\#}((k, l)_T) = A_{\#}(T_j((k', l'))) = T_j(A_{\#}((k', l'))) = T_j((k', l')A) = T_j(ak' + cl', bk' + dl').$$

There exist integers u, v such that $uk + vl = j$, then $\begin{bmatrix} k & l \\ -v & u \end{bmatrix} = j$ and $\det\left(\begin{bmatrix} k & l \\ -v & u \end{bmatrix} A\right) =$

$$j. \text{ Then } \det\left(\begin{bmatrix} k & l \\ -v & u \end{bmatrix} A\right) = \begin{bmatrix} ak + cl & bk + dl \\ -v' & u' \end{bmatrix} = (ak + cl)u' + (bk + dl)v' = j.$$

We also have $j|(ak + cl), j|(bk + dl)$. Thus $\gcd(ak + cl, bk + dl) = j$ and $ak + cl = j(ak' + cl'), bk + dl = j(bk' + dl')$. Then

$$A_{\#}((k, l)_T) = T_j(ak' + cl', bk' + dl') = (ak + cl, bk + dl)_T = ((k, l)A)_T.$$

So $GA_{\#}((k, l)_T) = G((k, l)A)_T = \theta_{(k,l)A} + \theta_{(k,l)A}^{-1}$ and $F_AG((k, l)_T) = F_A(\theta_{(k,l)} + \theta_{(k,l)}^{-1}) = \theta_{(k,l)A} + \theta_{(k,l)A}^{-1}$. We have $GA_{\#} = F_AG$ because all $(k, l)_T$ span the skein algebra. \square

$$\begin{array}{ccc}
SK_{q^{1/2}}(T^2) & \xrightarrow{G} & \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}} \xrightarrow{\rho_{u,v}} \text{End}(V) \\
\downarrow A_{\sharp} & & \downarrow F_A \quad \quad \quad \downarrow G_{\Lambda} \\
SK_{q^{1/2}}(T^2) & \xrightarrow{G} & \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}} \xrightarrow{\rho_{u,v}} \text{End}(V)
\end{array} \tag{7.8}$$

The following two Theorems give the intertwiners for the closed surface for all the diffeomorphisms. We will give explicit formulas for these intertwiners and their Trace in the following subsections.

Theorem 7.3.2. *In the diagram 7.8, suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $F_A = F_{A,+}$. Let $[\gamma] \in \mathcal{X}_{SL_2(\mathbb{C})}(T^2)$ with $\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and $\gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$ where $\lambda_1 = \lambda_1^a \lambda_2^b$, $\lambda_2 = \lambda_1^c \lambda_2^d$, and let u, v be two complex numbers such that $u^n = -\lambda_1$, $v^n = -\lambda_2$. We have the following conclusions:*

(a) $[\gamma]$ is invariant under A ;

(b) the classical shadow of $\rho_{u,v}G$ is $[\gamma]$;

(c) $\rho_{u,v}F_A \simeq \rho_{u,v}$.

(d) From (c), we know there exists an intertwiner $\Lambda_{n,+}$ such that $\rho_{u,v}F_A(Z) = \Lambda_{n,+}\rho_{u,v}(Z)\Lambda_{n,+}^{-1}$ for all $Z \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$. Then this intertwiner induces an intertwiner between two irreducible representations of the skein algebra.

Proof. (a) and (b) are already shown in the previous discussion.

To prove (c), we have

$$\begin{aligned}
\rho_{u,v}F_A(X^n) &= \rho_{u,v}F_A(\theta_{(n,0)}) = \rho_{u,v}(\theta_{(n,0)A}) \\
&= \rho_{u,v}(\theta_{(na,nb)}) = \rho_{u,v}((-1)^{ab}X^{na}Y^{nb}) \\
&= (-1)^{ab}u^{na}v^{nb}Id_V = (-1)^{ab}(-\lambda_1)^a(-\lambda_2)^bId_V \\
&= (-1)^{ab+a+b}\lambda_1^a\lambda_2^bId_V = -\lambda_1Id_V = u^nId_V.
\end{aligned}$$

Similarly we can show $\rho_{u,v}F_A(Y^n) = v^nId_V$, thus $\rho_{u,v}F_A \simeq \rho_{u,v}$.

(d) If $\lambda_1 \neq \pm 1$ or $\lambda_2 \neq \pm 1$, Theorem 7.2.2 implies that $\Lambda_{n,+}$ itself is the intertwiner between two irreducible representations of the skein algebra. If $\lambda_1 = \pm 1$ and $\lambda_2 =$

± 1 , Theorem 7.2.2 implies that V has only two irreducible subrepresentations V_1, V_2 and $\dim(V_1) = (n+1)/2, \dim(V_2) = (n-1)/2$. We have $\Lambda_{n,+}(V_1)$ is an irreducible subrepresentation of V and $\dim(\Lambda_{n,+}(V_1)) = (n+1)/2$. Then $\Lambda_{n,+}(V_1) = V_1$. This shows $\Lambda_{n,+}|_{V_1}$ is an intertwiner for V_1 . Similarly $\Lambda_{n,+}|_{V_2}$ is an intertwiner for V_2 . \square

Theorem 7.3.3. *In the diagram 7.8, suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $F_A = F_{A,-}$. Let $[\gamma] \in \mathcal{X}_{SL_2(\mathbb{C})}(T^2)$ with $\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and $\gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$ where $\lambda_1 = \lambda_1^{-a} \lambda_2^{-b}, \lambda_2 = \lambda_1^{-c} \lambda_2^{-d}$, and let u, v be two complex numbers such that $u^n = -\lambda_1^{-1}, v^n = -\lambda_2^{-1}$. We have the following conclusions:*

(a) $[\gamma]$ is invariant under A ;

(b) the classical shadow of $\rho_{u,v}G$ is $[\gamma]$;

(c) $\rho_{u,v}F_A \simeq \rho_{u,v}$.

(d) From (c), we know there exists an intertwiner $\Lambda_{n,-}$ such that $\rho_{u,v}F_A(Z) = \Lambda_{n,-}\rho_{u,v}(Z)\Lambda_{n,-}^{-1}$ for all $Z \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$. Then this intertwiner induces an intertwiner between two irreducible representations of the skein algebra.

Proof. The proof is the same as in Theorem 7.3.2. \square

Note that a rescaling of the intertwiner $\Lambda_{n,+}$ in Theorem 7.3.2 such that $|\det(\Lambda_{n,+})| = 1$ makes $|\text{Trace}\Lambda_{n,+}|$ independent of the choice of u and v . The same thing holds for the intertwiner in Theorem 7.3.3.

For the following discussion, we always require F_A to be $F_{A,+}$ unless specified otherwise (parallel results hold for $F_{A,-}$). From the above discussion, we know there exists an intertwiner $\Lambda_n \in \text{End}(V)$ such that the diagram 7.8 commutes, where $G_\Lambda(B) = \Lambda_n B \Lambda_n^{-1}, \forall B \in \text{End}(V)$. Next we are going to find an intertwiner $\bar{\Lambda}_n$ under the assumption in Theorem 7.3.2.

7.3.2 Calculation for intertwiners

Under the assumption of Theorem 7.3.2, we have $\rho_{u,v}F_A \simeq \rho_{u,v}$. For any $a \in V$ and $Z \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$, we use $Z \cdot a$ and $Z \star a$ to denote $\rho_{u,v}(Z)(a)$ and $\rho_{u,v}F_A(Z)(a)$

respectively. Then we are trying to find $\bar{\Lambda}_n \in \text{End}(V)$ such that $\bar{\Lambda}_n(X \cdot a) = X \star (\bar{\Lambda}_n(a))$ and $\bar{\Lambda}_n(Y \cdot a) = Y \star (\bar{\Lambda}_n(a))$ for all $a \in V$.

rem Assume $\gcd(b, n) = m$ and $n = n' m$. There exist two integers r, s such that $br + sn = m$. Then we have

$$\begin{aligned} & (v^{n'} b u^{n'(a-1)} q^{ab(n')^2/2})^m = v^{mn'} b u^{mn'(a-1)} q^{abm(n')^2/2} \\ & = v^{nb} u^{n(a-1)} q^{abn'/2} = (-\lambda_2)^b (-\lambda_1)^{a-1} (-1)^{abn'} \\ & = (-1)^{ab+b+a-1} \lambda_2^b \lambda_1^{a-1} = 1, \end{aligned} \quad (7.9)$$

and $q^{an'}$ is a primitive m -root of unity. Then there exists a unique integer $0 \leq k_0 \leq m - 1$ such that $(v^{n'} b u^{n'(a-1)} q^{ab(n')^2/2}) q^{an' k_0} = 1$ and $(v^{n'} b u^{n'(a-1)} q^{ab(n')^2/2}) q^{an' k} \neq 1$ for $k \neq k_0, 0 \leq k \leq m - 1$. We set $r_{k_0} = 1$ and $r_k = 0$ for $k \neq k_0, 0 \leq k \leq m - 1$, and define $r_{k+tb} = r_k v^{tb} u^{t(a-1)} q^{a(tk+bt^2/2)}, \forall 0 \leq k \leq m - 1, t \in \mathbb{Z}$, where we consider all indices modulo n . Since $\gcd(b, n) = m$ and $n = mn'$, we can reach all the indices. It is an easy check that $r_{k_1+t_1b} = r_{k_2+t_2b}$ if $k_1 + t_1b \equiv k_2 + t_2b \pmod{n}$. Then r_k is well-defined for each $0 \leq k \leq n - 1$.

It is easy to check that we have $r_{k+tb} = r_k v^{tb} u^{t(a-1)} q^{atk+abt^2/2}$ for all $k, t \in \mathbb{Z}$. Actually we have

$$r_{k_0+tb} = v^{tb} u^{t(a-1)} q^{a(tk_0+bt^2/2)}, \forall 0 \leq t \leq n' - 1,$$

and all other r_k are 0. We have $(v^b u^{a-1})^n = (v^n)^b (u^n)^{a-1} = (-\lambda_2)^b (-\lambda_1)^{a-1} = (-1)^{a+b-1} \lambda_1^{a-1} \lambda_2^b = (-1)^{a+b-1}$. Then we get $|r_k| = 0$ or 1 for all $k \in \mathbb{Z}$. From $br + sn = m$, we get $tm = tbr + tsn$ for all $t \in \mathbb{Z}$. Then we have

$$r_{k_0+tm} = r_{k_0+trb} = v^{trb} u^{tr(a-1)} q^{a(trk_0+bt^2r^2/2)}, \forall 0 \leq t \leq n' - 1,$$

and all other r_k are 0.

