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CUPPING IN THE COMPUTABLY ENUMERABLE DEGREES

TRAN HONG HANH

SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES

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Abstract

This thesis is mainly concerned with the cupping property in the computably enumerable (c.e.) degrees. In particular, we study major sub-degrees and the quotient structure $\mathbf{R}/\mathbf{NCup}$.

Chapter 1 contains a brief introduction to computability theory, the motivation, and the main results of the thesis. Some basic preliminaries and notations are also included in this chapter.

In Chapter 2, we study low major sub-degrees of a cuppable high degree avoiding a cone in the c.e. degrees. Using the existence of noncuppable degrees, Seetapun showed that every incomplete c.e. degree is a major sub-degree of a c.e. degree above it. Moreover, using a high noncuppable degree, one has a stronger result that every incomplete c.e. degree is a major sub-degree of a high c.e. degree above it. Consequently, there is a cuppable high c.e. degree $\mathbf{h}$ with a low major sub-degree $\mathbf{l} < \mathbf{h}$. Furthermore, this low major sub-degree $\mathbf{l}$ can be directly constructed avoiding the cone above a given nonzero c.e. degree $\mathbf{a}$ by using Sacks splitting theorem and Seetapun's idea. However, it is not straightforward to have that $\mathbf{h} \not\geq \mathbf{a}$. We will give a direct construction for a cuppable high c.e. $\mathbf{h}$ with a low major sub-degree $\mathbf{l}$ such that $\mathbf{h} \not\geq \mathbf{a}$.

In Chapter 3 and Chapter 4, we study n-cuppable degrees and the quotient structure $\mathbf{R}/\mathbf{NCup}$. Li, Wu, and Yang constructed a minimal pair in $\mathbf{R}/\mathbf{NCup}$. Following this work, Bie and Wu used another technique to construct a minimal pair in $\mathbf{R}/\mathbf{NCup}$ which is also a minimal pair in $\mathbf{M}/\mathbf{NCup}$. In particular, they constructed two cuppable c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ which form a minimal pair in $\mathbf{R}$ and there is no c.e. degree cupping both $\mathbf{a}$ and $\mathbf{b}$ to $\mathbf{0}'$. Given $1 \leq n \leq m$, we say that m different incomplete c.e. degrees $\mathbf{a}_1, \cdots, \mathbf{a}_m$ are *n-cuppable* if for any n different degrees $\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_n}$ among them, there is an incomplete c.e. degree $\mathbf{e}$ which cups each a_{i_k} $(1 \leq k \leq n)$ to $\mathbf{0}'$. Both the c.e. degrees constructed by Bie and Wu, and the c.e. degrees constructed by Li, Wu, and Yang are examples of two degrees that are 1-cuppable but not 2-cuppable. In Chapter 3, we generalize the technique of Bie and Wu to construction involving three degrees and construct three incomplete c.e. degrees which are 2-cuppable but not 3-cuppable. This result will be directly generalized to arbitrary n c.e. degrees. Thus, for any $n \geq 1$, there are degrees which are n-cuppable but not $(n+1)$-cuppable.

Li, Wu, and Yang claimed that the diamond lattice can be embedded in **R**/**NCup** preserving 0 and 1. There is no published proof of this fact in the literature. In Chapter 4, we will modify the technique of Bie and Wu to prove this result. In particular, we will construct two cuppable c.e. degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$ and for any c.e. degree $\mathbf{w} \in \mathbf{R}$, $\mathbf{a} \cup \mathbf{w} = \mathbf{b} \cup \mathbf{w} = 0' \rightarrow \mathbf{w} = \mathbf{0}'$.

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I am deeply indebted to my family for their devoted love.

During the pursuit of knowledge, I have learned more about myself and the philosophy of life, as Soare wrote: “Only when one’s own cup has first been filled with love abundantly can one then give love to another”.

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Chapter 1

Introduction

1.1 Overview

In the early 1920s, Hilbert proposed a program (see [68]), that called for a formalization of all mathematics which includes: (1) a finite system of axioms in which every mathematical statement can be expressed as a well-formed formula; (2) (consistency) a proof that the axioms of the system are consistent and this proof should be based on finitely many logic steps and finitely many objects; (3) (completeness) a proof that for any mathematical statement in the system's language, either this statement or its negation is provable from the axioms; (4) (decidability) there should be a procedure to determine the validity of any mathematical statement in the system's language. The program was initiated by Hilbert's second problem about proving the consistency of arithmetic. Nevertheless, Gödel's incompleteness theorems [16] in 1931 gave a negative answer for Hilbert's second problem and made a setback to Hilbert's program. These theorems stated that any consistent formal system T which includes elementary number theory is incomplete, and such a system T cannot prove its own consistency. Closely related to the characterization (4) above is the famous *Entscheidungsproblem*, a fundamental problem of mathematical logic proposed by Hilbert and Ackermann in 1928 [25]. The Entscheidungsproblem is about finding an algorithm that can decide whether a statement of first-order logic is valid or not. The refutation of Hilbert's second problem by Gödel's incompleteness theorems suggested a negative solution for the Entscheidungsproblem. However, the notion of an algorithm had not been precisely defined at that time, and until 1936, Church [6] (with λ-definable functions) and later independently Alan Turing [62] (with the Turing machines) gave a negative solution to this problem. These programs have motivated many mathematicians such as Gödel, Church, Kleene, and Turing, et al. to develop the computability theory we know nowadays.

To confirm the algorithmic unsolvability of a specific problem, one first has to provide an accurate mathematical definition of algorithm and then check that no algorithm can solve the problem. Before 1930s, there was no formal characterization for the informal class of all algorithms, also known as effectively calculable or computable functions. The program of capturing the informal class of intuitively computable functions was then focused intensely after Gödel's results in 1931. From 1931 to 1934, Church and Kleene, worked on λ-definable functions and showed that a function is effectively calculable iff it is λ-definable. However, this result was rejected by Gödel. In [16], Gödel used the notion of primitive recursive function and realized that this notion does not fully cover all effectively calculable functions. After a letter from Herbrand (see [18] and [54]), he extended the class of primitive functions to the class of general recursive functions, also called recursive functions and introduced them in his lecture at Princeton in 1934 (see [17]). Attracted by these concepts, Church then formulated in [5] and [7] (known as Church's Thesis by Kleene in [28]) that the class of effectively calculable functions and the class of recursive functions actually coincide. However, these papers still did not convince Gödel, although he did agree that every recursive function is effectively calculable. Due to the lack of mathematical definition for the notion of finite computation, we cannot come to the conclusion that every calculable function is recursive. But Turing did convince Gödel totally on this point. In his paper [61], Turing analyzed the informal notion of effectively calculable, proposed the formal definition of "Turing machine" and gave firm evidence for the equivalence of these two notions, known as the Turing's Thesis.

Theorem 1.1 (Turing's Thesis)**.** *A function is effectively calculable iff it can be computed by a Turing machine.*

A function computed by a Turing machine is called an *effectively computable* function. A function can be *partial* (i.e. the domain is a subset of $\mathbb{N}$) or *total* (i.e. the domain is $\mathbb{N}$). An effectively computable function is called *partial computable (p.c)* if it is a partial function; and *total computable* (or simply *computable*) if it is a total function.

By Gödel numbering, each Turing machine is coded to a unique number, called *index* of the machine. Inversely, each number codes a Turing machine. Let P_e be the Turing machine with the index e and denoted by φ_e the partial computable function computed by P_e. For each $e \in \omega$, let $W_e := \mathrm{dom}(\varphi_e)$. A set A is *computably enumerable (c.e.)* if $A = W_e$ for some $e \in \omega$, and A is *computable* if both A and it complement $\overline{A}$ are computably enumerable. It is known that A is c.e. iff either $A = \emptyset$ or it is the range of a total computable function.

Another fundamental concept of computability theory is *Turing reducibility*, which was proposed by Post based on *Turing's oracle machine*. Ideally, an *oracle Turing machine* is a Turing machine with an extra infinite "read-only" tape, called *oracle tape*, on which the characteristic function of some set B, called the *oracle*, is written (see [59]). We write Φ_e^A for the partial computable functional computed by the Turing machine P_e with oracle A. Turing introduced the oracle machine in [63], and left this subject forever. Oracle machine and relative computability went unnoticed until Post's significant paper [40] on c.e. sets and their decision problems. In that paper, Post developed Turing's idea of an oracle machine into Turing reducibility. Intuitively, a set A is *Turing reducible* to a set B (or is *computable* in B), denoted by $A \leq_T B$, if there is a Turing machine (or program) supplied with an oracle B which answers every question of the form "Is x in A?" by a computation asking at most finitely many questions of the form "Is y in B?". Turing reducibility induces an equivalence relation on the power set of $\mathbb{N}$: two sets A and B are Turing equivalence, denoted by $A \equiv_T B$ iff $A \leq_T B$ and $B \leq_T A$. The corresponding equivalence classes under this relation are called *Turing degrees*. This concept was introduced by Post in [41], as *degrees of unsolvability*. Kleene then inherited Post's notes and later published a joint paper [29], which laid the fundamentals on Turing reducibility and Turing degrees.

For a set A, we denote by $\deg(A) = \{B : B \equiv_T A\}$ the (Turing) degree of A. We also use boldface lower-case letters, for example, $\mathbf{a}, \mathbf{b}, \ldots$ to denote degrees. Let $\mathbf{D}$ be the class of all Turing degrees. The Turing reduction also induces a partial order on $\mathbf{D}$ as follows. A degree $\mathbf{a} = \deg(A)$ is said to be below a degree $\mathbf{b} = \deg(B)$, denoted by $\mathbf{a} \leq \mathbf{b}$ iff $A \leq_T B$. This definition is well-defined because for any sets A, B, C and D, if $C \equiv_T A$, and $D \equiv_T B$ then $C \leq_T D$. It is known that $(\mathbf{D}, \leq)$ is an upper semi-lattice. Here, the supremum of degrees $\mathbf{a} = \deg(A)$ and $\mathbf{b} = \deg(B)$, denoted by $\mathbf{a} \cup \mathbf{b}$, is the Turing degree of $A \oplus B := \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$. A (Turing) degree is said to be *computably enumerable (c.e.)* if it contains a c.e. set. Let $\mathbf{R}$ be the class of all c.e. degrees. With the restriction of the partial order $\leq$ above on c.e. degrees, $\mathbf{R}$ is also an upper semi-lattice.

The Halting problem is defined by $K = \{e : \varphi_e(e)\downarrow\}$. Given a set A, the *jump* of A, denoted by A', is the Halting problem relativized to A:

$$A' := K^A = \{e : \Phi_e^A(e)\downarrow\} = \{e : e \in W_e^A\}.$$

Correspondingly, the *jump* of a degree $\mathbf{a} = \deg(A)$, is defined by $\mathbf{a}' := \deg(A')$. This definition is well-defined because $A' \equiv_T B'$ if $A \equiv_T B$ for any sets A and B.

Inductively, for every $n \in \omega$, the *n-th jump* of A, denoted by $A^{(n)}$, is defined as follows.

(i) $A^{(0)} := A$,

(i) $\forall n \geq 1, A^{(n)} := (A^{(n-1)})'$.

We write $\mathbf{a}'$ for $\deg(A')$ if $\mathbf{a} = \deg(A)$. Similarly, for a degree $\mathbf{a}$, we have $\mathbf{a}^0 = \mathbf{a}$ and $\mathbf{a}^{(n)} = (\mathbf{a}^{(n-1)})'$ for all $n \geq 1$. The Turing jump induces an infinite hierarchy of degrees as follows.

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \cdots$$

Here, the degree $\mathbf{0} = \deg(\emptyset)$ is the least c.e. degree, the degree $\mathbf{0}' = \deg(K)$ is the greatest element in $\mathbf{R}$, and $\mathbf{0}''' = \deg(Tot)$, where $Tot = \{e : \varphi_e(x) \downarrow$ for all $x \in \omega\}$.

A set $A <_T \emptyset'$ is *low$_n$* if $\deg(A^{(n)}) = \mathbf{0}^{(n)}$ and is *high$_n$* if $\deg(A^{(n-1)}) = \mathbf{0}^{(n)}$. Correspondingly, a degree $\mathbf{a}$ is *low$_n$* if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$ and is *high$_n$* if $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$. Let $\mathbf{H}_n$ and $\mathbf{L}_n$ be the set of all high$_n$ c.e. degrees and low$_n$ c.e. degrees, respectively. For $n = 1$, low$_1$ and high$_1$ degrees (or sets) are simply called low and high degrees (or low and high sets). We have that the inclusions $\mathbf{L}_n \subset \mathbf{L}_{n+1}$ and $\mathbf{H}_n \subset \mathbf{H}_{n+1}$ are strict for any n.

Post noted that many important questions from various fields of mathematics could be considered as decision problems of c.e. sets, for example, Hilbert's tenth problem, the word problem, the isomorphism problem, and the conjugacy problem (see [24], [11]). Hilbert's tenth problem is finding an algorithm to determine whether a given Diophantine equation has an integer solution. The latter three problems are in group theory: the word problem asked for an algorithm to decide whether a word in a finitely presented group is identity; the isomorphism problem is finding an algorithm to determine whether two given finitely presented groups are isomorphic; and the conjugacy problem is finding an algorithm to determine whether two words in a finitely presented group are conjugate. In 1910, Dehn studied geometry interpretations for these three problems on CW-complexes and asked for an algorithm to determine whether a CW-complex is simply connected, i.e. whether a finite presentation of its fundamental group is isomorphic to the trivial group. These mentioned problems later were proven to be in general unsolvable, and in fact, some instances of these problems can have degree $\mathbf{0}'$ (see surveys [3, 10, 43] for the detailed references to these problems). However, in the 1940s, the examples of c.e. sets only included computable sets and complete sets (i.e. sets of degree $\mathbf{0}'$). Hence, Post asked a significant question, i.e. whether there is a c.e. degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ (see [40]). If this question has a negative answer, then all undecidable problems are of the same Turing equivalence class. In attempting to solve his problem, Post introduced simple sets. A c.e set A is *simple* if its complement $\overline{A}$ is infinite and $\overline{A}$ contains no infinite c.e. set, i.e. every infinite c.e. set has nonempty intersection with A. However, the simple sets constructed by Post are Turing complete. He then considered stronger notions, hypersimple sets and hyperhypersimple sets, but still could not provide an answer to his problem, since the sets he has constructed, are also Turing complete.

In 1954, Kleene and Post [29] constructed nonzero degrees below $\mathbf{0}'$, but still, these degrees are not c.e. degrees. In their construction, they introduced the *finite extension method* to construct incomparable sets A, B Turing reducible to $\mathbf{0}'$. In particular, to satisfy a condition such as $B \not\leq_T A$, we first break it into infinitely many sub-conditions (called requirements) $R_e : B \neq \Phi_e^A, e \in \omega$, and assign priority for all requirements. The set A (similarly for B) will be constructed in a sequence of stages such that $A = \bigcup_s \sigma_s$, where $\sigma_0 \preceq \sigma_1 \preceq \cdots$ are finite strings. At each stage s, σ_s is chosen using the oracle $\mathbf{0}'$, to meet a single requirement, says R_e. Namely, with σ_{s-1} has been defined and x be the least number such that $B(x)$ has not been defined before stage s, we ask a $\mathbf{0}'$-question

$$(\exists \rho)(\exists t)[\rho \succeq \sigma_{s-1} \wedge \Phi_{e,t}^{\rho}(x) \downarrow]?$$

If there is such a ρ, we let $\sigma_s = \rho$, $B(x) = 1 \dot{-} \Phi_{e,t}^{\rho}(x) \downarrow$ to satisfy R_e by a diagonalization $\Phi_e^A(x) \downarrow = \Phi_{e,t}^{\rho}(x) \downarrow \neq B(x) = 1 \dot{-} \Phi_{e,t}^{\rho}(x) \downarrow$. Otherwise, we let σ_s be an arbitrary $\rho \succeq \sigma_{s-1}$ and meet R_e by $\Phi^A(x) \uparrow \neq B(x)$. Once R_e is satisfied, it will remain forever, i.e. there is no injury. The construction is not computable, since it uses oracle $\mathbf{0}'$, so the sets A and B are not computable enumerable. Many facts about the structure $(\mathbf{D}, \leq)$ are proven by using the finite extension method. In particular, for any noncomputable degree $\mathbf{a}$, there is a degree $\mathbf{b} < \mathbf{a}'$ which is incomparable with $\mathbf{a}$. Moreover, there exist minimal pairs in $\mathbf{D}$ and hence, there are degrees $\mathbf{a}$ and $\mathbf{b}$ such that their greatest lower bound exists. Here, two noncomputable degrees $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair if

$$(\forall \mathbf{c} \in \mathbf{D})[(\mathbf{c} \leq_T \mathbf{a}) \wedge (\mathbf{c} \leq_T \mathbf{b}) \rightarrow \mathbf{c} = \mathbf{0}].$$

On the other hand, it was shown in the same paper [29] that there are degrees $\mathbf{b}$ and $\mathbf{c}$ with no greatest lower bound. Therefore, the structure of Turing degrees $(\mathbf{D}, \leq)$ is not a lattice.

The Post's problem was finally solved by Friedberg in 1957 [15] and independently by Muchnik in 1956 [39]. They invented the *finite injury priority method* to construct two incomparable c.e. sets. Consequently, $\mathbf{R}$ is not a linear structure.

Theorem 1.2 (Friedberg, Muchnik)**.** *There exist Turing incomparable c.e. sets.*

In the finite injury priority method, all requirements are given priority as in the finite extension method. However, the construction is computable instead of using an oracle. A c.e. set A will be constructed by an uniformly computable approximation $A = \bigcup_s A_s$, where $A_0 \subseteq A_1 \subseteq \cdots$ are finite sets. At the end of each stage s, A_s is defined to be the set of all numbers which have been enumerated into A. At each stage s, only action of the highest priority requirement which needs attention at this stage will be taken. Action is usually putting some numbers into a set, or putting a restraint to keep numbers out of a set. Each requirement has at most finitely many higher priority requirements. It may happens that a requirement R_e has taken action at stage s; but later, at a stage $s' > s$, a higher priority requirement $R_{e'}$ takes action which injures the previous action of R_e. Frieberg and Muchnik's idea to deal with such a situation is letting R_e be initialized, i.e. restart R_e, and protecting the action of $R_{e'}$ against injuries from later actions of requirements of lower priority than $R_{e'}$. In the construction of theorem 1.2, each requirement only injures lower priority requirements at most finitely many times. Inductively, every requirement will eventually be satisfied after finitely many injuries. A simpler application of the finite injury priority argument is Friedbeg's splitting theorem, where there is no injury among requirements.

Theorem 1.3 (Friedbeg Splitting Theorem)**.** *For any non-computable c.e. set A, there are disjoint noncomputable c.e. sets A_1 and A_2 such that $A_1 \cup A_2 = A$.*

Another application of the finite injury priority method is the construction of a low simple set. Moreover, combining the finite injury priority method with permitting and coding, one can construct a simple set of any c.e. degree.

Improving Friedberg and Muchnik's method in [15,39], Sacks used a finite injury priority argument of unbounded type to prove Sacks Splitting theorem (see [45]). In the proof, he introduced Sacks preservation strategy to achieve a requirement $B \neq \Phi_e^A$, where B is a given noncomputable c.e. set and A is the c.e. set to be constructed in the construction. Instead of breaking down the agreements between B and Φ^A, he preserved all computations $\Phi_e^A(x)[s]$ that agree with $B_s(x)$ at every stage s. Eventually, either there are only finitely many agreements between B and Φ_e^A (so, $B \neq \Phi_e^A$); or B must be computable. The latter contradicts the fact that B is noncomputable. Therefore, the requirement is satisfied. The construction used the finite injury priority argument, however the number of injuries of each requirement cannot be bounded by any computable function.

Theorem 1.4 (Sacks Splitting Theorem)**.** *Let* $\mathbf{a}$ *be a nonzero c.e. degree. Then there are Turing incomparable c.e. degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{a}$.

It is directly implied from Sacks Splitting Theorem that no c.e. degree can be minimal in $\mathbf{R}$, i.e. there is no noncomputable c.e. degree $\mathbf{a}$ such that there exists a noncomputable c.e. degree $\mathbf{b}$ below $\mathbf{a}$. This fact of $\mathbf{R}$ is different from the structure $\mathbf{D}$, in which there are minimal degrees proven by Spector [60] and by Sacks [44]. Sacks preservation strategy is a very useful technique; usually used in constructions of c.e. degrees avoiding a cone, and in negative requirements of infinite injury constructions.

Infinite injury constructions involve the infinite injury priority argument, another powerful technique in the Degree Theory. The infinity injury argument was first introduced by Shoenfield in [50] to prove a weak version of the thickness lemma. Independently, Sacks used the infinite injury priority argument combined with his preservation strategy to prove Sacks jump theorem, and then to prove a significant result that $(\mathbf{R}, \leq)$ is dense [44, 46, 47].

Theorem 1.5 (Sacks Density Theorem)**.** *For any c.e. degrees* $\mathbf{c} < \mathbf{a}$*, there is a c.e. degree* $\mathbf{b}$ *such that* $\mathbf{c} < \mathbf{b} < \mathbf{a}$.

In the infinite injury priority method, usually, there is a phenomenon as follows. A negative requirement, say N_e, may be injured infinitely times, making the restraints $r(e, s)$ of N_e have the infinite lim sup (i.e. $\limsup_s r(e, s) = \infty$), and hence, N_e can prevent a lower priority positive requirement P_i from putting numbers into A. One can make $\liminf_s r(e, s) = L < \infty$, so that there are infinitely many stages s at which $r(e, s) = L$ and P_i can put $x > L$ into A. However, it may happen that two infinitary negative requirements, says N_e and $N_{e'}$, with $\liminf_s r(e, s) < \infty$ and $\liminf_s r(e', s) < \infty$, together have a combine infinite lim inf of restraints, i.e. at almost all stages s, a positive requirement P_i of lower priority than N_e and $N_{e'}$ cannot put numbers into A by at least one of these two restraints $r(e, s)$, $r(e', s)$. There are several ways to overcome this difficulty. Sacks' approach was to let a follower of P_i (i.e. a number that P_i wants to put into A), pass one by one the restraints of higher priority negative requirements. This idea was formalized in the pinball machine model (see [36]). Another approach, suggested by Lachlan, was to make the restraints of all negative requirements of higher priority than P_i drop simultaneously at a stage s, allowing P_i to put numbers into A (see [30, 32, 57]). Another approach is the priority tree model [33]. In the tree model, each requirement has several strategies, each is assigned to a node in a tree, guessing about the outcomes of higher priority requirements. The tree model will be used throughout constructions in this thesis and will be described in Section 1.2.

Naturally, one may ask whether Sacks Density Theorem can be combined with Sacks splitting theorem, i.e. given any c.e. degree $\mathbf{c} < \mathbf{a}$, can we construct two c.e. degree $\mathbf{a}_0$ and $\mathbf{a}_1$ such that $\mathbf{c} < \mathbf{a}_0, \mathbf{a}_1 < \mathbf{a}$ and $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{a}$? Robinson [42] gave a positive answer to this question if $\mathbf{c}$ is low.

Theorem 1.6 (Robinson Splitting Theorem)**.** *Given two c.e. degrees* $\mathbf{a}$ *and* $\mathbf{c}$ *such that* $\mathbf{c} < \mathbf{a}$ *and* $\mathbf{c}$ *is low. Then there are incomparable low c.e. degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\mathbf{a} = \mathbf{a}_0 \cup \mathbf{a}_1$ *and* $\mathbf{c} < \mathbf{a}_i$ *for* $i \in \{0, 1\}$.

Furthermore, Harrington [52] proved that it also can be done when $\mathbf{a}$ is low$_2$, i.e. $\mathbf{a}'' := (\mathbf{a}')' = \mathbf{0}'$.

Theorem 1.7 (Harrington Splitting Theorem)**.** *Let* $\mathbf{a}$ *and* $\mathbf{c}$ *be low*$_2$ *c.e. degrees such that* $\mathbf{c} < \mathbf{a}$. *Then there exist c.e. degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\forall i \in \{0, 1\}, \mathbf{c} < \mathbf{a}_i < \mathbf{a}$ *and* $\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a}$.

However, Lachlan [33] proved that it is not possible to combine Sacks Splitting Theorem and Sacks Density Theorem in general.

Theorem 1.8 (Lachlan Non-splitting Theorem)**.** *There are c.e. degrees* $\mathbf{a} < \mathbf{b}$ *such that for any c.e. degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$*,*

$$[(\mathbf{a} \le \mathbf{a}_0, \mathbf{a}_1 \le \mathbf{b}) \wedge (\mathbf{b} \le \mathbf{a}_0 \cup \mathbf{a}_1)] \to [\mathbf{b} \le \mathbf{a}_0 \vee \mathbf{b} \le \mathbf{a}_1].$$

To prove his Non-splitting Theorem, Lachlan introduced a powerful and useful technique, called the $\mathbf{0}'''$- argument, a finite injury argument along the true path. Due to its enormous complexity, the method was referred to as the "monster method" in 1970s. Later, a better understanding was delivered in [21, 22] by Harrington. In [21], Harrington extended Lachlan's result to prove the following.

Theorem 1.9 (Harrington Non-splitting Theorem)**.** *There is an incomplete c.e. degree* $\mathbf{a}$ *such that* $\mathbf{0}'$ *is not splittable above* $\mathbf{a}$*, i.e. there are no incomplete c.e. degrees* $\mathbf{a}_0 > \mathbf{a}$ *and* $\mathbf{a}_1 > \mathbf{a}$ *such that* $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}'$*.*

The $\mathbf{0}'''$- argument had been widely used to prove many important results, such as Lachlan's Nonbounding Theorem [34], the undecidability of the first order theory of $\mathbf{R}$ by Harrington and Shelah [23], Slaman's theorems on the density of branching degrees [55] and Slaman triples [53].

Inspired by Sacks Density Theorem, Shoenfield conjectured in 1965 [51] that any embedding which preserves 0 and 1 (the least element and the greatest element, respectively) of a finite partial ordering P into $\mathbf{R}$ can be extended to an embedding of any $Q \supseteq P$ into $\mathbf{R}$ (here Q is also a finite partial ordering). Shoenfield even provided the following two consequences of the conjecture in the structure $(\mathbf{R}, \le)$.

C1. For any incomparable c.e. degrees $\mathbf{a}$ and $\mathbf{b}$, the greatest lower bound $\mathbf{a} \cap \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b}$ does not exist.

C2. For any c.e. degrees $\mathbf{0} < \mathbf{c} < \mathbf{a}$, there is a c.e. degree $\mathbf{b} < \mathbf{a}$ such that $\mathbf{b} \cup \mathbf{c} = \mathbf{a}$.

Consequence C1, and hence Shoenfield's conjecture, were first found to be incorrect by Lachlan [30] and Yates [64]. Two nonzero c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ form a *minimal pair* in $\mathbf{R}$ if their greatest lower bound is $\mathbf{0}$. Consequence C1 implies that there is no minimal pair in $\mathbf{R}$. However, Lachlan in 1966 [30] and independently, Yates in [64] proved the existence of minimal pairs in $\mathbf{R}$, thus, disproving C1. Moreover, Lachlan [30] and Yates [64] proved that there are two incomparable c.e. degrees with no infimum. Hence, $\mathbf{R}$ is not a lattice.

