KORENBLUM CONSTANTS FOR SOME FUNCTION SPACES

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Abstract. We study the Korenblum Maximum Principle on the weighted Fock spaces and the weighted Bergman spaces with exponential weights. First, we give explicit expressions for the upper bounds of Korenblum constants for the weighted Fock spaces. Then, we obtain upper bounds of such constants for the weighted Bergman spaces. Finally, we show a failure of the Korenblum Maximum Principle for weighted Bergman spaces $A^p_\alpha(\mathbb{D})$, where $0 < p < 1$, $\alpha > 0$.

1. Introduction

1.1. Basic definitions and notation.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\mathcal{O}(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$ endowed with the compact-open topology and $\mathcal{O}(\mathbb{C})$ be the space of entire functions on $\mathbb{C}$ endowed with the compact-open topology.

Definition 1.1. For $0 < p < \infty$, $\alpha \geq 0$, the weighted Hardy space $H^p_\alpha(\mathbb{D})$ consists functions $f(z) \in \mathcal{O}(\mathbb{D})$, for which

$$\|f\|_{H^p_\alpha} = \sup_{0 \leq r < 1} \left[(1-r)^\alpha \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \right] < \infty.$$

Note that when $\alpha = 0$, we have the Hardy space $H^p(\mathbb{D})$. Furthermore, while when $p = \infty$, we have the space $H^\infty(\mathbb{D})$ of bounded holomorphic functions on $\mathbb{D}$, where $\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)|$.

Definition 1.2. Let $0 < p < \infty$ and $\alpha > -1$. The weighted Bergman space $A^p_\alpha(\mathbb{D})$ with exponential weight $e^{-\frac{p\alpha}{2}|z|^2}$ consists of functions $f(z) \in \mathcal{O}(\mathbb{D})$, for which

$$\|f\|_{A^p_\alpha} = \left[\frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} \, dA(z) \right]^{\frac{1}{p}} < \infty,$$

where $dA(z) = dxdy = rdrd\theta$, $z = x + iy = re^{i\theta}$, is the Lebesgue measure on $\mathbb{C}$.

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If $p = \infty$, then $A^\infty(\mathbb{D}) = H^\infty(\mathbb{D})$. For $0 < p < \infty$ and $\alpha = 0$, we have the standard Bergman space which is commonly denoted as $A^p(\mathbb{D})$. If $p \geq 1$, $A^p(\mathbb{D})$ is also a Banach space. In particular, when $p = 2$, we have the classical Bergman space $A^2(\mathbb{D})$, i.e. the space of functions $f(z) \in \mathcal{O}(\mathbb{D})$, for which

$$\|f\|_{A^2} = \left[\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2 \, dA(z)\right]^{\frac{1}{2}} < \infty.$$

Definition 1.3. For $0 < p < \infty$, $\alpha > 0$, the weighted Fock space $\mathcal{F}^p_\alpha(\mathbb{C})$ with exponential weight $e^{-\frac{p\alpha}{2}|z|^2}$ consists of entire functions $f(z) \in \mathcal{O}(\mathbb{C})$, for which

$$\|f\|^p_{\mathcal{F}^p_\alpha} = \frac{p\alpha}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} \, dA(z) < \infty.$$

For the case $p = \infty$ and $\alpha > 0$, we have

$$\|f\|_{\mathcal{F}^\infty_\alpha} = \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

When $0 < p \leq \infty$ and $\alpha = 1$, we have the Fock space $\mathcal{F}^p(\mathbb{C})$. It is well known that $\mathcal{F}^p(\mathbb{C})$ with $1 \leq p \leq \infty$ is a Banach space while for $0 < p < 1$, $\mathcal{F}^p(\mathbb{C})$ is a complete metric space with distance $d(f, g) = ||f - g||^p_p$.

1.2. **Korenblum's conjecture and development.** In 1991, Boris Korenblum [6] stated the following conjecture for $A^2(\mathbb{D})$.

Conjecture 1.4. *There exists a numerical constant* c*,* $0 < c < 1$*, such that if* f *and* g *are holomorphic in* $\mathbb{D}$ *and* $|f(z)| \leq |g(z)|$ $(c < |z| < 1)$*, then* $\|f\|_{A^2} \leq \|g\|_{A^2}$*.*

We call c a Korenblum constant and denote by κ as the largest value of c. So far, the exact value of κ is still not known.