The following Lemma offers an explicit formula for the intertwiner constructed in Theorem 7.3.2 (d).

Lemma 7.3.4. *Under the assumption of Theorem 7.3.2, suppose $\bar{\Lambda}_n \in \text{End}(V)$ and*

$$\bar{\Lambda}_n(e_t) = \sum_{0 \leq k \leq n-1} (\bar{\Lambda}_n)_{k,t} e_k$$

for all $0 \leq t \leq n - 1$, where

$$(\bar{\Lambda}_n)_{k,t} = r_{k-td}(v^{(d-1)}u^c)^t q^{c(tk-dt^2/2)}.$$

Then $\bar{\Lambda}_n$ satisfies the conditions in Theorem 7.3.2 (d).

Proof. From direct calculations, we can get $\bar{\Lambda}_n(X \cdot e_t) = X \star (\bar{\Lambda}_n(e_t))$ and $\bar{\Lambda}_n(Y \cdot e_t) = Y \star (\bar{\Lambda}_n(e_t))$ for all $0 \leq t \leq n - 1$. \square

We have $(v^{d-1}u^c)^n = (v^n)^{d-1}(u^n)^c = (-\lambda_2)^{d-1}(-\lambda_1)^c = (-1)^{c+d-1}\lambda_1^c\lambda_2^{d-1} = (-1)^{c+d-1}$. Then we can get $|(\bar{\Lambda}_n)_{k,t}| = 0$ or 1 . We have $(\bar{\Lambda}_n)_{k,t} = 0$ if and only if $r_{k-td} = 0$. Then it is easy to show that $(\bar{\Lambda}_n)_{ld+km+k_0,l+tm}, 0 \leq l \leq m - 1, 0 \leq k, t \leq n' - 1$, are the only nonzero entries.

For each $0 \leq l \leq m - 1$, we define an $n' \times n'$ matrix B^l such that $(B^l)_{k,t} = (\bar{\Lambda}_n)_{ld+km+k_0,l+tm}$ for all $0 \leq k, t \leq n' - 1$. Then by Laplace expansion, we know $|\det(\bar{\Lambda}_n)| = \prod_{0 \leq l \leq m} |\det(B^l)|$.

From pure calculations, we can get $|\det(B^l)| = (n')^{n'/2}$. Then we have

$$|\det(\bar{\Lambda}_n)| = \prod_{0 \leq l \leq m} |\det(B^l)| = ((n')^{n'/2})^m = (n')^{mn'/2} = (n')^{n/2},$$

furthermore $|\det((n')^{-1/2}\bar{\Lambda}_n)| = 1$.

rem If $\tilde{\Lambda}_n$ is the intertwiner in Theorem 7.3.3, and we still suppose $\gcd(b, n) = m, br + sn = m, n = n'm$. We have

$$r_{k_0-tm} = (v^{-b}u^{-a-1})^{tr} q^{-trak_0+abt^2r^2/2}, \forall t \in \mathbb{Z}; r_k = 0, \text{ otherwise}$$

$$(\tilde{\Lambda}_n)_{k,t} = r_{k+td}(v^{-d-1}u^{-c})^t q^{-tck-cdt^2/2}, \forall 0 \leq k, t \leq n - 1,$$

where $0 \leq k_0 \leq m - 1$ such that

$$(v^{-b}u^{-a-1})^{n'} q^{ab(n')^2/2} q^{-n'ak_0} = 1.$$

Also we can get $|\det(\tilde{\Lambda}_n)| = (n')^{n/2}$.

7.3.3 On the trace of intertwiners

Bonahon-Wong-Yang only formulated the conjecture when the mapping tori are hyperbolic. So they considered surfaces with negative Euler characteristic because the mapping tori for the closed torus can never be hyperbolic. Since the simplicial volume of mapping tori for the closed torus is zero, see page 380 [FM11], we expect the corresponding limit to be zero. In Theorem 7.3.11, we can show the limit superior is zero for any diffeomorphism. But the limits are not zero for some cases, see Example 7.3.8. Some diffeomorphisms even do not have invariant characters that live in the Azumaya locus, but the intertwiners in Theorems 7.3.2 and 7.3.3 are very close to intertwiners constructed in [BWY21].

When we consider the intertwiners in Theorems 7.3.2 and 7.3.3, we fix the mapping class A and the A -invariant character $[\gamma]$. In this subsection we will use (l, s) to denote $\gcd(l, s)$ for any two integers l, s .

Theorem 7.3.5. *If we require $|\det(\Lambda_n)| = 1$ for the intertwiner in Theorem 7.3.2, then $|\text{Trace}\Lambda_n| \leq n^{3/2}$.*

Proof. Since any two intertwiners in Theorem 7.3.2 are different by a scalar multiplication or by conjugation and we require $|\det(\Lambda_n)| = 1$, the absolute value $|\text{Trace}\Lambda_n|$ is independent of the choice of intertwiners. Let $\Lambda_n = (n')^{-1/2}\bar{\Lambda}_n$, then $|\det(\Lambda_n)| = 1$. Since $|(\bar{\Lambda}_n)_{k,t}| = 0$ or 1 for all $0 \leq k, t \leq n-1$ and each row has exactly n' nonzero entries, we have the absolute value of every eigenvalue of $\bar{\Lambda}_n$ is not more than n' . Then

$$|\text{Trace}(\Lambda_n)| = |\text{Trace}((n')^{-1/2}\bar{\Lambda}_n)| \leq (n')^{-1/2}(nn') = (n')^{1/2}n \leq n^{3/2}.$$

□

Theorem 7.3.6. *If we require $|\det(\Lambda_n)| = 1$ for the intertwiner in Theorem 7.3.3, then $|\text{Trace}\Lambda_n| \leq n^{3/2}$.*

Proof. It is similar with the proof for Theorem 7.3.5. □

Lemma 7.3.7. *Let k be any integer, then we have*

$$\left| \sum_{0 \leq t \leq n-1} (-q^{\frac{1}{2}})^{kt^2} \right| = \sqrt{(k, n)n}.$$

Recall that $q^{1/2}$ is a primitive n -root of -1 .

Proof. In [MOO92], they proved this result for $k = 2$. Using the same trick, we can prove this generalized lemma. □

In the following of this section, we always assume $q^{1/2} = e^{\pi i/n}$ unless especially specified. Next we are going to calculate $\text{Trace}\Lambda_n$, where Λ_n is the intertwiner in Theorem 7.3.2 or 7.3.3 with $|\det\Lambda_n| = 1$. First we give detailed discussion on the invariant character. Recall that for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and a character $[\gamma] \in \mathcal{X}_{SL_2(\mathbb{C})}(T^2)$ with $\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and $\gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$, we have $[\gamma]$ is A -invariant if and only if $1 = \lambda_1^{a-1}\lambda_2^b, 1 = \lambda_1^c\lambda_2^{d-1}$ or $1 = \lambda_1^{a+1}\lambda_2^b, 1 = \lambda_1^c\lambda_2^{d+1}$.

We will provide a detailed discussion only for the case when $1 = \lambda_1^{a-1}\lambda_2^b, 1 = \lambda_1^c\lambda_2^{d-1}$. Suppose $\lambda_1 = \alpha_1 e^{i\theta_1}, \lambda_2 = \alpha_2 e^{i\theta_2}$, then we can get equations:

$$\begin{cases} 1 = \alpha_1^{a-1}\alpha_2^b \\ 1 = \alpha_1^c\alpha_2^{d-1} \end{cases} \quad \begin{cases} (a-1)\theta_1 + b\theta_2 = 2k_1\pi \\ c\theta_1 + (d-1)\theta_2 = 2k_2\pi \end{cases} \quad (7.10)$$

Since $u^n = -\lambda_1, v^n = -\lambda_2$, we can suppose $u = -\alpha_1^{\frac{1}{n}} e^{\frac{i\theta_1}{n}} q^{r_1}, v = -\alpha_2^{\frac{1}{n}} e^{\frac{i\theta_2}{n}} q^{r_2}$ where both r_1 and r_2 are integers. Then we have

$$\begin{aligned} u^{a-1}v^b &= (-1)^{a+b-1} q^{r_1(a-1)+r_2b} e^{\frac{i}{n}((a-1)\theta_1+b\theta_2)} = (-1)^{a+b-1} q^{r_1(a-1)+r_2b+k_1}, \\ u^c v^{d-1} &= (-1)^{c+d-1} q^{r_1c+r_2(d-1)} e^{\frac{i}{n}(c\theta_1+(d-1)\theta_2)} = (-1)^{c+d-1} q^{r_1c+r_2(d-1)+k_2}. \end{aligned} \quad (7.11)$$

Define $s_1 = r_1(a-1) + r_2b + k_1, s_2 = r_1c + r_2(d-1) + k_2$. Then, we have $u^{a-1}v^b = (-1)^{a+b-1} q^{s_1}, u^c v^{d-1} = (-1)^{c+d-1} q^{s_2}$.

From $1 = \lambda_1^{a-1}\lambda_2^b, 1 = \lambda_1^c\lambda_2^{d-1}$, we can get $1 = \lambda_1^{(a-1)c}\lambda_2^{bc}, 1 = \lambda_1^{(a-1)c}\lambda_2^{(a-1)(d-1)}$.

Thus we have

$$\lambda_2^{bc} = \lambda_2^{(a-1)(d-1)} = \lambda_2^{ad-(a+d)+1} \Rightarrow 1 = \lambda_2^{ad-bc-(a+d)+1} = \lambda_2^{2-(a+d)}.$$

If $a + d \neq 2$, we have λ_2 is a root of unity with $\lambda_2^{2-(a+d)} = 1$. Similarly we can show λ_1 is also a root of unity, with $\lambda_1^{2-(a+d)} = 1$, under the assumption $a + d \neq 2$.