A c.e. degree $\mathbf{a}$ is *cuppable* if there is an incomplete c.e. degree $\mathbf{b}$ cups $\mathbf{a}$ to $\mathbf{0}'$, i.e. $\mathbf{b} \cup \mathbf{a} = \mathbf{0}'$. A c.e. degree $\mathbf{a}$ is *noncuppable* if it is not cuppable.

Sacks Splitting Theorem implies that there are incomplete cuppable degrees. Consequence C2 implies that every nonzero c.e. degree is cuppable. However, this contradicts the existence of noncuppable degrees which was first proven by Yates (unpublished) and later by Cooper in [8], Harrington in [19].

Although being refuted, these consequences and Shoenfield's conjecture led to many important results and techniques in the study of c.e. degrees.

In [20], Harrington considered the *plus-cupping*, a much stronger cupping property. Say that a nonzero c.e. degree $\mathbf{a}$ is *plus-cupping* if for any c.e. degrees $\mathbf{c}, \mathbf{d}$ such that $\mathbf{0} < \mathbf{c} \leq \mathbf{a} \leq \mathbf{d}$, there is a c.e. degree $\mathbf{b} < \mathbf{d}$ such that $\mathbf{b} \cup \mathbf{c} = \mathbf{d}$.

Theorem 1.10 (Harrington Plus-cupping Theorem)**.** *There exist plus-cupping degrees.*

If a c.e. degree $\mathbf{a}$ has the plus-cupping property, for any c.e. degree $0 < \mathbf{c} < \mathbf{a}$, there is a c.e. degree $\mathbf{b} < \mathbf{a}$ such that $\mathbf{b} \cup \mathbf{c} = \mathbf{a}$, and hence, consequence C2 holds for $\mathbf{a}$. Another simpler version of plus-cupping was presented in [14] by Fejer and Soare, where the degree $\mathbf{d}$ in the definition of plus-cupping is restricted to $\mathbf{0}'$.

The dual notion of cuppable is cappable. Particularly, a c.e. degree $\mathbf{a}$ is *cappable* if it is either $\mathbf{0}$ degree or a half of a minimal pair. A c.e. degree $\mathbf{a}$ is *noncappable*, if it is not cappable. Let $\mathbf{M}$ and $\mathbf{NC}$ be the class of cappable degrees and the class of noncappable degrees, respectively. The existence of noncappable degrees was proven by Yates in [65]. Hence, $\mathbf{M}$ and $\mathbf{NC}$ form a nontrivial partition of $\mathbf{R}$. To study the automorphisms of the lattice of c.e. sets, Maass [38] introduced promptly simple sets. A c.e. set A is *promptly simple* if it is coinfinite and there is a computable function f and an enumeration $\{A_s\}_{s\in\omega}$ of A such that

$$\forall e \in \omega, |W_e| = \infty \to (\exists^\infty x)(\exists s)[x \in (W_{e,s} \setminus W_{e,s-1}) \cap A_{f(s)}].$$

A *promptly simple* degree is a c.e. degree containing a promptly simple set. Denote by $\mathbf{PS}$ the class of promptly simple degrees. A c.e. degree $\mathbf{a}$ is *low-cuppable* if there is a low c.e. degree $\mathbf{b}$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$. Let $\mathbf{LC}$ be the class of low-cuppable degrees. The classes $\mathbf{M}$ and $\mathbf{NC}$ have the following properties, proven in [1].

Theorem 1.11 (Ambos-spies, Jockusch, Shore, and Soare)**.** *In the structure of c.e. degrees* $(\mathbf{R}, \leq)$*,*

(i) $\mathbf{M}$ *is an ideal;*

(ii) $\mathbf{NC} = \mathbf{PS} = \mathbf{LC}$ *is a strong filter.*

Here, a subset $F \subseteq \mathbf{R}$ is a *strong filter* if it is closed upwards and for all $\mathbf{a}, \mathbf{b} \in F$, there is $\mathbf{c} \in F$ such that $\mathbf{c} \leq \mathbf{a}$ and $\mathbf{c} \leq \mathbf{b}$.

Since all noncappable degrees are low-cuppable, we have that all noncuppable degrees are cappable. Therefore, every c.e. degree is either cappable or cuppable, a result first proved by Harrington.

When characterizing properties of c.e. sets, Lachlan introduced in [31] the notion of major subset and proved that every c.e. set has a major subset. Similarly, Lachlan then suggested the concept of major sub-degree for the semi-lattice of c.e. degrees as follows. A c.e. degree $\mathbf{c}$ is called a *major sub-degree* of a c.e. degree $\mathbf{a}$ if $\mathbf{c} < \mathbf{a}$ and for every c.e. degree $\mathbf{b}$, $\mathbf{a} \cup \mathbf{b} = \mathbf{0}' \Rightarrow \mathbf{c} \cup \mathbf{b} = \mathbf{0}'$. In other words, a major sub-degree of $\mathbf{a}$ is a c.e. degree $\mathbf{c}$ below $\mathbf{a}$ and shares the same cupping partners with $\mathbf{a}$.

For a noncuppable c.e. degree $\mathbf{a}$, every c.e. degree $\mathbf{c}$ below $\mathbf{a}$ is a major sub-degree of $\mathbf{a}$. Therefore, major sub-degrees do exist. Moreover, applying pseudo jump operators, Jockusch and Shore in [26] showed that a c.e. degree and its major sub-degrees can be far from each other as follows.

Theorem 1.12 (Jockusch and Shore)**.** *There is a high c.e. degree* $\mathbf{h}$ *such that* $\mathbf{h}$ *is cuppable and has a low major sub-degree* $\mathbf{l} < \mathbf{h}$*.*

It is natural to consider whether every non-computable incomplete c.e. degree has a major sub-degree, as Lachlan proposed in 1967 (known as Lachlan's major sub-degree problem). During 70s and 80s, people put a lot of effort into this question but did not succeed. Until 1993, a little progress has been done in [2] with the following theorem.

Theorem 1.13 (Ambos-Spies, Lachlan and Soare)**.** *Given a noncomputable incomplete c.e. degree* $\mathbf{a}$ *and a c.e. degree* $\mathbf{b}$ *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$*, there is a c.e. degree* $\mathbf{c}$ *strictly below* $\mathbf{a}$ *such that* $\mathbf{c} \cup \mathbf{b} = \mathbf{0}'$*.*

This result can be regarded as a non-uniform version of Lachlan's major sub-degree conjecture.

In [49], Seetapun studied the local noncuppability and proved that low_2 degrees are locally noncuppable. A given c.e. degree $\mathbf{a}$ is *locally noncuppable* if there is a c.e. degree $\mathbf{c}$ strictly below $\mathbf{a}$ satisfying

$$\forall \mathbf{b} \in \mathbf{R}, (\mathbf{b} > \mathbf{c}) \wedge (\mathbf{a} \cup \mathbf{b} = \mathbf{0}') \rightarrow \mathbf{b} = \mathbf{0}'.$$

This c.e. degree $\mathbf{c}$ is a major sub-degree of $\mathbf{a}$.

Clearly, a c.e. degree is locally noncuppable iff it has a major sub-degree. Seetapun's result in [49] is a uniform version of Lachlan's major sub-degree problem for low_2 degrees.

Theorem 1.14 (Seetapun)**.** *Every nonzero c.e.* low_2 *degree has a major sub-degree.*

Also in [49], Seetapun showed that every c.e. degree $\mathbf{a} < \mathbf{0}'$ is a major sub-degree of some c.e. degree $\mathbf{b}$ above $\mathbf{a}$. This result used the existence of a noncuppable degree. Precisely, for a given c.e. degree $\mathbf{a}$, take a noncuppable $\mathbf{c}|\mathbf{a}$, then one can directly check that $\mathbf{a}$ is a major sub-degree of $\mathbf{b} := \mathbf{a} \cup \mathbf{c}$. Moreover, $\mathbf{b}$ can be high by choosing $\mathbf{c}$ as a noncuppable high c.e. degree. So, every incomplete c.e. degree is a major sub-degree of some high c.e. degree. This result immediately implies Jockusch and Shore's result in Theorem 1.12.

In Chapter 2 we study the major sub-degrees with the cone avoidance in the c.e. degrees. Particularly, we prove the following.

Theorem A. *Let* $\mathbf{a}$ *be a nonzero c.e. degree. Then there is a cuppable high c.e. degree* $\mathbf{h}$ *with a low major sub-degree* $\mathbf{l}$ *such that* $\mathbf{h} \ngeq \mathbf{a}$.

For a given incomplete c.e. degree $\mathbf{a}$, from Sacks splitting theorem and considering a noncuppable high degree $\mathbf{c}$ (as Seetapun's idea), immediately we have a cuppable low c.e. degree $\mathbf{l} \ngeq \mathbf{a}$ such that $\mathbf{l}$ is a major sub-degree of the cuppable high c.e. degree $\mathbf{h} := \mathbf{c} \cup \mathbf{l}$. In contrast to this, it is not straightforward to have $\mathbf{h}$ avoiding the cone above $\mathbf{a}$. We will give a direct construction to have that $\mathbf{h} \ngeq \mathbf{a}$ in the Theorem A.

One also has the motivation of finding a natural, degree theoretic semi-lattice satisfying Shoenfield conjecture. Let $\mathbf{NCup}$ be the class of all noncuppable degrees. Clearly, $\mathbf{NCup}$ is an ideal in $\mathbf{R}$. So, it is possible to consider the corresponding quotient structures $\mathbf{R}/\mathbf{M}$ and $\mathbf{R}/\mathbf{NCup}$. Both structures are upper semi-lattices with the greatest element and the least element.

The structure $\mathbf{R}/\mathbf{M}$ was studied in [48] by Schwarz with several structural properties compared to $\mathbf{R}$. In particular, the Friedberg-Muchnick theorem and the Sacks splitting are true in $\mathbf{R}/\mathbf{M}$, but there is no minimal pair as well as no minimal element in this structure. Hence, the diamond lattice cannot be embedded preserving 0 and 1 in $\mathbf{R}/\mathbf{M}$. Additionally, Sui and Zhang proved in [67] that the consequence C2 above is true in $\mathbf{R}/\mathbf{M}$. In [56], Slaman asked if the Shoenfield conjecture is true in this structure. Yi gave a negative answer to this question in [66].

For the structure $\mathbf{R}/\mathbf{NCup}$, it is easy to see that the greatest element in this structure is the equivalent class of $\mathbf{0}'$, which contains only degree $\mathbf{0}'$. In [37], Li, Wu, and Yang constructed two incomplete cuppable c.e. degrees $\mathbf{a}, \mathbf{b}$ such that there is no incomplete c.e. degree which cups both $\mathbf{a}, \mathbf{b}$ to $\mathbf{0}'$. The corresponding equivalent classes of $\mathbf{a}, \mathbf{b}$ then form a minimal pair in $\mathbf{R}/\mathbf{NCup}$. Hence, Shoenfield's conjecture does not hold in this structure.

Theorem 1.15 (Li, Wu, and Yang)**.** *There exist incomplete cuppable c.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that for any c.e. degree* $\mathbf{w} \in \mathbf{R}$,

$$(\mathbf{w} \cup \mathbf{a} = \mathbf{0}') \wedge (\mathbf{w} \cup \mathbf{b} = \mathbf{0}') \rightarrow \mathbf{w} = \mathbf{0}'.$$

Following the work of Li, Wu, and Yang, Bie and Wu constructed in [4], two c.e. degrees $\mathbf{a}, \mathbf{b}$ which satisfy all properties in Theorem 1.15 and form a minimal pair in $\mathbf{R}$.

Theorem 1.16 (Bie and Wu)**.** *There are two cuppable c.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$ *and for any c.e. degree* $\mathbf{w} \in \mathbf{R}$*,*

$$\mathbf{a} \cup \mathbf{w} = \mathbf{b} \cup \mathbf{w} = 0' \to \mathbf{w} = \mathbf{0}'.$$

It is clearly seen that **NCup** is also an ideal of **M**. Similar to the previous structure, **M**/**NCup** is an upper semi-lattice with least element $[\mathbf{0}] = \mathbf{NCup}$. However, this structure does not have the greatest element. The equivalence classes $[\mathbf{a}], [\mathbf{b}]$ of degrees $\mathbf{a}, \mathbf{b}$ in Theorem 1.16 then form a minimal pair in **M**/**NCup**. So, Shoenfield's conjecture is not true in this structure. Since the greatest element doesn't exist in **M**/**NCup**, the diamond lattice cannot be embedded preserving 0 and 1 into **M**/**NCup**.

Given $1 \leq n \leq m$. We say that m different incomplete c.e. degrees $\mathbf{a}_1, \cdots, \mathbf{a}_m$ are *n-cuppable* if for any n different degrees $\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_n}$ among them, there is an incomplete c.e. degree $\mathbf{e}$ such that

$$\forall 1 \leq k \leq n, \mathbf{e} \cup \mathbf{a}_{i_k} = \mathbf{0}'.$$

In Chapter 3 we study n-cuppable degrees. The degrees $\mathbf{a}$, $\mathbf{b}$ constructed in 1.15 and 1.16 are 1-cuppable but not 2-cuppable. The technique for ensuring no incomplete c.e. degree cups both $\mathbf{a}, \mathbf{b}$ to $\mathbf{0}'$ in [4] are different from the one used in [37]. Generalizing this technique to deal with three cuppable degrees, we will construct 2-cuppable degrees that are not 3-cuppable. In particular, we prove the following result in Chapter 3.

Theorem B. *There are cuppable c.e. degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *satisfying the following conditions.*

(i) There exist incomplete c.e. degrees $\mathbf{e}, \mathbf{g}, \mathbf{h}$ *such that*

$$(\mathbf{a} \cup \mathbf{e} = \mathbf{b} \cup \mathbf{e} = \mathbf{0}') \wedge (\mathbf{b} \cup \mathbf{g} = \mathbf{c} \cup \mathbf{g} = \mathbf{0}') \wedge (\mathbf{c} \cup \mathbf{h} = \mathbf{a} \cup \mathbf{h} = \mathbf{0}').$$

(ii) There is no incomplete c.e. degree that cups all $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *to* $\mathbf{0}'$*, i.e. for any c.e. degree* $\mathbf{w}$*,* $\mathbf{a} \cup \mathbf{w} = \mathbf{b} \cup \mathbf{w} = \mathbf{c} \cup \mathbf{w} = \mathbf{0}' \to \mathbf{w} = \mathbf{0}'$.

Applying directly the proof of this theorem to $n \geq 3$ cuppable c.e. degrees, we can construct n-cuppable degrees which are not $(n+1)$-cuppable for any $n \geq 3$. Thus, we have the following.

Theorem B'. *For any* $n \geq 1$*, there are degrees which are* n*-cuppable but not* $(n+1)$*-cuppable.*

Li, Wu, and Yang [37] claimed that the diamond lattice can be embedded into **R**/**NCup** preserving 0 and 1. There is no published proof of this fact in the literature. In Chapter 4, we will modify the technique in [4] to prove the claim. In particular, we will show that

Theorem C (Li, Wu, and Yang)**.** *There are two cuppable c.e. degrees* $\mathbf{a}, \mathbf{b}$ *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$ *and for any c.e. degree* $\mathbf{w} \in \mathbf{R}$*,*

$$\mathbf{a} \cup \mathbf{w} = \mathbf{b} \cup \mathbf{w} = 0' \to \mathbf{w} = \mathbf{0}'.$$

As a consequence, the equivalent classes $[\mathbf{0}], [\mathbf{a}], [\mathbf{b}]$ and $[\mathbf{0}']$ form the diamond lattice in the quotient structure **R**/**NCup**, as could be seen in figure 1.1.

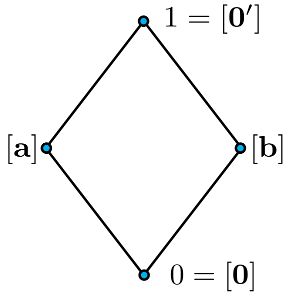

Figure 1.1: The diamond lattice in **R**/**NCup**

The questions about whether the Sacks splitting theorem and consequence C2 are true in **R**/**NCup** (and respectively, in **M**/**NCup**) are still open. All these mentioned works on the structures **R**, **R**/**M**, **R**/**NCup** and **M**/**NCup** are summarised on table 1.1.

All notations and terminologies in this thesis are adapted from [9], [58,59], [12] and [35]. To be simple, we sometimes omit subscripts when they are clear from the context. The following section is devoted to preliminaries that will be needed in the thesis.

1.2 Preliminaries

The set of natural numbers, denoted by $\mathbb{N}$ or $\omega = \{0, 1, 2, \cdots\}$, is the universe here. Lowercase letters $a, b, c, x, y, z, \cdots$ are used for numbers and upper-case letters $A, B, C, \cdots$ are for subsets of ω. Total functions are often denoted by lower-case letters such as $f, g, h, \cdots$. Besides, lower-case Greek letters $\varphi, \psi, \theta, \cdots$, and upper-case Greek letters $\Phi, \Psi, \Theta, \cdots$ usually stand for partial computable functions and partial computable functionals, respectively.

Property	**R**	**R/M**	**R/NCup**	**M/NCup**
Upper-semilattice	yes	yes	yes	yes
Sacks splitting	yes	yes	?	?
Least element	y:$\mathbf{0}$	y:**M**	y:**NCup**	y: **NCup**
Greatest element	y:$\mathbf{0}'$	y: $[\mathbf{0}']$	y:$[\mathbf{0}'] = \{\mathbf{0}'\}$	No
Minimal pair	y	no	yes	yes
Consequence C2	no($\mathbf{NCup} \neq \emptyset$)	y	?	?
Shoenfield's conjecture	no	no(by Yi)	no	no
Diamond lattice embedded preserving $0, 1$	No, by Lachlan's nondiamond theorem	No, since there is no minimal pair	yes	No, since there is no greatest element

Table 1.1: Properties of structures

A set A is identified with its characteristic function χ_A, i.e. for any number $n \in \omega$, $A(n) = 1$ if $n \in A$ and otherwise, $A(n) = 0$. The restriction of A to n, denoted by $A \upharpoonright n$, is the string $A(0)A(1) \dots A(n-1)$. For a total function f, let $f \upharpoonright n$ be the restriction to elements $y < x$. The standard pairing function $\langle \cdot, \cdot \rangle$ from $\omega \times \omega$ to ω is defined by $\langle x, y \rangle := \frac{1}{2}(x^2 + 2xy + y^2 + 3x + y)$ for all $x, y \in \omega$.

We fix a Gödel numbering, by which each Turing machine has a unique code number, also called a Gödel number or an index. Our numbering can be chosen so that it satisfies Lemma 1.1, Theorem 1.17, and Theorem 1.18 below.

Let P_e be the Turing program with index e and φ_e be the (partial) computable function computed by P_e. We say that $\varphi_e(x)$ *converges* to y and write $\varphi_e \downarrow = y$ if the program P_e with input x eventually halts after a computation consisting of finitely many steps and yields output y. Otherwise, if P_e with input x never halts, we write $\varphi_e(x) \uparrow$ and say that $\varphi_e(x)$ *diverges* or is *undefined*.

The following padding lemma says that each p.c. function can have infinitely many indices.

Lemma 1.1 (Padding Lemma)**.** *Each p.c. function φ_x has infinitely many indices. Moreover, there is an infinite set A_x of indices, which can be found effectively in x such that for any $y \in A_x, \varphi_y = \varphi_x$.*

The Enumeration theorem below is a crucial theorem in computability, saying that we can effectively enumerate all p.c. functions.

Theorem 1.17 (Enumeration Theorem or Universal Machine). *There exists a p.c function $\psi(e,x)$ satisfying $\forall e, x \in \omega, \psi(e,x) = \varphi_e(x)$. Since ψ is a p.c. function, there is an index i such that $\varphi_i(e,x) = \psi(e,x)$.*

We remind the reader that any numbering satisfying Theorem 1.17 is said to be *effective*.

The following are fundamental results.

Theorem 1.18 (s-1-1 Theorem). *There is a $1:1$ computable function $s(x,y)$ such that*

$$\forall x, y, z \in \omega, \varphi_{s(x,y)}(z) = \varphi_x(y,z).$$

We recall that a numbering satisfying Theorem 1.17 and Theorem 1.18 is called *acceptable*. Any acceptable numbering automatically satisfies Lemma 1.1.

For each $e \in \omega$, let $W_e := \mathrm{dom}(\varphi_e)$. The sequence $\{W_e\}_{e\in\omega}$ is an effective enumeration of all c.e. sets. We write $\varphi_{e,s}(x) = y$ or $\varphi_e(x)[s] = y$ if $x, y, e < s$ and y is the output in $< s$ steps of program P_e with input x (or of $\varphi_e(x)$ for short). If such a number y exists, we say that $\varphi_{e,s}(x)$ *converges* and denote as $\varphi_{e,s}(x)\downarrow$. Otherwise, we say that $\varphi_{e,s}(x)$ *diverges* or $\varphi_{e,s}(x)$ is *undefined* and write $\varphi_{e,s}(x)\uparrow$. Let $W_{e,s} := \mathrm{dom}(\varphi_{e,s})$.

Theorem 1.19 (Recursion Theorem). *For every computable function f, there is a number n, called a fixed point for f, such that*

$$\varphi_n = \varphi_{f(n)},$$

and hence,

$$W_n = W_{f(n)}.$$

Moreover, such a number n can be computed from an index for the function f.

This theorem was first proven by Kleene in [27] and has had many important applications. In [13], Fejer described clearly how to apply the recursion theorem to get an index for the c.e. set being built in the construction.

A set A is *Turing reducible* to a set B, denoted by $A \leq_T B$ if there is an oracle Turing program with oracle B (also called a Turing functional or Turing reduction) which computes A. We use upper-case Greek letters (for instance, $\Phi, \Gamma, \Theta \cdots$) for oracle Turing programs and write Φ^B for oracle Turing program Φ with oracle B.

Let $\{\tilde{P}_e\}_{e\in\omega}$ be an effective numbering of all oracle Turing programs.

We write $\Phi_e^A(x) = y$ or $\Phi_e^A(x)\downarrow= y$ and say that $\Phi_e^A(x)$ *converges* to y (or $\Phi_e^A(x)$ is defined and equal to y) if the oracle program $\tilde{P}_e$ with oracle A halts at input x and yields output y. Let $\varphi_e^A(x)$ be the maximum element that has been scanned on the oracle tape during the computation. We also say that $\varphi_e^A(x)$ *converges* (or $\varphi_e^A(x)$ is *defined*) and denote as $\varphi_e^A(x)\downarrow$. Clearly, $\Phi_e^A(x)$ is defined iff $\varphi_e^A(x)$ is defined. If $\Phi_e^A(x)$ (or $\varphi_e^A(x)$, respectively) is not defined, we write $\Phi_e^A(x)\uparrow$ (or $\varphi_e^A(x)\uparrow$, respectively) and say that it is *undefined* or *diverges*.

The function φ_e^A defined above is called the *use function* of Φ_e^A. In general, for an oracle Turing program represented by an uppercase Greek letter, we use the corresponding lowercase Greek letter to denote its use function.

If $\Phi_e^A(x) = y$ happens within s steps and e, x, y, and $\varphi_e^A(x) = u$ are all less than s, then we write

$$\Phi_{e,s}^A(x) = y \quad \text{and } \varphi_{e,s}^A(x) = y.$$

Otherwise, we say $\Phi_{e,s}^A(x)$ *diverges* (or $\Phi_{e,s}^A(x)$ is *undefined*) and write $\Phi_{e,s}^A(x)\uparrow$ and $\varphi_{e,s}^A(x)\uparrow$.

Let $W_e^A = \text{dom}(\Phi_e^A)$ and $W_{e,s}^A = \text{dom}(\Phi_{e,s}^A)$.

For an approximation $\{A_s\}_{s\in\omega}$ of a set A (i.e. $A(x) = \lim_s A_s(x)\forall x \in \omega$), we denote $\Phi_e^A(x)[s]$ to be the result of running Φ_e with oracle A_s on input x for s many steps.

The following conventions on any use function φ^A are adopted through the thesis.

(i) φ^A is strictly increasing, i.e. if $m < n$ and $\varphi^A(n)\downarrow, \varphi^A(m)\downarrow$, then $\varphi^A(n) < \varphi^A(m)$. Similarly, if $m < n$ and $\varphi^A(n)[s]\downarrow, \varphi^A(m)[s]\downarrow$, then $\varphi^A(n)[s] < \varphi^A(m)[s]$.

(ii) When A is being approximated, for any n and $s < t$, if $\varphi^A(n)[s]\downarrow$ and $\varphi^A(n)[t]\downarrow$, then $\varphi^A(n)[s]\downarrow \leq \varphi^A(n)[t]\downarrow$.

(iii) For any n and s such that $\varphi^A(n)[s]$ is defined, $\varphi^A(n)[s] \leq s$.

Proposition 1 (Use Principle)**.** *If $\Phi^A(x)\downarrow = y$ and $B \upharpoonright \varphi^A(x) = A \upharpoonright \varphi^A(x)$, then $\Phi^B(x)\downarrow = y = \Phi^A(x)\downarrow$.*

A set A is *limit computable* if there is a computable sequence $\{A_s\}_{s\in\omega}$ such that

$$\forall x \in \omega, A(x) = \lim_{s\to\infty} A_s(x).$$

The sequence $\{A_s\}_{s\in\omega}$ is also called a Δ_2-*approximation* for A.

Theorem 1.20 (Shoenfield Limit Lemma)**.** *Given a set A. The following are equivalent*

(i) A is limit computable,

(ii) $A \in \Delta_2$,

(iii) $A \leq_T \emptyset'$.

Throughout the thesis, we use the infinite injury method. The following are some basic terminology and notations about construction trees. For the details, please refer to [35, 58].

Let Λ be a finite set equipped with a linear order $<_\Lambda$. Consider the tree $\Lambda^{<\omega}$ consisting of finite strings on Λ with the lexicographical ordering induced from $(\Lambda, <_\Lambda)$. We use lower case Greek letters $\alpha, \beta, \gamma, \tau, \eta, \cdots$ to denote strings (also called *nodes*) in $\Lambda^{<\omega}$. For each node $\alpha \in \Lambda^{<\omega}$, we denote by $|\alpha|$ the length of α. Let λ be the empty string, and $\langle o \rangle$ be the string consisting of an element o alone. We denote by $\alpha\hat{\ }\beta$ the concatenation of string α followed by string β. We write $\alpha \prec \beta$ (or $\alpha \subset \beta$) for α is (strictly) an initial segment of β and $\alpha \preceq \beta$ (or $\alpha \subseteq \beta$) for $\alpha \prec \beta$ or $\alpha = \beta$. We say that α *is to the left* of β (or β is *to the right* of α), denoted by $\alpha < \beta$ (or $\beta > \alpha$), if there is a string γ and $o_\alpha <_\Lambda o_\beta$ such that $\gamma\hat{\ }\langle o_\alpha \rangle \preceq \alpha$ and $\gamma\hat{\ }\langle o_\beta \rangle \preceq \beta$. We denote by $\alpha \leq \beta$ that $\alpha < \beta$ or $\alpha \preceq \beta$.

