Newton-ADE FDTD Method for Time-Varying Plasma

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***Abstract*—A novel finite-difference time-domain (FDTD) method is proposed in this paper for electromagnetic (EM) wave propagation in time-varying plasma. It is formulated based on the fundamental Newton's equation of motion, which governs the relationships among the time-varying electron density, current density and electric field. Utilizing the auxiliary differential equation (ADE) FDTD scheme, the method is aptly referred to as the Newton-ADE FDTD method for time-varying plasma. Traditionally, the previous ADE FDTD methods directly apply the time-varying electron density for the plasma frequency in the update equations of current density and electric field. This is inadequate and incorrect in general time-varying plasma conditions. The formulation of the Newton-ADE FDTD method is provided and compared to that of the traditional ADE FDTD method. It is found that the key difference lies in the new compact term being the time-derivative of logarithm of electron density. Two discretization schemes for this term are provided. The Newton-ADE FDTD method is validated based on the matrix exponential method. Novel stability and convergence analyses are provided for the proposed method in time-varying dispersive media. These analyses show that our proposed method is stable and achieves second-order temporal accuracy. Numerical results are presented for various time-varying plasma conditions. It is demonstrated that when the absolute value for time-derivative of logarithm of electron density is large, e.g., greater than the collision frequency, the Newton-ADE FDTD method can provide correct results, while the traditional ADE FDTD method yields significant differences.**

***Index Terms*—Dispersive media, FDTD method, Newton's equation of motion, time-varying plasma.**

I. INTRODUCTION

FINITE-difference time domain (FDTD) method [1] is widely used for solving various electromagnetic (EM) waves propagation problems. For wave propagation in dispersive media, there have been numerous formulations of FDTD methods as well. These method include the recursive convolution (RC) method [2]–[5], the auxiliary differential equation (ADE) method [5]–[7], frequency-dependent Z transform method [8]–[10], matrix exponential method [11], JE convolution (JEC) method [12], [13], current density recursive convolution (CDRC) method [14], piecewise linear JE recursive convolution (PLJERC) [15], etc.

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Among the dispersive media, analyzing propagation of EM waves in static (non time-varying) plasma using FDTD method has drawn much attention [16]–[19]. However, in some practical applications, the plasma is not always static, but exhibits time-varying characteristics such as in time-varying plasma sheath [20]–[22]. Time-varying plasma sheath is produced by turbulence, which is caused by complex thermo-chemical reactions and flight dynamics. The time-varying plasma sheath will lead to rising bit error rate in communication system [23] and abnormal radar detection [24], etc. These important problems call for specific FDTD method to compute wave propagation in time-varying plasma sheath. In addition, the plasma produced by the amplitude modulated radio frequency plasma generator for EM analysis and measurement is generally time-varying [25]. Therefore, it is useful to develop the computational method for EM wave propagation in time-varying plasma. Many previous works adopt the FDTD method to solve this problem [26]–[28]. In particular, [26] adopted the FDTD method to analyze the electromagnetic scattering characteristics of blunt vehicle covered by two-dimensional time-varying plasma. In [27], JEC FDTD is used to study the radar cross section (RCS) of aircraft covered in 3D time-varying and spatially nonuniform plasma. Reference [28] studied the channel characterization for time-varying plasma sheath surrounding hypersonic vehicles using FDTD method.

In the previous FDTD methods for time-varying plasma, the treatment of time-varying plasma is by directly applying the time-varying electron density for the plasma frequency in the update equations of current density and electric field. As to be shown later, this is inadequate and incorrect under general time-varying plasma conditions. In fact, the relationships among the time-varying electron density, current density and electric field are governed by the fundamental Newton's equation of motion. Therefore, FDTD formulations for time-varying plasma should be derived based on such equation from the first principle. In this paper, we propose a novel ADE FDTD method for time-varying plasma based on the fundamental Newton's equation of motion. This method is aptly referred to as the Newton-ADE FDTD method. Its formulation is provided and compared to the traditional ADE FDTD method. It is found that the key difference lies in the new significant term being the time-derivative of logarithm of electron density. The Newton-ADE FDTD method is validated based on the matrix exponential method. The CPU time and memory cost of Newton-ADE FDTD method are further analyzed. Moreover, numerical results are presented for various time-varying plasma.

The present paper is organized as follows. Section II presents the formulations of Newton-ADE FDTD method and its key differences compared to the traditional ADE FDTD method. In Section III, the Newton-ADE FDTD method is validated through the comparison of simulation result of Newton-ADE FDTD method, matrix exponential method and traditional ADE FDTD method. In addition, numerical results under different scenes will be presented. Transmitted and reflected waves for time-varying plasma slab and RCS for metal cylinder covered by time-varying plasma will be considered under various time-varying plasma conditions. Section IV provides the conclusions and highlights the main contributions of this work.

II. Formulations of Newton-ADE FDTD for Time-varying Plasma

A. *Newton's Equation for Time-varying Plasma*

The electron density may exhibit time-varying characteristics in general cases. For time-varying plasma, the current density **J** is

$$\mathbf{J} = -eN_e(t)\mathbf{u}_e \,, \tag{1}$$

where $N_e(t)$ is the time-varying plasma electron density, e is electron charge and $\mathbf{u}_e$ is the electron speed. [Note that for brevity, the argument (t) has been omitted for **J** and $\mathbf{u}_e$ that are well understood to be time-varying like **E**, even when the electron density is constant.] Equation (1) can be rewritten as

$$\mathbf{u}_e = \frac{\mathbf{J}}{-eN_e(t)} \,, \tag{2}$$

The fundamental Newton's equation of motion that describes the electron movement can be expressed as

$$m_e \frac{d\mathbf{u}_e}{dt} = -e\mathbf{E} - m_e \nu \mathbf{u}_e \,, \tag{3}$$

where m_e is the electron mass and ν is the collision frequency (which may be time-varying also, but with its argument omitted for simplicity). Substituting (2) into (3), we obtain

$$m_e \frac{\frac{\partial \mathbf{J}}{\partial t} A(t) - \mathbf{J}\frac{\partial A(t)}{\partial t}}{A(t)^2} = -e\mathbf{E} - m_e \nu \frac{\mathbf{J}}{A(t)} \,, \tag{4}$$

where $A(t) = -eN_e(t)$. By simplifying (4), one can derive the update equation of current density **J** for time-varying plasma that incorporates the Newton's equation directly. Combining with the Maxwell's equations, the governing equations for wave propagation in time-varying isotropic plasma can be written as

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \,, \tag{5}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \,, \tag{6}$$

$$\frac{\partial \mathbf{J}}{\partial t} = \left(\frac{1}{N_e(t)} \frac{\partial N_e(t)}{\partial t} - \nu \right) \mathbf{J} + \frac{e^2}{m_e} N_e(t)\mathbf{E} \,, \tag{7}$$

B. *Formulation of Update Equation*

Since the equation of current density **J** is derived from the Newton's equation of motion for time-varying plasma, the corresponding FDTD method based on (5)–(7) is referred to as the Newton-ADE FDTD method. The term $(1/N_e(t))\cdot(\partial N_e(t)/\partial t)$ in equation (7) is a critical term that should be handled carefully during the discretization. There are two schemes to discretize this term as provided below:

(i) Finite-difference (FD) with time-average.