We look at the case when $(b, n) = 1$, and suppose $br + sn = 1$. Then we have

$$q^{br} = q, q^{\frac{br}{2}} = (-1)^s q^{\frac{1}{2}}, (-1)^s = (-1)^{br+1}.$$

When $(b, n) = 1$, we can choose $k_0 = 1$ and set $r_{k_0} = r_0 = 1$. Then we have

$$r_{tb} = v^{tb} u^{t(a-1)} q^{abt^2/2}, \forall t \in \mathbb{Z}.$$

For any $k \in \mathbb{Z}$, we have $k = krb + ksn$. Then we have

$$r_k = r_{krb} = v^{krb} u^{kr(a-1)} q^{abk^2r^2/2}.$$

From the above discussion, we know we can choose Λ_n to be $n^{-1/2} \overline{\Lambda}_n$. We have

$$\begin{aligned} (\overline{\Lambda}_n)_{t,t} &= r_{t-td} (v^{d-1} u^c)^t q^{ct^2 - cdt^2/2} = (-1)^{cdt} r_{t-td} q^{s_2 t} q^{ct^2 - cdt^2/2} \\ &= (-1)^{cdt} (v^b u^{a-1})^{r(t-td)} q^{abr^2(t-td)^2/2} q^{s_2 t} q^{ct^2 - cdt^2/2} \\ &= (-1)^{cdt} (-1)^{abr(t-td)} q^{s_1 r(t-td)} q^{abr^2(t-td)^2/2} q^{s_2 t} q^{ct^2 - cdt^2/2} \\ &= (-1)^{cdt} (-1)^{abr(t-td)} (q^{s_1 r(1-d)} q^{s_2})^t \left((q^{\frac{br}{2}})^{ar(1-d)^2} q^{c - \frac{1}{2}cd} \right)^{t^2} \\ &= (-1)^{cdt} (-1)^{abr(t-td)} (q^{s_1 r(1-d)} q^{s_2})^t \left((-1)^s q^{\frac{1}{2}} \right)^{ar(1-d)^2} q^{c - \frac{1}{2}cd} t^2 \\ &= (-1)^{cdt} (-1)^{abr(t-td)} (-1)^{tsar(1-d)^2} (q^{s_1 r(1-d)} q^{s_2})^t (q^{\frac{1}{2}ar} q^{-ard} q^{\frac{1}{2}ard^2} q^{c - \frac{1}{2}cd})^{t^2} \\ &= (-1)^{cdt} (-1)^{abr(t-td)} (-1)^{tsar(1-d)^2} (q^{s_1 r(1-d)} q^{s_2})^t (q^{\frac{1}{2}ar} q^{-r} q^{-rbc} q^{\frac{1}{2}rd} q^{\frac{1}{2}rbc} q^{c - \frac{1}{2}cd})^{t^2} \\ &= (-1)^{cdt} (-1)^{abr(t-td)} (-1)^{tsar(1-d)^2} (q^{s_1 r(1-d)} q^{s_2})^t (q^{\frac{1}{2}ar} q^{-r} q^{-c} q^{\frac{1}{2}rd} (-1)^s q^{\frac{1}{2}})^{cd} q^{c - \frac{1}{2}cd} t^2 \\ &= (-1)^{cdt} (-1)^{abr(t-td)} (-1)^{tsar(1-d)^2} (-1)^{tscd} (q^{s_1 r(1-d)} q^{s_2})^t (q^{\frac{1}{2}(a+d-2)r})^{t^2} \\ &= (-1)^{cdt} (-1)^{abr(t-td)} (-1)^{t(br+1)ar(1-d)} (-1)^{t(br+1)cd} (q^{s_1 r(1-d)} q^{s_2})^t (q^{\frac{1}{2}(a+d-2)r})^{t^2} \\ &= (-1)^{tar} (-1)^{tard} (-1)^{trd(ad+1)} (q^{s_1 r(1-d)} q^{s_2})^t (q^{\frac{1}{2}(a+d-2)r})^{t^2} \\ &= (-1)^{tar} (-1)^{trd} (q^{s_1 r(1-d)} q^{s_2})^t (q^{\frac{1}{2}(a+d-2)r})^{t^2} \\ &= ((-1)^{(a+d-2)r})^t (q^{s_1 r(1-d)} q^{s_2})^t (q^{\frac{1}{2}(a+d-2)r})^{t^2}. \end{aligned}$$

Since $q^{s_2} = q^{rs_2b}$, we have

$$(\overline{\Lambda}_n)_{t,t} = ((-1)^{(a+d-2)r})^t q^{\frac{r}{2}((a+d-2)t^2 + 2(s_1(1+d) - s_2b)t)}, \quad (7.12)$$

and

$$\text{Trace}\Lambda_n = n^{-\frac{1}{2}} \sum_{0 \leq t \leq n-1} ((-1)^{(a+d-2)r})^t q^{\frac{r}{2}((a+d-2)t^2+2(s_1(1+d)-s_2b)t)}. \quad (7.13)$$

rem Here we state the parallel results for $1 = \lambda_1^{a+1}\lambda_2^b$, $1 = \lambda_1^c\lambda_2^{d+1}$ and $u^n = -\lambda_1^{-1}$, $v^n = -\lambda_2^{-1}$.

Suppose $\lambda_1 = \alpha_1 e^{i\theta_1}$, $\lambda_2 = \alpha_2 e^{i\theta_2}$, then we can get equations:

$$\begin{cases} 1 = \alpha_1^{a+1}\alpha_2^b \\ 1 = \alpha_1^c\alpha_2^{d+1} \end{cases}$$

$$\begin{cases} (a+1)\theta_1 + b\theta_2 = 2k_1\pi \\ c\theta_1 + (d+1)\theta_2 = 2k_2\pi \end{cases}.$$

Since $u^n = -\lambda_1^{-1}$, $v^n = -\lambda_2^{-1}$, we can suppose $u = -\alpha_1^{-\frac{1}{n}} e^{-\frac{i\theta_1}{n}} q^{r_1}$, $v = -\alpha_2^{-\frac{1}{n}} e^{-\frac{i\theta_2}{n}} q^{r_2}$ where both r_1 and r_2 are integers. Then we have

$$\begin{aligned} u^{a+1}v^b &= (-1)^{a+b+1} q^{r_1(a+1)+r_2b-k_1}, \\ u^c v^{d+1} &= (-1)^{c+d+1} q^{r_1c+r_2(d+1)-k_2}. \end{aligned}$$

Similarly we set $s_1 = r_1(a+1) + r_2b - k_1$, $s_2 = r_1c + r_2(d+1) - k_2$, then we have $u^{a+1}v^b = (-1)^{a+b+1} q^{s_1}$, $u^c v^{d+1} = (-1)^{c+d+1} q^{s_2}$. If $a+d \neq -2$, we have $\alpha_1 = \alpha_2 = 1$.

For the case when $(b, n) = 1$ and $br + sn = 1$, we have

$$(\tilde{\Lambda}_n)_{t,t} = ((-1)^{(a+d+2)r})^t q^{\frac{r}{2}((a+d+2)t^2+2(s_1(1+d)-s_2b)t)},$$

and

$$\text{Trace}\Lambda_n = n^{-\frac{1}{2}} \sum_{0 \leq t \leq n-1} ((-1)^{(a+d+2)r})^t q^{\frac{r}{2}((a+d+2)t^2+2(s_1(1+d)-s_2b)t)}. \quad (7.14)$$

Example 7.3.8. Let $A = \begin{pmatrix} 2 & 1 \\ -7 & -3 \end{pmatrix}$. If we try to solve $1 = \lambda_1^{a-1}\lambda_2^b$, $1 = \lambda_1^c\lambda_2^{d-1}$, we can get equations:

$$\begin{cases} \theta_1 + \theta_2 = 2\pi \\ -7\theta_1 - 4\theta_2 = 2\pi \end{cases}. \quad (7.15)$$

We have $\theta_1 = -\frac{10\pi}{3}$, $\theta_2 = \frac{16\pi}{3}$, thus $\lambda_1 = e^{\frac{2\pi i}{3}}$, $\lambda_2 = e^{\frac{4\pi i}{3}}$. So we can set $u = -e^{-\frac{10\pi i}{3n}}$, $v = -e^{\frac{16\pi i}{3n}}$, then we have $u^{a-1}v^b = q$, $u^c v^{d-1} = -q$. We have $s_1 = s_2 = 1$. Since $b = 1$, we get $r = 1$. Then from equation 7.13, we have

$$\text{Trace}\Lambda_n = n^{-\frac{1}{2}} \sum_{0 \leq t \leq n-1} (-1)^t q^{\frac{1}{2}(-3t^2+10t)}.$$

Note that when n is a multiple of 3, we have $\text{Trace}\Lambda_n = 0$.

Example 7.3.9. Let A be the same matrix as above. But this time we try to solve $1 = \lambda_1^{a+1}\lambda_2^b$, $1 = \lambda_1^c\lambda_2^{d+1}$, then we get $\lambda_1^{2+a+d} = \lambda_2^{2+a+d} = 1$. Since $2 + a + d = 1$, so we have $\lambda_1 = \lambda_2 = 1$. We can set $u = v = -1$, then $s_1 = s_2 = 0$. From equation 7.13, we have

$$|\text{Trace}\Lambda_n| = n^{-\frac{1}{2}} \left| \sum_{0 \leq t \leq n-1} (-1)^t q^{\frac{1}{2}t^2} \right| = 1.$$

Lemma 7.3.10. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $(b, n) = 1$ and $|a + d| = 2$. Then we have the following statements:

(1) If $a + d = 2$ and Λ_n is the intertwiner obtained in Theorem 7.3.2 such that $|\det(\Lambda_n)| = 1$, we have $|\text{Trace}\Lambda_n| = \sqrt{n}$.

(2) If $a + d = -2$ and Λ_n is the intertwiner obtained in Theorem 7.3.3 such that $|\det(\Lambda_n)| = 1$, we have $|\text{Trace}\Lambda_n| = \sqrt{n}$.

Proof. We only prove the statement (1) (the proof for the statement (2) is similar).

Let $[\gamma] \in \mathcal{X}_{\text{SL}_2(\mathbb{C})}(T^2)$, with $\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and $\gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$, be any A -invariant character.