In each proof and construction throughout the thesis, we have a list of conditions to be satisfied (called *requirements*), say $\{R_e\}_{e\in\omega}$. We then design for each requirement a *strategy*, i.e. an effective procedure to satisfy the requirement. Each strategy has one or several outcomes. The set Λ is the set of symbols for all possible outcomes of strategies. We assign to each node $\alpha \in \Lambda^{<\omega}$ of length e the e-th requirement, say R_e. The priority tree T of construction will be a subtree of $\Lambda^{<\omega}$, which is inductively defined as follows.

- $\lambda \in T$;
- if $\alpha \in T$ is of length e (so α is assigned to the e-th requirement R_e), then for any o that is a possible outcome of R_e, $\alpha\hat{\ }\langle o \rangle \in T$.

Each node $\alpha \in T$ is a strategy of the requirement attached to α. If $\alpha \in T$ and $k < |\alpha|$, then $\alpha(k) = o \in \Lambda$ means that α guesses that the outcome of the strategy $\beta = \alpha \upharpoonright k$ is o. Each construction runs stage by stage. At a stage s, we will define *accessible* nodes inductively as follows.

- λ is accessible at stage s,
- If $\alpha \in T$ is accessible at stage s, $|\alpha| < s$, and the outcome of α at stage s is o, then $\beta = \alpha\hat{\ }\langle o \rangle$ is accessible at stage s.

Stage s is called an *α-stage* if α is accessible at stage s. At each stage s, let the *current true path* TP_s be the unique node of length s accessible at this stage. Let the *true path* TP be the unique path of T such that

$$\alpha \prec TP \text{ iff } \exists^\infty s(\alpha \preceq TP_s) \wedge \exists^{<\infty} s(\alpha < TP_s).$$

We say that a strategy α is on the true path if $\alpha \prec TP$.

Chapter 2

A low major sub-degree of a high c.e. degree

In this chapter, we give a direct construction of a high c.e. degree **h** with a low major sub-degree **l**, such that $\mathbf{h} \not\geq \mathbf{a}$ for a given nonzero c.e. degree **a**, as stated in Theorem A.

2.1 Requirements and strategies

Fix a noncomputable c.e. set $A \in \mathbf{a}$. We will construct c.e. sets H and L satisfying, for all $e, i \in \omega$, the following requirements.

$\mathcal{S} : K = \Gamma^{B \oplus L}$;
$\mathcal{P}_e : D \neq \Phi_e^B$;
$\mathcal{H}_e : Tot(e) = \lim_{n \to \infty} \Lambda^H(e, n)$;
$\mathcal{L}_e : (\exists^\infty s)\Phi_{e,s}^L(e) \downarrow \to \Phi_e^L(e) \downarrow$;
$\mathcal{N}_{e,i} : \Phi_i^{H \oplus L \oplus W_e} = K \oplus F \to \exists \Delta_{e,i}(K = \Delta_{e,i}^{L \oplus W_e})$;
$\mathcal{R}_e : A \neq \Phi_e^{H \oplus L}$.

Here, K is a fixed complete c.e. set and $Tot = \{e \in \omega : \Phi_e \text{ is total}\}$ is a Π_2-complete set. The auxiliary c.e. sets B, D, F and (partial) computable functionals $\Gamma, \Lambda, \Delta_{e,i}$ (for all $e, i \in \omega$) are constructed during the construction. All (partial) computable functionals $\{\Phi_e\}_{e \in \omega}$ and all c.e. sets $\{W_e\}_{e \in \omega}$ are effectively listed.We will omit the index i in order to simplify notations.

We check that if H and L are successfully constructed satisfying all the above requirements, then the c.e. degrees $\deg(H \oplus L)$ and $\deg(L)$ are desired c.e. degrees **h** and **l**, respectively, in Theorem A. Indeed, all requirements $\{\mathcal{P}_e\}_{e \in \omega}$ guarantee that B is incomplete, and from the requirement $\mathcal{S}$, the c.e. set L is cuppable. Requirements $\{\mathcal{L}_e\}_{e \in \omega}$ imply that L is low. By satisfaction of all the requirements $\{\mathcal{H}_e\}_{e \in \omega}$, the c.e. set H is high. Therefore, $H \oplus L$ is a high cuppable c.e. set. The requirements $\{\mathcal{N}_e\}_{e \in \omega}$ ensures that $\mathbf{l} = \deg(L)$

is a major sub-degree of $\mathbf{h} = \deg(H \oplus L)$ and finally, $H \oplus L \not\geq_T A$ is assured by the requirements $\{\mathcal{R}_e\}_{e\in\omega}$.

We use a priority tree for the construction. During the construction, whenever we pick a fresh number, this number is bigger than all numbers used so far. In the following, we discuss strategies for the above requirements.

2.1.1 The $\mathcal{S}$-strategy

The $\mathcal{S}$-strategy will construct a functional Γ as follows.

1. If there is the least number n such that $\Gamma^{B\oplus L}(n) \downarrow \neq K(n)$, put $\gamma(n)$ into B and undefine $\Gamma^{B\oplus L}(m)$ for any $m \geq n$.

2. If there is the least number n such that $\Gamma^{B\oplus L}(n) \uparrow$, define $\Gamma^{B\oplus L}(n) = K(n)$ with the use $\gamma(n)$ as a fresh large number.

To meet the requirement $\mathcal{S}$, we will ensure that Γ is total and its corresponding use function γ has the following basic properties.

(1) Whenever $\gamma(n)$ is defined, it will be a fresh number larger than all previously used numbers;

(2) For any number n and stage s, if $\Gamma^{B\oplus L}(n)[s] \downarrow$ then $\gamma(n)[s] \notin B_s \cup L_s$;

(3) For any number n and $m < n$, if $\gamma(n)[s] \downarrow$ then $\gamma(m)[s] \downarrow$ and $\gamma(m)[s] < \gamma(n)[s]$;

(4) If $\Gamma^{B\oplus L}(n)[s] \downarrow= 0$ and n enters K at stage $s+1$, then $\gamma(m)[s]$ is enumerated into B or L for some $m \leq n$;

(5) At stage s, $\Gamma^{B\oplus L}(n)$ is undefined iff there is a number $y \leq n$ such that $\gamma(y)[s]$ is put into B or L at this stage.

Here, the $\mathcal{S}$-strategy only puts numbers into B and it is not sufficient to satisfy the requirement $\mathcal{S}$. Otherwise, K is coded into B, and B will be complete. Construction of $\Gamma^{B\oplus L}$ involves strategies of the other requirements. The requirement $\mathcal{S}$ will have the highest priority among all requirements.

2.1.2 The $\mathcal{P}$-strategies

Basically, a $\mathcal{P}_e$ strategy α is going to apply the Friedberg-Muchnik strategy:

1) pick a fresh witness x,

2) wait for $\Phi_e^B(x) \downarrow= 0$,

3) put x into D, and stop.

However, the computation $\Phi_e^B(x)\downarrow= 0$ may be injured by the $\mathcal{S}$-strategy: when a small number n enters K, $\mathcal{S}$ puts $\gamma(n) \leq \phi_e^B(x)$ into B, making the computation $\Phi_e^B(x)\downarrow= 0$ becomes undefined. To protect this computation, we lift $\gamma-$uses exceeding $\phi_e^B(x)$. Precisely, we pick a *killing point* k (also called a *threshold*). If $\Phi_e^B(x)\downarrow= 0$ at stage s, we put $\gamma(k)[s]$ into L. Thereby, the uses $\gamma(n)$, $n \geq k$, are lifted beyond $\phi_e^B(x)[s]$ and the computation $\Phi_e^B(x)[s]\downarrow = 0$ will be unchanged unless a small number (less than or equal to k) enters K. Whenever an $n \leq k$ is enumerated to K, we reset witness x as fresh. Such a situation can only occur at most $k + 1$ times and hence α is eventually satisfied. So, α will perform as follows.

P1. Pick a killing point k as fresh.
Whenever $n \leq k$ enters K, go to P2.

P2. Pick a fresh witness $x > k$.

P3. Wait for $\Phi_e^B(x)[s]\downarrow = 0$.

P4. Put $\gamma(k)[s]$ into L, x into D, and stop.

The outcomes of α are finitary:

w: wait for P3 forever for some x. In this outcome, $D(x) = 0 \neq \Phi_e^B(x)$,

s: stop at P4 forever for some x, say from stage s onwards. In this outcome, there is no number less than or equal to k enters K after stage s and $\Phi_e^B(x)\downarrow= 0 \neq D(x) = 1$.

2.1.3 The $\mathcal{H}$-strategies

The $\mathcal{H}$-strategies will construct a partial computable functional Λ such that Λ^H is total and

$$\forall e, Tot(e) = \lim_{n\to\infty} \Lambda^H(e, n).$$

By Shoenfield's limit lemma, we have that $\emptyset'' \leq_T Tot \leq_T H'$, and hence, H is high.

For each $e \in \omega$, $\Lambda^H(e, \cdot)$ will be defined as follows. The priority tree T of the construction will be constructed to be finitely branching and all $\mathcal{H}_e$-strategies are located at a certain level of T. There are only finitely many $\mathcal{H}_e$-strategies and these nodes will work jointly to define $\Lambda^H(e, \cdot)$. We effectively number the $\mathcal{H}_e$-strategies in the lexicographical order and hence, for each $\mathcal{H}_e$-strategy τ, there is a unique number, denoted by $k(\tau)$, such that τ is the $k(\tau)$-th node in this numbering. Let

$$k(e) := \max\{k(\tau) : \tau \text{ is an } \mathcal{H}_e\text{-strategy on } T\},$$

and for each $n \in \omega$, define $\Lambda^H(e, n)[0] = 0$ with the use $\langle e, k(e), n\rangle$.
Consider an $\mathcal{H}_e$-strategy τ and a τ-stage s. We define

$$l(\tau, s) := \max\{x < s : \varphi_e(y)[s] \downarrow \forall y < x\},$$

and

$$m(\tau, s) := \max\{0, l(\tau, t) : t < s \text{ and } t \text{ is a } \tau\text{-stage}\}.$$

Stage s is called a τ-*expansionary* stage if $s = 0$ or $l(\tau, s) > m(\tau, s)$.

Let $s_\tau < s$ be the last stage at which τ is initialized. The strategy τ will only deal with $\Lambda^H(e, n)$ for $n > s_\tau$. In particular, when s is a τ-expansionary stage, τ puts $\langle e, k(\tau), n\rangle$ into H and redefines $\Lambda^H(e, n) = 1$ for each n such that $\max\{s_\tau + 1, m(\tau, s)\} \leq n < l(\tau, s)$.

For the strategy τ on the true path, there are two cases as follows.

- There are finitely many τ-expansionary stages. Let s_0 be the last τ-expansionary stage and $n_0 = l(\tau, s_0)$. We have $\varphi_e(n_0) \uparrow$ and for any $\mathcal{H}_e$-strategy τ', $l(\tau', s) \leq n_0$ (s is an arbitrary $\tau'-$stage). Hence, for any $n \geq n_0$, $\Lambda^H(e, n)$ will never be redefined during the construction, and so $\Lambda^H(e, n) \downarrow= 0$. Therefore, $\lim_{n\to\infty} \Lambda^H(e, n) = 0 = Tot(e)$.
- There are infinitely many τ-expansionary stages. For any $n > s_\tau$, at the first τ-expansionary stage $s_1 > s_\tau$ such that $n < l(\tau, s_1)$, τ redefines $\Lambda^H(e, n) = 1$ by putting $\langle e, k(\tau), n\rangle$ into H. Since s_τ is the last stage at which τ is initialised, no $\mathcal{H}_e$-strategy to the left of τ puts numbers into H to redefine $\Lambda^H(e, n)$ after stage s_τ, and hence, $\Lambda^H(e, n) \downarrow= \Lambda^H(e, n)[s_1] = 1$. Thus, $\lim_{n\to\infty} \Lambda^H(e, n) = 1 = Tot(e)$.

The strategy τ has two outcomes: i (for infinitely many expansionary stages) and f (for finitely many expansionary stages).

2.1.4 The $\mathcal{L}$-strategies

Fix an $\mathcal{L}_e$-strategy σ. The goal of $\mathcal{L}_e$ is to ensure that $\Phi_e^L(e)$ converges if there are infinitely many stages s at which $\Phi_e^L(e)[s] \downarrow$. Ideally, at stage s, if $\Phi_e^L(e)[s] \downarrow$, σ will put restraint $\phi_e^L(e)[s]$ on lower priority strategies, to preserve the computation $\Phi_e^L(e)[s] \downarrow$, making $\Phi_e^L(e) \downarrow= \Phi_e^L(e)[s]$.

However, there is an issue as follows. There could be infinitely many stages $s_1 < s_2 < \cdots < s_n < \cdots$ such that for each $i \geq 1$, $\Phi_e^L(e)[s_i] \downarrow$ and for any t which is a σ-stage, $\Phi_e^L(e)[t]\uparrow$. So, σ never sees any $\Phi_e^L(e)$-convergent computation to protect and then σ fails, even if it is on the true path.

To overcome this problem, we give the highest priority on the tree to the $\mathcal{L}$-strategies as follows. At each stage s, let $\mathcal{M}_s$ be the set of all nodes in the priority tree which were accessible before stage s and have not been canceled or initialized yet. Before doing any action on the priority tree at stage s, we first find the least $e < s$ such that $\Phi_e^L(e)[s]\downarrow$ and $\mathcal{L}_e$ has not been satisfied yet. If such a number exists, say e_0, we take the leftmost $\mathcal{L}_{e_0}$-strategy $\sigma' \in \mathcal{M}_s$, and initialise ξ for any strategy $\xi > \sigma'$. By this action, $\Phi_{e_0}^L(e_0)[s]\downarrow$ will be preserved forever, making $\Phi_{e_0}^L(e_0)[s] = \Phi_{e_0}^L(e_0)\downarrow$, unless σ' is initialised by a strategy $\xi' < \sigma'$, and there are only finitely many such injuries. Hence, $\mathcal{L}_e$ will eventually be satisfied. Note that for each strategy ξ on the true path, since there are only finitely many nodes to the left of ξ, the strategy ξ is initialized by such a situation at most finitely many times.

The $\mathcal{L}_e$- strategy σ has outcome 1.

2.1.5 The $\mathcal{N}$-strategies

Fix an $\mathcal{N}_e$-strategy β. We will construct a partial computable functional Δ_β such that if $\Phi^{H\oplus L\oplus W_e} = K \oplus F$ then $\Delta_\beta^{L\oplus W_e}$ is total and computes K correctly.

At a β-stage s, we say that a computation $\Phi^{H\oplus L\oplus W_e}(y)[s]\downarrow$ is *β-believable* if for any $\mathcal{H}_{e'}$-strategy $\tau \prec \tau^\frown\langle i\rangle \prec \beta$,

$$(n > s_\tau) \wedge (\langle e', k(\tau), n\rangle \leq \phi^{H\oplus L\oplus W_e}(y)[s]) \rightarrow \langle e', k(\tau), n\rangle \in H_{s-1},$$

where $s_\tau < s$ is the last stage at which τ has been initialised.

Recall that the *length of agreement* between $\Phi^{H\oplus L\oplus W_e}$ and $K \oplus F$ at a stage s is defined by

$$\begin{aligned} l(\beta, s) := \max\{x < s : \forall y < x, \Phi^{H\oplus L\oplus W_e}(y)[s]\downarrow \text{ is } \beta\text{-believable} \\ \text{and } \Phi^{H\oplus L\oplus W_e}(y)[s] = K_s \oplus F_s(y)\}. \end{aligned}$$

Let $m(\beta, s) := \max\{0, l(\beta, t) : t < s \text{ is a } \beta\text{-stage}\}$.

A stage s is called *β-expansionary* if $s = 0$ or $l(\beta, s) > m(\beta, s)$.

Ideally, we will define Δ_β when the length of agreement increases: at a β-expansionary stage s, if p is the least number such that $\Delta_\beta^{L\oplus W_e}(p)$ has not been defined yet and $2p+1 < l(\beta, s)$, then we set $\Delta_\beta^{L\oplus W_e}(p)[s] = K_s(p)$ with a fresh use $\delta(p)[s]$. After that, this computation is undefined only if $L \oplus W_e$

changes below $\phi(2p+1)[s]$. Therefore, if p enters K after stage s, we would like to force $L \oplus W_e$ to change below $\phi(2p+1)[s]$ so that $\Delta_\beta^{L\oplus W_e}(p)$ is undefined. The possible outcomes of β are i (for infinitely many β-expansionary stages) and f (for finitely many β-expansionary stages).

Interaction with $\mathcal{H}$-strategies

There is a problem as follows. At a β-expansionary stage s_1 with length of agreement $l(\beta, s_1) > 2p+1$, we define $\Delta_\beta^{L\oplus W_e}(p)[s_1] = K_{s_1}(p) = 0$ with use $\delta(p)[s_1] > \phi(2p+1)[s_1]$. After stage s_1, the set $L \oplus W_e$ never changes below $\delta(p)[s_1]$ and an $\mathcal{H}_{e'}$-strategy $\tau \succeq \beta\hat{\ }\langle i\rangle$ puts numbers into H, lifting $\phi(2p)[s_1]$ to a large number so that at a β-expansionary stage $s_2 > s_1$, we have

$$\Phi^{H\oplus L\oplus W_e}(2p)[s_2] = K_{s_2} \oplus F_{s_2}(2p) = K_{s_1} \oplus F_{s_1}(2p) = 0$$

with the new use $\phi(2p)[s_2] > \delta(p)[s_1] > \phi(2p+1)[s_1]$. Later, p enters K and W_e changes below $\phi(2p)[s_2]$, making

$$\Phi^{H\oplus L\oplus W_e}(2p)[s_3] = K_{s_3} \oplus F_{s_3}(2p) = K_{s_3}(p) = 1$$

at a β-expansionary stage $s_3 > s_2$. Since $L \oplus W_e$ never changes below $\delta(p)[s_1]$ after stage s_1, the computation $\Delta_\beta^{L\oplus W_e}(p)[s_1] = 0$ then cannot be rectified to compute $K(p) = 1$ correctly, and β is failed.

We apply the argument for the construction of a noncuppable degree to overcome this obstacle. The rough idea is as follows: when a τ-strategy would like to put a number below $\phi(2p) < \delta(p)$ into H, it first puts a number into F to force $L \oplus W_e$ to change below a small number so that $\Delta_\beta^{L\oplus W_e}(p)$ can be redefined later. Precisely, τ will do as follows. At stage s_0, when choosing a number $x = \langle e', k(\tau), n\rangle$ (to put into H), τ also takes a fresh number $z \notin F$. We call z the *attached* number of x corresponding to β. For any β-expansionary stage $s > s_0$, we require the length of agreement to be $l(\beta, s) > 2z+1$ so that $\delta(p)[s] > \max\{\phi(2z+1)[s], \phi(2p+1)[s]\}$ if $\Delta_\beta^{L\oplus W_e}(p)$ is defined at stage s (this delays the construction a little bit and makes no difference on the true outcome of β). At a β-expansionary stage $s_1 > s_0$, if τ would like to put $\langle e', k(\tau), n\rangle$ into H, it first puts z into F and waits for the next β-expansionary stage s_2. There are two possibilities. If there is no more β-expansionary stage after s_1, then β is satisfied and doesn't need to define the functional Δ_β. Otherwise, the next β-expansionary stage s_2 comes. Without loss of generality, we suppose that τ is not initialized during stages between s_0 and s_2. Note that by the choices of $\langle e', k(\tau), n\rangle$ and z, the $\Delta_\beta^{L\oplus W_e}$-computations defined before or during stage s_0 will not be affected from the action of putting $\langle e', k(\tau), n\rangle$ into H of τ. For a computation $\Delta_\beta^{L\oplus W_e}(p)$ defined at stage t after s_0 ($s_0 < t \leq s_1$), we have

$$\Phi^{H\oplus L\oplus W_e}(2z+1)[t] = K_t \oplus F_t(2z+1) = 0$$

and

$$\Phi^{H\oplus L\oplus W_e}(2z+1)[s_2] = K_{s_2} \oplus F_{s_2}(2z+1) = 1.$$

Since there is no number below $\phi(2z+1)[t]$ entering H during stages between t and s_2, it must be that $L \oplus W_e$ has changed below $\phi(2z+1)[t] < \delta(p)[t]$ at stage s_2, making $\Delta_\beta^{L\oplus W_e}(p)$ redefinable. Then τ now can put $\langle e', k(\tau), n\rangle$ into H and the problem is solved.

Next, we consider an $\mathcal{H}$-strategy working with all of its higher priority $\mathcal{N}$-strategies on the priority tree. Fix an $\mathcal{H}_e$-strategy τ, and suppose that

$$\beta_1\hat{\ }\langle i\rangle \prec \beta_2\hat{\ }\langle i\rangle \prec \cdots \prec \beta_n\hat{\ }\langle i\rangle \prec \tau,$$

where $\beta_j(1 \leq j \leq n)$ are the $\mathcal{N}$-strategies of higher priority than τ with outcome i. To be consistent with these β_j's strategies, when choosing $x = \langle e, k(\tau), n\rangle$ at stage s_0, for each $1 \leq j \leq n$, we take a fresh number z_j to be the attached number of x corresponding to β_j (as described above) such that $z_0 < z_1 < \cdots < z_n$. We define the *attached sequence* of x corresponding to β_j's to be $z(x) = (z_0, \cdots, z_n)$. We require that, for every $1 \leq j \leq n$, any β_j-expansionary stage $s' > s_0$ has length of agreement $l(\beta_j, s') > 2z_j + 1$ (again, this condition only delays the construction and doesn't make any change as regards the true outcome of each β_j, $1 \leq j \leq n$). At a τ-stage $s_1 \geq s_0$, if τ would like to put $\langle e, k(\tau), n\rangle$ into H, it first puts z_n into F and waits for the next β_n-expansionary stage. To be convenient, we make a link between τ and β_n, denoted as $(\beta_n - \tau)$. If there is no β_n-expansionary stage, then β_n has finitary outcome f and is satisfied. Otherwise, at the next β_n-expansionary stage $s_2 > s_1$, we go to τ via the link $(\beta_n - \tau)$ and then cancel this link, continue to put z_{n-1} into F, create the link $(\beta_{n-1} - \tau)$ and wait for the next β_{n-1}-expansionary stage. This action can be repeated at most n times, and if eventually, we cancel the link $(\beta_1 - \tau)$, we then can put x into H fulfilling our purposes. For each β_j $(1 \leq j \leq n)$, there must be changes on the corresponding c.e. set W_e of β_j, sufficient for the $\Delta_{\beta_j}-$computations defined at stages between s_0 and $s_1 + 1$ to be redefined later.

2.1.6 The$\mathcal{R}$-strategies

Fix an $\mathcal{R}_e$-strategy η. At an η-stage s, we say that a computation $\Phi_e^{H\oplus L}(y)[s]\downarrow$ is *η-believable* if for any $\mathcal{H}_{e'}$-strategy $\tau \prec \tau\hat{\ }\langle i\rangle \prec \eta$,

$$(n > s_\tau) \wedge (\langle e', k(\tau), n\rangle \leq \phi_e^{H\oplus L}(y)[s]) \rightarrow \langle e', k(\tau), n\rangle \in H_{s-1},$$

where $s_\tau < s$ is the last stage at which τ has been initialised.

The *length of agreements* between A and $\Phi_e^{H\oplus L}$ is

$$l(\eta, s) := \max\{x < s : \forall y < x, \Phi_e^{H\oplus L}(y)[s]\downarrow \text{ is } \eta\text{-believable}$$
$$\text{and } \Phi_e^{H\oplus L}(y)[s] = A_s(y)\}.$$

A stage s is called *η-expansionary* if $s = 0$ or $l(\eta, s) > l(\eta, t)$ for any η-stage $t < s$.

Let $r(\eta, s) = s$ be the restraint of η at each η-expansionary stage s. Note that at an η-expansionary stage s, the η-believable computations $\Phi_e^{H\oplus L}(y)[s]$ (where $y \leq l(\eta, s)$) are preserved from lower priority strategies of η by the restraint $r(\eta, s)$.

Let

$$T_\eta = \{s : s \text{ is an } \eta\text{-stage}\}.$$

If η is on the true path TP, we have $\lim_{s\in T_\eta} r(\eta, s) < \infty$. Indeed, let s_0 be a stage such that

(i) there is no strategy to the left of η accessible at any stage $t \geq s_0$,

(ii) there is no $\mathcal{H}_{e'}$-strategy $\tau \prec \tau\hat{}\langle i\rangle \prec \eta$ which is initialised at a stage $t \geq s_0$,

(iii) there is no $\mathcal{P}_j$-strategy $\alpha \prec \eta$ putting number into L after stage s_0,

(iv) η is not initialised by any $\mathcal{L}_i$-strategy $\sigma < \eta$ after stage s_0.

There are two cases as below.

- There are only finitely many η-expansionary stages, let s_1 be the last η-expansionary stage. Clearly, $\lim_{s\in T_\eta} r(\eta, s) = r(\eta, s_1) = s_1 < \infty$.

- There are infinitely many η-expansionary stages. For any $n \in \omega$, to compute $A(n)$, we wait for an η-stage $s > s_0$ such that $l(\eta, s) > n$; since the computation $\Phi_e^{H\oplus L}(n)[s]$ is η-believable and η is not initialised after stage s_0, then no number less than $\phi_e^{H\oplus L}(n)[s]$ is enumerated into H or L, and therefore we have that $A_s(n) = \Phi_e^{H\oplus L}(n)[s] = \Phi_e^{H\oplus L}(n) = A(n)$. Thus, $A = \Phi_e^{H\oplus L}$ is computable , that is a contradiction.

Thus, there are only finitely many η-expansionary stages and $\lim_{s\in T_\eta} r(\eta, s) < \infty$, implying that the requirement R_e is satisfied. Note that η will not injure infinitely many times any lower $\mathcal{H}_{e'}$-strategy τ' (η only prevents τ' from putting numbers less than $\lim_{s\in T_\eta} r(\eta, s)$ into H). We set 1 as the outcome of $\mathcal{R}_e$-strategy η.

2.2 The construction

Let $\Lambda = \{i < f < s < w < 1\}$ be the set of outcomes and let $\Lambda^{<\omega}$ be equipped with the lexicography order induced from the order in Λ.

The $\mathcal{S}$-strategy has the highest priority and the $\mathcal{P}, \mathcal{H}, \mathcal{L}, \mathcal{N}, \mathcal{R}$-strategies are set up on a priority tree $T \subseteq \Lambda^{<\omega}$ as follows. We effectively list all $\mathcal{P}, \mathcal{H}, \mathcal{L}, \mathcal{N}, \mathcal{R}$-requirements and assign to each node ξ of length e the e-th requirement on the list. The root λ is assigned to the requirement $\mathcal{P}_0$. The priority tree T of strategies is a subtree of $\Lambda^{<\omega}$ where each node $\xi \in T$ is a strategy of its assigned requirement and $\{\xi\hat{\ }\langle o\rangle : o$ ranges over all possible outcomes of $\xi\}$ is the set of all immediate successors of ξ.

We fix an effective enumeration $\{K_s\}_{s\in\omega}$ of K such that there is at most one number entering K at each stage. During the construction, whenever we pick a number, we choose a fresh one that is larger than all numbers which have been used before. All computations at a stage s are bounded by s.