The term $(1/N_e(t))\cdot(\partial N_e(t)/\partial t)$ can be directly discretized using central difference and time-average as

$$\frac{1}{\frac{N_e^{n+1} + N_e^n}{2}} \frac{N_e^{n+1} - N_e^n}{\Delta t} \,, \tag{8}$$

(ii) Finite-difference (FD) of $\ln(N_e)$

The term $(1/N_e(t))\cdot(\partial N_e(t)/\partial t)$ is first expressed as

$$\frac{1}{N_e(t)} \frac{\partial N_e(t)}{\partial t} = \frac{\partial \ln(N_e(t))}{\partial t} \,, \tag{9}$$

Defining L as the logarithm of electron density:

$$L(t) = \ln(N_e(t)) \,, \tag{10}$$

equation (9) can be discretized as

$$\frac{L^{n+1} - L^n}{\Delta t} \,, \tag{11}$$

The following derivation utilizes scheme (ii) as an illustration. Equation (7) is discretized using semi-implicit scheme given by

$$\frac{\mathbf{J}^{n+1} - \mathbf{J}^n}{\Delta t} = \frac{\mathbf{J}^{n+1} + \mathbf{J}^n}{2} \cdot \left(\frac{L^{n+1} - L^n}{\Delta t} - \nu \right) + \frac{e^2}{m_e} \frac{(N_e^{n+1} + N_e^{n})}{2} \frac{(\mathbf{E}^{n+1} + \mathbf{E}^n)}{2} \tag{12}$$

The current density **J** can then be derived as

$$\begin{aligned} J_x\Big|_{i+1/2,j,k}^{n+1} = & \frac{1}{1 - 0.5\left(L\Big|_{i+1/2,j,k}^{n+1} - L\Big|_{i+1/2,j,k}^{n} - \nu\Delta t \right)} \\ & \cdot \left\{ J_x\Big|_{i+1/2,j,k}^{n} \left[1 + 0.5\left(L\Big|_{i+1/2,j,k}^{n+1} - L\Big|_{i+1/2,j,k}^{n} - \nu\Delta t \right) \right] \right. \\ & \left. + \Delta t \frac{e^2}{m_e} \frac{N_e\Big|_{i+1/2,j,k}^{n+1} + N_e\Big|_{i+1/2,j,k}^{n}}{2} \frac{E_x\Big|_{i+1/2,j,k}^{n+1} + E_x\Big|_{i+1/2,j,k}^{n}}{2} \right\} \end{aligned} \tag{13}$$

$$\begin{aligned} J_y\Big|_{i,j+1/2,k}^{n+1} = & \frac{1}{1 - 0.5\left(L\Big|_{i,j+1/2,k}^{n+1} - L\Big|_{i,j+1/2,k}^{n} - \nu\Delta t \right)} \\ & \cdot \left\{ J_y\Big|_{i,j+1/2,k}^{n} \left[1 + 0.5\left(L\Big|_{i,j+1/2,k}^{n+1} - L\Big|_{i,j+1/2,k}^{n} - \nu\Delta t \right) \right] \right. \\ & \left. + \Delta t \frac{e^2}{m_e} \frac{N_e\Big|_{i,j+1/2,k}^{n+1} + N_e\Big|_{i,j+1/2,k}^{n}}{2} \frac{E_y\Big|_{i,j+1/2,k}^{n+1} + E_y\Big|_{i,j+1/2,k}^{n}}{2} \right\} \end{aligned} \tag{14}$$

$$\begin{aligned} J_z\Big|_{i,j,k+1/2}^{n+1} = & \frac{1}{1 - 0.5\left(L\Big|_{i,j,k+1/2}^{n+1} - L\Big|_{i,j,k+1/2}^{n} - \nu\Delta t \right)} \\ & \cdot \left\{ J_z\Big|_{i,j,k+1/2}^{n} \left[1 + 0.5\left(L\Big|_{i,j,k+1/2}^{n+1} - L\Big|_{i,j,k+1/2}^{n} - \nu\Delta t \right) \right] \right. \\ & \left. + \Delta t \frac{e^2}{m_e} \frac{N_e\Big|_{i,j,k+1/2}^{n+1} + N_e\Big|_{i,j,k+1/2}^{n}}{2} \frac{E_z\Big|_{i,j,k+1/2}^{n+1} + E_z\Big|_{i,j,k+1/2}^{n}}{2} \right\} \end{aligned} \tag{15}$$

Subsequently, (5) can be solved to obtain E_x^{n+1}. A semi-implicit scheme centered at n + 1/2 is also used. This requires the knowledge of $\mathbf{J}^{n+1/2}$ that can be expressed as

$$\mathbf{J}^{n+1/2} = \frac{\mathbf{J}^{n+1} + \mathbf{J}^{n}}{2}, \tag{16}$$

The update equations of E_x are derived from (5) as

$$E_x\Big|_{i+1/2,j,k}^{n+1} = E_x\Big|_{i+1/2,j,k}^{n} + \frac{\Delta t}{\varepsilon_0}\left(\frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\Big|_{i+1/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z}\right) - \frac{\Delta t}{\varepsilon_0}\frac{J_x\Big|_{i+1/2,j,k}^{n+1} + J_x\Big|_{i+1/2,j,k}^{n}}{2} \tag{17}$$