We use the same notation as in Remark 7.3.3. Then we have

$$\begin{aligned} s_1(1-d) + s_2b &= r_1(a-1)(1-d) + r_2b(1-d) + k_1(1-d) + r_1cb + r_2(d-1)b + k_2b \\ &= k_1(1-d) + k_2b. \end{aligned}$$

From equation 7.10, we can get

$$\begin{aligned} 2\pi((1-d)k_1 + k_2b) &= (1-d)2\pi k_1 + b2\pi k_2 \\ &= (1-d)(a-1)\theta_1 + b(1-d)\theta_2 + bc\theta_1 + b(d-1)\theta_2 = 0. \end{aligned}$$

Thus we have

$$(1-d)k_1 + k_2b = s_1(1-d) + s_2b = 0.$$

From equation 7.13, we know

$$\text{Trace}\Lambda_n = n^{-\frac{1}{2}} \sum_{0 \leq t \leq n-1} ((-1)^{(a+d-2)r})^t q^{\frac{r}{2}((a+d-2)t^2 + 2(s_1(1-d) + s_2b)t)} = n^{-\frac{1}{2}}n = \sqrt{n}.$$

□

The following Theorem shows the limit superior related to the trace of intertwiners for any diffeomorphism of the closed torus is zero, which equals the simplicial volume of the corresponding mapping torus.

Theorem 7.3.11. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any fixed element in the mapping class group for the closed torus, and let $[\gamma]$ be any fixed A -invariant character with $\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and $\gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$. If $1 = \lambda_1^{a-1}\lambda_2^b$ and $1 = \lambda_1^c\lambda_2^{d-1}$, let $\{\Lambda_n\}_{n \in 2\mathbb{Z}_{\geq 0} + 1}$ be intertwiners obtained in Theorem 7.3.2 such that $|\det(\Lambda_n)| = 1$ for all $n \in 2\mathbb{Z}_{\geq 0} + 1$. If $1 = \lambda_1^{a+1}\lambda_2^b$ and $1 = \lambda_1^c\lambda_2^{d+1}$, let $\{\Lambda_n\}_{n \in 2\mathbb{Z}_{\geq 0} + 1}$ be intertwiners obtained in Theorem 7.3.3 such that $|\det(\Lambda_n)| = 1$ for all $n \in 2\mathbb{Z}_{\geq 0} + 1$. Then we have*

$$\limsup_{\text{odd } n \rightarrow \infty} \frac{\log(|\text{Trace}\Lambda_n|)}{n} = 0.$$

Proof. Since $[\gamma]$ is A -invariant, we have $1 = \lambda_1^{a-1}\lambda_2^b$, $1 = \lambda_1^c\lambda_2^{d-1}$ or $1 = \lambda_1^{a+1}\lambda_2^b$, $1 = \lambda_1^c\lambda_2^{d+1}$. We look at the case when $1 = \lambda_1^{a-1}\lambda_2^b$, $1 = \lambda_1^c\lambda_2^{d-1}$. Then we can set Λ_n to be $a_n\bar{\Lambda}_n$ where $a_n = |\det(\bar{\Lambda}_n)|^{-\frac{1}{n}}$.

Case *I* when $b = 0$. In this case we know $|\det(\bar{\Lambda}_n)| = 1$ since $n' = 1$.

We have $A = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$. We first consider the case when $A = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$. If $c = 0$, it is trivial. So suppose $c \neq 0$. Since we have $\lambda_1^{a-1}\lambda_2^b = 1$, $\lambda_1^c\lambda_2^{d-1} = 1$,

then we get $\lambda_1^c = 1$. Suppose $\lambda_1 = e^{i\theta}$, then we get $\theta c = 2k\pi$ where k is an integer. Since $u^n = -\lambda_1$, we can choose $u = -e^{\frac{i\theta}{n}} q^r$ where r is an integer. Then we have

$$u^c = (-1)^c e^{\frac{ic}{n}} q^{cr} = (-1)^c e^{\frac{2k\pi i}{n}} q^{cr} = (-1)^c q^{k+cr}.$$

Note that $|\text{Trace}\Lambda_n|$ is independent of the choice of r .

From Remark 7.3.2 and Lemma 7.3.4, we can get Λ_n is a diagonal matrix, and

$$(\overline{\Lambda}_n)_{t,t} = (v^{(d-1)} u^c)^t q^{c(t^2-dt^2/2)}.$$

Then we have

$$\text{Trace}\Lambda_n = \sum_{0 \leq t \leq n-1} v^{t(d-1)} u^{tc} q^{ct^2/2} = \sum_{0 \leq t \leq n-1} (-1)^{ct} q^{ct^2/2} q^{(k+cr)t}.$$

Let $\{n_i\}_{i \in \mathbb{N}}$ be a subsequence of $2\mathbb{Z}_{\geq 0} + 1$ such that $(n_i, c) = 1$ for all i . Then for every i there exists r such that $k + cr \equiv 0 \pmod{n_i}$, thus

$$|\text{Trace}\Lambda_{n_i}| = \left| \sum_{0 \leq t \leq n_i-1} (-1)^{ct} q^{ct^2/2} \right| = \sqrt{(n_i, c)n_i} = \sqrt{n_i} \geq 1.$$

Thus we have

$$0 \leq \limsup_{\text{odd } n \rightarrow \infty} \frac{\log(|\text{Trace}\Lambda_n|)}{n}.$$

According to Theorem 7.3.5, we also have

$$\limsup_{\text{odd } n \rightarrow \infty} \frac{\log(|\text{Trace}\Lambda_n|)}{n} \leq \limsup_{\text{odd } n \rightarrow \infty} \frac{\log(n^{\frac{3}{2}})}{n} = 0.$$

We look at the case when $A = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}$. Then we get $(\lambda_1)^2 = 1$ and $\lambda_1 = \pm 1$.

We can choose $u = \pm 1$. From Remark 7.3.2 and Lemma 7.3.4 we can get

$$(\overline{\Lambda}_n)_{t,k} = r_{k-t} (v^{(d-1)} u^c)^t q^{c(tk-dt^2/2)},$$

where $r_k = 1$ if k is a multiple of n and it is zero otherwise. Then $(\overline{\Lambda}_n)_{t,t} \neq 0$ if and only if $r_{2t} \neq 0$ if and only if $n|(2t)$, which means there is only one nonzero diagonal element. Then we get $|\text{Trace}\Lambda_n| = 1$ for any n , which proves this special case.

Case II when $b \neq 0$.

Subcase (1) when $a + d \neq 2$. From the above discussion we know λ_1, λ_2 are both roots of unity, thus we can suppose $\lambda_1 = e^{i\theta_1}, \lambda_2 = e^{i\theta_2}$ and we can get equation 7.10 where $\theta_1, \theta_2, k_1, k_2$ are determined by γ . Since $u^n = -\lambda_1, v^n = -\lambda_2$, we can write $u = -e^{\frac{i\theta_1}{n}} q^{r_1}, v = -e^{\frac{i\theta_2}{n}} q^{r_1}$. Then we have equation 7.11. Note that $|\text{Trace}\Lambda_n|$ is independent of the choice of r_1, r_2 .

Since $b \neq 0$ and $2 - (a + d) \neq 0$, let $\{n_j\}_{j \in \mathbb{N}}$ be a subsequence of $2\mathbb{Z}_{\geq 0} + 1$ such that $(n_j, b) = (n_j, 2 - (a + d)) = 1$.

Since $\begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} = 2 - (a + d)$ and $(n_j, 2 - (a + d)) = 1$, the following equations always have solutions in \mathbb{Z}_{n_j}

$$\begin{cases} r_1(a-1) + r_2b + k_1 = 0 \\ r_1c + r_2(d-1) + k_2 = 0 \end{cases}.$$

Thus for every j , there always exist integers r_1, r_2 such that $s_1 = s_2 = 0$ in \mathbb{Z}_{n_j} . Then from equation 7.13, we have

$$\begin{aligned} |\text{Trace}\Lambda_{n_j}| &= n_j^{-\frac{1}{2}} \left| \sum_{0 \leq t \leq n_j-1} ((-1)^{(a+d-2)r})^t q^{\frac{r}{2}((a+d-2)t^2 + 2(s_1(1-d) + s_2b)t)} \right| \\ &= n_j^{-\frac{1}{2}} \left| \sum_{0 \leq t \leq n_j-1} ((-1)^{(a+d-2)r})^t q^{\frac{r}{2}((a+d-2)t^2)} \right| \\ &= n_j^{-\frac{1}{2}} \left| \sum_{0 \leq t \leq n_j-1} (-q^{\frac{1}{2}})^{r(a+d-2)t^2} \right| \\ &= n_j^{-\frac{1}{2}} \sqrt{(r(a+d-2), n_j)n_j} = 1. \end{aligned}$$

Then we have

$$0 \leq \limsup_{\text{odd } n \rightarrow \infty} \frac{\log(|\text{Trace}\Lambda_n|)}{n}.$$

From Theorem 7.3.5, we have

$$\limsup_{\text{odd } n \rightarrow \infty} \frac{\log(|\text{Trace}\Lambda_n|)}{n} = 0.$$

Subcase (2) when $a + d = 2$. Since $b \neq 0$, let $\{n_k\}_{k \in \mathbb{N}}$ be a subsequence of $2\mathbb{Z}_{\geq 0} + 1$ such that $(n_k, b) = 1$ for all k . From Lemma 7.3.10, we have $|\text{Trace}\Lambda_{n_k}| = \sqrt{n_k} \geq 1$.

Similarly we get

$$\limsup_{\text{odd } n \rightarrow \infty} \frac{\log(|\text{Trace} \Lambda_n|)}{n} = 0.$$

□

From now on we discuss the periodic mapping class. Recall that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is periodic if and only if $|a + d| \in \{0, 1\}$. Suppose $[\gamma]$ is an A -invariant character with $\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ and $\gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$. Then we have $1 = \lambda_1^{a-1} \lambda_2^b$, $1 = \lambda_1^c \lambda_2^{d-1}$ or $1 = \lambda_1^{a+1} \lambda_2^b$, $1 = \lambda_1^c \lambda_2^{d+1}$. For the case when $1 = \lambda_1^{a-1} \lambda_2^b$, $1 = \lambda_1^c \lambda_2^{d-1}$, the above discussion implies $1 = \lambda_1^{2-(a+d)} = \lambda_2^{2-(a+d)}$. So if $a + d = 0$, we have $\lambda_1^2 = \lambda_2^2 = 1$, and then $\lambda_1 = \pm 1, \lambda_2 = \pm 1$. Thus there is no A -invariant character living in the Azumaya locus if $\lambda_1 = \pm 1, \lambda_2 = \pm 1$. But we can still get intertwiners in Theorems 7.3.2 and 7.3.3 although $\lambda_1 = \pm 1, \lambda_2 = \pm 1$. Now we consider intertwiners if we choose $\lambda_1 = \lambda_2 = 1$.