For each stage s, let $\mathcal{M}_s$ be the set of all nodes which were accessible before stage s and have not been initialized at the beginning of stage s.

To be convenient, we attach parameters to each strategy. Particularly, each $\mathcal{P}$-strategy α has parameters $k(\alpha), x(\alpha)$ corresponding to the killing point k and the witness x, respectively. An $\mathcal{H}_e$-strategy τ has parameter $s(\tau)$ marking the stage at which τ is accessible after being initialised. An $\mathcal{N}$-strategy β will be attached a parameter $n(\beta)$ for marking the length of agreements. The construction is as follows.

Stage 0**.** All sets are empty. Initialize all strategies. Define $\Lambda^H(e,n)[0] = 0$

Stage $s+1$**.**
a. The $\mathcal{S}$-strategy defines $\Gamma^{B\oplus L}$ as follows.

S1. If n enters K, making $\Gamma^{B\oplus L}(n)[s]\downarrow\neq K_{s+1}(n)$, put $\gamma(n)[s]$ into B and undefine $\Gamma^{B\oplus L}(y)$, for any $y \geq n$.

S2. Else, for the least number n such that $\Gamma^{B\oplus L}(n)\uparrow$, define $\Gamma^{B\oplus L}(n)[s+1] = K_{s+1}(n)$ with the use $\gamma(n)[s+1]$ as a fresh number.

b. Working on the priority tree.

Step 1. If there is the least number $e_0 < s$ such that $\Phi^L_{e_0}(e_0)[s+1]\downarrow$ and there is no $\mathcal{L}_{e_0}$-strategy satisfied yet, take the leftmost $\mathcal{L}_{e_0}$-node $\sigma \in \mathcal{M}_{s+1}$, initialise all $\xi > \sigma$ and declare that $\mathcal{L}_{e_0}$ is satisfied via σ at stage $s+1$.

Step 2. If n enters K, and for any $\mathcal{P}$-strategy α such that $k(\alpha)$ has been defined with $n \leq k(\alpha)$, reset $x(\alpha)$ as undefined and initialise all lower priority strategies.

Step 3. Compute the current true path TP_{s+1} inductively: $\lambda \prec TP_{s+1}$; if $\xi \preceq TP_{s+1}$ and ξ is of length $s+1$, let $\xi = TP_{s+1}$, initialise all strategies to the right of TP_{s+1} and go to **Step 4**; otherwise, do the actions for ξ and find its immediate successor $\xi' \preceq TP_{s+1}$ as follows. There are five cases.

Case 1. ξ is a $\mathcal{P}_e$-strategy α. Run the following α-program.

α1.
- $*$ If $k(\alpha)\uparrow$, set it as fresh.
- $*$ Go to α2.

α2. If $x(\alpha)\downarrow$ go to α3.
Else,
- $*$ set $x(\alpha)\uparrow$ as a fresh number,
- $*$ let $\alpha\hat{}\langle w\rangle = \xi' \preceq TP_{s+1}$ and initialise all lower priority strategies.

α3. Parameters $k(\alpha), x(\alpha)$ are defined:
If $\Phi_e^B(x(\alpha))[s+1]\downarrow= 0$ and $x(\alpha)\in D$, let $\alpha\hat{}\langle s\rangle = \xi' \preceq TP_{s+1}$.
If $\Phi_e^B(x(\alpha))[s+1]\downarrow= 0$ and $x(\alpha)\notin D$,
- $*$ put $\gamma(k(\alpha))[s+1]$ into L and undefine $\Gamma^{B\oplus L}(y)\forall y\geq k(\alpha)$,
- $*$ put $x(\alpha)$ into D,
- $*$ let $\alpha\hat{}\langle s\rangle = \xi' \preceq TP_{s+1}$ and initialise all lower priority strategies.

Otherwise, i.e. $\neg(\Phi_e^B(x(\alpha))[s+1]\downarrow= 0)$, let $\alpha\hat{}\langle w\rangle = \xi' \preceq TP_{s+1}$.

Case 2. ξ is an $\mathcal{H}_e$-strategy τ.

τ1. If τ was not accessible at any stage t such that $s_\tau < t < s+1$, let $s(\tau) = s+1$ (here $s_\tau < s+1$ is the last stage at which τ was initialised).

τ2. If $s+1$ is a τ-expansionary stage, let $\tau\hat{}\langle i\rangle = \xi' \preceq TP_{s+1}$. Otherwise, let $\tau\hat{}\langle f\rangle = \xi' \preceq TP_{s+1}$.

Case 3. ξ is an $\mathcal{L}_e$-strategy σ. Let $\sigma\hat{}\langle 1\rangle = \xi' \preceq TP_{s+1}$. If $\Phi_e^L(e)[s+1]\downarrow$ and $\mathcal{L}_e$ has not been satisfied yet, initialise all lower priority strategies and declare that $\mathcal{L}_e$ is satisfied via σ.

Case 4. ξ is an $\mathcal{N}_e$-strategy β.

β1. If $n(\beta)\uparrow$, set $n(\beta) := 0$ and go to β2.

β2. If $l(\beta, s+1) > 2n(\beta)+1$, let $\beta\hat{}\langle i\rangle = \xi_{i+1} \preceq TP_{s+1}$ and go to β3. Otherwise, let $\beta\hat{}\langle f\rangle = \xi_{i+1} \preceq TP_{s+1}$.

β3. If there is a link $(\beta - \tau)$, where τ is an $\mathcal{H}_j$-strategy with a number, say $x = \langle e', k(\tau), n\rangle$, waiting to be enumerated into H, do the followings.
- $*$ Go to τ, cancel the link $(\beta - \tau)$.

- Without loss of generality, suppose that

$$\beta_1 \hat{}\ \langle i\rangle \prec \cdots \prec \beta_{j_0} \hat{}\ \langle i\rangle \prec \cdots \prec \beta_l \hat{}\ \langle i\rangle \prec \tau,$$

where $\beta = \beta_{j_0}$, $1 \leq j_0 \leq l$, and $x = \langle e', k(\tau), n\rangle$ has the attached sequence $z(x) = (z_1, \cdots, z_{j_0}, \cdots, z_l)$ corresponding to the $\mathcal{N}$-strategies β_j's (when x is chosen by τ, its attached sequence $z(x)$ is defined immediately, as in **Step 4** below). If $j_0 = 1$, i.e. all $z_1, \cdots, z_l$ are all in F, put x into H; otherwise, put the next attached number z_{j_0-1} into F, create a new link $(\beta_{j_0-1} - \tau)$.
- Initialise all strategies to the right of τ and go to stage $s + 2$.

Otherwise, i.e. if no lower $\mathcal{H}$-strategy τ links to β, do the following, for which we distinguish two subcases.
(Subcase 1) If there is $p \leq n(\beta)$ with $\Delta_\beta^{L\oplus W_e}(p)[t] \downarrow\neq K_{s+1}(p)$ (where $t \leq s$ is the last β-stage that we have defined $\Delta_\beta^{L\oplus W_e}(p)[t]$), do the following.

(a) Choose the least such p.
(b) If $W_e \upharpoonright \delta(p)[t]$ has changed, undefine $\Delta_\beta^{L\oplus W_e}(q)$, $\forall q \geq p$.

(Subcase 2) Otherwise, for the least $p \leq n(\beta)$ such that $\Delta_\beta^{L\oplus W_e}(p)$ has not been defined yet, set $\Delta_\beta^{L\oplus W_e}(p)[s+1] = K_{s+1}(p)$ with the use $\delta(p)[s+1]$ as fresh.

Case 5. ξ is an $\mathcal{R}_e$-strategy η. Let $\eta \hat{}\ \langle 1\rangle = \xi' \preceq TP_{s+1}$. If $s+1$ is an η-expansionary stage, initialize all lower priority strategies.

Step 4.

(i) For each $\mathcal{H}_e$-strategy τ which was accessible before or at stage $s+1$, and has not been initialised yet, pick $\langle e, k(\tau), s+1\rangle$. Without loss of generality, suppose that $\beta_1, \cdots, \beta_n$ are $\mathcal{N}$-strategies above τ with outcome i, i.e. $\beta_1 \hat{}\ \langle i\rangle \prec \cdots \prec \beta_n \hat{}\ \langle i\rangle \preceq \tau$. Then for each $1 \leq j \leq n$, take a fresh numbers z_j as an attached number of $\langle e, k(\tau), s+1\rangle$ corresponding to β_j such that $z_1 < z_2 < \cdots < z_n$. Define the attached sequence of $\langle e, k(\tau), s+1\rangle$ corresponding to the $\mathcal{N}$-strategies $\beta_1, \cdots, \beta_n$ to be $z(\langle e, k(\tau), s+1\rangle) = (z_1, z_2, \cdots, z_n)$.

Update for each $\mathcal{N}_e$-strategy β

$$n(\beta) := \max\{\lceil \frac{m(\beta, s+1)}{2} \rceil, z+1 : z \text{ is a number corresponding to } \beta \text{ taken by some } \tau \succeq \beta \hat{}\ \langle i\rangle \text{ during stage } s+1 \text{ (if any)}\}.$$

(ii) Along the current true path TP_{s+1}, choose the smallest number $x = \langle e, k(\tau), m\rangle \notin H$ (where τ is an $\mathcal{H}_e$-strategy such that $\tau\hat{}\langle i\rangle \preceq TP_{s+1}$). Without loss of generality, suppose that $\beta_1\hat{}\langle i\rangle \prec \cdots \prec \beta_n\hat{}\langle i\rangle \preceq \tau$ are all $\mathcal{N}$-strategies above τ with outcome i along TP_{s+1} and $z(x) = z(\langle e, k(\tau), m\rangle) = (z_1, \cdots, z_n)$ is the attached sequence of x corresponding to $\beta_1, \cdots, \beta_n$. Put z_n into F and create the link $(\beta_n - \tau)$.

Go to stage $s+2$.

2.3 Verification

Recall that the *true path* $TP = \liminf_s TP_s$.

Lemma 2.1. *Every node ξ on the true path TP satisfies the following properties.*

(i) There is a stage s_0 such that ξ is not initialised after s_0 and for every stage $t \geq s_0$, either $\xi \prec TP_t$ or $\xi < TP_t$. Clearly, no node to the left of ξ is accessible after stage s_0.

(ii) The node ξ is accessible at infinitely many stages, i.e. $\exists^\infty s : \xi \preceq TP_s$.

(iii) There is a unique $o \in \Lambda$ (the true outcome of ξ) such that $\xi\hat{}\langle o\rangle \prec TP$.

Proof. We prove the lemma by induction on the length of $\xi \prec TP$.

For $\xi = \lambda \prec TP$, clearly λ has the highest priority on the tree and $\lambda \preceq TP_s$ for all $s \geq 0$, so λ is never initialised after stage 0 and (ii) is satisfied. We have that λ is the $\mathcal{P}_0$-strategy, then only the strategy $\mathcal{S}$ can injure λ by putting $\gamma(n) \leq \phi_e^B(x)$ into B when a small number n enters K after $\Phi_e^B(x)\downarrow = 0$. However, such an injury happens at most $k(\lambda)+1$ times. Hence, either λ waits for $\phi_e^B(x)\downarrow = 0$ forever for some x and has true outcome w (so, $\lambda\hat{}\langle w\rangle \prec TP$) or it eventually sees $\Phi_e^B(x)\downarrow = 0$, puts $\gamma(k(\lambda))$ into L (in this case it successfully lifts the use $\gamma(n) > \phi_e^B(x)$ for any $n \geq k$ and no number $n \leq k$ enters K later, making this computation $\Phi_e^B(x)\downarrow = 0$ correct forever), λ puts x into D and stops with the true outcome s (so, $\lambda\hat{}\langle s\rangle \prec TP$). Once λ reaches it true outcome, it never puts numbers into D or L.

Suppose inductively that the nodes $\lambda = \xi_0 \prec \xi_1 \prec \cdots \prec \xi_{n-1} \prec TP$ satisfy $(i-iii)$, where $\xi_{i+1} = \xi_i\hat{}\langle o_i\rangle$ with $o_i \in \Lambda$ is the true outcome of ξ_i for any $0 \leq i \leq n-2$. Let o_{n-1} be the true outcome of ξ_{n-1} and $\xi_n = \xi_{n-1}\hat{}\langle o_{n-1}\rangle$. We will prove that ξ_n also satisfies $(i-iii)$.

Let s be the least stage such that $\xi_n \prec TP_s$ and from stage s onwards, (i) satisfied for every $\xi_i, 0 \leq i \leq n-1$. Since no strategy $\xi_i (0 \leq i \leq n-1)$ is initialised from stage s onwards, we have the following.

- No $\mathcal{P}, \mathcal{L}, \mathcal{R}$-strategy $\xi < \xi_{n-1}$ acts from stage s onwards;
- No strategy to the left of ξ_{n-1} is accessible from stage s onwards.

In the following, we prove that there is a stage $t > s$ after which ξ_n is never initialized. The other parts of the properties (i-ii) clearly hold for ξ_n.We consider four cases as follows.

Case 1. ξ_{n-1} is a $\mathcal{P}_e$-strategy α. Since α is not initialised after stage s, from the construction we have that $k(\alpha)\downarrow= k(\alpha)[s^+]$, where s^+ is the next α-stage after s. In addition, after stage s, α is only injured by $\mathcal{S}$ at most $k(\alpha)[s^+]+1$ times. Therefore, there is a stage s' such that

- either α waits for $\Phi_e^B(x)\downarrow= 0$ forever from a stage $s' > s$ onwards and has the true outcome $o_{n-1} = w$;
- or eventually $\Phi_e^B(x)[s']\downarrow= 0$, α puts $\gamma(k(\alpha))$ into L (in this case it successfully lifts all uses $\gamma(n) > \phi_e^B(x)\forall n \geq k(\alpha)$ and no number $n \leq k(\alpha)$ enters K, making $\Phi_e^B(x)\downarrow= \Phi_e^B(x)[s']\downarrow= 0$), puts x into D and has the true outcome $o_{n-1} = s$.

In both cases, α never acts after stage s' and hence, $\xi_n = \xi_{n-1}\hat{}\langle o_{n-1}\rangle$ is never initialised after stage s'.

Case 2. ξ_{n-1} is an $\mathcal{H}_e$-strategy τ or an $\mathcal{N}_e$-strategy β. From the construction, after taking any action related to τ (or β, respectively), we only initialise the nodes to the right of τ (or β, respectively). Therefore, $\xi_n = \xi_{n-1}\hat{}\langle o_{n-1}\rangle \prec TP$ is never initialised after stage s.

Case 3. ξ_{n-1} is an $\mathcal{L}_e$-strategy σ. If there is the least stage $s' \geq s$ such that $\Phi_e^L(e)[s']\downarrow$ and $\mathcal{L}_e$ has not been satisfied yet, then $\sigma \preceq TP_{s'}$ (because no ξ_i $(0 \leq i \leq i-1)$ is initialised after stage $s \leq s'$). At the beginning of stage s', we initialise all strategies $\xi > \xi_{n-1}$ and declare that $\mathcal{L}_e$ is satisfied via ξ_{n-1} at stage s'. From the next ξ_{n-1}-stage $t > s'$ onwards, $\xi_n = \xi_{n-1}\hat{}\langle 1\rangle$ is never initialised. If $\Phi_e^L(e)[s']\downarrow$ and $\mathcal{L}_e$ has been already satisfied before stage s or at any stage $\Phi_e^L(e)[t]\uparrow \forall t \geq s$, then the strategy ξ_{n-1} never acts, and hence, we have that, from stage s onwards, ξ_n is never initialised.

Case 4. ξ_{n-1} is an $\mathcal{R}_e$-strategy η. Note that there are only finitely many η-expansionary stages, as otherwise, A would be computable as follows. To compute $A(n)$, we wait for an η-stage $s_0 > s$ such that $l(\eta, s_0) > n$; since the computation $\Phi_e^{H\oplus L}(n)[s_0]$ is η-believable and η is not initialised after stage s, then no number less than $\phi_e^{H\oplus L}(n)[s_0]$ is enumerated into H or L, and therefore we have $A_s(n) = \Phi_e^{H\oplus L}(n)[s_0] = \Phi_e^{H\oplus L}(n) = A(n)$. Thus, $A = \Phi_e^{H\oplus L}$ is computable, a contradiction.

Let $s' \geq s$ be the last η-expansionary stage, then after stage s', η never acts. So, ξ_n is never initialised after stage s'.

□

Lemma 2.2. *For each $e \in \omega$, the requirement $\mathcal{L}_e$ is satisfied.*

Proof. Fix $e \in \omega$. Suppose that $\Phi_e^L(e)$ converges infinitely many often, i.e. there are infinitely many stages s such that $\Phi_e^L(e)[s] \downarrow$. Consider the $\mathcal{L}_e$-strategy $\sigma \prec TP$. By lemma 2.1 above, let s be the least stage after which $\sigma\hat{\ }\langle 1\rangle \prec TP$ is never initialised. Consider the least stage $s' > s$ such that $\Phi_e^L(e)[s'] \downarrow$.

If $\mathcal{L}_e$ is not satisfied at s', then by the choice of stage s, we have that $\sigma \prec TP_{s'}$ and at the beginning of stage s', we declare $\mathcal{L}_e$ is satisfied via σ at stage s', and since $\sigma\hat{\ }\langle 1\rangle$ is never initialised after stage s, $\mathcal{L}_e$ is satisfied via σ at stage s' will never change.

Otherwise, if $\mathcal{L}_e$ has been satisfied via an $\mathcal{L}_e$-strategy $\sigma^* \leq \sigma$, since $\sigma\hat{\ }\langle 1\rangle$ is never initialised after stage s, $\mathcal{L}_e$ is satisfied via σ^* forever. □

Lemma 2.3. *For every e, requirement $\mathcal{P}_e$ is satisfied.*

Proof. Fix $e \in \omega$. Consider the $\mathcal{P}_e$-strategy $\alpha \prec TP$. Let s be the least stage at which $\alpha\hat{\ }\langle o\rangle \prec TP$ is accessible and α is never initialized after stage s. From the construction, $k(\alpha) \downarrow= k(\alpha)[s], x(\alpha) \downarrow= x(\alpha)[s]$ and no number $x \leq k$ enters K after stage s. There are two cases:

- the true outcome $o = w$, α waits for $\Phi_e^B(x(\alpha)) \downarrow= 0$ forever from stage s onwards and $D(x(\alpha)) = 0 \neq \Phi_e^B(x(\alpha))$;
- the true outcome $o = s$, α puts $\gamma(k(\alpha))$ into L (in this case it successfully lifts all uses $\gamma(n) > {\phi_e}^B(x(\alpha))\forall n \geq k)$, puts x into D and $\Phi_e^B(x(\alpha)) \downarrow= \Phi_e^B(x(\alpha))[s] \downarrow= 0 \neq 1 = D(x(\alpha))$.

Hence, $\mathcal{P}_e$ is satisfied. □

Lemma 2.4. *For every $e \in \omega$, $Tot(e) = \lim_{n\to\infty} \Lambda^H(e, n)$. Hence, H is high.*

Proof. Fix $e \in \omega$. Consider the $\mathcal{H}_e$-strategy $\tau \prec TP$. Let s be the least stage after which τ is never initialised, and let $s_0 > s$ be the next τ-stage. We have, $s_\tau = s$ and $s(\tau) \downarrow= s_0$.

If there are finitely many τ-expansionary stages, then there is the least number x such that $\varphi_e(x) \uparrow$. So, $Tot(e) = 0$. On the other hand, let $s_1 \geq s_0$ be the least τ-stage at which $l(\tau, s_0) = x$. Clearly, for any $s > s_1$, and for any $\mathcal{H}_e$-strategy τ', $l(\tau', s) = x$. Hence, for any $n \geq x$, $\Lambda^H(e, n)$ will not be redefined by any $\mathcal{H}_e$-strategy after stage s_0, and so $\Lambda^H(e, n) \downarrow= 0 \forall n \geq x$, implying $\lim_{n\to\infty} \Lambda^H(e, n) = 0 = Tot(e)$.

If there are infinitely many τ-expansionary stages, then φ_e is total, so $Tot(e) = 1$. For any $n \geq s_0$, since τ is not initialised after stage s, when τ picks $x = \langle e, k(\tau), n\rangle$ at stage $t > s$ and chooses the attached sequence $(z_1, \cdots, z_n)$ for x, this sequence will never be reset after that. From the construction, there is a τ-expansionary stage $s_1 > t$ big enough such that at stage s_1, x is the smallest number which is not in H and is having the attached sequence. Then at stage s_1, τ starts putting the attached numbers one by one into F, and at stage $s_2 > s_1$, when all attached numbers have been enumerated into F, x will be put into H, and τ then redefines $\Lambda^H(e,n) = 1$. Thus, eventually, for any $n \geq s_0$, τ redefines $\Lambda^H(e,n) = 1$, making $\lim_{n\to\infty} \Lambda^H(e,n) = 1 = Tot(e)$.

□

Lemma 2.5. *For every $e \in \omega$, the requirement $\mathcal{N}_e$ is satisfied.*

Proof. Fix $e \in \omega$. Suppose that $\Phi^{H\oplus L\oplus W_e} = K \oplus F$. Consider the $\mathcal{N}_e$-strategy $\beta \prec TP$. Then there are infinitely many β-expansionary stages and $\beta\hat{}\langle i\rangle \prec TP$. We will prove by induction on p that for any $p \in \omega$, $\Delta_\beta^{L\oplus W_e}(p)\downarrow = K(p)$.

Let s be the least stage at which $\beta\hat{}\langle i\rangle$ is accessible and β is never initialised after stage s.

For $p = 0$, if $\Delta_\beta^{L\oplus W_e}(0)\uparrow$ at stage s, then from the construction, $\Delta_\beta^{L\oplus W_e}(0)[s] = K_s(0)$ with the use $\delta(0)[s]$ as a fresh number. If 0 never enters to K, this computation remains correct, i.e. $\Delta_\beta^{L\oplus W_e}(0)\downarrow = \Delta_\beta^{L\oplus W_e}(0)[s] = K_s(0) = K(0)$. Otherwise, if 0 enters K at stage $t > s$, let $s_1 \geq t$ be the next β-expansionary stage. We have

$$\Phi^{H\oplus L\oplus W_e}(0)[s] = K_s \oplus F_s(0) = K_s(0) = 0, \tag{2.1}$$

and

$$\Phi^{H\oplus L\oplus W_e}(0)[s_1] = K_{s_1} \oplus F_{s_1}(0) = K_{s_1}(0) = 1. \tag{2.2}$$

By the choice of s and from the construction, computation $\Phi^{H\oplus L\oplus W_e}(0)[s]$ is β-believable, there is no small number enumerated into H by $\mathcal{H}$-strategies above β. In addition, any $\mathcal{H}_j$-strategy τ which is of lower priority than β or to the right of β , is only concerned with numbers $\langle j, k(\tau), n\rangle$ for $n \geq s$ after stage s. Hence, from stage s to stage s_1, no number smaller or equal to $\phi(0)[s]$, is enumerated into H. Therefore, at stage s_1, equation 2.2 is achieved because $L \oplus W_e$ has changed below $\phi(0)[s]$, allowing $\Delta_\beta^{L\oplus W_e}(0)[s]$ to be undefined. At the next β-expansionary stage $s_2 > s_1$, we can rectify $\Delta_\beta^{L\oplus W_e}(0)[s_2] = 1 = \Delta_\beta^{L\oplus W_e}(0)\downarrow = K(0)$ with the use $\delta(0)[s_2]$ as fresh.

Suppose that $\Delta_\beta^{L\oplus W_e}$ correctly computes K up to $p - 1 \geq 0$, i.e. there is the least stage $s_0 \geq s$ such that

$$\Delta_\beta^{L\oplus W_e}(q)\downarrow = \Delta_\beta^{L\oplus W_e}(q)[s_0]\downarrow = K_{s_0}(q) = K(q) \forall q \leq p - 1.$$

From the construction, at the next β-expansionary stage $s_1 > s_0$, $\Delta_\beta^{L\oplus W_e}(p)\uparrow$ and we define $\Delta_\beta^{L\oplus W_e}(p)[s_1] = K_{s_1}(p)$ with the use $\delta(p)[s_1]$ as fresh. If p

never enters K after stage s_1, this computation is correct, i.e., $\Delta_\beta^{L\oplus W_e}(p)\downarrow = \Delta_\beta^{L\oplus W_e}(p)[s_1] = K_{s_2}(p) = K(p)$. Otherwise, suppose that p enters K at stage $s' > s_1$ and $s_2 \geq s'$ is the next β-expansionary stage. We have

$$\Phi^{H\oplus L\oplus W_e}(2p)[s_1] = K_{s_1} \oplus F_{s_1}(2p) = K_{s_1}(p) = 0, \tag{2.3}$$

and

$$\Phi^{H\oplus L\oplus W_e}(2p)[s_2] = K_{s_2} \oplus F_{s_2}(2p) = K_{s_2}(p) = 1. \tag{2.4}$$

If no number below $\phi(l(\beta, s_1))$ enters H at stages between s_1 and s_2, then clearly, equation (2.4) provides that at stage s_2, $L \oplus W_e$ has changed below $\phi(2p)[s_1] < \delta(p)[s_1]$. Otherwise, let t ($s_1 < t < s_2$) be the first stage that a small number $y = \langle j, k(\tau), n\rangle$ is enumerated into H by an $\mathcal{H}_j$-strategy $\tau \succeq \beta\hat{}\langle i\rangle$, where $y = \langle j, k(\tau), n\rangle \leq \phi(l(\beta, s_1))$. Suppose that

$$\beta_1\hat{}\langle i\rangle \prec \cdots \prec \beta_{j_0}\hat{}\langle i\rangle \preceq \cdots \preceq \beta_n\hat{}\langle i\rangle \preceq \tau,$$

where $\beta = \beta_{j_0}$. From the construction, τ puts y into H at stage t only when t is a β_1-expansionary stage and τ has just cancelled the link $(\beta_1 - \tau)$ before stage t. So, before t, there are β-expansionary stages $t_1 < t_2$ such that

- τ puts the attached number z_{j_0} (corresponding to $\beta = \beta_{j_0}$) of y into F at stage t_1;
- and at stage t_2, we go from β to τ via the link $(\beta - \tau)$, we cancel this link, and then we continue to put the next attached number into F.

Since $y < \phi(l(\beta, s_1))$, the attached sequence of y has been chosen before stage s_1. From the construction, after τ puts z_{j_0} into F till y enumerated into H at stage t, $\beta_{j_0} = \beta_j$ will not define its functional Δ_β, so $s_1 < t_1$. We have

$$\Phi^{H\oplus L\oplus W_e}(2z_{j_0}+1)[s_1] = K_{s_1} \oplus F_{s_1}(2z_{j_0}+1) = F_{s_1}(z_{j_0}) = 0, \tag{2.5}$$

$$\Phi^{H\oplus L\oplus W_e}(2z_{j_0}+1)[t_2] = K_{t_2} \oplus F_{t_2}(2z_{j_0}+1) = F_{t_2}(z_{j_0}) = 1, \tag{2.6}$$

Since β is not initialised after stage s, and all computations $\Phi^{H\oplus L\oplus W_e} \upharpoonright l(\beta, s_1)$ at stage s_1 are β-believable, no number smaller than or equal to $\phi(l(\beta, s_1))$ is enumerated into H by $\mathcal{H}$-strategies of higher priority than β. Hence, from stage s_1 to stage t_2, no $x \leq \phi(l(\beta, s_1))$ enters H, and 2.6 is obtained just because at stage $t_2 < s_2$, $L \oplus W_e$ has changed below $\phi(2z_{j_0}+1)[s_1] < \delta(p)[s_1]$. So, $\Delta_\beta^{L\oplus W_e}(p)$ can be redefined correctly at stage s_2.