By applying (13) into (17), respectively, the update equations of E_x in time-varying plasma read

$$E_x\Big|_{i+1/2,j,k}^{n+1} = \left[1 + \frac{(\Delta t)^2 e^2}{8\varepsilon_0 m_e}\cdot\frac{N_e\Big|_{i+1/2,j,k}^{n+1} + N_e\Big|_{i+1/2,j,k}^{n}}{1 - 0.5(L\Big|_{i+1/2,j,k}^{n+1} - L\Big|_{i+1/2,j,k}^{n} - \nu\Delta t)}\right]^{-1} \cdot\left\{\left[1 - \frac{(\Delta t)^2 e^2}{8\varepsilon_0 m_e}\cdot\frac{N_e\Big|_{i+1/2,j,k}^{n+1} + N_e\Big|_{i+1/2,j,k}^{n}}{1 - 0.5(L\Big|_{i+1/2,j,k}^{n+1} - L\Big|_{i+1/2,j,k}^{n} - \nu\Delta t)}\right]\cdot E_x\Big|_{i+1/2,j,k}^{n} + \frac{\Delta t}{\varepsilon_0}\left(\frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\Big|_{i+1/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z}\right) - \frac{\Delta t}{\varepsilon_0}\left[\frac{1}{1 - 0.5(L\Big|_{i+1/2,j,k}^{n+1} - L\Big|_{i+1/2,j,k}^{n} - \nu\Delta t)}\right] J_x\Big|_{i+1/2,j,k}^{n}\right\} \tag{18}$$

E_y and E_z have the similar derivation by permuting the indices in (18).

Finally, the update equations of **H** can be written as

$$H_x\Big|_{i,j+1/2,k+1/2}^{n+1/2} = H_x\Big|_{i,j+1/2,k+1/2}^{n-1/2} - \frac{\Delta t}{\mu_0}\left(\frac{E_z\Big|_{i,j+1,k+1/2}^{n} - E_z\Big|_{i,j,k+1/2}^{n}}{\Delta y} - \frac{E_y\Big|_{i,j+1/2,k+1}^{n} - E_y\Big|_{i,j+1/2,k}^{n}}{\Delta z}\right) \tag{19}$$

H_y and H_z can be similarly derived as the same manner.

C. *Comparison with the Traditional Method*

For comparison, the update equations of current density **J** for the traditional ADE FDTD method [26]–[28] in time-varying plasma is discussed in the following. The relationship between current density **J** and electric field **E** in frequency domain is

$$\mathbf{J}(\omega) = \sigma(\omega)\mathbf{E}(\omega) = \frac{\varepsilon_0\omega_p^2(\omega)}{j\omega + \nu}\mathbf{E}(\omega), \tag{20}$$

Here, **J**, **E**, σ and ω_p are the parameters in frequency domain, where σ is the conductivity and ω_p is the plasma frequency. In the usual ADE FDTD method, (20) is converted to time domain as

$$\frac{\partial \mathbf{J}}{\partial t} = -\nu\mathbf{J} + \varepsilon_0\omega_p^2\mathbf{E}, \tag{21}$$

The traditional ADE FDTD method takes ω_p a function of time and the above equation can be rewritten as

$$\frac{\partial \mathbf{J}}{\partial t} = -\nu\mathbf{J} + \varepsilon_0\omega_p^2(t)\mathbf{E}, \tag{22}$$

where $\omega_p(t)$ refers to the plasma frequency as

$$\omega_p(t) = \sqrt{\frac{N_e(t)e^2}{\varepsilon_0 m_e}}, \tag{23}$$

By applying (23), (22) can be written as

$$\frac{\partial \mathbf{J}}{\partial t} = -\nu\mathbf{J} + \frac{e^2}{m_e}N_e(t)\mathbf{E}, \tag{24}$$

Equations (20)-(24) are the derivation of traditional ADE FDTD method. Note that the traditional FDTD methods shown above are essentially taking the Fourier-transformed quantities (e.g., ω_p, N_e) as functions of time. This procedure is in fact incorrect, since one should take Fourier transforms for all equations and quantities in time from the outset, instead of taking the Fourier transforms for only partial equations and quantities (treating ω_p and N_e initially as time-independent), and then varying subsequently the transformed quantities with time [$\omega_p(t)$, $N_e(t)$]. Therefore, the correct discretization is to go back to the fundamental time-domain equations from the first principle, which leads to Newton-ADE FDTD method proposed here. In addition, it is worth noting that Z-Transform, RC and other methods, which also start from frequency domain methods, cannot be used directly under time-varying plasma sheath scene, either. For example, in the Z-transform method, applying the bilinear transformation relation

$$j\omega \to \frac{2}{\Delta t}\frac{1 - z^{-1}}{1 + z^{-1}}, \tag{25}$$

the constitutive relation (20) can be written in z domain as

$$J(z) = \frac{\varepsilon_0\omega_p^2}{\frac{2}{\Delta t}\frac{1 - z^{-1}}{1 + z^{-1}} + V_e}E(z), \tag{26}$$

Further utilizing $z^{-m}J(z) \leftrightarrow J^{(n-m)}$ and upon some manipulations, (26) will lead to the same update equation of **J** in discrete time domain as the traditional ADE-FDTD method. Hence, the results of Z-transform method for computing wave propagating through time varying plasma are the same as the traditional ADE-FDTD method, which are incorrect for both methods.

Comparing our Newton-ADE FDTD and the traditional ADE FDTD methods, one can see that the coefficients of **J** on the right-hand side of (7) is $(1/N_e(t))\cdot(\partial N_e(t)/\partial t)-\nu$, while the coefficients of **J** on the right-hand side of (24) is $-\nu$. The term $(1/N_e(t))\cdot(\partial N_e(t)/\partial t)$ is the key difference between these two methods. Based on (9) and (10), we denote this term as the time-derivative of $\ln(N_e)$. In certain time-varying plasma conditions, the simulation results of the proposed Newton-ADE FDTD method and the traditional ADE FDTD method would yield significant differences, which will be discussed in the following.

III. Numerical Results and Discussions

A. *Validation of the Newton-ADE FDTD Method*

Here we consider the EM waves propagation through 1D sinusoidally varying plasma slab to validate our Newton-ADE FDTD method. The simulation model is shown in Fig. 1 and the plasma slab is 10-cm-thick. To minimize the impact of reflection from the domain boundary, their locations are set to be sufficiently far away from the plasma slab. The spatial step Δ is 0.005 m and the time step Δt is 1.67×10^{-11} s. The Courant number is 1 in the simulation. The source excitation is a Gaussian pulse expressed in the time domain as

$$f(t)=\exp\left[-\frac{4\pi(t-t_0)^2}{\tau^2}\right], \qquad (27)$$

where $\tau = 1\times10^{-9}$ s and $t_0 = 3\times10^{-9}$ s.