Theorem 7.3.12. *Let A be a periodic mapping class, and let Λ_n be the intertwiner obtained in Theorem 7.3.2 or 7.3.3 by using the trivial A -invariant character, that is, $\lambda_1 = \lambda_2 = 1$, and we require $|\det(\Lambda_n)| = 1$. We have the following conclusions:*

(1) *If $a + d = 1$ and Λ_n is obtained in Theorem 7.3.2, we have $|\text{Trace}(\Lambda_n)| = 1$ for any odd n .*

(2) *If $a + d = -1$ and Λ_n is obtained in Theorem 7.3.3, we have $|\text{Trace}(\Lambda_n)| = 1$ for any odd n .*

(3) *If $a + d = 0$ and Λ_n is obtained in Theorem 7.3.2 or 7.3.3, we have $|\text{Trace}(\Lambda_n)| = 1$ for any odd n .*

Proof. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(b, n) = m$, $br + sn = m$ and $n = mn'$. Since $\lambda_1 = \lambda_2 = 1$, we can set $u = v = -1$. From the previous discussion we know

$$(\overline{\Lambda}_n)_{k,t} = (-1)^{cdt} r_{k-td} q^{c(tk-dt^2/2)}$$

for all $0 \leq k, t \leq n - 1$, where

$$r_{tm} = (-1)^{abrt} q^{abt^2r^2/2}, \forall 0 \leq t \leq n' - 1,$$

and all other r_k are 0. For the l -th column, we have $\{(\bar{\Lambda}_n)_{ld+km,l}\}_{0 \leq k \leq n'-1}$ are the only nonzero entries. Then the l -th column contains a nonzero diagonal entry if and only if $ld + km \equiv l \pmod{n}$ for some $0 \leq k \leq n' - 1$. It is easy to show $ld + km \equiv l \pmod{n}$ for some $0 \leq k \leq n' - 1$ if and only if $m|(ld - l)$.

Now we suppose $(m, d - 1) = 1$. Then the l -th column of $\bar{\Lambda}_n$ contains a nonzero diagonal entry if and only if $m|l$. Thus $(\bar{\Lambda}_n)_{tm,tm}, 0 \leq t \leq n' - 1$, are the only nonzero diagonal entries.

$$\begin{aligned} (\bar{\Lambda}_n)_{tm,tm} &= (-1)^{cdtm} r_{tm-tmd} q^{c(t^2m^2-dt^2m^2/2)} \\ &= (-1)^{cdtm} (-1)^{abr(t-d)} q^{abr^2(1-d)^2t^2/2} q^{c(t^2m^2-dt^2m^2/2)}. \end{aligned}$$

After a similar calculation as in Remark 7.3.3, we can get

$$(\bar{\Lambda}_n)_{tm,tm} = (-1)^{(ar+dr-2r)t} (q^{\frac{m}{2}})^{(ar+dr-2r)t^2}.$$

Then we have

$$\text{Trace} \bar{\Lambda}_n = \sum_{0 \leq t \leq n'-1} (-1)^{(ar+dr-2r)t} (q^{\frac{m}{2}})^{(ar+dr-2r)t^2}.$$

Similarly if $(m, d + 1) = 1$, we have

$$\text{Trace} \tilde{\Lambda}_n = \sum_{0 \leq t \leq n'-1} (-1)^{(ar+dr+2r)t} (q^{\frac{m}{2}})^{(ar+dr+2r)t^2}.$$

(1) Since the intertwiner is obtained in Theorem 7.3.2, we can set $\Lambda_n = (n')^{-\frac{1}{2}} \bar{\Lambda}_n$. We have $d - 1 = -a$ because $a + d = 1$. Then we get $(d - 1, m) = 1$ because $(a, b) = 1$ and m is a divisor of b . Then from the above discussion, we get

$$\begin{aligned} |\text{Trace} \Lambda_n| &= (n')^{-\frac{1}{2}} \left| \sum_{0 \leq t \leq n'-1} (-1)^{(ar+dr-2r)t} (q^{\frac{m}{2}})^{(ar+dr-2r)t^2} \right| \\ &= (n')^{-\frac{1}{2}} \left| \sum_{0 \leq t \leq n'-1} (-1)^{-rt} (q^{\frac{m}{2}})^{-rt^2} \right| = (n')^{-\frac{1}{2}} \sqrt{(-r, n') n'} = 1. \end{aligned}$$

(2) The proof is similar with (1).

(3) First we show $(m, d-1) = (m, d+1) = 1$ if $a+d=0$. From $ad-bc=1$, we get $-bc=d^2+1$. Suppose $p|m$ and $p|d-1$, then $p|(d^2+1)$ and $p|(d^2-1)$. Thus we get $p|2$, which means $p=1$ because $(m, 2) = 1$. Similarly we can show $(m, d+1) = 1$. If $\Lambda_n = (n')^{-\frac{1}{2}}\bar{\Lambda}_n$, then we

$$\begin{aligned} |\text{Trace}\Lambda_n| &= (n')^{-\frac{1}{2}} \left| \sum_{0 \leq t \leq n'-1} (-1)^{(ar+dr-2r)t} (q^{\frac{m}{2}})^{(ar+dr-2r)t^2} \right| \\ &= (n')^{-\frac{1}{2}} \left| \sum_{0 \leq t \leq n'-1} (q^m)^{-rt^2} \right| = (n')^{-\frac{1}{2}} \sqrt{(-r, n')n'} = 1. \end{aligned}$$

If $\Lambda_n = (n')^{-\frac{1}{2}}\tilde{\Lambda}_n$, similarly we can show $|\text{Trace}\Lambda_n| = 1$. □

7.4 The volume conjecture for surface diffeomorphisms: periodic case

7.4.1 Preliminaries for the volume conjecture for periodic surface diffeomorphisms

If we want to formulate the parallel conjecture for periodic diffeomorphisms as in [BWY21; BWY22], we have to find a good invariant character that lives in the smooth part of $\mathfrak{X}_{\text{SL}_2(\mathbb{C})}(S)$.

Lemma 7.4.1 ([Con20]). *Let $A, B \in \text{SL}_2(\mathbb{C})$. If $\text{Trace}([A, B]) = 2$ where $[A, B] = ABA^{-1}B^{-1}$, then $G = \langle A, B \rangle \leq \text{SL}_2(\mathbb{C})$ is not free of rank two where $\langle A, B \rangle$ is the group generated by A, B .*

Lemma 7.4.2. *Let G be a subgroup of $\text{SL}_2(\mathbb{C})$ freely generated by two elements, let R be the subalgebra of $\text{Mat}(2, \mathbb{C})$ generated by G , where $\text{Mat}(2, \mathbb{C})$ is the algebra of all 2 by 2 complex matrices. Then $R = \text{Mat}(2, \mathbb{C})$.*

Proof. Suppose G is freely generated by A, B . We know there exists $X \in \text{GL}(2, \mathbb{C})$ such that $XAX^{-1} = \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix}$, $XBX^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then XGX^{-1} is a free subgroup generated by XAX^{-1} and XBX^{-1} , and XRX^{-1} is the subalgebra generated

by XGX^{-1} . Since $XRX^{-1} = Mat(2, \mathbb{C})$ if and only if $R = Mat(2, \mathbb{C})$, we can assume $A = \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix}$, $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

I. Suppose $v = 0$. Then $A = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ and $u^2 \neq 1$, otherwise we have $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which contradicts the fact that G is freely generated by A, B . We also can get $b \neq 0$ and $c \neq 0$. Otherwise we have $\text{Trace}([A, B]) = 2$, which contradicts the fact that A, B freely generate G by Lemma 7.4.1. Since $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R$ and $u \neq \pm 1$, we have $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in R$. Then $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in R$. From multiplication, we get $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \in R$, which implies $R = Mat(2, \mathbb{C})$ since $b \neq 0, c \neq 0$.

II. Suppose $v \neq 0$. In this case we should have $c \neq 0$, otherwise we have $\text{Trace}([A, B]) = 2$, which is a contradiction.

If $u = \pm 1$, then $A = \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \in R$. Remember we also have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R$, which implies $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in R$. Furthermore we have $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$ because $v \neq 0$. From $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$, we can get $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in R$, and also $\begin{pmatrix} a-d & 0 \\ c & 0 \end{pmatrix} \in R$. From multiplication, we get $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a-d & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \in R$, which implies $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$ because $c \neq 0$. We have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$, then

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R.$$

Remember we also have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$ and $c \neq 0$, which implies $R = Mat(2, \mathbb{C})$.

If $u \neq \pm 1$. We have $\begin{pmatrix} u - u^{-1} & v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ 0 & u^{-1} - u \end{pmatrix} \in R$, so $\begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -k \\ 0 & 1 \end{pmatrix} \in R$ where $k = v/(u - u^{-1})$. Then from multiplication, we get

$$\begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + kc & b + kd \\ 0 & 0 \end{pmatrix} \in R.$$

Next we want to show $b + kd \neq k(a + kc)$. Suppose on the contrary. Then we have $b + kd = k(a + kc) = ka + k^2c$. With $k = v/(u - u^{-1})$, we can get

$$b + \frac{dv}{u - u^{-1}} = \frac{av}{u - u^{-1}} + \frac{cv^2}{(u - u^{-1})^2} \implies 2b = bu^2 + bu^{-2} + dvu - dvu^{-1} - avu + avu^{-1} - cv^2.$$

Then we get

$$\begin{aligned} \text{Trace}([A, B]) &= ad + acv + cdvu^{-1} + c^2v^2 - cbu^2 - cdv - cbu^{-2} - cau^{-1}v + ad \\ &= 2ad - c(-auv - dvu^{-1} - cv^2 + bu^2 + dvv + bu^{-2} + au^{-1}v) = 2ad - 2cb = 2. \end{aligned}$$

Since $\text{Trace}([A, B]) = 2$ is a contradiction, we have $b + kd \neq k(a + kc)$. We can get $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$ because $\begin{pmatrix} a + kc & b + kd \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix} \in R$. We also have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$, then we can get $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$. So $R = \text{Mat}(2, \mathbb{C})$ because $c \neq 0$.