Thus, requirement $\mathcal{N}_e$ is satisfied. □

Lemma 2.6. *For each $e \in \omega$, requirement $\mathcal{R}_e$ is satisfied.*

Proof. Fix $e \in \omega$. Consider the $\mathcal{R}_e$-strategy $\eta \prec TP$. Suppose that $A = \Phi_e^{H \oplus L}$. Let s be the least stage after which η is never initialised. We have:

(i) no strategy to the left of η is accessible at any stage $t \geq s$,

(ii) no $\mathcal{P}_j$-strategy $\alpha \prec \eta$ puts number into L after stage s.

There are infinitely many η-expansionary stages. To compute $A(n)$, we wait for an η-stage $s' > s$ such that $l(\eta, s') > n$; since the computation $\Phi_e^{H \oplus L}(n)[s']$ is η-believable and η is not initialised after stage s, then no number less than $\phi_e^{H \oplus L}(n)[s']$ is enumerated into H or L, and therefore we have $A_{s'}(n) = \Phi_e^{H \oplus L}(n)[s'] = \Phi_e^{H \oplus L}(n) = A(n)$. Thus, $A = \Phi_e^{H \oplus L}$ is computable, a contradiction. □

Lemma 2.7. *The $\mathcal{S}$-requirement is satisfied.*

Proof. We prove the lemma by induction on n.

For $n = 0$, at the beginning of stage 1, we define $\Gamma^{B \oplus L}(0)$ with fresh use $\gamma(0)[1]$. After that, since $\mathcal{S}$ has the highest priority, all killing points chosen by the $\mathcal{P}$-strategies are large numbers and will not affect $\Gamma^{B \oplus L}(0)$. Therefore, the computation $\Gamma^{B \oplus L}(0)$ remains unchanged and computes $K(0)$ correctly unless 0 enters K at a stage $s > 1$. If the latter happens, $\gamma(0)[1]$ is enumerated to B and $\Gamma^{B \oplus L}(0)$ is undefined at the beginning of stage s. At stage $s+1$, $\Gamma^{B \oplus L}(0)$ will be redefined correctly forever, i.e. $\Gamma^{B \oplus L}(0) \downarrow = \Gamma^{B \oplus L}(0)[s+1] = K(0)$.

Suppose that $\Gamma^{B \oplus L}(m) \downarrow = K(m) \forall 0 \leq m < n$. Let s be the first stage at which $\Gamma^{B \oplus L}(m) \downarrow = \Gamma^{B \oplus L}(m)[s] \downarrow = K_s(m) = K(m)$, $\forall m < n$. From the construction, at the end of stage s, we have $\Gamma^{B \oplus L}(n) \uparrow$. At the beginning of stage $s+1$, via step S2, we define $\Gamma^{B \oplus L}(n)[s+1] = K_{s+1}(n)$ with $\gamma(n)[s+1]$ as fresh. By the choice of stage s, no $\gamma(m)[s]$ (for any $m < n$) will be put into B or L after stage s. From the construction, $\gamma(n)[s+1]$ is enumerated to B at a stage $t > s+1$ only if $n \in K_t \setminus K_{t-1}$. In addition, only the $\mathcal{P}$-strategy which takes its killing point $k = n$, may put $\gamma(n)$ into L, and this action occurs at most two times. Hence, there is a stage $s' \geq s+1$ such that $\gamma(n)$ is not enumerated into L after s'. If at stage $s'+1$, $\Gamma^{B \oplus L}(n) \uparrow$, then we define $\Gamma^{B \oplus L}(n)[s'+1] = K_{s'+1}(n)$. So, without loss of generality, we can suppose that $\Gamma^{B \oplus L}(n)[s'+1] \downarrow = K_{s'+1}(n)$ and there is no $\mathcal{P}$-strategy that puts $\gamma(n)[s'+1]$ into L after stage $s'+1$. Then the computation $\Gamma^{B \oplus L}(n)[s'+1]$ remains unchanged unless n enters K at a stage $s" > s'+1$. If the latter happens, $\Gamma^{B \oplus L}(n)$ will be rectified at stage $s"+1$ and will be correct forever. Thus, $\Gamma^{B \oplus L}(n) \downarrow = K(n)$. □

Chapter 3

n-cuppable degrees

Given $1 \leq n \leq m$, recall that m different incomplete c.e. degrees $\mathbf{a}_1, \cdots, \mathbf{a}_m$ are *n-cuppable* if for any n different degrees $\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_n}$ among them, there is an incomplete c.e. degree $\mathbf{e}$ such that

$$\forall 1 \leq k \leq n, \mathbf{e} \cup \mathbf{a}_{i_k} = \mathbf{0}'.$$

Li, Wu, and Yang in [37] and Bie and Wu in [4] constructed two cuppable c.e. degrees $\mathbf{a}, \mathbf{b}$ such that there is no incomplete c.e. degree cupping both $\mathbf{a}$ and $\mathbf{b}$ to $\mathbf{0}'$. These degrees $\mathbf{a}$, $\mathbf{b}$ are 1-cuppable degrees that are not 2-cuppable. The two papers used a method of constructing noncuppable degrees but in different manners to ensure the nonexistent of an incomplete c.e. degree cupping both $\mathbf{a}$ and $\mathbf{b}$ to $\mathbf{0}'$. In this chapter, we will prove Theorem B by applying the technique employed in [4]. The theorem gives an instance of degrees that are 2-cuppable but not 3-cuppable. In particular, we will construct three cuppable c.e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ with the following properties.

(i) There are incomplete c.e. degrees $\mathbf{e}, \mathbf{g}, \mathbf{h}$ such that

$$(\mathbf{a} \cup \mathbf{e} = \mathbf{b} \cup \mathbf{e} = \mathbf{0}') \wedge (\mathbf{b} \cup \mathbf{g} = \mathbf{c} \cup \mathbf{g} = \mathbf{0}') \wedge (\mathbf{c} \cup \mathbf{h} = \mathbf{a} \cup \mathbf{h} = \mathbf{0}').$$

(ii) There is no incomplete c.e. degree cupping all $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to $\mathbf{0}'$, i.e.

$$\forall \mathbf{w} \in \mathbf{R}(\mathbf{a} \cup \mathbf{w} = \mathbf{b} \cup \mathbf{w} = \mathbf{c} \cup \mathbf{w} = \mathbf{0}' \rightarrow \mathbf{w} = \mathbf{0}').$$

The property (ii) is obtained by applying directly the technique in [4]. However, unlike the case of constructing two c.e. degrees that are 1-cuppable but not 2-cuppable, the desired functionals to satisfy (i) are not independent and will be discussed thoroughly in the next section. For any $n \geq 3$, the construction of n-cuppable degrees which are not $(n+1)$-cuppable then follows analogously the case of 2-cuppable degrees which are not 3-cuppable.

3.1 Requirements and strategies

We will construct c.e. sets A, B, C satisfying, for all $e, i = \langle i_1, i_2, i_3, e\rangle \in \omega$, the following requirements:

$$S : (\Gamma_1^{A\oplus E} = \Gamma_2^{B\oplus E} = K) \wedge (\Omega_1^{B\oplus G} = \Omega_2^{C\oplus G} = K) \wedge (\Delta_1^{A\oplus H} = \Delta_2^{C\oplus H} = K),$$

$$P_e : \Phi_e^E \neq D, \quad Q_e : \Phi_e^H \neq D, \quad J_e : \Phi_e^G \neq D,$$

$$N_i : \Psi^{A\oplus W_e} = \Psi^{B\oplus W_e} = \Psi^{C\oplus W_e} = K \oplus F \Rightarrow K = \Theta_i^{W_e}.$$

Here, K is a fixed complete set. We fix an effective enumeration $\{K_s\}_{s\in\omega}$ of K such that there is at most one number entering K at each stage s. All partial computable functionals Φ's and Ψ's are effectively listed. Other partial computable functionals (Γ_1, Γ_2, Δ_1, Δ_2, Ω_1, Ω_2, Θ) and c.e. sets E, H, G, D, F are built in the construction.

Each requirement N_i actually is considered with three functionals Ψ_{i_1}, Ψ_{i_2}, Ψ_{i_3}, and the functional Θ_i constructed during the construction. Here, we will sometimes omit the indices in order to simplify notations.

Requirements P_e, Q_e and J_e (for all $e \in \omega$) imply that the c.e. sets E, H, G are incomplete. Hence, requirement S gives that c.e. sets A, B, C are cuppable. The corresponding c.e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of c.e. sets A, B, C then fulfill two conditions (i-ii) of Theorem B and so, the degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are 2-cuppable but not 3-cuppable. In particular, the S-requirement and the P, Q, J-requirements guarantee (i), giving that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are 2-cuppable. Meanwhile, all N-requirements ensure (ii), implying that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are not 3-cuppable.

In the following we discuss about strategies for the requirements. All P, Q, J, N-strategies will be processed on a priority tree. The S-strategy has the highest priority and will not to put on the tree.

3.1.1 The S-strategy

We will construct functionals Γ_1, Γ_2, Δ_1, Δ_2, Ω_1, Ω_2 so that they are eventually total and compute K correctly. Ideally, the functional Γ_1 (Γ_2, respectively) is defined as follows.

(i) If $\Gamma_1^{A\oplus E}(n) \downarrow\neq K(n)$ ($\Gamma_2^{B\oplus E}(n) \downarrow\neq K(n)$, respectively), put $\gamma_1(n)$ ($\gamma_2(n)$, respectively) into E to undefine $\Gamma_1^{A\oplus E}(m)$ ($\Gamma_2^{B\oplus E}(m)$, respectively) for any $m \geq n$.

(ii) Otherwise, for the least number n such that $\Gamma_1^{A\oplus E}(n)\uparrow$ ($\Gamma_2^{B\oplus E}(n)\uparrow$, respectively), define $\Gamma_1^{A\oplus E}(n)=K(n)$ ($\Gamma_2^{B\oplus E}(n)=K(n)$, respectively) with the use $\gamma_1(n)$ ($\gamma_2(n)$, respectively) as a fresh large number.

The functionals $\Delta_i,\Omega_i (i=0,1)$ are constructed analogously as above, but putting numbers into H to undefine the Δ-functionals and putting numbers into G to undefine the Ω-functionals in item (i).

However, the constructions of Γ_1 and Γ_2 may disrupt each other. For example, a computation $\Gamma_1(n)$ can be undefined by infinitely many γ_2-uses enumerated into E. Such a problem can also happen to other pairs of functionals Δ_1 and Δ_2, Ω_1 and Ω_2. To avoid these scenarios, we will define the functionals together, i.e. we will define $\Gamma_1^{A\oplus E}(n)$ (respectively, $\Gamma_2^{B\oplus E}(n)$, $\Delta_1^{A\oplus H}(n)$, $\Delta_2^{C\oplus H}(n)$, $\Omega_1^{B\oplus G}(n)$ or $\Omega_2^{C\oplus G}(n)$) only if all computations $\Gamma_1^{A\oplus E}(n-1)$, $\Gamma_2^{B\oplus E}(n-1)$, $\Delta_1^{A\oplus H}(n-1)$, $\Delta_2^{C\oplus H}(n-1)$, $\Omega_1^{B\oplus G}(n-1)$ and $\Omega_2^{C\oplus G}(n-1)$ have been already defined. In particular, if $\gamma_1(n)[s]\downarrow$ and $\gamma_2(m)[s]\leq\gamma_1(n)[s]$ is enumerated into E for some $m>n$ at stage s, then $\gamma_1(n)$ will be defined before $\gamma_2(m)$. By this setting, such an injury coming from γ_2-uses (as has occurred at stage s) can only happen at most finitely many times.

So, the functionals $\Gamma_1,\Gamma_2,\Delta_1,\Delta_2,\Omega_1,\Omega_2$ are defined at the beginning of each stage as follows.

i. If n enters K then

(a) if $\Gamma_1^{A\oplus E}(n)\downarrow\neq K(n)=1$, then put $\gamma_1(n)$ to E and undefine $\Gamma_1^{A\oplus E}(q)$ for all $q\geq n$;

(b) if $\Gamma_2^{B\oplus E}(n)\downarrow\neq K(n)=1$, then put $\gamma_2(n)$ to E and undefine $\Gamma_2^{B\oplus E}(q)$ for all $q\geq n$;

(c) if $\Delta_1^{A\oplus H}(n)\downarrow\neq K(n)=1$, then put $\delta_1(n)$ to H and undefine $\Delta_1^{A\oplus H}(q)$ for all $q\geq n$;

(d) if $\Delta_2^{C\oplus H}(n)\downarrow\neq K(n)=1$, then put $\delta_2(n)$ to H and undefine $\Delta_2^{C\oplus H}(q)$ for all $q\geq n$;

(e) if $\Omega_1^{B\oplus G}(n)\downarrow\neq K(n)=1$, then put $\omega_1(n)$ to G and undefine $\Omega_1^{B\oplus G}(q)$ for all $q\geq n$;

(f) if $\Omega_2^{C\oplus G}(n)\downarrow\neq K(n)=1$, then put $\omega_2(n)$ to G and undefine $\Omega_2^{C\oplus G}(q)$ for all $q\geq n$.

ii. If $p=0$ or $p>0$ is the largest number such that for any $m<p$,

$$\Gamma_1^{A\oplus E}(m)\downarrow=\Gamma_2^{B\oplus E}(m)\downarrow=\Delta_1^{A\oplus H}(m)\downarrow=\Delta_2^{C\oplus H}(m)\downarrow \\ =\Omega_1^{BG}(m)\downarrow=\Omega_2^{C\oplus G}(m)\downarrow=K(m),$$

then

(a) if $\Gamma_1^{A\oplus E}(p)\uparrow$, then define $\Gamma_1^{A\oplus E}(p) = K(p)$ with the use $\gamma_1(p)$ as a fresh number;

(b) if $\Gamma_2^{B\oplus E}(p)\uparrow$, then define $\Gamma_2^{B\oplus E}(p) = K(p)$ with the use $\gamma_2(p)$ as a fresh number;

(c) if $\Delta_1^{A\oplus H}(p)\uparrow$, then define $\Delta_1^{A\oplus H}(p) = K(p)$ with the use $\delta_1(p)$ as a fresh number;

(d) if $\Delta_2^{C\oplus H}(p)\uparrow$, then define $\Delta_2^{C\oplus H}(p) = K(p)$ with the use $\delta_2(p)$ as a fresh number;

(e) if $\Omega_1^{B\oplus G}(p)\uparrow$, then define $\Omega_1^{B\oplus G}(p) = K(p)$ with the use $\omega_1(p)$ as a fresh number;

(f) if $\Omega_2^{C\oplus G}(p)\uparrow$, then define $\Omega_2^{C\oplus G}(p) = K(p)$ with the use $\omega_2(p)$ as a fresh number.

Here, the S-strategy only puts numbers into E, H and G, and this is not sufficient to satisfy the requirement S. Otherwise, K will be coded into E, H, G, and then these sets E, G, H are complete. As usual, we ensure certain basic properties for the use function γ_1 of Γ_1 during the construction. Below are the properties of γ_1, which are analogous to those properties of the other use functions $\gamma_2, \delta_1, \delta_2, \omega_1, \omega_2$ of the functionals $\Gamma_2, \Delta_1, \Delta_2, \Omega_1, \Omega_2$, respectively.

(1) Whenever $\gamma_1(n)$ is defined, it will be a fresh number larger than all previously used numbers;

(2) For any number n and stage s, if $\Gamma_1^{A\oplus E}(n)[s]\downarrow$ then $\gamma_1(n)[s] \notin A_s \cup E_s$;

(3) For any number n and $m < n$, if $\gamma_1(n)[s]\downarrow$ then $\gamma_1(m)[s]\downarrow$ and $\gamma_1(m)[s] < \gamma_1(n)[s]$;

(4) If $\Gamma_1^{A\oplus E}(n)[s]\downarrow$ and n enters K at stage $s+1$, then $\gamma_1(m)[s]$ is enumerated into E or A at stage $s+1$ for some $m \leq n$;

(5) At stage s, $\Gamma_1^{A\oplus E}(n)$ is undefined iff there is a number $y \leq \gamma_1(n)[s]$ which either enters E or enters A.

Clearly, if properties $(1-5)$ are satisfied and $\Gamma_1^{A\oplus E}$ is total, then $\Gamma_1^{A\oplus E} = K$. A similar conclusion can be drawn for the other functionals $\Gamma_2, \Delta_1, \Delta_2, \Omega_1, \Omega_2$. So, the requirement S is satisfied.

3.1.2 The P, Q, J-strategies

The P, Q, J-strategies are similar and we will describe below a P-strategy, say a P_e-strategy α on the priority tree. Similarly to the $\mathcal{P}$-strategies for cupping described in the previous chapter, α processes as follows.

P1. Pick a killing point (also called a *threshold*) k as fresh.
Let $m := 0$. Whenever $n \leq k$ enters K, set $m := m + 1$ and go to P2.

P2. Pick a fresh witness $x_m > k$.

P3. Wait for $\Phi_e^E(x_m)[s] \downarrow = 0$.

P4. Put $\gamma_1(k)[s]$ into A, $\gamma_2(k)[s]$ into B, x_m into D, and stop.

Here, if $\Phi_e^E(x) \downarrow = 0$ at stage s, we put $\gamma_1(k)[s]$ into A and $\gamma_2(k)[s]$ into B so that the uses $\gamma_1(n), \gamma_2(n), \forall n \geq k$ are lifted above $\varphi^E(x)[s]$ and the computation $\Phi_e^E(x)[s] \downarrow = 0$ will be unchanged unless a small number n (less than or equal to k) enters K and S puts $\gamma_1(n), \gamma_2(n)$ into E. Whenever an $n \leq k$ is enumerated into K, we reset the witness x as fresh. Such a situation can only occur at most $k + 1$ times and eventually there is a stage t after that α has a witness x and there is no $n \leq k$ entering K. Then either $\Phi_e^E(x) \downarrow = 0 \neq D(x) = 1$ remains forever or $\Phi_e^E(x) \uparrow \neq D(x) = 0$. Hence, the requirement P_e is satisfied.

The outcomes of α are finitary outcomes as follows.

w: wait for P3 forever for some $0 \leq m \leq k$, $D(x_m) = 0 \neq \Phi^E(x_m)$,

s: stop at P4 forever for some $0 \leq m \leq k$, say from stage s onwards, there is no number less than or equal to k enters K after stage s and $\Phi_e^E(x_m) \downarrow = 0 \neq D(x_m) = 1$.

3.1.3 The N-strategies

Fix an N_e-strategy β on the priority tree. We will construct a partial computable functional Θ_β (or Θ for short) such that if $\Psi^{A\oplus W_e} = \Psi^{B\oplus W_e} = \Psi^{C\oplus W_e} = K \oplus F$ then $\Theta_\beta^{W_e}$ is a total computable functional which computes K correctly.

Recall that the *length of agreement* at stage s is defined by

$$l(\beta, s) := \max\{x < s : \Psi^{A\oplus W_e}(y)[s] = \Psi^{B\oplus W_e}(y)[s] = \Psi^{C\oplus W_e}(y)[s] = K_s \oplus F_s(y) \forall y \leq x\}.$$

A stage s is called a *β-expansionary* stage if $s = 0$ or $l(\beta, s) > l(\beta, t)$ for any β-stage $t < s$.

Let

$$\psi(x)[s] := \max\{\psi^{A\oplus W_e}(x)[s], \psi^{B\oplus W_e}(x)[s], \psi^{C\oplus W_e}(x)[s]\} \text{ for any } x \leq l(e, s).$$

Ideally, we will define Θ when the length of agreement increases: at a β-expansionary stage s, if p is the least number that $\Theta^{W_e}(p)$ has not been defined yet and $2p+1 < l(\beta, s)$, then we set $\Theta^{W_e}(p)[s] = K_s(p)$ with the use $\theta(p)[s]$ as fresh. After that, this computation is undefined only if $W_e \upharpoonright \psi(2p+1)[s]$ changes. Therefore, if p enters K after stage s, we would like to force W_e to change below $\psi(2p+1)[s]$ so that $\Theta^{W_e}(p)$ can be redefined later. The possible outcomes of β are i (for infinitely many β-expansionary stages) and f (for finitely many β-expansionary stages).

3.1.4 Interactions between N-strategies and P, Q, J-strategies

There may be an obstacle as follows. At a β-expansionary stage s_1 with the length of agreement $l(\beta, s_1) > 2p+1$, we define $\Theta^{W_e}(p)[s_1] = K_{s_1}(p) = 0$ with use $\theta_e(p)[s_1] > \psi(2p+1)[s_1]$. After stage s_1, W_e never changes below $\theta_e(p)[s_1]$ and P, Q, J-strategies below $\beta\hat{}\langle i\rangle$ put numbers into A, B, C, lifting $\psi(2p)[s_1]$ to large numbers so that at a β-expansionary stage $s_2 > s_1$, we have

$$\begin{aligned}\Psi^{A\oplus W_e}(2p)[s_2] = \Psi^{B\oplus W_e}(2p)[s_2] = \Psi^{C\oplus W_e}(2p)[s_2] \\ = K_{s_2} \oplus F_{s_2}(2p) = K_{s_1} \oplus F_{s_1}(2p) = 0\end{aligned}$$

with the new use $\psi(2p)[s_2] > \theta_e(p)[s_1] > \psi(2p+1)[s_1]$. Later, p enters K and W_e changes below $\psi(2p)[s_2]$, making

$$\Psi^{A\oplus W_e}(2p)[s_3] = \Psi^{B\oplus W_e}(2p)[s_3] = \Psi^{C\oplus W_e}(2p)[s_3] = K_{s_3} \oplus F_{s_3}(2p) = 1$$

at a β-expansionary stage $s_3 > s_2$. Since W_e never changes below $\theta_e(p)[s_1]$ after stage s_1, the computation $\Theta^{W_e}(p)[s_1] = 0$ then cannot be rectified to compute $K(p) = 1$ correctly, and N_e fails.

Similar to the interactions between the $\mathcal{N}$-strategies and the $\mathcal{H}$-strategies described in the previous chapter, we apply the argument for the construction of a noncuppable degree to overcome this obstacle. Roughly speaking, when a lower P, Q, J-strategy $\alpha \succeq \beta\hat{}\langle i\rangle$ puts numbers below $\psi(2p) < \theta_e(p)$ into A, B, C, it also puts a number (called an *attached* number, as in the proof of Theorem A) into F so that W_e is forced to change below a number small enough to redefine $\Theta^{W_e}(p)$ if needed. In detail, we regulate the P, Q, J-strategies below $\beta\hat{}\langle i\rangle$ as follows.

Consider a P_i-strategy $\alpha \succeq \beta\hat{}\langle i\rangle$. When picking a witness x at stage s_0, α also takes an attached number $z \notin F$. For any β-expansionary stage $s > s_0$, we require the length of agreement $l(\beta, s) > 2z+1$ to be so that $\theta(p)[s] > \max\{\psi(2z+1)[s], \psi(2p+1)[s]\}$ if $\Theta^{W_e}(p)$ is defined at stage s (this delays the construction a little bit but it makes no difference if there are infinitely many β-expansionary stages, on the other hand, if there are only finitely many β-expansionary stages then β is simply satisfied). At a

β-expansionary stage $s_1 > s_0$, if α enumerates numbers into A and B, it also puts z into F. There are two possibilities as follows. If there is no more β-expansionary stage after s_1, then β is satisfied and doesn't need to define the functional Θ. Otherwise, we will have a bigger β-expansionary stage s_2. Note that by the choices of x and z as fresh large numbers at stage s_0, the Θ^{W_e}-computations defined before or during stage s_0 will not be affected from the action of putting numbers into A and B performed by α. For a computation $\Theta^{W_e}(p)$ defined at stage t $(s_0 < t \leq s_1)$, we have

$$\begin{aligned}\Psi^{A\oplus W_e}(2z+1)[t] = \Psi^{B\oplus W_e}(2z+1)[t] = \Psi^{C\oplus W_e}(2z+1)[t] \\ = K_t \oplus F_t(2z+1) = 0\end{aligned}$$

and

$$\begin{aligned}\Psi^{A\oplus W_e}(2z+1)[s_2] = \Psi^{B\oplus W_e}(2z+1)[s_2] = \Psi^{C\oplus W_e}(2z+1)[s_2] \\ = K_{s_2} \oplus F_{s_2}(2z+1) = 1.\end{aligned}$$

Since there is no number below $\psi(2z+1)[t]$ entering C during stage t till stage s_2, it must be that W_e has changed below $\psi(2z+1)[t] < \theta(p)[t]$ at stage s_2, making $\Theta^{W_e}(p)$ undefined. So, the obstacle is solved.

Fix a P, Q, J-strategy α. In the priority tree, suppose that

$$\beta_1\hat{\ }\langle i\rangle \prec \beta_2\hat{\ }\langle i\rangle \prec \cdots \prec \beta_n\hat{\ }\langle i\rangle \prec \alpha,$$

where $\beta_j(1 \leq j \leq n)$ are the N-strategies of higher priority than α with outcome i. Without loss of generality, we assume that α is a P_e-strategy. To be consistent with these β_j's strategies, when we choose a witness $x(\alpha)$ and take an attached number $z(\alpha)$ at stage s_0, we then require that, for every $1 \leq j \leq n$, the next β_j-expansionary stage s' has the length of agreement $l(\beta_j, s') > 2z(\alpha)+1$. Note that α will put $z(\alpha)$ in to F and puts numbers into A, B at the same time, hence, we need only one attached number for all β_j and we don't need to create the links $(\beta_j - \alpha)$ as in the construction of Theorem A. Let $k(\alpha)$ be the killing point of α. If there is a stage $s_1 > s_0$ at which α is accessible and α puts numbers into A, B, then it also puts $z(\alpha)$ into F and puts $x(\alpha)$ into D. So, α is satisfied, unless α is reset by some number $x \leq k(\alpha)$ entering K (when this happens, we wait for the next stage such that α is accessible again and choose a new witness x' and a new attached number z'). For each j, clearly, s_1 is a β_j-expansionary stage and there are also two possibilities: if there is no β_j-expansionary stage after s_1, then β_j is satisfied and has outcome f; otherwise, there is the next β_j-expansionary stage s_2 and since there is no small number entering C during stages between s_1 and s_2, there must be a change on the c.e. set W_e relative to β_j, which is sufficient to undefine all $\Theta_{\beta_j}-$computations defined after stage s_0, and hence Θ_{β_j} can be redefined correctly.

Thus, α will be modified as follows.

P1. Pick a killing point k as fresh.
Let $m := 0$. Whenever $n \le k$ enters K, set $m := m + 1$ and go to P2'.

P2'. Pick a fresh witness $x_m > k$ and a new attached number $z_m > x_m$.

P3. Wait for $\Phi_e^E(x_m)[s] \downarrow= 0$.

P4'. Do the following.