The sinusoidally varying electron density is given by

$$N_e(t)=N_{e,avg}\left[1+\Delta_{N_e}\cdot\sin(2\pi f_0 t)\right], \qquad (28)$$

where $N_{e,avg}$ is the average electron density, Δ_{Ne} is the electron density variation ratio, f_0 is the variation center frequency: $N_{e,avg}$ is 3×10^{16} m^{-3}, Δ_{Ne} is 0.3, f_0 is 600 MHz and collision frequency is $2\pi\times80$ Mrad/s. The collision frequency can also be a time-varying parameter if desired but it is fixed for simpler comparison in this study.

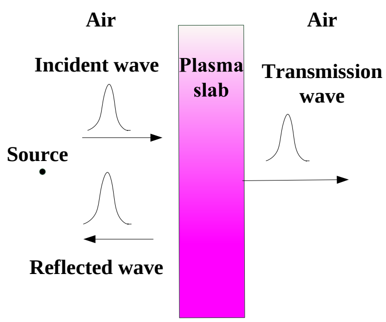


Fig. 1. Schematic of Gaussian pulse propagation through time-varying plasma slab

The validation of the Newton-ADE FDTD method is ascertained independently using the matrix exponential method. The matrix exponential is a general mathematical method applicable for any set of first order differential equations. It is formulated here with the same spatial discretization in the Yee's cell. An initial condition due to the incident Gaussian source in (27) is initialized within the spatial domain. The corresponding simulation results using the Newton-ADE FDTD method, the traditional ADE FDTD method (Ref.[26]–[28]) and the matrix exponential method are shown in Fig. 2. The results of Newton-ADE FDTD and matrix exponential methods are in good agreement, while those of the traditional ADE FDTD method show significant differences. Therefore, Newton-ADE FDTD method is more accurate and credible than the traditional ADE FDTD method.

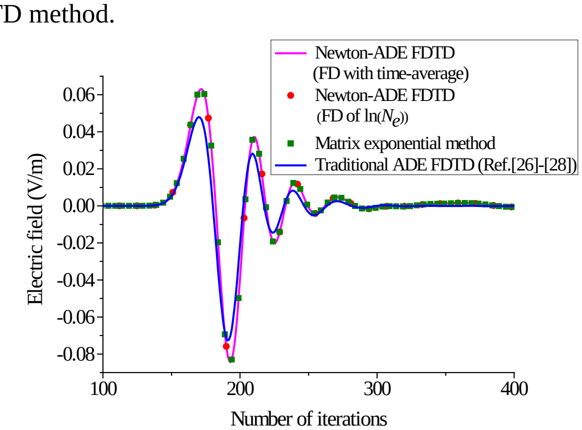


Fig. 2 .Comparison of simulation result using different methods

B. *Stability and Convergence Analyses*

To ascertain the stability, we apply the von Neumann method and compute the eigenvalues of the overall amplification matrix. The overall amplification matrix, M is computed by cascading the amplification matrix leftward at each time index, n up to one period of the sinusoidally varying electron density via

$$M = M_{n_p}\cdots M_2\cdot M_1 . \qquad (29)$$

Here, M_n is the amplification matrix at time index n, and n_p is the time index at one period of the sinusoidally varying electron density. Fig. 3 shows the eigenvalues plot of the overall amplification matrix using random samples of spatial modes. The Courant number is 1. One can see that all eigenvalues are within the semi unit circle, which indicates that our method is stable as long as the time step is within the Courant stability limit of the explicit FDTD method. Also, we have increased the time domain simulation duration to cover large number of iterations, e.g., 10000 and beyond. The corresponding simulation results are shown in Fig. 4. It can be found that using the Newton-ADE FDTD method, the simulation will not become unstable.

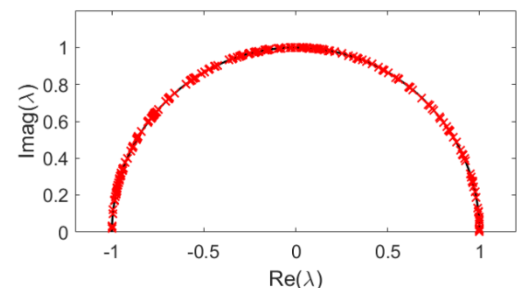


Fig. 3. Eigenvalues plot of the Newton-ADE method.

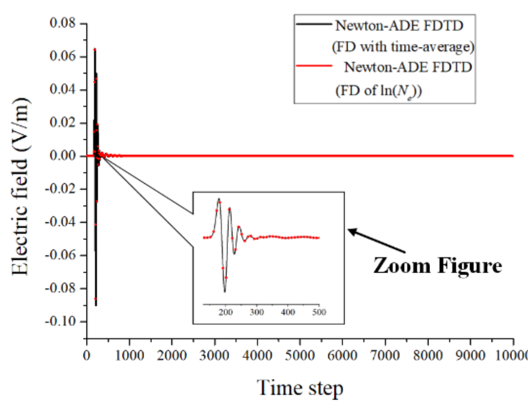


Fig. 4. The simulation of transmitted wave for sinusoidally varying electron density with f_0 = 600 MHz when number of iterations are 10000

For convergence analysis, we consider the same plasma slab within the free space with perfect electric conductor (PEC) boundaries. The normalized norm error is computed at decreasing time step specified through the geometric time subdivision q, where $\Delta t = \Delta t_{CFL}/2^q$ and Δt_{CFL} is the time step of the Courant stability limit of the explicit FDTD method. To compute the error, the numerical results of the Newton-ADE FDTD and traditional ADE-FDTD are subtracted from the temporally exact solution of the matrix exponential with the same spatial discretization. The overall matrix exponential is obtained by cascading the matrix exponential leftward at each time index. The norm-2 error is then computed and normalized with the norm-2 of the exact solutions. Fig. 5 shows the normalized norm error versus the geometric time subdivision q of both methods. It can be seen that the Newton-ADE FDTD method exhibits a temporal order of two. On the other hand, the traditional ADE-FDTD has generally high errors at all q, and the errors generally do not converge. This again ascertains the correctness and accuracy of our proposed Newton-ADE FDTD method over the traditional ADE-FDTD method.