Thereafter, a number of papers were devoted to the topic. The Korenblum Maximum Principle has since then become one of the important properties of function spaces (see, e.g., [18]). Latest developments showed greater diversity and interest in the topic in other function spaces as well as interesting results pertaining to the failure of Korenblum Maximum Principle in some function spaces (see, e.g., [1, 5]).

In order to avoid ambiguity, for the rest of the paper, we denote the largest Korenblum constant, unless specified otherwise, by $\kappa_{A^p_\alpha}$ for $A^p_\alpha(\mathbb{D})$, $\kappa_{\mathcal{F}^p_\alpha}$ for $\mathcal{F}^p_\alpha(\mathbb{C})$, etc.

It is worthy to note that Korenblum [6] proved with the counterexample provided by Martin that $\kappa_{A^2} < \frac{1}{\sqrt{2}}$. A series of partial results were then produced along the way by Korenblum, O'Neil, Richards and Zhu [8], Korenblum and Richards [7], Matero [9], Schwick [11] and others.

The conjecture was then only first proven of its existence in 1999 by Hayman [3] with $c = 0.04$ for $A^2(\mathbb{D})$. In the same year, Hinkkanen [4] proved the existence of c for $A^p(\mathbb{D})$, $1 \leq p < \infty$, with $c = 0.15724$. Since then, several works have been carried out in finding lower bounds and upper bounds of Korenblum constants.

Furthermore, several results expanded the topic into finding c for other function spaces as well. It is also worthy to mention that in 1998, Hayman and Danikas [2] also worked on Korenblum Maximum Principle for $H^p(\mathbb{D})$. In 2003 – 2004, Wang [12, 13, 14] published several results on the upper bounds of c. In the subsequent years, improvements were made by Schuster [10] and Wang [15, 16, 17]. In 2012, Zhu [18] showed that it is possible to choose some c for the Korenblum Maximum Principle to hold in $\mathcal{F}^p_\alpha(\mathbb{C})$ where $\alpha > 0$ and $p \geq 1$.

Recently, Božin and Karapetrović [1] showed a failure in Korenblum Maximum Principle for Bergman space $A^p(\mathbb{D})$, $0 < p < 1$. Lastly, in latest developments, Lou and Hu [5] also disproved the Korenblum Maximum Principle for general Fock Space $\mathcal{F}^p_\alpha(\mathbb{C})$ where $0 < p < 1$, $\alpha > 0$.

2. Preliminaries

In this section, we recall two results from [1, 6] respectively, which are closely related to our main results for $A^p_\alpha(\mathbb{D})$ in the next section. We also prove Lemma 2.3 which will be used to prove our main result.

Theorem 2.1. *([6]) Let $c > \frac{1}{\sqrt{2}}$. There exist functions f and g in $A^2(\mathbb{D})$ such that $|f(z)| \leq |g(z)|$ for all $c < |z| < 1$, but $\|f\|_{A^2} > \|g\|_{A^2}$. Therefore, $\kappa_{A^2} \leq \frac{1}{\sqrt{2}}$.*

Theorem 2.2. *([1]) Let $0 < p < 1$ and $0 < c < 1$. There exist functions f and g in $A^p(\mathbb{D})$ such that $|f(z)| < |g(z)|$ for all $c < |z| < 1$ and $\|f\|_{A^p} > \|g\|_{A^p}$.*

Lemma 2.3. *Let $0 < p < 1$, $\alpha > 0$ and $0 < c < 1$. Then there exist positive integer n and $0 < \delta < 1$, such that*

$$\begin{aligned} &2\delta^{np+2}\bigg(\int_0^1 ue^{-\frac{p\alpha}{2}\delta^2u^2}\,du + \int_1^{\frac{1}{\delta}} u^{np+1}e^{-\frac{p\alpha}{2}\delta^2u^2}\,du\bigg) \\ &\qquad > \left(1+\left(\frac{\delta}{c}\right)^n\right)^p \left(\frac{2}{p\alpha}\right)^{\frac{np}{2}+1} \int_0^{\frac{p\alpha}{2}} u^{\frac{np}{2}}e^{-u}\,du. \end{aligned} \tag{2.1}$$

Proof. For $0 < p < 1$, we can choose $n \in \mathbb{N}$ sufficiently large such that $n > np + 2$.