- Put z_m into F.
- Put $\gamma_1(k)[s]$ into A, $\gamma_2(k)[s]$ into B.
- Put x_m into D, and stop.

An N_e-strategy β will do the following.

N1. Set $n := 0$.

N2. Wait for a β-expansionary stage s with $l(\beta, s) > 2n + 1$.

N3. (Case 1) If there is $p \le n$ with $\Theta^{W_e}(p) \downarrow\neq K(p)$, then

(a) choose the least such p;

(b) if $W_e \upharpoonright \theta(p)$ has changed, undefine $\Theta^{W_e}(q)$ and $\theta(q), \forall q \ge p$.

(Case 2) Otherwise, for the least $p \le n$ that $\Theta^{W_e}(p)$ has not been defined yet, set $\Theta^{W_e}(p)[s] = K_s(p)$ with the use $\theta(p)[s]$ as a fresh number.

N4. Update

$$n := \max\left\{\lceil\frac{l(e,s)}{2}\rceil, z+1 : z \text{ taken during stage } s \text{ (if any)}\right\}$$

and go back to N2.

3.2 The construction

Let $\Lambda = \{i < f < s < w\}$ be the set of outcomes and $\Lambda^{<\omega}$ be equipped with the lexicographic order induced from the order in Λ. All P, Q, J, N-requirements are effectively listed and each node $\alpha \in \Lambda^{<\infty}$ of length e is assigned to the e-th requirement on the list. Here, λ is assigned to the requirement P_0. The tree of strategy T is a subtree of $\Lambda^{<\omega}$ where each node $\alpha \in T$ is a strategy of its assigned requirement and

$\{\alpha\hat{}\langle o\rangle : o$ ranges over all possible outcomes of $\alpha\}$ is the set of all immediate successors of α.
For a more convenient description of the strategies, we assign parameters to each strategy. In particular, each P, Q- or J-strategy α has parameters $k(\alpha), m(\alpha), x(\alpha)$ and $z(\alpha)$ corresponding to its killing point k, counter m, witness x and attached number z, respectively. An N-strategy α has a parameter $n(\alpha)$ marking the length of agreements.

The construction is as follows.

Stage 0. All sets are empty.

Stage $s+1$.
a. The S-strategy implements in turn these following steps to define functionals Γ_1, Γ_2, Δ_1, Δ_2, Ω_1, Ω_2:

S1. If $n \in K_{s+1} \setminus K_s$ then

(a) if $\Gamma_1^{A\oplus E}(n)[s]\downarrow \neq K_{s+1}(n) = 1$, then put $\gamma_1(n)[s]$ to E and undefine $\Gamma_1^{A\oplus E}(q)$ for $q \geq n$;
(b) if $\Gamma_2^{B\oplus E}(n)[s]\downarrow \neq K_{s+1}(n) = 1$, then put $\gamma_2(n)[s]$ to E and undefine $\Gamma_2^{B\oplus E}(q)$ for $q \geq n$;
(c) if $\Delta_1^{A\oplus H}(n)[s]\downarrow \neq K_{s+1}(n) = 1$, then put $\delta_1(n)[s]$ to H and undefine $\Delta_1^{A\oplus H}(q)$ for $q \geq n$;
(d) if $\Delta_2^{C\oplus H}(n)[s]\downarrow \neq K_{s+1}(n) = 1$, then put $\delta_2(n)[s]$ to H and undefine $\Delta_2^{C\oplus H}(q)$ for $q \geq n$;
(e) if $\Omega_1^{B\oplus G}(n)[s]\downarrow \neq K_{s+1}(n) = 1$, then put $\omega_1(n)[s]$ to G and undefine $\Omega_1^{B\oplus G}(q)$ for $q \geq n$;
(f) if $\Omega_2^{C\oplus G}(n)[s]\downarrow \neq K_{s+1}(n) = 1$, then put $\omega_2(n)[s]$ to G and undefine $\Omega_2^{C\oplus G}(q)$ for $q \geq n$.

S2. If $p = 0$, or p is the largest number such that for any $m < p$,

$$\Gamma_1^{A\oplus E}(m)[s]\downarrow = \Gamma_2^{B\oplus E}(m)[s]\downarrow = \Delta_1^{A\oplus H}(m)[s]\downarrow = \Delta_2^{C\oplus H}(m)[s]\downarrow = \Omega_1^{B\oplus G}(m)[s]\downarrow = \Omega_2^{C\oplus G}(m)[s]\downarrow = K_{s+1}(m),$$

then

(a) if $\Gamma_1^{A\oplus E}(p)\uparrow$, then define $\Gamma_1^{A\oplus E}(p)[s+1] = K_{s+1}(p)$ with the use $\gamma_1(p)[s+1]$ as fresh;
(b) if $\Gamma_2^{B\oplus E}(p)\uparrow$, then define $\Gamma_2^{B\oplus E}(p)[s+1] = K_{s+1}(p)$ with the use $\gamma_2(p)[s+1]$ as fresh;
(c) if $\Delta_1^{A\oplus H}(p)\uparrow$, then define $\Delta_1^{A\oplus H}(p)[s+1] = K_{s+1}(p)$ with the use $\delta_1(p)[s+1]$ as fresh;

(d) if $\Delta_2^{C\oplus H}(p)\uparrow$, then define $\Delta_2^{C\oplus H}(p)[s+1] = K_{s+1}(p)$ with the use $\delta_2(p)[s+1]$ as fresh;

(e) if $\Omega_1^{B\oplus G}(p)\uparrow$, then define $\Omega_1^{B\oplus G}(p)[s+1] = K_{s+1}(p)$ with the use $\omega_1(p)[s+1]$ as fresh;

(f) if $\Omega_2^{C\oplus G}(p)\uparrow$, then define $\Omega_2^{C\oplus G}(p)[s+1] = K_{s+1}(p)$ with the use $\omega_2(p)[s+1]$ as fresh.

b. **Step 1.** For any P, Q, J-strategy α, if $k(\alpha)$ has been defined and $n \leq k(\alpha)$, then set $m(\alpha) = m(\alpha) + 1$ and reset $x(\alpha), z(\alpha)$ as undefined.

Step 2. Compute the current true path TP_{s+1} (as the longest accessible node): starting with $\lambda \preceq TP_{s+1}$; if $\tau \preceq TP_{s+1}$ and is of length $s+1$, set $\tau = TP_{s+1}$ and go to **Step 3**, otherwise, find the immediate successor of τ on TP_{s+1} as follows. There are four cases.

Case 1. τ is a P_e-strategy α. Run the following P-program.

α1. If $m(\alpha)\uparrow$, then

- set $m(\alpha) = 0$,
- let $\alpha\hat{}\langle w\rangle \preceq TP_{s+1}$ and initialise all lower priority strategies.

Otherwise, go to α2.

α2.
- If $k(\alpha)\uparrow$, then set it as fresh.
- Go to α3.

α3. If $x(\alpha)\downarrow$ and $z(\alpha)\downarrow$, go to α4.
Else,

- set $x(\alpha)\uparrow$ and $z(\alpha)\uparrow$ as fresh numbers,
- let $\alpha\hat{}\langle w\rangle \preceq TP_{s+1}$ and initialise all lower priority strategies.

α4. All parameters are defined, i.e. $m(\alpha)\downarrow, k(\alpha)\downarrow, x(\alpha)\downarrow, z(\alpha)\downarrow$,

- if $\Phi_e^E(x(\alpha))[s+1]\downarrow= 0$ and $z(\alpha)\downarrow\in F$, then let $\alpha\hat{}\langle s\rangle \preceq TP_{s+1}$;
- if $\Phi_e^E(x(\alpha))[s+1]\downarrow= 0$ and $z(\alpha)\downarrow\notin F$, then go to α5.;
- otherwise, let $\alpha\hat{}\langle w\rangle \preceq TP_{s+1}$.

α5. Do the following:

- put $z(\alpha)$ into F,
- put $\gamma_1(k(\alpha))[s+1]$ into A and $\gamma_2(k(\alpha))[s+1]$ into B,
- undefine $\Gamma_1^{A\oplus E}(y), \Gamma_2^{B\oplus E}(y)$ for any $y \geq k(\alpha)$,
- undefine $\Delta_1^{A\oplus H}(y)$ if $\delta_1(y)[s+1]\downarrow\geq \gamma_1(k(\alpha))[s+1]$,
- undefine $\Omega_1^{B\oplus G}(y)$ if $\omega_1(y)[s+1]\downarrow\geq \gamma_2(k(\alpha))[s+1]$,
- put $x(\alpha)$ into D,

- let $\alpha\hat{}\langle s\rangle \preceq TP_{s+1}$,
- initialise all lower strategies.

Case 2. τ is a Q_e-strategy η. Run the P-program above with α replaced by η, $\Phi_e^E(x(\alpha))[s+1]$ replaced by $\Phi_e^H(x(\eta))[s+1]$ and step $\alpha 5$ replaced by step $\eta 5$ as follows.

$\eta 5$. Do the following:

- put $z(\eta)$ into F,
- put $\delta_1(k(\eta))[s+1]$ into A and $\delta_2(k(\eta))[s+1]$ into C,
- undefine $\Delta_1^{A\oplus H}(y), \Delta_2^{C\oplus H}(y)$ for any $y \geq k(\eta)$,
- undefine $\Gamma_1^{A\oplus H}(y)$ if $\gamma_1(y)[s+1] \downarrow\geq \delta_1(k(\eta))[s+1]$,
- undefine $\Omega_2^{C\oplus G}(y)$ if $\omega_2(y)[s+1] \downarrow\geq \delta_2(k(\eta))[s+1]$,
- put $x(\eta)$ into D,
- let $\eta\hat{}\langle s\rangle \preceq TP_{s+1}$,
- initialise all lower strategies.

Case 3. τ is a J_e-strategy ν. Run the P-program with α replaced by ν, $\Phi_e^E(x(\alpha))[s+1]$ replaced by $\Phi_e^G(x(\nu))[s+1]$ and step $\alpha 5$ replaced by step $\nu 5$ as follows.

$\nu 5$. Do the following:

- put $z(\nu)$ into F,
- put $\omega_1(k(\nu))[s+1]$ into B and $\omega_2(k(\nu))[s+1]$ into C,
- undefine $\Omega_1^{B\oplus G}(y), \Omega_2^{C\oplus G}(y)$ for any $y \geq k(\nu)$,
- undefine $\Gamma_2^{B\oplus E}(y)$ if $\gamma_2(y)[s+1] \downarrow\geq \omega_1(k(\nu))[s+1]$,
- undefine $\Delta_2^{C\oplus H}(y)$ if $\delta_2(y)[s+1] \downarrow\geq \omega_2(k(\nu))[s+1]$,
- put $x(\nu)$ into D,
- let $\nu\hat{}\langle s\rangle \preceq TP_{s+1}$,
- initialise all lower strategies.

Case 4. τ is an N_e-strategy β. Run the following N-program.

$\beta 1$. If $n(\beta)\downarrow$, then go to $\beta 2$.
Else, set $n(\beta) := 0$ and go to $\beta 2$.

$\beta 2$. If $l(\beta, s+1) > 2n(\beta)+1$, then let $\beta\hat{}\langle i\rangle \preceq TP_{s+1}$ and go to $\beta 3$.
Otherwise, let $\beta\hat{}\langle f\rangle \preceq TP_{s+1}$.

$\beta 3$. (Subcase 1) If there is $p \leq n(\beta)$ with $\Theta^{W_e}(p)[t] \downarrow\neq K(p)$ and $W_e \upharpoonright \theta(p)[t]$ has changed (here, $t \leq s$ is the last β-stage that we have defined $\Theta^{W_e}(p)[t]$), then do the following.

(a) Choose the least such p,

(b) undefine $\Theta_\beta^{W_e}(q)$, for any $q \geq p$.

(Subcase 2) For the least $p \leq n(\beta)$ such that $\Theta^{W_e}(p)$ has not been defined yet, set $\Theta^{W_e}(p)[s+1] = K_{s+1}(p)$ with fresh use $\theta(p)[s+1]$.

Step 3. Do the following.

E1. For each N-strategy β with $\beta' := \alpha\hat{}\langle i\rangle \preceq TP_{s+1}$, update

$$n(\beta) := \max\{\lceil \frac{l(\beta, s+1)}{2} \rceil, z+1 : z \text{ taken during stage } s+1 \text{ (if any)}\}.$$

E2. Initialise all nodes to the right of TP_{s+1}.

E3. Go to stage $s+2$.

3.3 Verification

Lemma 3.1. *For each node $\tau \prec TP$,*

(i) There is a stage s_0 such that τ is not initialised after s_0 and for every stage $t \geq s_0$, either $\tau \prec TP_t$ or $\tau < TP_t$. Clearly, no node to the left of τ is accessible after stage s_0.

(ii) The node τ is accessible at infinitely many stages, i.e. $\exists^\infty s : \tau \preceq TP_s$.

(iii) If τ is a P, Q, J-strategy then there is a stage s_0 after which τ never puts numbers into A, B, C, D or F.

(iv) There is unique $o \in \Lambda$ (called the true outcome of τ) such that $\tau\hat{}\langle o\rangle \prec TP$.

Proof. We prove the lemma by induction on the length of $\tau \prec TP$.

For $\tau = \lambda \prec TP$, clearly λ has the highest priority on the tree and $\lambda \preceq TP_s$ $\forall s \geq 0$, so λ is never initialised after stage 0 and (ii) is satisfied.

We have that λ is the P_0-strategy with killing point k, it is injured only by the S-strategy at most $k+1$ times. Hence, it eventually reaches its true outcome o at a stage s_0, and never puts numbers into A, B, C, D or F, i.e. $\lambda\hat{}\langle o\rangle \prec TP_t$ for any $t > s_0$ and $(i-iv)$ hold for $\lambda\hat{}\langle o\rangle$.

Suppose inductively that the nodes $\lambda = \tau_0 \prec \tau_1 \prec \cdots \prec \tau_{n-1} \prec TP$ satisfy $(i-iv)$, where $\tau_{i+1} = \tau_i\hat{}\langle o_i\rangle$ with $o_i \in \Lambda$ is the true outcome of τ_i for any $0 \leq i \leq n-2$. Let o_{n-1} be the true outcome of τ_{n-1} and $\tau_n = \tau_{n-1}\hat{}\langle o_{n-1}\rangle$. We will prove that τ_n also satisfies $(i-iv)$.

Since α_{n-1} fulfils (iii), we have $\alpha_n \prec TP$ and α_n satisfies $(i),(ii)$. It remains to prove (iii) for α_n.

Let s be the first stage such that τ_n is accessible at s and after stage s,

(a) there is no number enumerated into A, B, C, D or F by any P, Q, J-strategy $\tau \prec \tau_n$;

(b) no node to the left of τ_n is accessible.

There are two cases.

Case 1. τ_n is a P, Q, J-strategy α. Clearly, α will never be cancelled or initialised after stage s and hence, $k(\alpha) \downarrow = k(\alpha)[s^+]$, where s^+ is the next α-stage of s. Since $\alpha = \tau_{n-1}\hat{\ } \langle o_{n-1} \rangle \prec TP$, (i-ii) hold for α. In addition, α is only injured by S at most $k(\alpha)[s^+] + 1$ times. Therefore, eventually at a stage $s' > s$, α has the true outcome o ($\alpha\hat{\ } \langle o \rangle \prec TP$), stays there forever and never puts numbers into A, B, C, D or F after stage s'.

Case 2. τ_n is an N_e-strategy β. Then β is is not initialised after stage s. We have:

- either β has the true outcome f: $\beta\hat{\ } \langle f \rangle$ is accessible at stage $s_0 \geq s$ and for any stage $t > s_0$, $\beta\hat{\ } \langle f \rangle \prec TP_t$.
- or it reaches the true outcome i at infinitely many β-stages: let $t_0 \geq s$ be the first stage that $\beta\hat{\ } \langle i \rangle$ is accessible, we have that $\beta\hat{\ } \langle i \rangle \preceq TP_t$ for any β-expansionary stage $t \geq t_0$ and for any $t > t_0$ that $\beta\hat{\ } \langle i \rangle$ is not accessible, $\beta\hat{\ } \langle i \rangle < TP_t$.

□

Lemma 3.2. *Every strategy $\tau \prec TP$ satisfies its corresponding requirement.*

Proof. Let τ of length e be a node on the true path TP. From lemma 3.1, there is the first τ-stage s after which τ is never initialised, i.e.,

(a) no node to the left of τ is accessible after stage s,

(b) no P, Q, J-strategy above τ on the priority tree puts numbers into A, B, C, D or F.

We consider two cases: τ is a P_e-strategy α and τ is an N_e-strategy β. For τ is a Q_e-strategy η or a J_e-strategy ν, the verification is analogous to the case of a P_e-strategy α.

Case 1. τ is a P_e-strategy α. Since α will never be initialised after stage s, from the construction we have that $k(\alpha)\downarrow= k(\alpha)[s^+]$, where s^+ is the next α-stage after s. Let $s_1 > s$ be an α-stage such that there is no $n \leq k(\alpha)$ entering K from stage s_1 onwards. Then, $m(\alpha)\downarrow= m(\alpha)[s_1]$, $x(\alpha)\downarrow= x(\alpha)[s_1^+]$, where s_1^+ is the next α-stage after s_1. If $\alpha\widehat{\ }\langle w\rangle \prec TP$ then $D(x(\alpha)) = 0 \neq \Phi_e^E(x(\alpha))$. If $\alpha\widehat{\ }\langle s\rangle \prec TP$ then $D(x(\alpha)) = 1 \neq \Phi_e^E(x(\alpha))\downarrow= 0$. Hence, P_e is satisfied.

Case 2. τ is an N_e-strategy β. Suppose that $\Psi^{A\oplus W_e} = \Psi^{B\oplus W_e} = \Psi^{C\oplus W_e} = K \oplus F$. So, there are infinitely many β-expansionary stages. We will prove by induction on p that for any $p \in \omega, \Theta^{W_e}(p)\downarrow= K(p)$. Thus, the functional Θ^{W_e} is total and computes K correctly, implying that N_e is satisfied.

Under these assumptions, there are infinitely many β-expansionary stages and $\beta\widehat{\ }\langle i\rangle \prec TP$. Let $s' \geq s$ be the first β-expansionary stage after stage s.

For $p = 0$, at stage s', step $\beta3$ (Subcase 2) in the construction is applicable for 0 and we define $\Theta^{W_e}(0)[s'] = K_{s'}(0)$ with fresh use $\theta(0)[s']$, so $\theta(0)[s'] > \psi(0)[s']$. If 0 never enters K after stage s', then $K(0) = K_{s'}(0) = \Theta^{W_e}(0)[s'] = \Theta^{W_e}(0)$. Otherwise, suppose that 0 enters K at stage $t > s'$. Let $s_1 \geq t$ be the next β-expansionary stage after stage t. We have

$$\begin{aligned}\Psi^{A\oplus W_e}(0)[s'] = \Psi^{B\oplus W_e}(0)[s'] = \Psi^{C\oplus W_e}(0)[s'] \\ = K_{s'} \oplus F_{s'}(0) = K_{s'}(0) = 0,\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\Psi^{A\oplus W_e}(0)[s_1] = \Psi^{B\oplus W_e}(0)[s_1] = \Psi^{C\oplus W_e}(0)[s_1] \\ = K_{s_1} \oplus F_{s_1}(0) = K_{s_1}(0) = K_t(0) = 1.\end{aligned} \tag{3.2}$$

Note that β is not initialised after stage s and by the choice of stage s',

- any P, Q, J-strategy node below β has parameters larger than $\theta(0)[s']$,
- all parameters of P, Q, J-strategies which are to the right of $\beta\widehat{\ }\langle i\rangle$ and accessible after stage s', are larger than $\theta(0)[s']$,
- all P, Q, J-strategies which are to the right of $\beta\widehat{\ }\langle i\rangle$ and were accessible before stage s', have been already cancelled at the end of stage s'.

Together with properties (a-b) above, it follows that no number smaller than or equal to $\theta(0)[s'] > \psi(0)[s']$ enters A, B or C after stage s. Thus, at stage s_1, we get equation (3.2) only because W_e has changed below $\psi(0)[s']$. This change ensures that $\Theta^{W_e}(0)$ may be undefined by step $\beta 3$ (Subcase 1) at stage s_1. At the next β-expansionary stage s_2 after s_1, we have that step $\beta 3$ (Subcase 2) applies to 0 and we redefine $\Theta^{W_e}(0)[s_2] = K_{s_2}(0) = 1 = K(0)$. This computation then remains correct from stage s_2 onwards, so $\Theta^{W_e}(0) \downarrow= K(0) = 1$.

Suppose that Θ^{W_e} correctly computes K up to $p - 1 \geq 0$. We will prove $\Theta^{W_e}(p) \downarrow= K(p)$. Let $s_0 \geq s'$ be the first β-expansionary stage satisfying

$$\Theta^{W_e}(q) \downarrow= \Theta^{W_e}(q)[s_0] \downarrow= K_{s_0}(q) = K(q) \forall q \leq p - 1$$

and let s_1 be the next β-expansionary stage after s_0. We have

$$l(\beta, s_0) > 2n(\beta)[s_0] + 1 \geq 2(p-1) \Rightarrow n(\beta)[s_1] \geq \lceil \frac{l(\beta, s_0)}{2} \rceil \geq p.$$

Hence, at stage s_1, from the construction we have that $\Theta^{W_e}(p)[s_1] = K_{s_1}(p)$ with use $\theta(p)[s_1] > \psi(l(\beta, s_1))$. If p never enters K after stage s_1 then $\Theta^{W_e}(p) \downarrow= \Theta^{W_e}(p)[s_1] = K_{s_1}(p) = K(p)$. Otherwise, suppose p enters K at stage $\overline{s} > s_1$. Let s_2 and s_3 be two consecutive β-expansionary stages such that $s_1 \leq s_2 < \overline{s} \leq s_3$. We will show that W_e has changed below $\theta(p)[s_1]$ at a stage $\tilde{s}$, where $s_1 < \tilde{s} \leq s_3$. Therefore, at stage s_3, $\Theta^{W_e}(p)$ may be undefined by the step $\beta 3$ (Subcase 1) in the construction. At the next β-expansionary stage s_4 after s_3, we have that step $\beta 3$ (Subcase 2) is applicable for p and we then redefine $\Theta^{W_e}(p)[s_4] = K_{s_4}(p) = 1 = K(p)$. This computation remains forever, i.e. $\Theta^{W_e}(p) = \Theta^{W_e}(p)[s_4] = 1 = K(p)$, and the proof is complete.

Clearly, a P, Q, J-strategy below β can put numbers smaller than $\theta(p)[s_1]$ into A, B or C during a stage t, where $s_1 \leq t < s_3$, only if t is a β-expansionary stage and all parameters of this P, Q, J-strategy are chosen before stage s_1. Note that

$$\begin{aligned} \Psi^{A \oplus W_e}(2p)[s_1] = \Psi^{B \oplus W_e}(2p)[s_1] &= \Psi^{C \oplus W_e}(2p)[s_1] \\ &= K_{s_1} \oplus F_{s_1}(2p) = K_{s_1}(p) = 0, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \Psi^{A \oplus W_e}(2p)[s_3] = \Psi^{B \oplus W_e}(2p)[s_3] &= \Psi^{C \oplus W_e}(2p)[s_3] \\ &= K_{s_3} \oplus F_{s_3}(2p) = K_{s_3}(p) = K_{\overline{s}}(p) = 1. \end{aligned} \tag{3.4}$$

If there is no strategy which puts numbers smaller than $\theta(p)[s_1]$ into A, B or C during β-expansionary stages between s_1 and s_3, then (3.4) is obtained only by a change of W_e below $\psi(l(\alpha, s_1))$ at a stage $\widetilde{s}$, where $s_1 < \widetilde{s} \leq s_3$. Otherwise, without loss of generality, suppose that the first such strategy $\alpha \succeq \beta$ be a P-strategy (α puts $\gamma_1(k(\tau))[t]$ into A, $\gamma_2(k(\tau))[t]$ into B at β-expansionary stage t, where $s_1 \leq t < s_3$ and $\min\{\gamma_1(k(\tau))[t],\ \gamma_2(k(\tau))[t]\} \leq \theta(p)[s_1]$). Arguing as above, all parameters of α were chosen before stage s_1. Hence, $2z(\alpha)+1 < l(\beta, s_1)$ and $\theta(p)[s_1] > \psi(2z(\alpha)+1)[s_1]$. From the construction, α also puts $z(\alpha)$ into F at stage t. We have

$$\begin{aligned}\Psi^{A\oplus W_e}(2z(\alpha)+1)[s_1] = \Psi^{B\oplus W_e}(2z(\alpha)+1)[s_1] = \Psi^{C\oplus W_e}(2z(\alpha)+1)[s_1] \\ = K_{s_1} \oplus F_{s_1}(2z(\alpha)+1) = F_{s_1}(z(\alpha)) = 0,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\Psi^{A\oplus W_e}(2z(\tau)+1)[t_1] = \Psi^{B\oplus W_e}(2z(\tau)+1)[t_1] = \Psi^{C\oplus W_e}(2z(\tau)+1)[t_1] \\ = K_{t_1} \oplus F_{t_1}(2z(\tau)+1) = F_{t_1}(z(\tau)) = 1,\end{aligned} \tag{3.6}$$

where t_1 is the next β-expansionary stage after t and so, $t < t_1 \leq s_3$. By the choice of α and t, from stage s_1 till stage t, there is no number less than or equal to $\theta(p)[s_1]$ entering C. Moreover, any strategy accessible at a stage between t and t_1, is to the right of $\beta\hat{}\langle i\rangle$ and it parameters are all bigger than $\theta(p)[s_1]$. Thus, from stage s_1 to stage t_1, no number smaller than or equal to $\theta(p)[s_1]$ enters C. Hence, it is clearly seen from equations (3.5) and (3.6) that W_e has changed below $\psi(2z(\tau)+1)[s_1] < \theta(p)[s_1]$ at some stage $\tilde{s}$, where $s_1 < \tilde{s} \leq t_1 \leq s_3$, making $\Theta^{W_e}(p)$ rectified correctly.

□

Lemma 3.3. *Requirement S is satisfied.*

Proof. We will prove by induction that for any $n \geq 0$,

$$\begin{aligned}\Gamma_1^{A\oplus E}(n)\downarrow = \Gamma_2^{B\oplus E}(n)\downarrow = \Delta_1^{A\oplus H}(n)\downarrow = \Delta_2^{C\oplus H}(n)\downarrow \\ = \Omega_1^{B\oplus G}(n)\downarrow = \Omega_2^{C\oplus G}(n)\downarrow = K(n).\end{aligned} \tag{3.7}$$

The arguments are based on properties $(1-5)$ of the functionals Γ_1, Γ_2, Δ_1, Δ_2, Ω_1, Ω_2.