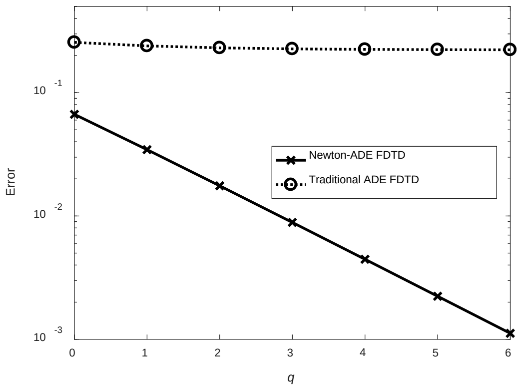


Fig. 5 Normalized norm error versus the geometric time subdivision q, where $\Delta t = \Delta t_{CFL}/2^q$

C. *Execution Time and Memory Cost*

Next, the execution time and memory cost are compared between the traditional and Newton-ADE FDTD methods. The simulation geometry is the 3D time-varying plasma terminated by PEC. The spatial steps are $\Delta x=\Delta y=\Delta z=0.0075$m, and the number of grids in the simulation domain is 100×100×100. The time step Δt is 1.43×10^{-11} s and the total number of iterations is 2000. As shown in Table I, the two methods require the same amount of memory. There are 3 variables for each magnetic field H and current density J in 3D simulation. For electric field E, because semi-implicit FDTD method is adopted here, not only the values of E_x, E_y and E_z at the current time are needed, but also those in the previous time step are to be recorded. Both Newton-ADE FDTD and traditional ADE FDTD methods use 12 variables. Therefore, the memory usage of these two methods is the same. In addition, the CPU time of Newton-ADE FDTD method is comparable with that of the traditional ADE FDTD method, albeit slightly larger. This is because the calculation of extra term $1/N_e(t))\cdot(\partial N_e(t)/\partial t)$ in Newton-ADE FDTD method would consume a little more time, since this term has been omitted in the traditional FDTD method. Actually the CPU time for both methods is very close, and the difference can be neglected.

TABLE I
CPU Time and Memory

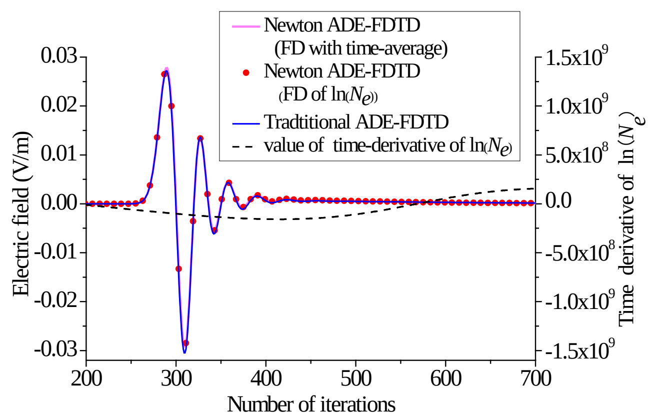
CPU Time		Memory	
Traditional ADE-FDTD	Newton ADE-FDTD	Traditional ADE-FDTD	Newton ADE-FDTD
414 s	433 s	91.55 MB	91.55 MB

D. *Analysis of Wave Propagation in 1D Time-varying Plasma Slab*

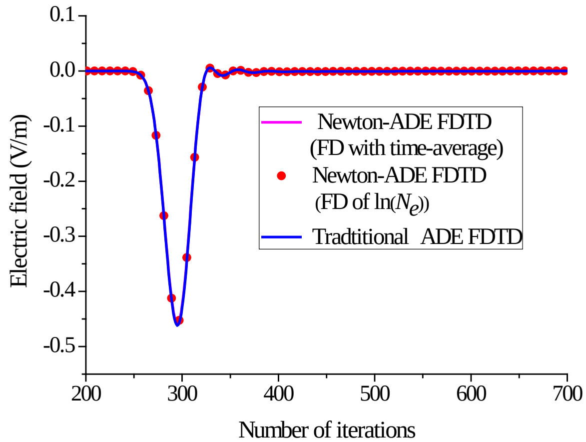
In this subsection, we consider a 1D time-varying plasma slab. The simulation can be regarded as a scenario for near-space hypersonic communication blackout problem. Two common variations of time-varying plasma including the sinusoidally varying and exponentially decreasing electron densities are adopted to study the characteristics of EM wave propagation through 1D time-varying plasma slab.

1) Sinusoidally varying electron density

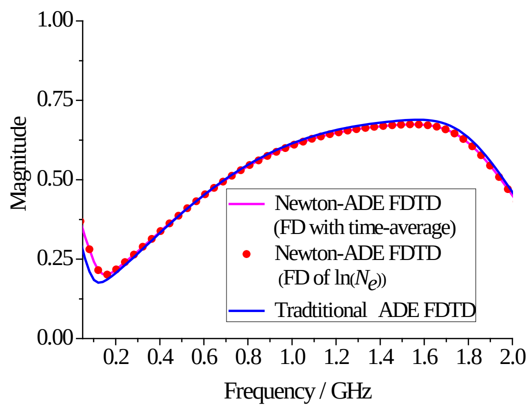
The simulation results are shown in Fig. 6 and Fig. 7 when the electron density variation center frequency f_0 is 80 MHz and 600MHz, respectively. The collision frequency is 2π×100 Mrad/s, $N_{e,avg}$ is 3×10^{16} m^{-3} and Δ_{Ne} is 0.3. The Courant number is 1 in the simulation. In Fig. 6(a) and Fig. 6(b), the transmitted and reflected waves are shown in time domain, which are computed using both Newton-ADE FDTD and traditional ADE FDTD methods. For the former, both discretizations using FD with time-average and FD of $\ln(N_e)$ are adopted. Also included in Fig. 6(a) is the value of time-derivative of $\ln(N_e)$. The maximum amplitude of time-derivative of $\ln(N_e)$ is small, around 2π×20 Mrad/s. The simulation results of transmitted and reflected waves show slight difference between the Newton-ADE FDTD and traditional ADE FDTD methods. For the transmitted and reflected waves in frequency domain shown in Fig. 6 (c) and Fig. 6 (d), their curves are similar and the difference is also slight, which is consistent with the characteristics of the time domain results. This is because the absolute value of time-derivative of $\ln(N_e)$ (in dash line) in Fig. 6(a) is smaller than the collision frequency of 2π×100 Mrad/s. Therefore, in this case, the term $1/N_e(t))\cdot(\partial N_e(t)/\partial t)-\nu$ in (7) is close to $-\nu$ in (24).