□

Proposition 7.4.3. *Let $\gamma : \pi_1(S) \rightarrow \text{SL}_2(\mathbb{C})$ be a representative of an element in the character variety $\mathfrak{X}_{\text{SL}_2(\mathbb{C})}(S)$. Then γ is irreducible if $\text{Im}\gamma$ contains a subgroup of $\text{SL}_2(\mathbb{C})$ freely generated by two elements. In particular γ is irreducible if S has negative Euler characteristic and γ is injective.*

Proof. Lemma 7.4.2.

□

7.4.2 Statement of the conjecture

To get the intertwiner, we first have to get a φ -invariant smooth character $\gamma \in \mathfrak{X}_{\mathrm{PSL}_2(\mathbb{C})}(S)$. At page 371 of [FM11] it is proved that every periodic diffeomorphism fixes a point in the Teichmüller space. This means there is a discrete and faithful group homomorphism $\bar{\gamma} : \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ such that $\bar{\gamma}\varphi_*$ is conjugate to $\bar{\gamma}$ by an element in $\mathrm{PSL}(2, \mathbb{R})$ where φ_* is the isomorphism from $\pi_1(S)$ to $\pi_1(S)$ induced by φ .

Since $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$, we can regard $\bar{\gamma}\varphi_*$ and $\bar{\gamma}$ as two elements in $\mathfrak{X}_{\mathrm{PSL}_2(\mathbb{C})}(S)$. Then $\bar{\gamma}\varphi_*$ is conjugate to $\bar{\gamma}$ by an element in $\mathrm{PSL}_2(\mathbb{C})$. Thus $\bar{\gamma}$ can be extended to a group homomorphism from $\pi_1(M_\varphi)$ to $\mathrm{PSL}_2(\mathbb{C})$, we use $\hat{\gamma}$ to denote this homomorphism. Then we can lift $\hat{\gamma}$ to a group homomorphism $\tilde{\gamma}$ from $\pi_1(M_\varphi)$ to $\mathrm{SL}_2(\mathbb{C})$. The restriction of $\tilde{\gamma}$ to $\pi_1(S)$ is φ -invariant, and we use γ to denote this group homomorphism. Note that γ is a group homomorphism from $\pi_1(S)$ to $\mathrm{SL}_2(\mathbb{C})$. Let ε be the projection from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{PSL}_2(\mathbb{C})$, then we have $\varepsilon\tilde{\gamma} = \hat{\gamma}$, furthermore we have

$$\varepsilon\gamma = \varepsilon\tilde{\gamma}|_{\pi_1(S)} = \hat{\gamma}|_{\pi_1(S)} = \bar{\gamma}.$$

Since $\bar{\gamma}$ is injective, we have γ is injective. From Proposition 7.4.3, we know γ is irreducible. Thus we get a φ -invariant smooth character $\gamma \in \mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(S)$. From now on, we use γ_φ to denote γ and $\overline{\gamma_\varphi}$ to denote $\bar{\gamma}$.

For every puncture v in S , we know $\mathrm{Trace}\gamma_\varphi(\alpha_v) = \pm 2$ where α_v is the loop going around puncture v . If $\mathrm{Trace}\gamma_\varphi(\alpha_v) = 2$, we choose $p_v = -(q + q^{-1})$. Then

$$T_n(p_v) = (-q)^n + (-q^{-1})^n = -1 - 1 = -\mathrm{Trace}\gamma_\varphi(\alpha_v).$$

If $\mathrm{Trace}\gamma_\varphi(\alpha_v) = -2$, we choose $p_v = 1 + 1$. Then

$$T_n(p_v) = 1^n + 1^n = 1 + 1 = -\mathrm{Trace}\gamma_\varphi(\alpha_v).$$

Since $\mathrm{Trace}\gamma_\varphi(\alpha_v) = \mathrm{Trace}\gamma_\varphi(\varphi(\alpha_v)) = \mathrm{Trace}\gamma_\varphi(\alpha_{\varphi(v)})$, we have $p_v = p_{\varphi(v)}$. So now we have everything we want. Then we obtain the Kauffman bracket intertwiner $\Lambda_{\varphi, \gamma_\varphi}^q$ associated to these data. We require $|\det(\Lambda_{\varphi, \gamma_\varphi}^q)| = 1$. With the fixed S , φ , γ_φ and $\{p_v\}_v$, we have $|\mathrm{Trace}\Lambda_{\varphi, r}^q|$ is only related to q .

Conjecture 7.4.4. *Suppose S is an oriented surface with negative Euler characteristic, and φ is a periodic diffeomorphism for S . Let γ_φ be the φ -invariant smooth character defined as in the second paragraph of this subsection. For each puncture v , let p_v be the complex number defined as in the third paragraph of this subsection. Let $q_n = e^{2\pi i/n}$ with $(q_n)^{1/2} = e^{\pi i/n}$. Then*

$$\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace} \Lambda_{\varphi, r_\varphi}^{q_n}| = 0.$$

7.4.3 Proofs for the conjecture for some special cases

In the remaining part of this chapter, we will present some results related to our conjecture. Especially, we will give a proof for our conjecture when the surface S is the once punctured torus.

In the following Theorem, we use the periodic property of the diffeomorphisms to prove that the limit in Conjecture 7.4.4 is less than or equal to zero if it exists.

Theorem 7.4.5. *If $\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace} \Lambda_{\varphi, r_\varphi}^{q_n}|$ exists, the limit is less than or equal to zero.*

Proof. Let $\rho : SK_{q_n^{1/2}}(S) \rightarrow \text{End}(V)$ be an irreducible representation of the skein algebra associated to γ_φ and weight system $\{p_v\}_v$. From the definition of intertwiners $\Lambda_{\varphi, r_\varphi}^{q_n}$, we know

$$\rho\varphi_\#(X) = \Lambda_{\varphi, r_\varphi}^{q_n} \circ \rho(X) \circ (\Lambda_{\varphi, r_\varphi}^{q_n})^{-1}$$

for all $X \in SK_{q_n^{1/2}}(S)$. We have

$$\rho(\varphi^2)_\#(X) = \rho\varphi_\#(\varphi_\#(X)) = \Lambda_{\varphi, r_\varphi}^{q_n} \circ \rho\varphi_\#(X) \circ (\Lambda_{\varphi, r_\varphi}^{q_n})^{-1} = (\Lambda_{\varphi, r_\varphi}^{q_n})^2 \circ \rho\varphi_\#(X) \circ (\Lambda_{\varphi, r_\varphi}^{q_n})^{-2}.$$

Then it is easy to show that, with any integer j , we have

$$\rho(\varphi^j)_\#(X) = (\Lambda_{\varphi, r_\varphi}^{q_n})^j \circ \rho\varphi_\#(X) \circ (\Lambda_{\varphi, r_\varphi}^{q_n})^{-j}.$$

Since φ is periodic, there exists a positive integer k such that $\varphi^k = Id_S$. Then we have

$$\rho(X) = \rho(\varphi^k)_\#(X) = (\Lambda_{\varphi, r_\varphi}^{q_n})^k \circ \rho(X) \circ (\Lambda_{\varphi, r_\varphi}^{q_n})^{-k}$$

for all $X \in SK_{q^{1/2}}(S)$. We must have $(\Lambda_{\varphi, r_\varphi}^{q_n})^k = \lambda I$ because ρ is irreducible, where I is the identity matrix and λ is a nonzero complex number. But we require $|\det(\Lambda_{\varphi, r_\varphi}^{q_n})| = 1$, thus $|\lambda| = 1$. Actually we can always choose a good $\Lambda_{\varphi, r_\varphi}^{q_n}$ such that $(\Lambda_{\varphi, r_\varphi}^{q_n})^k = I$. Since $x^k - 1$ has no multiple roots, then $\Lambda_{\varphi, r_\varphi}^{q_n}$ is always diagonalizable. All its eigenvalues are k -roots of unity. Then

$$\text{Trace}\Lambda_{\varphi, r_\varphi}^{q_n} = \sum_{0 \leq i \leq n-1} \lambda_i,$$

where $\lambda_i^k = 1$ for all $0 \leq i \leq n-1$.

We have $|\text{Trace}\Lambda_{\varphi, r_\varphi}^{q_n}| \leq n$. So if the limit exists, we have the limit is less than or equal to zero. \square

From the proof of Theorem 7.4.5, we know $|\text{Trace}\Lambda_{\varphi, r_\varphi}^{q_n}|$ is simply the absolute value of the sum of roots of unity. We are only concerned with how small $|\text{Trace}\Lambda_{\varphi, r_\varphi}^{q_n}|$ can be because of Theorem 7.4.5. Actually this problem was already asked by Myerson [Mye86] and Tao [Tao]. For any two positive integers k, n , let $f(n, k)$ be the least absolute value of a nonzero sum of n (not necessarily distinct) k -th roots of unity. Myerson gave the lower bound for all positive integers k, n

$$f(n, k) \geq n^{-k}. \quad (7.16)$$

According to [LL00], we know $\text{Trace}\Lambda_{\varphi, r}^{q_n} \neq 0$ if the order of φ is 2^m for some positive integer m .

Theorem 7.4.6. *If φ is of order 2^m where m is any positive integer, for any surface with negative Euler characteristic, we have*

$$\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace}\Lambda_{\varphi, r_\varphi}^{q_n}| = 0.$$

Proof. Since for any odd n , we have $\text{Trace}\Lambda_{\varphi, r}^{q_n} \neq 0$. Then

$$n^{-k} \leq f(n, k) \leq |\text{Trace}\Lambda_{\varphi, r}^{q_n}|,$$

where $k = 2^m$. So we get

$$\frac{1}{n} \log n^{-k} \leq \frac{1}{n} \log |\text{Trace}\Lambda_{\varphi, r_\varphi}^{q_n}| \leq \frac{1}{n} \log n.$$

Then $\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace} \Lambda_{\varphi, r_\varphi}^{q_n}| = 0$. \square

Proposition 7.4.7. *If φ is of order p^m where p is any positive prime number and m is any positive integer, for any surface with negative Euler characteristic, we have*

$$\limsup_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace} \Lambda_{\varphi, r_\varphi}^{q_n}| = 0.$$

Proof. The proof is similar with Theorem 7.4.6. \square

For the following discussion, we will use some notations and terminologies in [BWY21]. Suppose the surface S has at least one puncture, that is, it has ideal triangulations. Let τ be an ideal triangulation of S , and let φ be any periodic map of S . Suppose $\tau = \tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)} = \varphi(\tau)$ is an ideal triangulation sweep. Since φ fixes a point in the Teichmüller space, there exists a periodic edge weight system $a = a^{(0)}, a^{(1)}, \dots, a^{(k)} = a \in (\mathbb{R}_{>0})^e$ where a is the shear parameter corresponding to this fixed point in the Teichmüller space. Then $[\overline{\gamma}_\varphi] \in \mathcal{X}_{\text{PSL}_2(\mathbb{C})}(S)$ is the character associated to the weight system a . From the above discussion, we know $[\overline{\gamma}_\varphi]$ can be lift to a smooth φ -invariant character $[\gamma_\varphi]$ in $\mathfrak{X}_{\text{SL}_2(\mathbb{C})}(S)$.