For $n = 0$, at stage 1 of the construction, we define

$$\begin{aligned}\Gamma_1^{A\oplus E}(0)[1] = \Gamma_2^{B\oplus E}(0)[1] = \Delta_1^{A\oplus H}(0)[1] = \Delta_2^{C\oplus H}(0)[1] \\ = \Omega_1^{B\oplus G}(0)[1] = \Omega_2^{C\oplus G}(0)[1] = K_1(0).\end{aligned}$$

Since all later parameters of P, Q, J, N-strategies are chosen as fresh large numbers, these computations will be unchanged and equals $K(0)$ unless 0 enters K at a stage $s > 1$. If the latter happens, at the beginning of stage s, we process $S1$. to undefine all $\Gamma_1^{A\oplus E}(0)[1]$, $\Gamma_2^{B\oplus E}(0)[1]$, $\Delta_1^{A\oplus H}(0)[1]$, $\Delta_2^{C\oplus H}(0)[1]$, $\Omega_1^{B\oplus G}(0)[1]$, $\Omega_2^{C\oplus G}(0)[1]$. At step $S2$ of stage s, we redefine

$$\begin{aligned}\Gamma_1^{A\oplus E}(0)[s] = \Gamma_2^{B\oplus E}(0)[s] = \Delta_1^{A\oplus H}(0)[s] = \Delta_2^{C\oplus H}(0)[s] \\ = \Omega_1^{B\oplus G}(0)[s] = \Omega_2^{C\oplus G}(0)[s] = 1 = K(0),\end{aligned}$$

with fresh uses and these computation remain correct forever since all the parameters of the other strategies are reset as well.

Assuming (3.7) to hold for all $m \leq n-1$, we will prove it for n.

Let s be the first stage at which $\forall m \leq n-1$, the computations $\Gamma_1^{A\oplus E}(m)$, $\Gamma_2^{B\oplus E}(m)$, $\Delta_1^{A\oplus H}(m)$, $\Delta_2^{C\oplus H}(m)$, $\Omega_1^{B\oplus G}(m)$, $\Omega_2^{C\oplus G}(m)$ are correct.

Thus,

(a) after stage s, all uses $\gamma_1(m)[s]$, $\gamma_2(m)[s]$, $\delta_1(m)[s]$, $\delta_2(m)[s]$, $\omega_1(m)[s]$, $\omega_2(m)[s]$ have been already defined and never change for any $m \leq n-1$;

(b) from the construction, not all uses $\gamma_1(n)[s]$, $\gamma_2(n)[s]$, $\delta_1(n)[s]$, $\delta_2(n)[s]$, $\omega_1(n)[s]$, $\omega_2(n)[s]$ have been defined at the end of stage s.

Also from the construction, after step S2 of stage $s+1$, we have that all the uses $\gamma_1(n)[s+1]$, $\gamma_2(n)[s+1]$, $\delta_1(n)[s+1]$, $\delta_2(n)[s+1]$, $\omega_1(n)[s+1]$, $\omega_2(n)[s+1]$ have been defined (denote L to be the set of these uses). If n enters K at a stage $s' > s+1$ then at step $S2$ of stage s', all uses in L are reset as fresh numbers. So, without loss of generality, we can assume that $n \notin K_{t+1} \setminus K_t$ for any $t \geq s+1$, i.e. $K_{s+1}(n) = K(n)$.

After stage $s+1$, since all γ, δ, ω-uses at numbers below n have already converged, any P, Q, J-strategy which puts numbers into A, B or C (for example, a P_e-strategy α puts $\gamma_1(k(\alpha))[t]$ into A and $\gamma_2(k(\alpha))[t]$ into B at stage $t \geq s+1$) must have its killing point $k \geq n$.

By the choice of stage s and from the construction, if a γ, δ, ω-use at k ($k > n$) is defined at stage $s+1$, then it has been in fact defined before stage s (for example, if $\gamma_1(k)[s+1]\downarrow$, then $\gamma_1(k)[s+1] = \gamma_1(k)[s] = \gamma_1(k)[t]$ for some $t < s$). Let $p \geq n$ be the largest number such that there is a γ, δ, ω-use at p which has been defined at the end of stage s and let

$$U = \{\gamma_i(x)[s]\downarrow, \delta_i(x)[s]\downarrow, \omega_i(x)[s]\downarrow : i = 0, 1; n \leq x \leq p\}.$$

We have that any γ, δ, ω-use at n which is being defined during step S2 of stage $s+1$, is larger than $\max U$. Therefore, the uses in L can be undefined after stage $s+1$ only if some uses in $U \cup L$ are enumerated into the corresponding oracles via

(a) actions of S in step S1 when a number $n < x \leq p$ enters K after stage $s+1$ (for example,if $n+1$ enters K and S puts $\gamma_1(n+1)[s] \in U$ into E, then $\gamma_1(n+1)$ and $\gamma_2(n)$ are undefined);

(b) or actions of a P, Q, J-strategy with the killing point $n \leq k \leq p$ (for example, if a P_e-strategy α with $k(\alpha) = n$ puts $\gamma_1(n)[s] = \gamma_1(n)[s+1] \in U \cap L$ into A and $\gamma_2(n)[s+1] \in L$ into B, making $\delta_1(n)$ and $\omega_1(n)$ undefined).

If at stage $t > s+1$, (a) or (b) happens, then during step S2 of stage t, we reset the γ, δ, ω-uses at n, which have been undefined, as fresh numbers. Since $U \cup L$ is finite, the γ, δ, ω-uses at n can only be undefined at most finitely many times. Therefore, there is a stage $t \geq s+1$ such that all uses $\gamma_1(n)[t]$, $\gamma_2(n)[t]$, $\delta_1(n)[t]$, $\delta_2(n)[t]$, $\omega_1(n)[t]$, $\omega_2(n)[t]$ have been defined and from stage t onwards,

- there is no $n \leq k \leq p$ entering K,
- there is no P, Q, J-strategy with killing point $n \leq k \leq p$ putting numbers into A, B or C.

So, we have

$$\begin{aligned} K(n) &= \Gamma_1^{A\oplus E}(n)[t] = \Gamma_1^{A\oplus E}(n)\downarrow \\ &= \Gamma_2^{B\oplus E}(n)[t] = \Gamma_2^{B\oplus E}(n)\downarrow \\ &= \Delta_1^{A\oplus H}(n)[t] = \Delta_1^{A\oplus H}(n)\downarrow \\ &= \Delta_2^{C\oplus H}(n)[t] = \Delta_2^{C\oplus H}(n)\downarrow \\ &= \Omega_1^{B\oplus G}(n)[t] = \Omega_1^{B\oplus G}(n)\downarrow \\ &= \Omega_2^{C\oplus G}(n)[t] = \Omega_2^{C\oplus G}(n)\downarrow . \end{aligned}$$

□

Chapter 4

Lattice embeddings in $\mathbf{R}/\mathbf{NCup}$

In the previous Chapter 3, we studied n-cuppable degrees for $n \geq 1$. Examples of degrees that are 1-cuppable but not 2-cuppable were given in [37] and [4]. In particular, in [37], Li, Wu, and Yang constructed two incomplete cuppable c.e. degrees $\mathbf{a}, \mathbf{b}$ such that there is no incomplete c.e. degree which cups both $\mathbf{a}, \mathbf{b}$ to $\mathbf{0}'$. Consequently, the corresponding equivalent classes $[\mathbf{a}]$, $[\mathbf{b}]$ form a minimal pair in $\mathbf{R}/\mathbf{NCup}$. So Shoenfield's conjecture is not true in the structure $\mathbf{R}/\mathbf{NCup}$. Following the work in [37], Bie and Wu [4] constructed c.e. degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$ and $\mathbf{a}, \mathbf{b}$ are also 1-cuppable but not 2-cuppable. So $[\mathbf{a}], [\mathbf{b}]$ form a minimal pair in $\mathbf{R}/\mathbf{NCup}$, and moreover, they form a minimal pair in $\mathbf{M}/\mathbf{NCup}$. Hence, Shoenfield's conjecture is also not true in $\mathbf{M}/\mathbf{NCup}$. Since $\mathbf{M}/\mathbf{NCup}$ does not have the greatest element, the diamond lattice cannot be embedded preserving 0 and 1 into this structure. Li, Wu, and Yang claimed that it is possible to have such an embedding into $\mathbf{R}/\mathbf{NCup}$. However, there is no published proof of this fact in the literature. In this chapter, we apply the techniques from [4] (different from the techniques in [37]) to prove this claim, as stated in Theorem C. In particular, we construct two c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$ and there is no c.e. degree cupping both $\mathbf{a}$ and $\mathbf{b}$ to $\mathbf{0}'$. Clearly, the c.e. degrees $\mathbf{a}, \mathbf{b}$ satisfying Theorem C are 1-cuppable but not 2-cuppable.

4.1 Diamond lattice in $\mathbf{R}/\mathbf{NCup}$

Consider c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ in Theorem C and their corresponding equivalence classes $[\mathbf{a}]$ and $[\mathbf{b}]$ in $\mathbf{R}/\mathbf{NCup}$.

Clearly, $[\mathbf{a}] \vee [\mathbf{b}] = [\mathbf{0}']$. To see that $[\mathbf{a}] \wedge [\mathbf{b}] = [\mathbf{0}]$, we need the following lemma.

Lemma 4.1 ([37]). *Given two cuppable c.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$.

If

$$\forall \mathbf{w} \in \mathbf{R}(\mathbf{w} \cup \mathbf{a} = \mathbf{w} \cup \mathbf{b} = \mathbf{0}' \to \mathbf{w} = \mathbf{0}') \tag{4.1}$$

then $\mathbf{a}$ *and* $\mathbf{b}$ *are incomplete and* $[\mathbf{a}] \wedge [\mathbf{b}] = [\mathbf{0}]$*, i.e.* $[\mathbf{a}]$ *and* $[\mathbf{b}]$ *form a minimal pair in* $\mathbf{R}/\mathbf{NCup}$*.*

Proof.

Assume that we have property (4.1).

If $\mathbf{a} = \mathbf{0}'$ then for a cupping candidate $\widetilde{\mathbf{b}} < \mathbf{0}'$ of $\mathbf{b}$, we have $\mathbf{a} \cup \widetilde{\mathbf{b}} = \mathbf{0}' = \mathbf{b} \cup \widetilde{\mathbf{b}}$. It follows from property (4.1) that $\widetilde{\mathbf{b}} = \mathbf{0}'$, a contradiction. Hence, $\mathbf{a}$ is incomplete and so is $\mathbf{b}$.

It remains to prove that $[\mathbf{a}] \wedge [\mathbf{b}] = [\mathbf{0}]$. Let $[\mathbf{w}]$ be an element below both $[\mathbf{a}]$ and $[\mathbf{b}]$ in $\mathbf{R}/\mathbf{NCup}$. Suppose that $[\mathbf{w}] \neq [\mathbf{0}]$, i.e. $\mathbf{w}$ is cuppable. Then, there are $\widehat{\mathbf{b}} < \mathbf{0}'$ and $\mathbf{e}, \widehat{\mathbf{e}} \in \mathbf{NCup}$ such that $\mathbf{w} \cup \widehat{\mathbf{b}} = \mathbf{0}'$, $\mathbf{w} \leq \mathbf{a} \cup \mathbf{e}$ and $\mathbf{w} \leq \mathbf{b} \cup \widehat{\mathbf{e}}$. We have that $\mathbf{0}' = \mathbf{w} \cup \widehat{\mathbf{b}} \leq \mathbf{a} \cup \mathbf{e} \cup \widehat{\mathbf{e}} \cup \widehat{\mathbf{b}} \leq \mathbf{0}'$. Therefore, $\mathbf{a} \cup (\mathbf{e} \cup \widehat{\mathbf{e}} \cup \widehat{\mathbf{b}}) = \mathbf{0}'$. Similarly, $\mathbf{b} \cup (\mathbf{e} \cup \widehat{\mathbf{e}} \cup \widehat{\mathbf{b}}) = \mathbf{0}'$. As $\mathbf{e} \cup \widehat{\mathbf{e}} \cup \widehat{\mathbf{b}} \in \mathbf{R}$, it follows from property (4.1) that $\mathbf{e} \cup \widehat{\mathbf{e}} \cup \widehat{\mathbf{b}} = \mathbf{0}'$. Since $\mathbf{e}, \widehat{\mathbf{e}} \in \mathbf{NCup}$, and so is $\mathbf{e} \cup \widehat{\mathbf{e}}$, we must have $\widehat{\mathbf{b}} = \mathbf{0}'$, a contradiction. Thus, $[\mathbf{w}] = [\mathbf{0}]$ and hence, $[\mathbf{a}] \wedge [\mathbf{b}] = [\mathbf{0}]$. □

4.2 Construction of a diamond lattice in $\mathbf{R}/\mathbf{NCup}$

4.2.1 Requirements and strategies

From the previous lemma, to prove Theorem C, we construct c.e. sets A and B satisfying, for any $e, i = \langle i_1, i_2, e\rangle \in \omega$, the following requirements:

$$S : \Gamma^{A \oplus B} = K,$$

$$P_e : \Phi_e^A \neq D, \quad Q_e : \Phi_e^B \neq D,$$

$$N_i : \Psi_{i_1}^{A \oplus W_e} = \Psi_{i_2}^{B \oplus W_e} = K \oplus F \to K = \Omega_i^{W_e}.$$

Here, K is a fixed complete c.e. set. We fix an effective enumeration $\{K_s\}_{s\in\omega}$ of K such that there is at most one number entering K at each stage. All (partial) computable functionals Φ_e's and Ψ's are effectively listed. (Partial) computable functionals $\Gamma, \Omega_i (i \in \omega)$, and c.e. sets D, F are built in the construction. We will sometimes omit the indices i_1, i_2 in the requirement N_i in order to simplify notations.

We check that the c.e. degrees $\deg(A)$ and $\deg(B)$ are the desired c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ of Theorem C. Indeed, the requirements $\{P_e\}_{e\in\omega}$ and the requirement $\{Q_e\}_{e\in\omega}$ imply that A and B are incomplete. Hence, requirement S ensures that A and B are cuppable and $\deg(A) \cup \deg(B) = \mathbf{0}'$. Finally, the requirements $\{N_i\}_{i\in\omega}$ ensure that there is no incomplete c.e. degree which cups both deg(A) and deg(B) to $\mathbf{0}'$.

During the construction, whenever we pick a number, we choose a *fresh* number, i.e. a number which is larger than all numbers which have been used so far in the construction. We give the requirement S the highest priority and use a priority tree for the P, Q, N-requirements.

Before setting up the tree, we discuss strategies for these requirements and their interactions.

The S-strategy

The S-strategy will define functional $\Gamma^{A\oplus B}$ as follows.

S1. If $\Gamma^{A\oplus B}(n)\downarrow\neq K(n)$, enumerate $\gamma(n)$ to $A\cup B$ for the least such n and undefine all $\Gamma^{A\oplus B}(y), y\geq n$.

S2. Otherwise, for the least number n with $\Gamma^{A\oplus B}(n)\uparrow$, define $\Gamma^{A\oplus B}(n):=K(n)$ with the use $\gamma(n)$ as a fresh number.

As usual, we will ensure the following basic properties for the use function γ of Γ:

(1) Whenever $\gamma(n)$ is defined, it will be assigned a fresh number;

(2) For any number n and stage s, if $\Gamma^{A\oplus B}(n)[s]\downarrow$ then $\gamma(n)[s]\notin A_s\cup B_s$;

(3) For any number n and $m<n$, if $\gamma(n)[s]\downarrow$ then $\gamma(m)[s]\downarrow$ and $\gamma(m)[s]<\gamma(n)[s]$;

(4) If $\Gamma^{A\oplus B}(n)[s]\downarrow$ and n enters K at stage $s+1$, then $\gamma(m)[s]$ is enumerated in A or B at stage $s+1$ for some $m\leq n$;

(5) At stage s, $\Gamma^{A\oplus B}(n)$ is undefined iff either $\gamma(m)$ is enumerated into A or B for some $m\leq n$.

If $\Gamma^{A\oplus B}$ is total and all properties (1-5) are satisfied, then $\Gamma^{A\oplus B}$ computes K correctly.

TheP, Q-strategies

Fix a P_e-strategy α. The idea for α, as discussed in previous chapters, is described in the following.

P1. Pick a fresh killing point k. Whenever $n\leq k$ enters K, goes to P2.

P2. Pick a fresh witness $x>k$.

P3. Wait for $\Phi_e^A(x)[s]\downarrow=0$.

P4. Put $\gamma(k)[s]$ into B, x into D, and stop.

At stage s, to preserve the computation $\Phi_e^A(x)[s]\downarrow= 0$, α puts $\gamma(k)[s]$ into B so that all uses $\gamma(n)$, $n \geq k$, are lifted to numbers larger than $\phi_e^A(x)[s]$. Consequently, the computation $\Phi_e^A(x)[s]\downarrow= 0$ remains unchanged unless a number $n \leq k$ enters K and S puts $\gamma(n) \leq \phi_e^A(x)[s]$ into A. When this happens, α refreshes its witness x as a fresh number and starts over. Such a situation can occur at most $k+1$ times and hence α is eventually satisfied.

The outcomes of P_e are:

w: wait at P3 forever for some x, then $D(x) = 0 \neq \Phi_e^A(x)$,

s: stop at P4 forever from some stage s onward for some x, then there is no number less than k entering K after stage s and hence, $\Phi_e^A(x)\downarrow= 0 \neq D(x) = 1$.

A Q_e-strategy β acts similarly to α, by swapping A and B correspondingly. When α puts $\gamma(k)[s]$ into B, it may destroy a computation $\Phi_i^B(x)\downarrow= 0$ of a lower priority Q_i-strategy $\beta \succ \alpha$ (and similarly, a Q-strategy can injure a lower P-strategy by the action of putting γ-uses into A). However, there are only finitely many such injuries because α acts at most finitely many times. When such an injury happens, we simply initialize β.

The N-strategies

Fix an N_e-strategy η. We would like to construct a (partial) computable functional Ω_η such that if $\Psi^{A\oplus W_e} = \Psi^{B\oplus W_e} = K \oplus F$, then $\Omega_\eta^{W_e}$ is total and computes K correctly.

We recall the *length of agreement*

$$l(\eta, s) = \max\{x < s : \Psi^{A\oplus W_e}(y)[s] = \Psi^{B\oplus W_e}(y)[s] = K_s \oplus F_s(y)\forall y \leq x\}.$$

A stage s is an η-*expansionary* stage if $s = 0$ or $l(\eta, s) > l(\eta, t)$ for any η-stage $t < s$.

For any $x \leq l(\eta, s)$, let $\psi(x)[s] := \max\{\psi^{A\oplus W_e}(x)[s], \psi^{B\oplus W_e}(x)[s]\}$.

We will define $\Omega_\eta^{W_e}$ at η-expansionary stages. For instance, at an η-expansionary stage s, if there is the least number p that $2p+1 < l(\eta, s)$ and $\Omega_\eta^{W_e}(p)\uparrow$, we define $\Omega_\eta^{W_e}(p)[s] := K_s(p)$ with the use $\omega(p)[s]$ as a fresh number. After that, the computation $\Omega_\eta^{W_e}(p)$ is undefined only if W_e changes below $\psi(2p+1)[s]$. So, if p enters K after stage s, making $\Omega_\eta^{W_e}(p)[s]\downarrow\neq K(p)$, we need an appropriate change on W_e to undefine $\Omega_\eta^{W_e}(p)$.

The strategy η has two outcomes: i for infinitely many η-expansionary stages and f for finitely many η-expansionary stages.

The interactions between N-strategies andP,Q-strategies

There is a problem as follows. At an η-expansionary stage s we define $\Omega_\eta^{W_e}(p)[s] = K_s(p) = 0$ with use $\omega(p)[s] > \psi(2p+1)[s]$ (here, $l(\eta, s) > 2p+1$) and W_e never changes below $\omega(p)[s]$ after stage s. However, at an η-expansionary stage $s' > s$, a lower priority P-strategy $\alpha \succeq \eta\hat{}\langle i\rangle$ puts a number below $\psi(2p)[s]$ into B, and later S puts a number below $\psi(2p)[s]$ into A, p enters K so that at the next η-expansionary stage s'' after s',

$$\Psi^{A\oplus W_e}(2p)[s''] = \Psi^{B\oplus W_e}(2p)[s''] = K_{s''} \oplus F_{s''}(2p) = K_{s''}(p) = K(p) = 1;$$

then the computation $\Omega_\eta^{W_e}(p)[s] \downarrow= 0 \neq 1 = K(p)$ may not be redefined correctly.

As usual, we use the method for the construction of noncuppable degrees to overcome this problem. Ideally, whenever a lower priority P-strategy $\alpha \succeq \eta\hat{}\langle i\rangle$ would like to put some number $y \leq \psi(2p)$ into B, it forces W_e to change below $\omega(p)$ so that $\Omega_\eta^{W_e}(p)$ can be rectified later. Precisely, α will do as follows. At an η-expansionary stage s_0, when α picks a witness x and a killing point k, it also takes an *attached* number $z \notin F$ (as in the proof of Theorem A), preparing for its action of putting $\gamma(k)$ into B. We require every η-expansionary stage t after s_0 to have the length of agreement $l(\eta, t) > 2z+1$ so that $\omega(p)[t] > \psi(2z+1)[t]$ for any computation $\Omega_\eta^{W_e}(p)[t]$ defined during stage t (note that this only delays the construction a little bit). Suppose that at an η-expansionary stage $s_1 > s_0$, $\Phi_\alpha^A(x)[s_1] \downarrow= 0$ and α would like to put x into D and put $\gamma(k)$ into B to protect $\Phi_\alpha^A(x) \downarrow= 0$. Clearly, the $\Omega_\eta^{W_e}$-computations defined before stage s_0 will not be injured by this action of α. So we only care about the $\Omega_\eta^{W_e}$-computations defined after stage s_0. Consider a computation $\Omega_\eta^{W_e}(p)$ defined at stage t, $s_0 < t \leq s_1$. At stage s_1, α immediately puts $\gamma(k)$ into B to preserve $\Phi_\alpha^A(x) \downarrow= 0$, puts x into D to achieve the diagonalization $\Phi_\alpha^A(x) = 0 \neq 1 = D(x)$, and declares that from now on, if $n \leq k$ enters K, then $\gamma(n)$ is enumerated into B until a higher priority P, Q-strategy makes a diagonalization. There are two possibilities as follows.

- α is visited at stage $s_2 > s_1$. Then α will put z into F. At the next η-expansionary stage $s_3 > s_2$, since we have not put any small number into the A-side from stage s_1 till stage s_3, it must be that W_e has changed below $\psi^{A\oplus W_e}(2z+1)[s_1]$ by stage s_3.

- α is never accessible after stage s_1. If the next η-expansionary stage $s_2' > s_1$ comes, we have that no small number has been enumerated into A since stage s_1 and hence, at stage s_2', for any $y < l(\eta, s_1)$ such that $\Psi^{A\oplus W_e}(y)[s_2'] \neq \Psi^{A\oplus W_e}(y)[s_1]$, W_e has changed below $\psi^{A\oplus W_e}(y)[s_1]$.

Thus, to force W_e to change below $\omega(p)[t]$ so that $\Omega_\eta^{W_e}(p)$ can be redefined if p enters K after stage t, we have to guarantee that from stage s_0 to stage s_1,

(a) either no number $y \leq \psi(2p+1)[t]$ is enumerated into A at stages between s_0 and s_1;

(b) or if there is a number $y \leq \psi(2p+1)[t]$ entering A at a stage between s_0 and s_1, then at stage s_1, W_e had changed below $\omega(p)[t]$.

Note that (b) can happen when a lower priority Q-strategy β (with witness x', killing point $k' > k$, and the attached number z') makes a diagonalization before α, and then a small number $\gamma(n)$ is enumerated into A by S before stage s_1. Precisely, at stage s' ($s_0 < t \leq s' < s_1$), $\Phi^B(x') \downarrow = 0$, β puts $\gamma(k')$ into A, puts x' into D, and declares that if $n \leq k'$ enters K then $\gamma(n)$ is enumerated into A. At stage s_1' ($s' < s_1' < s_1$), a number $n \leq k'$ with $\gamma(n) \leq \psi(2p+1)[t]$ enters K, then S puts $\gamma(n)$ into A to correct the computation $\Gamma^{A\oplus B}(n)$, making $\psi^{A\oplus W_e}(2p+1)$ lifted to a large number.

To deal with such a situation, we will prepare an auxiliary number y_n for $\gamma(n)$ such that $\omega(p)[t] > \psi(2y_n+1)[t]$, and when $\gamma(n)$ is enumerated into A at stage s_1', we put y_n into F. By this way, at the next η-expansionary stage s_2' ($s_1' < s_2' \leq s_1$), because we have not put any small number into the B-side yet and the length of agreements of η has been recovered at stage s_2', W_e has changed below $\psi^{B\oplus W_e}(2y_n+1)[t] \leq \psi(2y_n+1)[t] < \omega(p)[t]$.

Next, we consider a P-strategy working with all of its higher priority N-strategies on the priority tree. Fix a P-strategy α and suppose that

$$\eta_1\hat{\ }\langle i\rangle \prec \eta_2\hat{\ }\langle i\rangle \prec \cdots \prec \eta_n\hat{\ }\langle i\rangle \preceq \alpha,$$

where $\eta_j (1 \leq j \leq n)$ are the N-strategies of higher priority than α with outcome i. To be consistent with these η_j's strategies, when we choose a witness x, a killing point k at stage s_0, we take attached numbers (making an *attached sequence*, as in the proof of theorem A) $z_1 < z_2 \cdots < z_n$ corresponding to $\eta_1, \eta_2, \cdots, \eta_n$ and then require, for every $1 \leq j \leq n$, the next η_j-expansionary stage s' to have the length of agreement $l(\eta_j, s') > 2z_j + 1$ (this condition only delays the construction a little bit and makes no difference on the true outcome of each η_j). At a stage $s_1 > s_0$, $\Phi^A(x) \downarrow = 0$, α puts $\gamma(k)$ into B, puts x into D, declares that if $n \leq k$ enters K then $\gamma(n)$ is enumerated into B. At the next α-stage, α first puts z_n into F, creates a link $(\eta_n - \alpha)$ and wait for the next η_n-expansionary stage. If no η-expansionary stage appears, then η has outcome f and is satisfied. Otherwise, the next η-expansionary stage $s_2 > s_1$ comes, and since there is no small number enters A, W_e has already changed below $\psi(2z_n+1)$ which is enough to guarantee that all the Ω_{η_n}-computations which have been defined after stage s_0 can be redefined later. At this stage s_2, we go from η_n to α via the link $(\eta_n - \alpha)$, cancel this link, continue to put z_{n-1} into F, create a link $(\eta_{n-1} - \alpha)$ and wait for the next η_{n-1}-expansionary stage. This action can be repeated at most n times and eventually when we go to α from η_1 and cancel the link $(\eta_1 - \alpha)$, there is no link left and for each $\eta_j (1 \leq j \leq n)$, we have that W_e has changed below a number small enough to guarantee that the Ω_{η_j}-computations which have been defined after stage s_0 can be redefined later. Clearly, after making the diagonalization at stage s_1, α is satisfied unless some number smaller or equal to k enters K, and if this happens, α will be reset with a new witness and a new attached sequence.

4.2.2 The construction

We first set up the priority tree of strategies for P, Q, N-requirements as follows. Consider $\Lambda = \{i < f < s < w\}$ as the set of outcomes and $\Lambda^{<\omega}$ with the lexicographical ordering induced from the order in Λ. We effectively list all P, Q, N-requirements and assign to each node $\tau \in \Lambda^{<\omega}$ of length e the e-th requirement, for every $e \in \omega$. Here, the root λ is assigned to the requirement P_0. The priority tree of strategies T is a subtree of $\Lambda^{<\omega}$, where each node $\tau \in T$ is considered as a strategy of its corresponding requirement and τ has nodes $\tau\hat{\ }\langle o \rangle$ (with o ranging over all possible outcomes of τ) as its immediate successors.