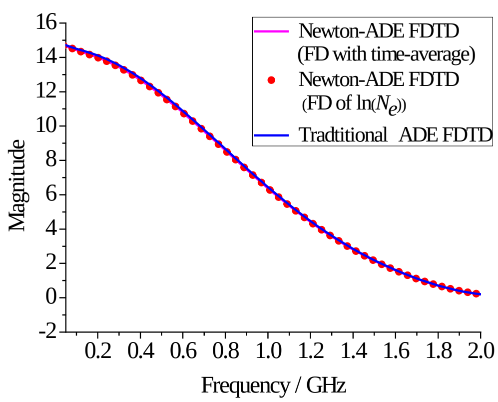
(a)



(b)



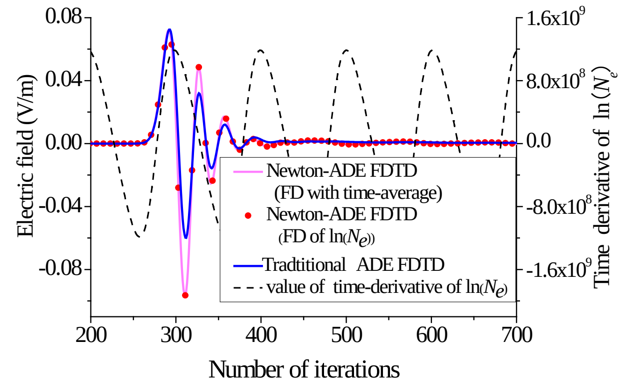
(c)



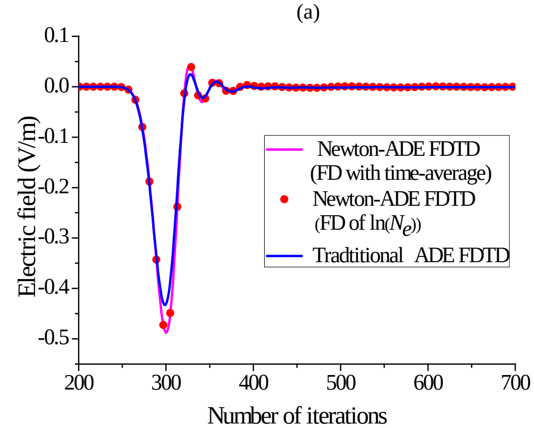
(d)

Fig. 6 . (a) Transmitted wave in time domain (b) Reflected wave in time domain (c) Transmitted wave in frequency domain (d) Reflected wave in frequency domain for sinusoidally varying electron density with f_0 = 80 MHz

Next, if the time-derivative of ln(N_e) is greater than the collision frequency, the simulation results between the two methods would yield significant differences. The variation center frequency f_0 is increased to 600 MHz in this simulation. Here, the maximum amplitude of time-derivative of ln(N_e) is around 2π×188 Mrad/s. It can be found that the absolute value of time-derivative of ln(N_e) shown in Fig. 7 (a) is greater than the collision frequency of 2π×100 Mrad/s. Fig. 7 (a) and Fig. 7 (b) show the simulation results of transmitted and reflected waves in time domain. Again for the Newton-ADE FDTD method, both discretizations using FD with time-average and FD of ln(N_e) still have the same curves. However, their simulation results show significant differences compared to the traditional ADE FDTD method. In addition, the transmitted and reflected waves in frequency domain also show significant differences in Fig. 7 (c) and Fig. 7 (d). It can be found from the above analysis that when the absolute value for time-derivative of logarithm of electron density is greater than the collision frequency, the Newton-ADE FDTD method can provide correct results, while the traditional ADE FDTD method yields incorrect results with significant differences.



(a)



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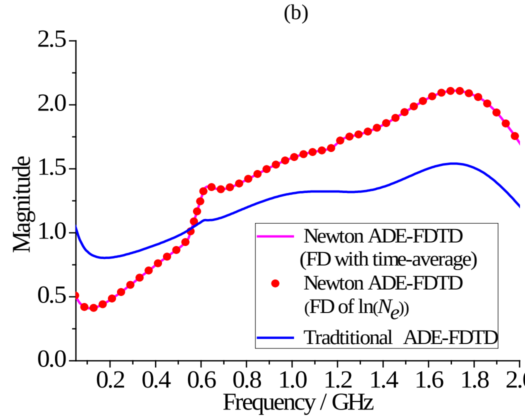

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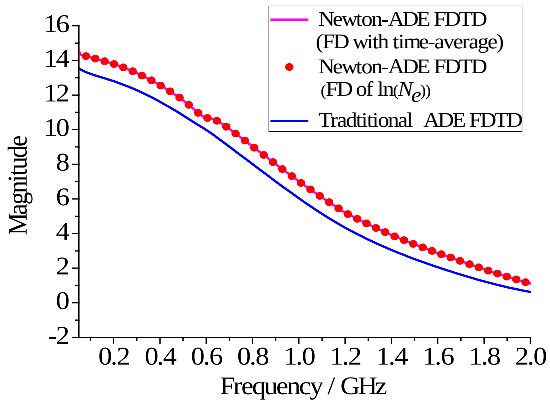


(d)

Fig. 7. (a) Transmitted wave in time domain (b) Reflected wave in time domain (c) Transmitted wave in frequency domain (d) Reflected wave in frequency domain for sinusoidally varying electron density with f_0 = 600 MHz

2) Exponentially decreasing electron density

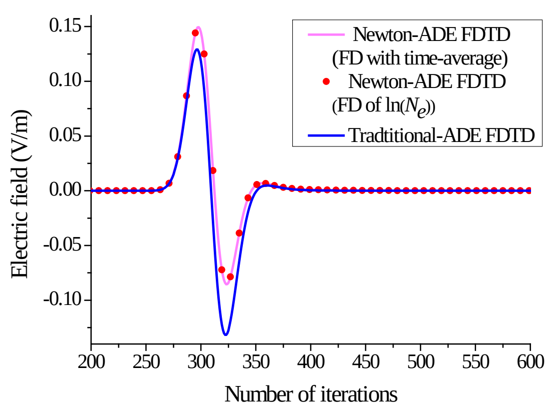
We now consider the exponentially decreasing electron density. The time-varying electron density is given by

$$N_e(t) = \begin{cases} N_{e,\max} & (t < t_s) \\ N_{e,\max}\exp(-\frac{t-t_s}{k}) & (t \ge t_s) \end{cases}, \quad (30)$$