On one hand, for any $\delta > 0$, by the Bernoulli's Inequality, we have

$$\left(1+\left(\frac{\delta}{c}\right)^n\right)^p \leq 1 + p\left(\frac{\delta}{c}\right)^n.$$

On the other hand, note that

$$\begin{aligned}
&\int_0^1 ue^{-\frac{p\alpha}{2}\delta^2u^2}\,du + \int_1^{\frac{1}{\delta}} u^{np+1}e^{-\frac{p\alpha}{2}\delta^2u^2}\,du \\
&= \int_0^1 (u-u^{np+1})e^{-\frac{p\alpha}{2}\delta^2u^2}\,du + \int_0^{\frac{1}{\delta}} u^{np+1}e^{-\frac{p\alpha}{2}\delta^2u^2}\,du \\
&\underbrace{=}_{(x=\frac{p\alpha}{2}\delta^2u^2)} \int_0^1 (u-u^{np+1})e^{-\frac{p\alpha}{2}\delta^2u^2}\,du + \frac{1}{2}\Big(\frac{2}{p\alpha\delta^2}\Big)^{\frac{np}{2}+1}\int_0^{\frac{p\alpha}{2}} x^{\frac{np}{2}}e^{-x}\,dx.
\end{aligned}$$

Consequently, to get (2.1), it suffices to prove

$$\begin{aligned}
2\delta^{np+2}\Big(\int_0^1 (u-u^{np+1})e^{-\frac{p\alpha}{2}\delta^2u^2}\,du + \frac{1}{2}\Big(\frac{2}{p\alpha\delta^2}\Big)^{\frac{np}{2}+1}\int_0^{\frac{p\alpha}{2}} x^{\frac{np}{2}}e^{-x}\,dx\Big)& \\
> \Big(1+p\Big(\frac{\delta}{c}\Big)^n\Big)\Big(\frac{2}{p\alpha}\Big)^{\frac{np}{2}+1}\int_0^{\frac{p\alpha}{2}} u^{\frac{np}{2}}e^{-u}\,du&,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&2\delta^{np+2}\int_0^1 (u-u^{np+1})e^{-\frac{p\alpha}{2}\delta^2u^2}\,du \\
&\qquad > p\Big(\frac{\delta}{c}\Big)^n\Big(\frac{2}{p\alpha}\Big)^{\frac{np}{2}+1}\int_0^{\frac{p\alpha}{2}} u^{\frac{np}{2}}e^{-u}\,du. \qquad (2.2)
\end{aligned}$$

Now we prove (2.2). Note that p, α, c are fixed and also $n > np+2$. We can choose $\delta \in (0,1)$ sufficiently small so that

$$0 < \delta^{n-(np+2)} < \frac{c^n}{(\frac{2}{p\alpha})^{\frac{np}{2}+1}\displaystyle\int_0^{\frac{p\alpha}{2}} u^{\frac{np}{2}}e^{-u}\,du}, \qquad (2.3)$$

which gives

$$p > p\Big(\frac{\delta^{n-(np+2)}}{c^n}\Big)\Big(\frac{2}{p\alpha}\Big)^{\frac{np}{2}+1}\int_0^{\frac{p\alpha}{2}} u^{\frac{np}{2}}e^{-u}\,du.$$

Now notice that

$$\lim_{\delta\to 0^+} 2\left(\int_0^1 (u-u^{np+1})e^{-\frac{p\alpha}{2}\delta^2u^2}\,du\right) = 2\int_0^1 (u-u^{np+1})\,du = \frac{np}{np+2} > p,$$

and hence we can choose $0<\delta<1$ sufficiently small so that

$$2\left(\int_0^1 (u-u^{np+1})e^{-\frac{p\alpha}{2}\delta^2u^2}\,du\right) > p. \qquad (2.4)$$

So choosing $0<\delta<1$ for which both (2.3) and (2.4) hold, we obtain (2.2). The lemma is proved. □

3. MAIN RESULTS

In this section, we work on obtaining the following results.

1) For Fock spaces $\mathcal{F}^p_\alpha(\mathbb{C})$:
   - The explicit expression for upper bounds of $\kappa_{\mathcal{F}^p_\alpha}$ in $\mathcal{F}^p_\alpha(\mathbb{C})$, $p \geq 1$, $\alpha > 0$.
2) For Bergman spaces $A^p_\alpha(\mathbb{D})$:
   - The explicit expression for the upper bounds of $\kappa_{A^p_\alpha}$ in $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha \geq 0$.
   - A failure of the Korenblum Maximum Principle for $A^p_\alpha(\mathbb{D})$, $0 < p < 1$, $\alpha > 0$.