For convenience, in the construction, we attach parameters to strategies. In particular, each P-strategy α has parameters $k(\alpha)$, $x(\alpha)$, $z(\alpha)$ corresponding to the killing point k, the witness x and the attached sequence $z = (z_1, \cdots, z_n)$, respectively. Similarly, a Q-strategy β has parameters $k(\beta)$, $x(\beta)$, $z(\beta)$. An N-strategy η will be attached with a parameter $m(\eta)$ for marking the length of agreements. If a strategy is initialized, we set its parameters as undefined and remove the corresponding declaration to put numbers into A (for a Q-strategy) or into B (for a P-strategy).

The construction is as follows.

Stage 0**.** All sets are empty and all strategies are initialized.

Stage $s+1$**.**

a. The S-strategy defines $\Gamma^{A \oplus B}$ as follows.

S1. If n enters K, find the highest priority P, Q-strategy α such that $k(\alpha)$ has been defined and $n \leq k(\alpha)$. If such α is found, initialize all strategies having lower priority than α and do the following.

– If α has been making declaration, put $\gamma(n)[s]$ (if it is defined) into B if α is a P-strategy and put $\gamma(n)[s]$ into A if α is a Q-strategy. Then undefine all $\Gamma^{A\oplus B}(y)\forall y \geq n$. Also put all η-attached number y_{n_η} of $\gamma(n)$ (if any) into F (where η is an N-strategy).
– Otherwise, reset $x(\alpha), z(\alpha)$ to be undefined. If $\gamma(n)[s]\downarrow$, put it into A and undefine all $\Gamma^{A\oplus B}(y)\forall y \geq n$.

If there is no P, Q-strategy α with killing point $k(\alpha) \geq n$, put $\gamma(n)[s]$ (if it has been defined) into A and undefine $\Gamma^{A\oplus B}(y)$, $\forall y \geq n$.

S2. Otherwise, for the least number p with $\Gamma^{A\oplus B}(p)\uparrow$, define $\Gamma^{A\oplus B}(p)[s+1] = K_{s+1}(p)$ with fresh use $\gamma(p)[s+1]$;

b. Working on the priority tree for P, Q, N-strategies.
Compute the current true path TP_{s+1} inductively: $\lambda = \tau_0$; if $\tau \preceq TP_{s+1}$ is of length $s+1$, let $\tau = TP_{s+1}$, initialise all strategies to the right of TP_{s+1} and go to stage $s+2$; otherwise, initialise all strategies to the right of τ and compute its immediate successor $\tau \prec \tau' \preceq TP_{s+1}$ as below. There are three cases.

Case 1. τ is a P_e-strategy α. Proceed as follows.

$\alpha 1$.
* If $k(\alpha)\uparrow$, set it as fresh.
* Go to $\alpha 2$.

$\alpha 2$. If $x(\alpha)\downarrow$ go to $\alpha 3$.
Otherwise,
* set $x(\alpha)\uparrow$ as fresh, and choose its attached sequence $z(\alpha) = (z_1, \cdots, z_n)$ corresponding to $\eta_1, \cdots, \eta_n$, which are the N-strategies with outcome i above α (where $\eta_1\hat{}\langle i\rangle \prec \cdots \prec \eta_n\hat{}\langle i\rangle \preceq \alpha$).
* update $m(\eta_j) = \max\{m(\eta), z_j + 1\}$ for each $\eta_j\hat{}\langle i\rangle \preceq \alpha$ $(1 \leq j \leq n)$.
* let $\alpha\hat{}\langle w\rangle = \tau' \preceq TP_{s+1}$ and initialise all lower priority strategies.

$\alpha 3$.
* If $\Phi_e^A(x(\alpha))[s+1]\downarrow= 0$, and $x(\alpha) \in D$, then
  · let $\alpha\hat{}\langle s\rangle = \tau' \preceq TP_{s+1}$,
  · if $z_n \notin F$ (where $z(\alpha) = (z_1, \cdots, z_n)$) and α has been declaring to enumerate γ-uses into B, then put z_n into F, create the link $(\eta_n - \alpha)$, initialise all lower priority strategies, and go to stage $s+2$.
* If $\Phi_e^A(x(\alpha))[s+1]\downarrow= 0$, and $x(\alpha) \notin D$, then put $\gamma(k(\alpha))$ into B, put x into D and declare that when $n \leq k(\alpha)$ enters K, $\gamma(n)$ is enumerated into B. Let $\alpha\hat{}\langle s\rangle = \tau' \preceq TP_{s+1}$, initialize all lower priority strategies, and go to stage $s+2$.

∗ Otherwise, let $\alpha\hat{\ }\langle w\rangle = \tau' \preceq TP_{s+1}$.

Case 2. τ is a Q_e-strategy β. Run β-program analogous to α with step $\alpha 3$ replaced by $\beta 3$. as follows.

$\alpha 3$.
∗ If $\Phi_e^B(x(\beta))[s+1]\downarrow = 0$, and $x(\beta) \in D$, let $\beta\hat{\ }\langle s\rangle = \tau' \preceq TP_{s+1}$.
  · If $z_n \notin F$ (where $z(\beta) = (z_1, \cdots, z_n)$) and β has been declaring to enumerate γ-uses into A, put z_n into F, create the link $(\eta_n - \beta)$, initialise all lower priority strategies and go to stage $s+2$.

∗ If $\Phi_e^B(x(\beta))[s+1]\downarrow = 0$, and $x(\beta) \notin D$, then put $\gamma(k(\beta))$ into A, put x into D and declare that when $n \le k(\beta)$ enters K, $\gamma(n)$ is enumerated into A. Let $\beta\hat{\ }\langle s\rangle = \tau' \preceq TP_{s+1}$, initialize all lower priority strategies, and go to stage $s+2$.

∗ Otherwise, let $\beta\hat{\ }\langle w\rangle = \tau' \preceq TP_{s+1}$.

Case 3. τ is an N_e-strategy η. Run the following N-program.

$\eta 1$. If $m(\eta)\downarrow$, go to $\eta 2$.
Else, set $m(\eta) := 0$ and go to $\eta 2$.

$\eta 2$. If $l(\eta, s+1) > 2m(\eta) + 1$,

∗ reset $m(\eta) := \left\lceil \dfrac{l(\eta, s+1)}{2} \right\rceil$,

∗ let $\eta\hat{\ }\langle i\rangle = \tau' \preceq TP_{s+1}$ and go to $\eta 3$.

Otherwise, let $\eta\hat{\ }\langle f\rangle = \tau' \preceq TP_{s+1}$.

$\eta 3$. If there is a link $(\eta - \alpha)$, where α is a P, Q-strategy with a witness $x(\alpha)$ and an attached sequence $(z_1, \cdots, z_n)$ of $x(\alpha)$ (where z_{j_0} corresponds to $\eta = \eta_{j_0}$, $1 \le j_0 \le n$), do the following:

∗ go to α, cancel the link $(\eta - \alpha)$, let $TP_{s+1} = \alpha$, initialise all strategies to the right of α and go to stage $s+2$;

∗ if $j_0 > 1$, put the next attached number z_{j_0-1} into F, create a new link $(\eta' - \alpha)$ (where $\eta' \prec \eta'\hat{\ }\langle i\rangle \prec \eta \prec \eta\hat{\ }\langle i\rangle \preceq \alpha$ and η' is an N-strategy corresponding to z_{j_0-1} in the attached sequence $z(\alpha)$ of $x(\alpha)$), go to stage $s+2$.

Otherwise, i.e. no lower P, Q-strategy links to η, do the following, distinguishing the two possible following subcases.

(Subcase 1) If there is $p \le m(\eta)$ with $\Omega_\eta^{W_e}(p)[t]\downarrow \ne K_{s+1}(p)$ (where $t \le s$ is the last η-stage at which we have defined $\Omega_\eta^{W_e}(p)[t]$), do the following.

(a) Choose the least such p.

(b) If $W_e \upharpoonright \omega(p)[t]$ has changed, undefine $\Omega_\eta^{W_e}(q)$, $\forall q \geq p$.

(Subcase 2) Otherwise, for the least $p \leq m(\eta)$ such that $\Omega_\eta^{W_e}(p)$ has not been defined yet, do the following.

(a) If there is $\gamma(n) \leq \psi(2p+1)[s+1]$ that has not been attached any number by η before, then attach to $\gamma(n)$ a fresh number y_{n_η} for every such $\gamma(n)$. Update $m(\eta) := \max\{m(\eta), y_{n_\eta} : \gamma(n) \leq \psi(2p+1)[s+1]\}$.

(b) Otherwise, every $\gamma(n) \leq \psi(2p+1)[s+1]$ has an η-attached number y_{n_η}, set $\Omega_\eta^{W_e}(p)[s+1] = K_{s+1}(p)$ with fresh use $\omega(p)[s+1]$.

4.2.3 Verification

Lemma 4.2. *For each node $\tau \prec TP$,*

(i) There is a stage s_0 such that τ is not initialised after s_0 and for every stage $t \geq s_0$, either $\tau \prec TP_t$ or $\tau < TP_t$. Clearly, no node to the left of τ is accessible after stage s_0.

(ii) The node τ is accessible at infinitely many stages.

(iii) If τ is a P, Q-strategy then there is a stage s_0 after that τ never puts numbers into A, B, D or F.

(iv) There is unique $o \in \Lambda$ (the true outcome of τ) such that $\tau\hat{\ }\langle o\rangle \prec TP$.

Proof. We prove the lemma by induction on the length of $\tau \prec TP$.

For $\tau = \lambda \prec TP$, clearly λ has the highest priority on the tree and $\lambda \preceq TP_s$ $\forall s \geq 0$, so λ is never initialised after stage 0 and $(i - ii)$ is satisfied.

We have that λ is a P_0-strategy with killing point k. So, it is injured only by the S-strategy at most $k+1$ times. Hence, it eventually reaches its true outcome o at a stage s_0, and never puts numbers into A, B, D or F, i.e. $\lambda\hat{\ }\langle o\rangle \prec TP_t$ for any $t > s_0$ and $(i - iv)$ hold for $\lambda\hat{\ }\langle o\rangle$.

Suppose inductively that the nodes $\lambda = \tau_0 \prec \tau_1 \prec \cdots \prec \tau_{n-1} \prec TP$ satisfy $(i - iv)$, where $\tau_{i+1} = \tau_i\hat{\ }\langle o_i\rangle$ with $o_i \in \Lambda$ is the true outcome of τ_i for any $0 \leq i \leq n-1$. Let $\tau_n = \tau_{n-1}\hat{\ }\langle o_{n-1}\rangle$. We will prove that τ_n also satisfies $(i - iv)$.

Let s be the first stage at which τ_n is accessible at s and after stage s,

(a) there is no number enumerated into A, B, D or F by a P, Q-strategy $\tau \prec \tau_n$;

(b) there is no accessible node to the left of τ_n.

There are three cases.

Case 1. τ_n is a P_e-strategy α. Clearly, α will never be initialised after stage s and hence, $k(\alpha)\downarrow= k(\alpha)[s^+]$, where s^+ is the next α-stage of s. Since $\alpha = \tau_{n-1}\hat{\ }\langle o_{n-1}\rangle \prec TP$, (i-ii) holds for α. In addition, α is only injured by S at most $k(\alpha)[s^+]+1$ times. Therefore, eventually at a stage $s' > s$, α has the true outcome o ($\alpha\hat{\ }\langle o\rangle \prec TP$), stays there forever and never puts numbers into B, D or F after stage s'.

Case 2. τ_n is a Q_e-strategy β. Similarly to case 1, β finally has the true outcome o, and satisfies (i-iv).

Case 3. τ_n is an N_e-strategy η. Then η is not initialised after stage s. We have:

- either η has true outcome f: $\eta\hat{\ }\langle f\rangle$ is accessible at stage $s_0 \geq s$ and for any stage $t > s_0$, $\eta\hat{\ }\langle f\rangle \prec TP_t$.
- or it reaches the true outcome i at infinitely many η-stages: let $t_0 \geq s$ be the first stage that $\eta\hat{\ }\langle i\rangle$ is accessible, we have that $\eta\hat{\ }\langle i\rangle \preceq TP_t$ for any η-expansionary stage $t \geq t_0$ and for any $t > t_0$ at which $\eta\hat{\ }\langle i\rangle$ is not accessible, we have that $\eta\hat{\ }\langle i\rangle < TP_t$.

□

Lemma 4.3. *For every $\tau \prec TP$, the requirement corresponding to τ is satisfied.*

Proof. Fix $\tau \prec TP$ of length e. From lemma 4.2, let $\tau\hat{\ }\langle o\rangle \prec TP$ and s_0 be the first stage such that

(a) $\tau\hat{\ }\langle o\rangle \preceq TP_{s_0}$,

(b) no strategy to the left of $\tau\hat{\ }\langle o\rangle$ is accessible after stage s_0,

(c) no P-strategy $\alpha \preceq \tau$ ever puts numbers into B, D or F during or after stage s_0,

(d) no Q-strategy $\beta \preceq \tau$ ever puts numbers into A, D or F during or after stage s_0.

There are two cases as follows.

Case 1. τ is a P-strategy α. Without loss of generality, suppose that α is a P_e-strategy. Clearly, α will never be initialised after stage s_0 and we have $k(\alpha)\downarrow= k(\alpha)[s_0]$, $x(\alpha)\downarrow= x(\alpha)[s_0]$. If $\alpha\hat{\ }\langle w\rangle \prec TP$, then $D(x(\alpha)) = 0 \neq \Phi_e^A(x(\alpha))$. If $\alpha\hat{\ }\langle s\rangle \prec TP$ then $D(x(\alpha)) = 1 \neq 0 = \Phi_e^A(x(\alpha))$. Thus, P_e is satisfied.

Case 2. τ is anN_e-strategy η. Suppose that $\Psi^{A\oplus W_e} = \Psi^{B\oplus W_e} = K \oplus F$. Then there are infinitely many η-expansionary stages and $o = i$. Note that s_0 is the first η-expansionary stage such that η will not be initialised during or after stage s_0. We will prove by induction on $p \in \omega$ that for every $p \in \omega, \Omega_\eta^{W_e}(p) \downarrow = K(p)$. Hence, $\Omega_\eta^{W_e}$ is total and computes K correctly.

For $p = 0$, if $\Omega_\eta^{W_e}(0) \uparrow$ at stage s_0, then from the construction and since $\Psi^{A\oplus W_e} = \Psi^{B\oplus W_e} = K \oplus F$, there is a stage $t \geq s_0$, such that every $\gamma(n) \leq \psi(1)[t]$ already has η-attached number y_{n_η}, and we define $\Omega_\eta^{W_e}(0)[t] = K_t(0)$ with the use $\omega(0)[t]$ as a fresh number. If 0 never enters K, this computation remains correct, i.e. $\Omega_\eta^{W_e}(0) \downarrow = \Omega_\eta^{W_e}(0)[t] = K_t(0) = K(0)$. Otherwise, if 0 enters K at stage $t' > t$, let $s_1 \geq t'$ be the next η-expansionary stage. We have

$$\Psi^{A\oplus W_e}(0)[t] = \Psi^{B\oplus W_e}(0)[t] = K_t \oplus F_t(0) = K_t(0) = 0, \tag{4.2}$$

and

$$\Psi^{A\oplus W_e}(0)[s_1] = \Psi^{B\oplus W_e}(0)[s_1] = K_{s_1} \oplus F_{s_1}(0) = K_{s_1}(0) = 1. \tag{4.3}$$

If no number less than or equal to $\psi(0)[t] \leq \psi(1)[t]$ is enumerated into A or B during stage t till stage s_1, then at stage s_1, equation 4.3 is achieved because W_e has changed below $\omega(0)[t]$, allowing $\Omega_\eta^{W_e}(0)[t]$ to be undefined. At an η-expansionary stage $s_2 > s_1$ such that every $\gamma(n) \leq \psi(1)[s_2]$ already has η-attached number y_{n_η}, we can rectify $\Omega_\eta^{W_e}(0)[s_2] = 1 = \Omega_\eta^{W_e}(0) \downarrow = K(0)$ with use $\omega(0)[s_2] = \omega(0)$ as fresh.

If there is a small number enumerated into A or B, without loss of generality, suppose that $y = \gamma(n) \leq \psi(1)[t]$ is the first number enumerated into A at stage s' between t and s_1. Let $s'_1 > s'$ be the next η-expansionary stage ($s'_1 \leq s_1$). Note that for any P, Q-strategy α to the right of $\eta\hat{}\langle i\rangle$, if α is visited after stage t, then $k(\alpha) > \omega(0)[t]$. From the construction, if α makes a diagonalization after stage t and at a stage $t_1 > t$, a number l enters K with $\gamma(l) \leq \psi(1)[t]$, then α will be reset, and its declaration is removed. So the P, Q-strategies to the right of $\eta\hat{}\langle i\rangle$ never change the side (either A or B) into which a small use $\gamma(l) \leq \psi(1)[t]$ is enumerated if l enters K.

If n enters K and $y = \gamma(n)$ is enumerated into A by S, then from the construction, S also puts y_{n_η} into F at stage s' (note that $\gamma(n) \leq \psi(1)[t]$ and hence, it has η-attached number y_{n_η} before stage t). So we have

$$\begin{aligned} \Psi^{A\oplus W_e}(2y_{n_\eta}+1)[t] &= \Psi^{B\oplus W_e}(2y_{n_\eta}+1)[t] \\ &= K_t \oplus F_t(2y_{n_\eta}+1) = F_t(y_{n_\eta}) = 0, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\Psi^{A\oplus W_e}(2y_{n_\eta}+1)[s_1'] &= \Psi^{B\oplus W_e}(2y_{n_\eta}+1)[s_1'] \\ &= K_{s_1'}\oplus F_{s_1'}(2y_{n_\eta}+1) = F_{s_1'}(y_{n_\eta}) = 1,\end{aligned} \tag{4.5}$$

From the construction, between stages s' and $s_1' \leq s_1$, there is no small number enumerated into B. Hence, at stage s_1', 4.5 is obtained just because W_e has changed below $\psi(2y_{n_\eta}+1)[t] < \omega(p)[t]$, allowing $\Omega_\eta^{W_e}(p)$ to be rectified later.

If $y = \gamma(n)$ is enumerated into A by a lower priority Q-strategy $\beta \succeq \eta\hat{}\langle i\rangle$ to make its diagonalization, then Q declares from stage s' onward that $\gamma(m)$ is enumerated into A if $m \leq k(\beta) = n$ enters K. Then from stage s' to stage s_1, no number smaller than or equal to $\psi(2p+1)[t]$ enters A. Hence, we also have that W_e has already changed below $\omega(p)[t]$, and the computation $\Omega_\eta^{W_e}$ can eventually be rectified correctly.

Suppose that $\Omega_\eta^{W_e}$ correctly computes K up to $p-1 \geq 0$, i.e. there is a stage $s \geq s_0$ such that

$$\Omega_\eta^{W_e}(q)\downarrow = \Omega_\eta^{W_e}(q)[s]\downarrow = K_s(q) = K(q) \forall q \leq p-1.$$

Let $s_1 \geq s_0$ be the smallest such stage. From the construction, we have that at the end of stage s_1, $\Omega_\eta^{W_e}(p)\uparrow$. Since $\Psi^{A\oplus W_e} = \Psi^{B\oplus W_e} = K\oplus F$, there is an η-expansionary stage $s_2 > s_1$ such that every $\gamma(n) \leq \psi(2p+1)[s_2]$ has already an η-attached number y_{n_η}. We then define $\Omega_\eta^{W_e}(p)[s_2] = K_{s_2}(p)$ with the use $\omega(p)[s_2]$ as fresh at stage s_2.

If p never enters K after stage s_2, this computation is correct, i.e., $\Omega_\eta^{W_e}(p)\downarrow = \Omega_\eta^{W_e}(p)[s_2] = K_{s_2}(p) = K(p)$. Otherwise, suppose that p enters K at stage $s' > s_2$ and $s_3 \geq s'$ is the next η-expansionary stage. We will prove that W_e has changed below $\omega(p)[s_2]$ at a stage s, where $s_2 < s \leq s_3$, allowing $\Omega_\eta^{W_e}(p)$ to be undefined at stage s_3. Hence, at an η-expansionary stage $s_4 > s_3$ such that every $\gamma(n) \leq \psi(2p+1)[s_4]$ has $\eta-$attached number already, we then redefine $\Omega_\eta^{W_e}(p)[s_4] = 1 = K(p)$ and this computation remains correct forever. Indeed, we have

$$\Psi^{A\oplus W_e}(2p)[s_2] = \Psi^{B\oplus W_e}(2p)[s_2] = K_{s_2}\oplus F_{s_2}(2p) = K_{s_2}(p) = 0, \tag{4.6}$$

and

$$\Psi^{A\oplus W_e}(2p)[s_3] = \Psi^{B\oplus W_e}(2p)[s_3] = K_{s_3}\oplus F_{s_3}(2p) = K_{s_3}(p) = 1. \tag{4.7}$$

If no number below $\psi(2p+1)[s_2]$ enters A or B during stages between s_2 and s_3, clearly, equation (4.7) yields that at stage s_3, W_e has changed below $\psi(2p)[s_2] < \omega(p)[s_2]$. Otherwise, without loss of generality, let t ($s_2 < t < s_3$) be the first stage at which a small number $y = \gamma(n)$ is enumerated into A and let $t' > t$ be the next η-expansionary stage.

If n enters K with $\gamma(n) \leq \psi(2p+1)[s_2]$ and the S strategy puts $\gamma(n)$ into A at stage t, then we also put the η-attached number y_{n_η} of $\gamma(n)$ into F. We have that

$$\begin{aligned} \Psi^{A\oplus W_e}(2y_{n_\eta}+1)[s_2] &= \Psi^{B\oplus W_e}(2y_{n_\eta}+1)[s_2] \\ &= K_{s_2} \oplus F_{s_2}(2y_{n_\eta}+1) = F_{s_2}(y_{n_\eta}) = 0, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \Psi^{A\oplus W_e}(2y_{n_\eta}+1)[t'] &= \Psi^{B\oplus W_e}(2y_{n_\eta}+1)[t'] \\ &= K_{t'} \oplus F_{t'}(2y_{n_\eta}+1) = F_{t'}(y_{n_\eta}) = 1. \end{aligned} \tag{4.9}$$

From the construction, since no small number is enumerated into B at stages between s_2 to t' yet, we have that 4.9 is achieved at stage t' due to a change of W_e below $\psi(2y_{n_\eta}+1)[s_2] < \omega(p)[s_2]$.

If $\gamma(n)$ is enumerated into A by a Q-strategy $\beta \succeq \eta\hat{\ }\langle i\rangle$ at stage t, suppose that $\eta_1\hat{\ }\langle i\rangle \prec \cdots \prec \eta_n\hat{\ }\langle i\rangle \preceq \beta$ are all N-strategies with outcome i above β and $\eta = \eta_{j_0}$, $1 \leq j_0 \leq n$. From the construction, $x(\beta)$ with the attached sequence $z(\beta) = (z_1, \cdots, z_n)$ was chosen before stage s_2. When β puts $\gamma(n)$ into A at stage t, it also makes a declaration on putting numbers into A. If the link $(\eta - \beta)$ is not created between stage t and s_3, by β-declaration, no small number is enumerated into B between stages t and s_3, and hence, if there is no small number enumerated into B (by the S strategy) between stage s_2, and t, we have that at stage s_3, W_e has already changed below $\psi(2p+1)[s_2]$. If there is a small use $\gamma(l)$ enumerated into B (by S) during stages s_2 till t, then with the same argument as for the case of $\gamma(n)$ above, we have that at stage t, W_e has already changed below $\psi(2y_{l_\eta}+1)[s_2]$ (where y_{l_η} is the η-attached number of $\gamma(l)$).

If the link $(\eta - \beta)$ is created at a stage t'' between t and s_3, then at stage t'', β puts z_{j_0} into F. Again, at the next η-expansionary stage t_1 ($t'' < t_1 \leq s_3$), W_e has already changed below $\psi(2z_{j_0}+1)[s_2] < \omega(p)[s_2]$.

Thus, the requirement N_e is satisfied.

□

Lemma 4.4. *Requirement S is satisfied.*

Proof. We prove by induction on n that $\Gamma^{A\oplus B}(n)\downarrow = K(n) \forall n \in \omega$.

For $n = 0$, at the beginning of stage 1, we define $\Gamma^{A\oplus B}(0)[1]$ and $\gamma(0)[1]$. After that, all killing points are chosen as fresh large numbers and will not affect $\Gamma^{A\oplus B}(0)[1]$. Therefore, the computation $\Gamma^{A\oplus B}(0)[1]$ remains unchanged and computes $K(0)$ correctly unless 0 enters K at a stage $s > 1$. If the latter happens, then $\gamma(0)[1]$ is enumerated into A or B and $\Gamma^{A\oplus B}(0)$ is undefined at the beginning of stage s. At stage $s+1$, $\Gamma^{A\oplus B}(0)$ will be redefined correctly forever, i.e. $\Gamma^{A\oplus B}(0)\downarrow = \Gamma^{A\oplus B}(0)[s+1] = K(0)$.

Suppose that $\Gamma^{A\oplus B}(m)\downarrow = K(m) \forall 0 \leq m < n$. Let s be the first stage at which $\Gamma^{A\oplus B}(m)\downarrow = \Gamma^{A\oplus B}(m)[s]\downarrow = K_s(m) = K(m)$, $\forall m < n$. From the construction, at the end of stage s, we have $\Gamma^{A\oplus B}(n)\uparrow$. At the beginning of stage $s+1$, we define $\Gamma^{A\oplus B}(n)[s+1] = K_{s+1}(n)$ with $\gamma(n)[s+1]$ as a fresh number. We have that no $\gamma(m)[s]$ (for any $m < n$) will be put into A or B after stage s. From the construction, $\gamma(n)[s+1]$ is enumerated into A or B at a stage $t > s+1$ only if $n \in K_t \setminus K_{t-1}$. In addition, only the P, Q-strategy which takes its killing point $k = n$, may put $\gamma(n)$ into B or A, and this action occurs at most two times. Hence, there is a stage $s' \geq s+1$ such that $\gamma(n)$ is not enumerated into B by a P-strategy (or into A by a Q-strategy) after s'. If at stage $s'+1$, $\Gamma^{A\oplus B}(n)\uparrow$, then in step S2 we define $\Gamma^{A\oplus B}(n)[s'+1] = K_{s'+1}(n)$. So, without loss of generality, we can suppose that $\Gamma^{A\oplus B}(n)[s'+1]\downarrow = K_{s'+1}(n)$ and there is no P, Q-strategy which puts $\gamma(n)[s'+1]$ into B or A after stage $s'+1$. Then the computation $\Gamma A \oplus B(n)[s'+1]$ remains unchanged unless n enters K at a stage $s'' > s'+1$. If the latter happens, $\Gamma^{A\oplus B}(n)$ will be rectified at stage $s''+1$ and will be correct forever. Thus, $\Gamma^{A\oplus B}(n)\downarrow = K(n)$. □

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