We also have $a_{i_1} a_{i_2} \dots a_{i_j} = 1$, where $e_{i_1}, e_{i_2}, \dots, e_{i_j}$ are all the edges connecting to a common vertex, because a corresponds to a complete hyperbolic structure. If $\text{Trace} \gamma_\varphi(\alpha_v) = 2$, set $h_v = q^2$. Then $h_v^n = 1$ and $p_v^2 = h_v + h_v^{-1} + 2$. If $\text{Trace} \gamma_\varphi(\alpha_v) = -2$, set $h_v = 1$. Then $h_v^n = 1$ and $p_v^2 = h_v + h_v^{-1} + 2$. Obviously $h_v = h_{\varphi(v)}$ for any puncture v . Proposition 15 in [BWY21] implies that we can obtain an interwiner $\overline{\Lambda}_{\varphi, \overline{r_\varphi}}^q$ with $|\det(\overline{\Lambda}_{\varphi, \overline{r_\varphi}}^q)| = 1$. According to Theorem 16 in [BWY21], we have $|\text{Trace} \overline{\Lambda}_{\varphi, \overline{r_\varphi}}^q| = |\text{Trace} \Lambda_{\varphi, r_\varphi}^q|$.

For the once punctured torus $S_{1,1}$, we only have one puncture v . Let $\alpha = K_2, \beta = K_1$ denote two elements in $\pi_1(S_{1,1})$, see Figure 7.3. It is well-known that α, β freely generate $\pi_1(S_{1,1})$. Let α_v be the loop around v . Then $\alpha_v = \beta \alpha \beta^{-1} \alpha^{-1}$. From Lemma 7.4.1, we have

$$\text{Trace} \gamma_\varphi(\alpha_v) = \text{Trace} \gamma_\varphi(\beta \alpha \beta^{-1} \alpha^{-1}) = \text{Trace} \gamma_\varphi(\beta) \gamma_\varphi(\alpha) \gamma_\varphi(\beta)^{-1} \gamma_\varphi(\alpha)^{-1} \neq 2$$

because γ_φ is injective. Thus we must have $\text{Trace} \gamma_\varphi(\alpha_v) = -2$, which means $h_v = 1$.

Lemma 7.4.8. *Let the surface be $S_{1,1}$, then Conjecture 7.4.4 holds for φ being*

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

or $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. We only prove the case when $\varphi = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ (the proof for the other one is similar with this one). Let τ be the ideal triangulation in Figure 7.2. Then $\varphi(\tau)$ is the following ideal triangulation in Figure 7.4.

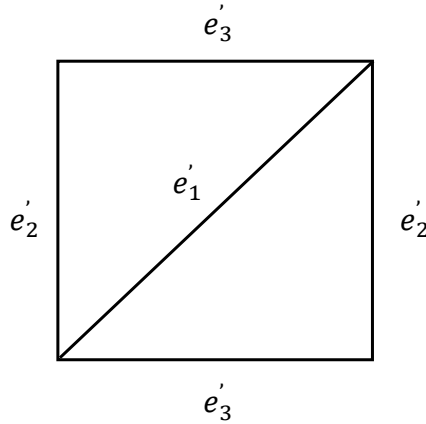


FIGURE 7.4

Thus from τ to $\varphi(\tau)$ is relabeling. Suppose the shear parameter for τ is $a^\tau = (a_1, a_2, a_3)$, then the shear parameter for $\varphi(\tau)$ is $a^{\varphi(\tau)} = (a_3, a_1, a_2)$. From $a^\tau = a^{\varphi(\tau)}$, we get $a_1 = a_2 = a_3$. Since we also have $a_1^2 a_2^2 a_3^2 = 1$ and $a_i \in \mathbb{R}_{>0}$, we have $a_1 = a_2 = a_3 = 1$.

Recall that the Chekhov-Fock algebra associated to the ideal triangulation τ is $\mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}]$, where Y_i corresponds to edge e_i for $i = 1, 2, 3$. The algebra $\mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}]$ is generated by Y_1, Y_2, Y_3 and subject to relations:

$$Y_1 Y_2 = q^4 Y_2 Y_1, Y_2 Y_3 = q^4 Y_3 Y_2, Y_3 Y_1 = q^4 Y_1 Y_3, Y_i Y_i^{-1} = Y_i^{-1} Y_i = 1.$$

Define the irreducible representation ρ of $\mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}]$ as $\rho_{1,1,1}$ in Lemma 7.2.7, that is, set $y_1 = y_2 = y_3 = 1$ in equation 7.7. Then

$$\rho(Y_1^n) = Id_V = a_1 Id_V, \rho(Y_2^n) = Id_V = a_2 Id_V, \rho(Y_3^n) = Id_V = a_3 Id_V$$

and

$$\rho(H_v) = \rho([Y_1^2 Y_2^2 Y_3^2]) = Id_V = h_v Id_V.$$

It is easy to calculate that $\Phi_{\tau\varphi(\tau)}^{q_n} \Psi_{\varphi(\tau)\tau}^{q_n}$ is actually an isomorphism from $\mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}]$ to itself and

$$\Phi_{\tau\varphi(\tau)}^{q_n} \Psi_{\varphi(\tau)\tau}^{q_n}(Y_1) = Y_3, \Phi_{\tau\varphi(\tau)}^{q_n} \Psi_{\varphi(\tau)\tau}^{q_n}(Y_2) = Y_1, \Phi_{\tau\varphi(\tau)}^{q_n} \Psi_{\varphi(\tau)\tau}^{q_n}(Y_3) = Y_2.$$

We use ρ' to denote the irreducible representation $\rho \Phi_{\tau\varphi(\tau)}^{q_n} \Psi_{\varphi(\tau)\tau}^{q_n}$. Then ρ is isomorphic to ρ' .

For each $0 \leq k \leq n-1$, set

$$v_k = \sum_{0 \leq i \leq n-1} q_n^{k^2+i^2+4ik+i-k} w_i.$$

Then we have

$$\rho'(Y_1)(v_k) = q_n^{4k} v_k, \rho'(Y_2)(v_k) = q_n^{-2k} v_{k+1}, \rho'(Y_3)(v_k) = q_n^{-2k} v_{k-1}.$$

Define invertible operator Λ for V such that $\Lambda(w_k) = v_k, \forall 0 \leq k \leq n-1$. Then, for all $0 \leq k \leq n-1$, we have

$$\rho'(Y_1)(\Lambda(w_k)) = \rho'(Y_1)(v_k) = q_n^{4k} v_k = \Lambda(q_n^{4k} w_k) = \Lambda(\rho(Y_1)w_k).$$

Thus we get $\rho'(Y_1) = \Lambda \circ \rho(Y_1) \circ \Lambda^{-1}$. Similarly we can show $\rho'(Y_2) = \Lambda \circ \rho(Y_2) \circ \Lambda^{-1}$ and $\rho'(Y_3) = \Lambda \circ \rho(Y_3) \circ \Lambda^{-1}$. Thus Λ is the intertwiner. As a matrix, we have $\Lambda_{i,k} = q_n^{k^2+i^2+4ik+i-k}$.

From pure calculate, we get $|\det(\Lambda)| = n^{\frac{n}{2}}$. Thus we can set $\overline{\Lambda}_{\varphi, \overline{r\varphi}}^{q_n} = n^{-\frac{1}{2}} \Lambda$. Then

$$|\text{Trace} \Lambda_{\varphi, r\varphi}^{q_n}| = |\text{Trace} \overline{\Lambda}_{\varphi, \overline{r\varphi}}^{q_n}| = n^{-\frac{1}{2}} \sum_{0 \leq i \leq n-1} q_n^{6i^2} = n^{-\frac{1}{2}} \sqrt{(6, n)n} = \sqrt{(6, n)}.$$

Obviously we get

$$\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace} \Lambda_{\varphi, r\varphi}^{q_n}| = 0.$$

□

rem In the proof of Lemma 7.4.8, when we try to find the periodic edge weight system for the triangulation sweep $\tau, \varphi(\tau)$, we require $a_i \in \mathbb{R}_{>0}$ because we want to get the fixed character corresponding to a point in the Teichmüller space. Actually we still get the same intertwiner Λ as in Lemma 7.4.8 without requiring $a_i \in \mathbb{R}_{>0}$, that is, for any periodic edge weight system, the intertwiner we get is Λ . This means Lemma 7.4.8 still holds when we choose any other φ -invariant smooth character (without the restriction for only choosing the one corresponding to a fixed point in the Teichmüller space). Readers can check the same arguments hold for Theorems 7.4.6 and 7.4.12.

Let ϕ be a pseudo-Anosov map for S , and let f be any diffeomorphism for surface S . Then $f\phi f^{-1}$ is also a pseudo-Anosov map. Then we have the following conclusion:

Lemma 7.4.9. *Let ϕ be any pseudo-Anosov map for S , and let f be any diffeomorphism for S . If Conjecture 7.1.3 holds for ϕ , then it also holds for $f\phi f^{-1}$.*

Proof. We will use the same notations as in Conjecture 7.1.3. Let f_*^{-1} be the isomorphism from $\pi_1(S)$ to itself induced by f^{-1} . Then $[\gamma f_*^{-1}]$ is a smooth $f\phi f^{-1}$ -invariant character. Set $\theta'_v = \theta_{f^{-1}(v)}$, then θ'_v are invariant under the action of $f\phi f^{-1}$ and

$$\text{Trace} \gamma f_*^{-1}(\alpha_v) = \text{Trace} \gamma(\alpha_{f^{-1}(v)}) = -e^{\theta_{f^{-1}(v)}} - e^{-\theta_{f^{-1}(v)}} = -e^{\theta'_v} - e^{-\theta'_v}.$$

Set $p'_v = e^{\frac{\theta'_v}{n}} + e^{-\frac{\theta'_v}{n}} = e^{\frac{\theta_{f^{-1}(v)}}{n}} + e^{-\frac{\theta_{f^{-1}(v)}}{n}} = p_{f^{-1}(v)}$, then

$$T_n(p'_v) = -\text{Trace} \gamma f_*^{-1}(\alpha_v).$$

Recall that we use $f_{\#}^{-1}$ to denote the isomorphism from $SK_{q^{1/2}}(S)$ to itself induced by f^{-1} . Let ρ be the irreducible representation associated to $[\gamma]$ and puncture weights p_v . Then $\rho f_{\#}^{-1}$ is an irreducible representation associated to character $[\gamma f_*^{-1}]$ and puncture weights p'_v .