3.1. Fock Spaces $\mathcal{F}^p_\alpha(\mathbb{C})$.

We first consider the general Fock space $\mathcal{F}^p_\alpha(\mathbb{C})$, where $p \geq 1$, $\alpha > 0$ and find an *upper bound* for $\kappa_{\mathcal{F}^p_\alpha}$.

Theorem 3.1. *Let $p \geq 1$, $\alpha > 0$ and*

$$c > \sqrt[p]{\Big(\frac{2}{p\alpha}\Big)^{\frac{p}{2}} \Gamma\Big(\frac{p}{2}+1\Big)}.$$

There exist functions f and g in $\mathcal{F}^p_\alpha(\mathbb{C})$, such that $|f(z)| < |g(z)|$ for any $|z| > c$, but $\|f\|^p_{\mathcal{F}^p_\alpha} > \|g\|^p_{\mathcal{F}^p_\alpha}$. Therefore,

$$\kappa_{\mathcal{F}^p_\alpha} \leq \sqrt[p]{\Big(\frac{2}{p\alpha}\Big)^{\frac{p}{2}} \Gamma\Big(\frac{p}{2}+1\Big)}.$$

Proof. Consider functions $f(z) = c,\ g(z) = z\ \in \mathcal{F}^p_\alpha(\mathbb{C})$, we have

$$|f(z)| = |c| < |z| = |g(z)|, \text{ for any } |z| > c.$$

On the other hand,

$$\begin{aligned}
&\|f\|^p_{\mathcal{F}^p_\alpha} - \|g\|^p_{\mathcal{F}^p_\alpha} \\
&= \frac{p\alpha}{2\pi}\int_{\mathbb{C}} c^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z) - \frac{p\alpha}{2\pi}\int_{\mathbb{C}} |z|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z) \\
&= \frac{p\alpha}{2\pi}\left[\int_0^{2\pi}\int_0^{\infty} c^p e^{-\frac{p\alpha}{2}r^2} r\, dr\, d\theta - \int_0^{2\pi}\int_0^{\infty} r^p e^{-\frac{p\alpha}{2}r^2} r\, dr\, d\theta\right] \\
&= \frac{p\alpha}{2\pi}\Bigg[\int_0^{2\pi}\Big(-\frac{c^p}{p\alpha}\Big)\int_0^{\infty}\Big(-p\alpha r e^{-\frac{p\alpha}{2}r^2}\Big)\, dr\, d\theta \\
&\underbrace{-}_{(u=\frac{p\alpha r^2}{2})} \int_0^{2\pi}\int_0^{\infty}\Big(\frac{2u}{p\alpha}\Big)^{\frac{p}{2}}\frac{e^{-u}}{p\alpha}\, du\, d\theta\Bigg] \\
&= -c^p\left[e^{-\frac{p\alpha}{2}r^2}\right]_0^{\infty} - \Big(\frac{2}{p\alpha}\Big)^{\frac{p}{2}}\int_0^{\infty} u^{\frac{p}{2}} e^{-u}\, du \\
&= c^p - \Big(\frac{2}{p\alpha}\Big)^{\frac{p}{2}}\Gamma\Big(\frac{p}{2}+1\Big) > 0.
\end{aligned}$$

Therefore, in order for the Korenblum Maximum Principle to hold, we must have

$$\kappa_{\mathcal{F}^p_\alpha} \leq \sqrt[p]{\Big(\frac{2}{p\alpha}\Big)^{\frac{p}{2}}\Gamma\Big(\frac{p}{2}+1\Big)}.$$

$\square$

As an immediate consequence of Theorem 3.1, we get the following result for the classical Hilbert-Fock space $\mathcal{F}^2(\mathbb{C})$.

Corollary 3.2. *Let $c > 1$. There exist functions f and g in $\mathcal{F}^2(\mathbb{C})$ such that $|f(z)| < |g(z)|$ for all $|z| > c$, but $\|f\|^2_{\mathcal{F}^2} > \|g\|^2_{\mathcal{F}^2}$. Therefore, $\kappa_{\mathcal{F}^2} \leq 1$.*

It is also interesting to note that when $p = 1$ and $\alpha = \frac{1}{2}$, we have

$$\kappa_{\mathcal{F}^1_{0.5}} \leq \sqrt{4}\cdot\Gamma\bigg(\frac{3}{2}\bigg) = \sqrt{\pi}.$$

3.2. **Bergman Spaces $A^p_\alpha(\mathbb{D})$.** The results about the failures of Korenblum Maximum Principle in $A^p(\mathbb{D})$, $0 < p < 1$ [1] and in $\mathcal{F}^p_\alpha(\mathbb{C})$, $0 < p < 1$, $\alpha > 0$ [5] inspire us to be interested in a question whether there is any failure of the Korenblum Maximum Principle for the weighted Bergman space $A^p_\alpha(\mathbb{D})$, $0 < p < 1$, $\alpha \neq 0$.