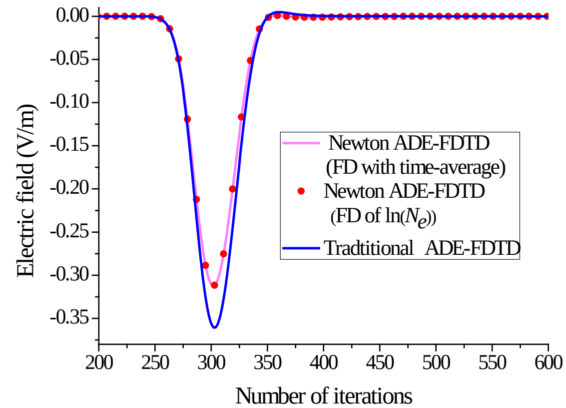
where $N_{e,\max}$ is the maximum electron density, t_s is the time when the electron density begins to decrease from the maximum value and k is a variable that controls the change rate. In our case, $N_{e,\max}$ is 5×10^{16} m^{-3} and t_s is 200 time steps. The collision frequency ν is 2π×150 Mrad/s. Here the plasma slab is 10-cm-thick, the spatial step size Δ is 0.005 m and the time step Δt is 1.67×10^{-11} s. The same Gaussian pulse excitation is described by (27). From (30), the time-derivative of ln(N_e) can be obtained as

$$\begin{aligned}\frac{\partial L(t)}{\partial t} &= \frac{1}{N_e(t)}\frac{\partial N_e(t)}{\partial t} \\ &= -\frac{1}{k}N_{e,\max}\exp(-\frac{t-t_s}{k})\frac{1}{N_{e,\max}\exp(-\frac{t-t_s}{k})} = -\frac{1}{k}\end{aligned}. \quad (31)$$

The simulation results of transmitted and reflected waves for exponentially decreasing electron density with various 1/k are shown in Fig. 8 and Fig. 9. In Fig. 8, the absolute value of time-derivative of ln(N_e) is 2π×150 Mrad/s, which is comparable to the collision frequency. The results for Newton-ADE FDTD method show differences compared to the traditional ADE FDTD method.



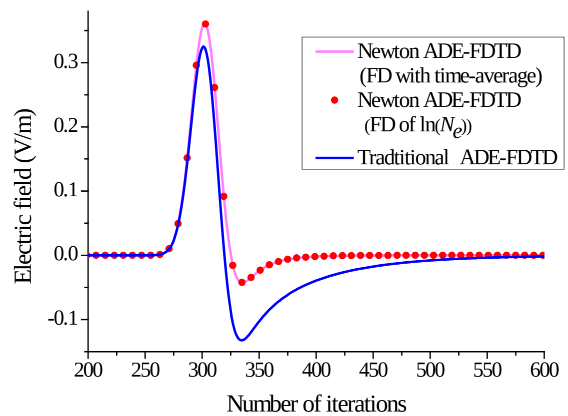
(a)



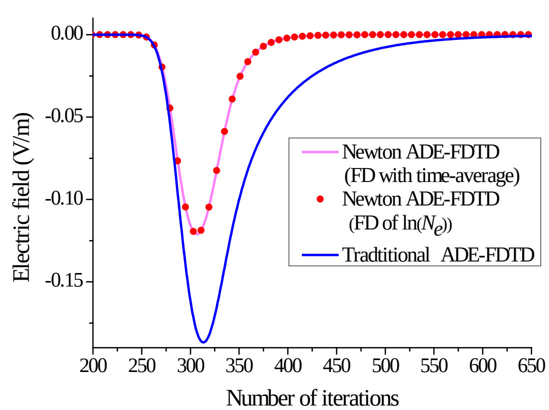
(b)

Fig. 8. (a) Transmitted and (b) reflected waves for exponentially decreasing electron density with 1/k = 2π×150 Mrad/s

In Fig. 9, the differences become more significant because the absolute value of time-derivative of ln(N_e) is 2π×300 Mrad/s, which is greater than the collision frequency of 2π×150 Mrad/s. As previously analyzed, for Newton-ADE FDTD method, both discretizations using FD with time-average and FD of ln(N_e) have similar simulation results and their curves overlap.



(a)



(b)

Fig. 9. (a) Transmitted and (b) reflected waves for exponentially decreasing electron density with 1/k =2π×300 Mrad/

E. *Analysis of RCS for 3D Metal Cylinder Covered by Time-varying Plasma*

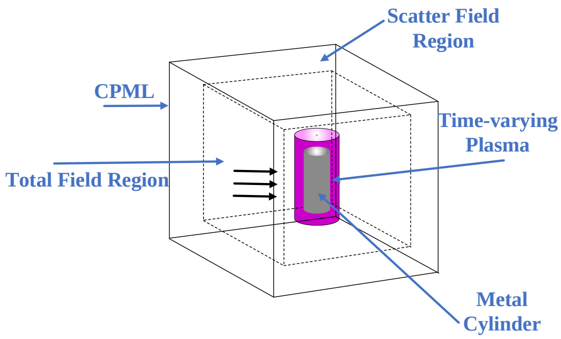


Fig. 10. Geometry for RCS of 3D metal cylinder covered by time-varying plasma

Finally, we consider the RCS for 3D metal cylinder covered by time-varying plasma. This scenario can mimic a radar detection of high-speed target covered by plasma sheath and other complex situation. The electron density variation is exponentially decreasing which is described by (27). $N_{e,\max}$ is 5 × 10^{16} m^{-3}, t_s is 30 time steps and the collision frequency is 2π×80 Mrad/s. The computational geometry is shown in Fig. 10. The radius of metal cylinder is 0.1 m and the length of cylinder is 0.2 m. The metal cylinder is covered by the plasma and the plasma thickness is 0.05 m. The spatial step $\Delta x=\Delta y=\Delta z$ is 0.0075 m and the time step Δt is 1.25× 10^{-11} s. The plane wave source is modulated Gaussian pulse given as

$$f(t) = -20\cos(2\pi f_0 t)e^{(-4\pi(t-t_0)^2/\tau^2)}, \qquad (32)$$

where τ is 2.8125 ns, f_0 is 1 GHz and t_0 is 2.25 ns. The plane wave is injected to the 3D total field boundary. The bistatic RCS can be calculated as

$$RCS(f) = 10\lg\left[4\pi r^2\left|\frac{E_s(f)}{E_i(f)}\right|\right], \qquad (33)$$

where $E_s(f)$ is the Fourier transform of time domain scattered field $E_s(t)$. $E_i(f)$is the Fourier transform of incident wave $E_i(t)$. The RCS results of 3D metal cylinder covered by time-varying plasma calculated by both Newton-ADE FDTD and traditional ADE-FDTD methods are shown in Fig. 11.