With the assumption for Conjecture 7.1.3, we have

$$\rho \phi_{\#}(X) = \Lambda_{\phi,r}^{q_n} \circ \rho(X) \circ (\Lambda_{\phi,r}^{q_n})^{-1}$$

for any element $X \in SK_{q^{1/2}}(S)$ and $|\det(\Lambda_{\phi,r}^{q_n})| = 1$. Then we get

$$\rho_{\#}^{f^{-1}}(f\phi f^{-1})_{\#}(X) = \rho_{\#}^{f^{-1}}f_{\#}\phi_{\#}f_{\#}^{-1}(X) = \rho_{\#}\phi_{\#}f_{\#}^{-1}(X) = \Lambda_{\phi,r}^{q_n} \circ \rho(f_{\#}^{-1}(X)) \circ (\Lambda_{\phi,r}^{q_n})^{-1}.$$

Thus we get $\Lambda_{f\phi f^{-1}, r f_*^{-1}}^{q_n} = \Lambda_{\phi,r}^{q_n}$, and

$$\begin{aligned} \lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace} \Lambda_{f\phi f^{-1}, r f_*^{-1}}^{q_n}| &= \lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace} \Lambda_{\phi,r}^{q_n}| \\ &= \frac{1}{4\pi} \text{vol}_{\text{hyp}}(M_{\phi}) = \frac{1}{4\pi} \text{vol}_{\text{hyp}}(M_{f\phi f^{-1}}). \end{aligned}$$

□

From [BWY22], we know Conjecture 7.1.3 holds for $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Corollary 7.4.10. *Conjecture 7.1.3 holds for all $f\phi f^{-1}$ where f is any element in $GL(2, \mathbb{Z})$.*

Let φ be a periodic map for S , and let g be any diffeomorphism for S . Then $g\varphi g^{-1}$ is also a periodic map. The same discussion as in Lemma 7.4.9 implies the following conclusion.

Lemma 7.4.11. *Let φ be any periodic map for S , and let g be any diffeomorphism for S . If Conjecture 7.4.4 holds for φ , then it also holds for $g\varphi g^{-1}$.*

The following Theorem shows Conjecture 7.4.4 holds for the once punctured torus. This confirms the relation between the intertwiner and the simplicial volume of the corresponding mapping torus.

Theorem 7.4.12. *Conjecture 7.4.4 holds for the once punctured torus.*

Proof. Let φ be any periodic map for $S_{1,1}$. Then the order of φ could be 1, 2, 3, 4 or 6. According to Theorem 7.4.6, Conjecture 7.4.4 holds if the order of φ is 2 or 4.

If the order of φ is 1, then φ is just the identity map. In this case, we can just choose the intertwiner to be the identity operator. Then Conjecture 7.4.4 holds trivially.

We look at the case when the order of φ is 3 or 6. For these two cases, we have $|\text{Trace} \varphi| = 1$. According to [Kar22], we know there exists an element $g \in GL(2, \mathbb{Z})$

such that $\varphi = g \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} g^{-1}$ or $\varphi = g \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} g^{-1}$. From Lemmas 7.4.8 and 7.4.11, we get Conjecture 7.4.4 holds for these two cases.

□

rem From the proof of Theorem 7.4.6, we know if we can show $\text{Trace}\Lambda_{\varphi,r}^{q_n} \neq 0$ after n is big enough, then we can prove

$$\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace}\Lambda_{\varphi,r}^{q_n}| = 0.$$

rem From subsection 7.1.4, we know the periodic edge weight system $a = a^{(0)}, a^{(1)}, \dots, a^{(k)} = a$ for the ideal triangulation sweep $\tau = \tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)} = \varphi(\tau)$ and φ -invariant puncture weights h_v can give us the intertwiner $\bar{\Lambda}_{\varphi,\bar{\gamma}}^q$ such that

$$\bar{\rho} \circ \Phi_{\tau\varphi(\tau)}^q \circ \Psi_{\varphi(\tau)\tau}^q(X) = \bar{\Lambda}_{\varphi,\bar{\gamma}}^q \circ \bar{\rho}(X) \circ (\bar{\Lambda}_{\varphi,\bar{\gamma}}^q)^{-1}$$

for every $X \in \mathcal{T}_\tau^q$.

It is easy to verify that $(\Phi_{\tau\varphi(\tau)}^q \circ \Psi_{\varphi(\tau)\tau}^q)^m = (\Phi_{\tau\varphi^m(\tau)}^q \circ \Psi_{\varphi^m(\tau)\tau}^q)$, and

$$a = a^{(0)}, a^{(1)}, \dots, a^{(k)}, \dots, a^{(0)}, a^{(1)}, \dots, a^{(k)} = a$$

is the periodic edge weight system for the ideal triangulation sweep

$$\begin{aligned} \tau = \tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)} = \varphi(\tau), \varphi(\tau^{(1)}), \dots, \varphi(\tau^{(k)}) = \varphi^2(\tau), \dots, \\ \varphi^{m-1}(\tau) = \varphi^{m-1}(\tau^{(0)}), \varphi^{m-1}(\tau^{(1)}), \dots, \varphi^{m-1}(\tau^{(k)}) = \varphi^m(\tau) \end{aligned}$$

and $h_{\varphi^m(v)} = h_v$.

Suppose φ is periodic with order m , then

$$(\bar{\Lambda}_{\varphi,\bar{\gamma}}^q)^m \circ \bar{\rho}(X) \circ (\bar{\Lambda}_{\varphi,\bar{\gamma}}^q)^{-m} = \bar{\rho} \circ \Phi_{\tau\varphi^m(\tau)}^q \circ \Psi_{\varphi^m(\tau)\tau}^q(X) = \bar{\rho}(X)$$

for every $X \in \mathcal{T}_\tau^q$. Then $(\bar{\Lambda}_{\varphi,\bar{\gamma}}^q)^m$ is a scalar matrix since $\bar{\rho}$ is irreducible. Actually we can choose good $\bar{\Lambda}_{\varphi,\bar{\gamma}}^q$ such that $(\bar{\Lambda}_{\varphi,\bar{\gamma}}^q)^m$ is the identity matrix. We have all the eigenvalues of $\bar{\Lambda}_{\varphi,\bar{\gamma}}^q$ are m -roots of unity, and $|\text{Trace}\bar{\Lambda}_{\varphi,\bar{\gamma}}^q| = 0$ or $|\text{Trace}\bar{\Lambda}_{\varphi,\bar{\gamma}}^q| \geq n^{-m}$.

From lemma 11 in [BWY21], we know the complex dimension of the space of all periodic edge weight systems for the fixed ideal triangulation is more than or equal to 1. Thus this space is connected. In a local open subset of this space, we can choose φ -invariant puncture weights such that these puncture weights smoothly vary according to periodic edge weight systems. Then we have $|\text{Trace}\overline{\Lambda}_{\varphi,\overline{\gamma}}^q|$ smoothly varies according to periodic edge weight systems in a local open subset by using the similar argument in complement 10 in [BW19]. Since this space is connected and 0 is an isolated point in the image, we have $|\text{Trace}\overline{\Lambda}_{\varphi,\overline{\gamma}}^q| = 0$ for all periodic edge weight systems with the chosen puncture weights, or $|\text{Trace}\overline{\Lambda}_{\varphi,\overline{\gamma}}^q| \geq n^{-m}$ for all periodic edge weight systems with the chosen puncture weights.

If we can find one periodic edge weight system with the chosen puncture weights such that

$$\lim_{n \text{ odd} \rightarrow \infty} \frac{1}{n} \log |\text{Trace}\overline{\Lambda}_{\varphi,\overline{\gamma}}^{qn}| = 0,$$

we can conclude that the above equation is true for every periodic edge weight system with the chosen puncture weights.

List of Author's Publications and Preprints

Journal Articles

- **Z. Wang**, Representation-reduced stated skein modules and algebras, *Journal of Algebra* 661 (2025): 831-852.
- **Z. Wang**, Kauffman bracket intertwiners and the volume conjecture, accepted by *Algebraic & Geometric Topology* (2024). (arXiv: arXiv:2212.01069).
- **Z. Wang**, Finiteness and dimension of stated skein modules over Frobenius, accepted by *Journal of Knot Theory and Its Ramifications* (2024). (arXiv: arXiv:2212.01069).
- H. Karuo, **Z. Wang**, Finiteness conjecture for 3-manifolds obtained from handlebodies by attaching 2-handles, accepted by *Algebraic & Geometric Topology* (2024), (arXiv: arXiv:2401.00262).
- **Z. Wang**, Stated SL_n -skein modules, roots of unity, and TQFT, accepted by *Israel Journal of Mathematics* (2025), (arXiv: arXiv:2401.09995).

Preprints

- H.K. Kim, T.T.Q. Lê, **Z. Wang**, Frobenius homomorphisms for stated SL_n -skein modules, 2025, arXiv:2504.08657.
- H.K. Kim, **Z. Wang**, Naturality of SL_n quantum trace maps for surfaces, 2024, arXiv:2412.16959.

- H. Karuo, **Z. Wang**, Center of stated $SL(n)$ -skein algebras, 2024, [arXiv:2408.12520](#).
- H.K. Kim, **Z. Wang**, The Unicity Theorem and the center of the SL_3 -skein algebra, 2024, [arXiv:2407.16812](#).
- **Z. Wang**, The classical limit for stated SL_n -skein modules, 2024, [arXiv:2401.14753](#).
- **Z. Wang**, On Frobenius algebras obtained from stated skein algebras, 2023, [arXiv:2310.13116](#).
- **Z. Wang**, On stated $SL(n)$ -skein modules, 2023, [arXiv:2307.10288](#).

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