It turns out that the failure exists and the result below not only proves this fact, but also generalizes Theorem 2.2 for any $\alpha > 0$.

Theorem 3.3. *Let $0 < p < 1$ and $\alpha > 0$. Suppose $0 < c < 1$. Then there exist functions f and g in $A^p_\alpha(\mathbb{D})$ such that $|f(z)| < |g(z)|$ for any z with $c < |z| < 1$ and $\|f\|_{A^p_\alpha} > \|g\|_{A^p_\alpha}$.*

Proof. We follow the scheme of [1, 5]. Choose $0 < \delta < 1$ and $n \in \mathbb{N}$ so that $n > np + 2$ and δ satisfy Lemma 2.3. Then we consider the following functions f and g in in $A^p_\alpha(\mathbb{D})$

$$f(z) = \frac{1}{1+(\frac{\delta}{c})^n}(z^n+\delta^n), \quad g(z) = z^n.$$

First we note that if $c < |z| < 1$, then $\dfrac{\delta^n}{c^n+\delta^n} > \dfrac{\delta^n}{|z|^n+\delta^n}$, which gives $\dfrac{c^n}{c^n+\delta^n} < \dfrac{|z|^n}{|z|^n+\delta^n}$. That is,

$$\frac{c^n(|z|^n+\delta^n)}{c^n+\delta^n} < |z|^n, \text{ for all } c < |z| < 1.$$

Then with $c < |z| < 1$, we have

$$|f(z)| = \frac{c^n}{c^n+\delta^n}|z^n+\delta^n| \leq \frac{c^n(|z|^n+\delta^n)}{c^n+\delta^n} < |z|^n = |g(z)|.$$

Next we prove $\|f\|_{A^p_\alpha} > \|g\|_{A^p_\alpha}$, or equivalently,

$$\int_{\mathbb{D}} |z^n + \delta^n|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z) > \left(1 + \left(\frac{\delta}{c}\right)^n\right)^p \int_{\mathbb{D}} |z|^{np} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z).$$

Clearly, it suffices to show that

(3.1)

$$\begin{aligned}&\int_{\mathbb{D}_1} |z^n + \delta^n|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z) + \int_{\mathbb{D}_2} |z^n + \delta^n|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z) := J_1 + J_2\\ &> \left(1 + \left(\frac{\delta}{c}\right)^n\right)^p \int_{\mathbb{D}} |z|^{np} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z),\end{aligned}$$

where $\mathbb{D}_1 = \{z \in \mathbb{C} : |z| < \delta\}$ and $\mathbb{D}_2 = \{z \in \mathbb{C} : \delta < |z| < 1\}$.

- *Estimates for* J_1.

Since $|\frac{z}{\delta}| < 1$ for $z \in \mathbb{D}_1$, by the binomial series, we have

$$\begin{aligned}J_1 &= \int_{\mathbb{D}_1} |z^n + \delta^n|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z) = \delta^{np} \int_{\mathbb{D}_1} \left|1 + \left(\frac{z}{\delta}\right)^n\right|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)\\ &= \delta^{np} \int_{\mathbb{D}_1} \left(1 + \left(\frac{z}{\delta}\right)^n\right)^{\frac{p}{2}} \left(1 + \left(\frac{\overline{z}}{\delta}\right)^n\right)^{\frac{p}{2}} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)\\ &= \delta^{np} \int_{\mathbb{D}_1} \sum_{k=0}^{\infty} \binom{p/2}{k} \left(\frac{z}{\delta}\right)^{nk} \sum_{j=0}^{\infty} \binom{p/2}{j} \left(\frac{\overline{z}}{\delta}\right)^{nj} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z).\end{aligned}$$