In Fig. 11 (a), the absolute value of $-1/k$ (time-derivative of $\ln(N_e)$) is 2π×30 Mrad/s, which is smaller than the collision frequency of 2π×80 Mrad/s. From the update equation of electric field in (18), we see that the discretization of time-derivative of $\ln(N_e)$ has slight effect on the electric field. Hence, the RCS results in Fig. 11 (a) show only slight differences. In Fig. 11 (b), the absolute values of $-1/k$ (time-derivative of $\ln(N_e)$) is increased to 2π×200 Mrad/s, which is greater than the collision frequency of 2π×80 Mrad/s. The simulation results of Newton-ADE FDTD and traditional ADE FDTD methods do show significant differences. The results in 3D simulation here are also consistent with the previous analyses of 1D time-varying plasma slab.

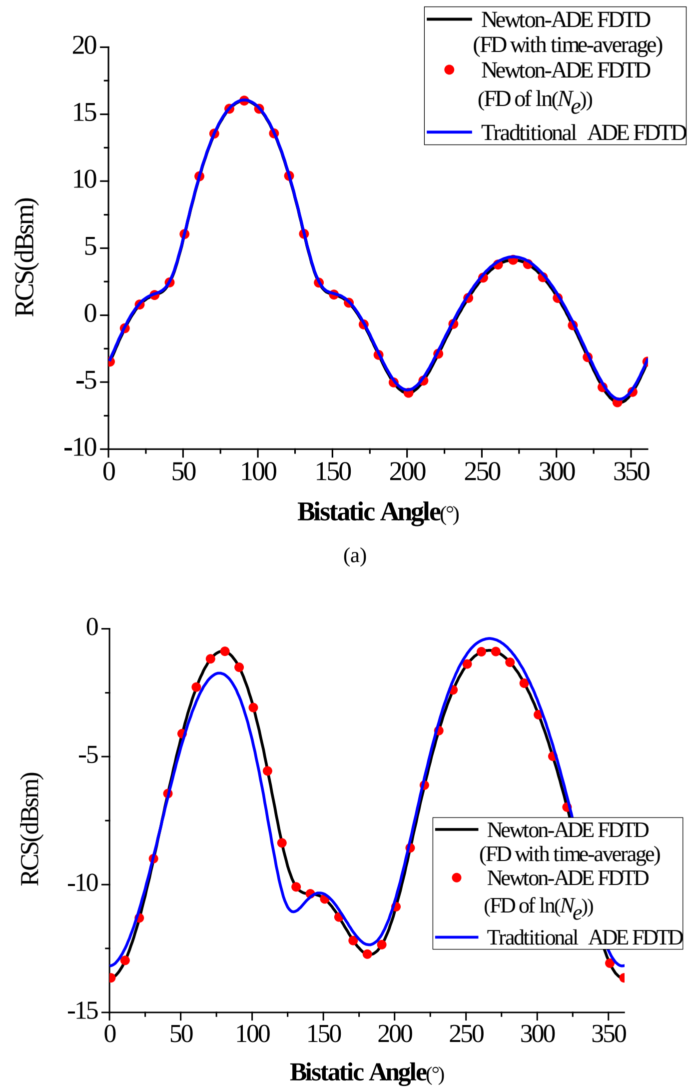


Fig. 11 The RCS results of 3D metal cylinder covered by time-varying plasma (a) exponentially decreasing electron density with $1/k$ = 2π×30 Mrad/s (b) exponentially decreasing electron density with $1/k$ = 2π×200 Mrad/s

IV. Conclusions

This paper has presented a novel Newton-ADE FDTD method for time-varying plasma, based on the fundamental Newton's equation of motion. The main contributions of this work are highlighted as follows:

i) *New correct formulations*: To our knowledge, this paper is the first one that attempts to work out the correct formulations of FDTD method for solving the propagation of electromagnetic waves in time-varying plasma. The previous analyses of time-varying plasma have only adopted the traditional ADE-FDTD method, which is inadequate and incorrect in general time-varying plasma conditions. In fact, the relationships among the time-varying electron density, current density and electric field are governed by the fundamental Newton's equation of motion. The Newton-ADE FDTD method proposed in this paper can solve the time-varying plasma problems correctly based on such equation from the first principle. The method has been validated based on the matrix exponential method.

ii) *New stability and convergence analyses.* We have formulated new stability and convergence analyses for FDTD method in time-varying dispersive media, which is generally challenging due to its time-varying nature. The amplification matrix and matrix exponential are cascaded leftward at each time index to obtain the overall matrices. It has been shown that our proposed method is stable and achieve second-order temporal accuracy.

iii) *New compact term for* $(1/N_e(t))\cdot(\partial N_e(t)/\partial t)$. The formulation of the Newton-ADE FDTD method has been provided and compared to that of the traditional ADE FDTD method. We have found that the key difference lies in the new significant term $(1/N_e(t))\cdot(\partial N_e(t)/\partial t)$, which can be compactly written as $\partial\ln(N_e(t))/\partial(t)$, i.e., the time-derivative of logarithm of electron density. There is no such term being identified in the previous FDTD methods.

iv) *Significance of* $\partial\ln(N_e(t))/\partial(t)$*:* Numerical results have been presented for EM wave propagation through time-varying plasma slab and RCS for metal cylinder covered by time-varying plasma. It has been demonstrated that when the absolute value for time-derivative of logarithm of electron density is large, e.g., greater than the collision frequency, the Newton-ADE FDTD method can provide correct results, while the traditional ADE FDTD method yields significant differences. Wave propagation through time-varying plasma including the term $\partial\ln(N_e(t))/\partial(t)$ is important for the communication problem of hypersonic vehicle enveloping plasma sheath and abnormal radar detection of high speed target covered by plasma sheath.

v) *Comparable computation cost*: The CPU time and memory cost of Newton-ADE FDTD method have been further analyzed. The results show that usage of CPU time and memory for both Newton-ADE FDTD and traditional ADE FDTD methods are comparable. By comparing the simulation results of Newton-ADE FDTD method with the matrix exponential method, the Newton-ADE FDTD method has been validated. The Newton-ADE FDTD method proposed in this paper will benefit the study of wave propagation in time-varying plasma and its subsequent engineering applications.

The proposed Newton-ADE FDTD method herein could be extended for efficient implicit FDTD methods such as the alternating direction implicit (ADI), locally one-dimensional (LOD) and Crank-Nicolson methods to achieve unconditional stability [29]-[32].

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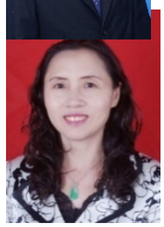

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