Note that

$$\begin{aligned}&\int_{\mathbb{D}_1} z^{nk}\overline{z}^{nj} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z) = \int_0^{\delta}\int_0^{2\pi} r^{n(k+j)+1} e^{in(k-j)\theta} e^{-\frac{p\alpha}{2}r^2} d\theta dr\\ &= 2\pi \int_0^{\delta} r^{2nk+1} e^{-\frac{p\alpha}{2}r^2} dr \underbrace{=}_{(u=\frac{r}{\delta})} 2\pi \int_0^1 \delta^{2nk+2} u^{2nk+1} e^{-\frac{p\alpha}{2}\delta^2 u^2}\, du,\end{aligned}$$

and hence

$$\begin{aligned}J_1 &= \int_{\mathbb{D}_1} |z^n + \delta^n|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)\\ &= \delta^{np} \sum_{k=0}^{\infty} \binom{p/2}{k}^2 \frac{1}{\delta^{2nk}} \int_{\mathbb{D}_1} z^{nk}\overline{z}^{nk} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)\\ &= 2\pi\delta^{np+2} \sum_{k=0}^{\infty} \binom{p/2}{k}^2 \int_0^1 u^{2nk+1} e^{-\frac{p\alpha}{2}\delta^2 u^2}\, du.\end{aligned}$$

- *Estimates for* J_2.

Similarly, since $|\frac{\delta}{z}| < 1$ for $z \in \mathbb{D}_2$, by binomial series, we have

$$J_2 = \int_{\mathbb{D}_2} |z^n + \delta^n|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z) = \int_{\mathbb{D}_2} |z|^{np} \left|1 + \left(\frac{\delta}{z}\right)^n\right|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)$$

$$= \int_{\mathbb{D}_2} |z|^{np} \left(1 + \left(\frac{\delta}{z}\right)^n\right)^{\frac{p}{2}} \left(1 + \left(\frac{\delta}{\overline{z}}\right)^n\right)^{\frac{p}{2}} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)$$

$$= \int_{\mathbb{D}_2} |z|^{np} \sum_{k=0}^{\infty} \binom{p/2}{k} \left(\frac{\delta}{z}\right)^{nk} \sum_{j=0}^{\infty} \binom{p/2}{j} \left(\frac{\delta}{\overline{z}}\right)^{nj} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z).$$

Note that

$$J_2 = \int_{\mathbb{D}_2} |z|^{np} \frac{1}{z^{nk}} \frac{1}{\overline{z}^{nj}} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)$$

$$= \int_{\delta}^{1} \int_{0}^{2\pi} r^{np+1} \frac{1}{r^{nk} e^{ink\theta}} \frac{1}{r^{nj} e^{-inj\theta}} e^{-\frac{p\alpha}{2}r^2}\, d\theta\, dr$$

$$= \int_{\delta}^{1} \int_{0}^{2\pi} r^{np-n(k+j)+1} e^{-\frac{p\alpha}{2}r^2} \frac{1}{e^{in(k-j)\theta}}\, d\theta\, dr = 2\pi \int_{\delta}^{1} r^{np-2nk+1} e^{-\frac{p\alpha}{2}r^2}\, dr$$

$$\underbrace{=}_{(u=\frac{r}{\delta})} 2\pi \int_{1}^{\frac{1}{\delta}} \delta^{np-2nk+2} u^{np-2nk+1} e^{-\frac{p\alpha}{2}\delta^2 u^2}\, du,$$

and hence

$$J_2 = \int_{\mathbb{D}_2} |z^n + \delta^n|^p e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)$$

$$= \sum_{k=0}^{\infty} \binom{p/2}{k}^2 \delta^{2nk} \int_{\mathbb{D}_2} |z|^{np} \frac{1}{z^{nk}} \frac{1}{\overline{z}^{nk}} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z)$$

$$= 2\pi \delta^{np+2} \sum_{k=0}^{\infty} \binom{p/2}{k}^2 \int_{1}^{\frac{1}{\delta}} u^{np-2nk+1} e^{-\frac{p\alpha}{2}\delta^2 u^2}\, du.$$

- *Estimates for the right-hand side of* (3.1).

$$\int_{\mathbb{D}} |z|^{np} e^{-\frac{p\alpha}{2}|z|^2}\, dA(z) = \int_0^{2\pi} \int_0^1 r^{np+1} e^{-\frac{p\alpha}{2}r^2}\, dr\, d\theta$$

$$\underbrace{=}_{(u=\frac{p\alpha}{2}r^2)} \pi \left(\frac{2}{p\alpha}\right)^{\frac{np}{2}+1} \int_0^{\frac{p\alpha}{2}} u^{\frac{np}{2}} e^{-u}\, du.$$

So (3.1) becomes

$$2\pi\delta^{np+2} \sum_{k=0}^{\infty} \binom{p/2}{k}^2 \left(\int_0^1 u^{2nk+1} e^{-\frac{p\alpha}{2}\delta^2 u^2}\, du + \int_1^{\frac{1}{\delta}} u^{np-2nk+1} e^{-\frac{p\alpha}{2}\delta^2 u^2}\, du\right)$$

$$> \pi \left(1 + \left(\frac{\delta}{c}\right)^n\right)^p \left(\frac{2}{p\alpha}\right)^{\frac{np}{2}+1} \int_0^{\frac{p\alpha}{2}} u^{\frac{np}{2}} e^{-u}\, du. \tag{3.2}$$

Indeed, (3.2) is true, because of Lemma 2.3:

$$
\begin{aligned}
&2\pi\delta^{np+2}\sum_{k=0}^{\infty}\binom{p/2}{k}^2\left(\int_0^1 u^{2nk+1}e^{-\frac{p\alpha}{2}\delta^2u^2}\,du+\int_1^{\frac{1}{\delta}} u^{np-2nk+1}e^{-\frac{p\alpha}{2}\delta^2u^2}\,du\right)\\
&>2\pi\delta^{np+2}\left(\int_0^1 ue^{-\frac{p\alpha}{2}\delta^2u^2}\,du+\int_1^{\frac{1}{\delta}} u^{np+1}e^{-\frac{p\alpha}{2}\delta^2u^2}\,du\right)\\
&>\pi\left(1+\left(\frac{\delta}{c}\right)^n\right)^p\left(\frac{2}{p\alpha}\right)^{\frac{np}{2}+1}\int_0^{\frac{p\alpha}{2}} u^{\frac{np}{2}}e^{-u}\,du.
\end{aligned}
$$

□

Lastly, we have the explicit expression for the upper bounds in the space $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha \geq 0$.

Theorem 3.4. *Let $p \geq 1$, $\alpha \geq 0$. Consider the Bergman space $A^p_\alpha(\mathbb{D})$.*

1) *For $\alpha = 0$, suppose*

$$\left(\frac{2}{p+2}\right)^{\frac{1}{p}} < c < 1.$$

2) *For $\alpha > 0$, suppose*

$$\sqrt[p]{\frac{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}}\int_0^{\frac{p\alpha}{2}} u^{\frac{p}{2}}e^{-u}du}{(1-e^{-\frac{p\alpha}{2}})}} < c < 1.$$

There exist functions f and g in $A^p_\alpha(\mathbb{D})$ such that $|f(z)| < |g(z)|$ for all $c < |z| < 1$, but $\|f\|_{A^p_\alpha} > \|g\|_{A^p_\alpha}$.

Proof. As in the previous theorems, consider the functions $f(z) = c$ and $g(z) = z$ from $A^p_\alpha(\mathbb{D})$. Obviously, for all $c < |z| < 1$, we have $|f(z)| = c < |z| = |g(z)|$.

- Case 1: $\alpha = 0$. We have

$$
\begin{aligned}
\|f\|^p_{A^p_\alpha} - \|g\|^p_{A^p_\alpha} &= \frac{1}{\pi}\left[\int_0^{2\pi}\int_0^1 c^p r\,dr\,d\theta - \int_0^{2\pi}\int_0^1 r^{p+1}\,dr\,d\theta\right]\\
&= \frac{1}{\pi}\left[\int_0^{2\pi}\frac{c^p}{2}\,d\theta - \int_0^{2\pi}\frac{1}{p+2}\,d\theta\right] = c^p - \frac{2}{p+2} > 0.
\end{aligned}
$$

- Case 2: $\alpha > 0$. We have

$$\begin{aligned}
&\|f\|_{A^p_\alpha} - \|g\|_{A^p_\alpha}\\
&= \frac{1}{\pi}\left[\int_0^{2\pi}\int_0^1 rc^p e^{-\frac{p\alpha}{2}r^2}\,drd\theta - \int_0^{2\pi}\int_0^1 r^{p+1}e^{-\frac{p\alpha}{2}r^2}\,drd\theta\right]\\
&= -\frac{2c^p}{p\alpha}\int_0^1 -p\alpha r e^{-\frac{p\alpha}{2}r^2}\,drd\theta - 2\int_0^{\frac{p\alpha}{2}}\left(\frac{2u}{p\alpha}\right)^{\frac{p}{2}+\frac{1}{2}} e^{-u}\frac{1}{\sqrt{2p\alpha u}}\,du\\
&(\text{where } u = \frac{p\alpha}{2}r^2)\\
&= -\frac{2c^p}{p\alpha}\left[e^{-\frac{p\alpha}{2}r^2}\right]_0^1 - \left(\frac{2}{p\alpha}\right)^{\frac{p}{2}+1}\int_0^{\frac{p\alpha}{2}} u^{\frac{p}{2}}e^{-u}\,du\\
&= \frac{2c^p}{p\alpha}(1-e^{-\frac{p\alpha}{2}}) - \left(\frac{2}{p\alpha}\right)^{\frac{p}{2}+1}\int_0^{\frac{p\alpha}{2}} u^{\frac{p}{2}}e^{-u}\,du\\
&> \left(\frac{2}{p\alpha}\right)^{\frac{p}{2}+1}\int_0^{\frac{p\alpha}{2}} u^{\frac{p}{2}}e^{-u}\,du - \left(\frac{2}{p\alpha}\right)^{\frac{p}{2}+1}\int_0^{\frac{p\alpha}{2}} u^{\frac{p}{2}}e^{-u}\,du = 0.
\end{aligned}$$

□

Remark 3.5. Clearly, in order to have the Korenblum Maximum Principle for $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha \geq 0$, we must have

$$\kappa_{A^p_\alpha} \leq \begin{cases} \left(\frac{2}{p+2}\right)^{\frac{1}{p}}, & \alpha = 0,\\ \sqrt[p]{\dfrac{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}}\displaystyle\int_0^{\frac{p\alpha}{2}} u^{\frac{p}{2}}e^{-u}du}{(1-e^{-\frac{p\alpha}{2}})}}, & \alpha > 0. \end{cases}$$

However, for $p = 2$, $\alpha = 0$, $\kappa_{A^2} \leq \frac{1}{\sqrt{2}}$ is already known and in fact, Korenblum, with Rainer Martin's counterexample, first proved it in 1991 as shown in Theorem 2.1. Nevertheless, this result not only coincides with his result but also obtains the upper bounds for rest of the $p \geq 1$.

4. Concluding remarks and open questions

In this paper, we obtained explicit expressions for the upper bounds of $\mathcal{F}^p_\alpha(\mathbb{C})$ (Theorem 3.1) and $A^p_\alpha(\mathbb{D})$, where $p \geq 1$ and $\alpha \geq 0$ (Theorem 3.4). We also proved that the Korenblum Maximum Principle fails for $A^p_\alpha(\mathbb{D})$, $0 < p < 1$, $\alpha > 0$, thereby obtaining greater closure to the Korenblum Maximum Principle for the Bergman spaces.

Theorems 3.1, and 3.4 led us to the following questions which call for investigation.

Question 4.1. *Let $c = 1$. Do there exist functions $f(z)$ and $g(z)$ in $\mathcal{F}^2(\mathbb{C})$ for which $|f(z)| < |g(z)|$ with all $|z| > c$ and $\|f\|_{\mathcal{F}^2} > \|g\|_{\mathcal{F}^2}$?*

Question 4.2. *Let $p \geq 1$, $\alpha > 0$ and*

$$c = \sqrt[p]{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}} \Gamma\left(\frac{p}{2}+1\right)}.$$

Do there exist functions $f(z)$ and $g(z)$ in $\mathcal{F}^p_\alpha(\mathbb{C})$ for which $|f(z)| < |g(z)|$ with all $|z| > c$ and $\|f\|_{\mathcal{F}^p_\alpha} > \|g\|_{\mathcal{F}^p_\alpha}$?

Question 4.3. *Let $p \geq 1$, $\alpha \geq 0$ and*

$$c = \begin{cases} \left(\frac{2}{p+2}\right)^{\frac{1}{p}}, & \alpha = 0 \\ \sqrt[p]{\dfrac{\left(\frac{2}{p\alpha}\right)^{\frac{p}{2}} \displaystyle\int_0^{\frac{p\alpha}{2}} u^{\frac{p}{2}} e^{-u} du}{(1 - e^{-\frac{p\alpha}{2}})}}, & \alpha > 0. \end{cases}$$

Do there exist functions $f(z)$ and $g(z)$ in $A^p_\alpha(\mathbb{D})$ for which $|f(z)| < |g(z)|$ with all $c < |z| < 1$ and $\|f\|_{A^p_\alpha} > \|g\|_{A^p_\alpha}$?

Remark 4.4. Note that solving Question 4.3 indirectly generalizes the counterexample by Martin in [6] for $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha \geq 0$. Note also that the case of $A^p_\alpha(\mathbb{D})$ where $-1 < \alpha < 0$ and $0 < p < \infty$ still remains unsolved.

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