

Fractional Delay Filters Based on Generalized Cardinal Exponential Splines

Ayush Bhandari and Pina Marziliano

Abstract—Fractional delay filters (FDFs) play an important role in certain areas of digital signal processing and communication engineering, where it is desirable to generate delays that are of the order of a fraction of the sampling period. In this paper, we advocate the use of generalized cardinal exponential splines—a class of compactly supported functions that is much richer than the existing B-spline family—for designing precision FDFs. One advantage of using generalized cardinal exponential splines is that it provides ready access to several spline families and other kernels which could be used for FDF design. The B-spline and Lagrange interpolator based FDFs are a special case of our proposition. We also discuss a design example and show that it is possible to design filters that have lower interpolation errors as compared to its B-spline counterparts.

I. INTRODUCTION

THE prefix “digital” when concatenated with “signal processing” creates a paradigm shift, which as we know, has been a catalyst in the modern day digital revolution—something that will continue to influence the development of technology. The inherent tradeoff in the process of digitization is the loss of control over the continuum that exists in the analog world. This *loss in control* is quite conspicuous to a fractional delay filter designer whose constant endeavor is to generate a delay that corresponds to a fraction of a sampling period.

Continuous control of fractional delay could be a desirable property of sampling rate conversion systems and the excellent expository article [1] is replete with examples of areas where FDFs play a key role. Assuming that the sampling interval is normalized to unity and the samples correspond to a continuous-time bandlimited signal, say $s(t)$, a sample delay of $\tau \in [0, 1)$ —from an available sample sequence, say $s[k] \in \ell_2, k \in \mathbb{Z}$ —can be achieved by interpolating $s[k]$ and resampling its τ -shifted/delayed version, which leads to (Shannon’s sampling theorem),

$$s_\tau[k] = s(k - \tau) = \sum_{n=-\infty}^{+\infty} s[n] \text{sinc}(k - \tau - n) \quad (1)$$

with $\text{sinc}(t) = \sin(\pi t)/(\pi t)$. Following (1), our immediate objective is to minimize the Euclidean distance (or the interpolation error) between the sampled version of $s(t - \tau)$ and its

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A. Bhandari is with the Temasek Laboratories@Nanyang Technological University, Singapore 637553 (e-mail: ayushbhandari@ntu.edu.sg).

P. Marziliano is with the Division of Information Engineering, School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: epina@ntu.edu.sg).

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FDF-based approximation, $s_\tau[k]$. In a theoretical setting, it is possible to accomplish a zero error, however, in practice, this might be an over ambitious goal. This is attributed to the recipe in (1) and more than that, the slow decay rate of the $\text{sinc}(\cdot)$ function and its infinite length. A quick remedy to this problem is to replace $\text{sinc}(\cdot)$ by a generic interpolant, say $\varphi(t) \in L_2, t \in \mathbb{R}$ which has a smoother spectrum (implies that $\varphi(t)$ decays quickly). Let $\hat{\varphi}(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} \varphi(t) e^{-j\omega t} dt$ denote the (Fourier) spectrum of $\varphi(t)$. For $\varphi(t)$ to be an admissible interpolant [2], it is imperative that its spectrum aliases white or equivalently,

$$\varphi(t)|_{t=k, k \in \mathbb{Z}} = \delta_k \stackrel{\text{Fourier}}{\longleftrightarrow} \sum_{n=-\infty}^{+\infty} \hat{\varphi}(\omega + 2\pi n) = 1 \quad (2)$$

where δ_k is the Kronecker-delta and we assume that the sampling period is unity. Windowing of $\text{sinc}(t)$ (cf. [1] and references therein) is one of the simplest ways of obtaining such a $\varphi(t)$. Other methods of designing FDFs have been discussed in [3]–[7]. FDFs based on Schoenberg’s B-splines [8], [9] have been discussed in [10]–[12] and the results are very interesting in that, one is able to engender accurate fractional delays with ease of implementation, and furthermore, the interpolation errors for spline-based FDFs are lesser in comparison to other standard filters.

In this paper, we want to highlight the fact that one can design precision FDFs with comparable computation complexity and lesser interpolation errors. Our design is based on expansions of form (1), but, in a latent sense as we use generalized interpolation. Our discussion is based on a richer class of splines—the family of generalized cardinal exponential splines (GenCESP) [13]. In the remainder of this paper, we discuss our methodology and present a design example.

II. FRACTIONAL DELAY FILTERS BASED ON GENERALIZED CARDINAL EXPONENTIAL SPLINES

A. Generalized Interpolation—Basic Setup

The cardinal series expansion in (1) can be written as a linear combination of some compactly supported basis functions

$$s(t) = \sum_{k=-\infty}^{+\infty} c[k] \varphi(t - k) \quad (3)$$

provided that $s(t)$ belongs to some shift-invariant space $V(\varphi) = \text{span}\{\varphi(t - k)\}_{k=-\infty}^{+\infty}$ where $V(\varphi) \subset L_2$. Also, in order to guarantee stability of (3) and one-to-one correspondence between $s(t) \in L_2$ and $c[k] \in \ell_2, \forall k \in \mathbb{Z}$, we would want that the integer translates of φ form a Riesz basis of $V(\varphi)$ [2]. In (3), $c[k]$ may no longer be the samples of $s(t)$ and $\varphi(t)$ may not satisfy the condition in (2). The space of bandlimited functions or $V(\text{sinc})$ is a special case of $V(\varphi)$ and for this case, $c[k] = s(t)|_{t=k}, \forall k \in \mathbb{Z}$. Sampling (3) yields $s[k] = (c * \varphi)[k]$ and the

TABLE I
SPECIAL CASES OF GENCESP BASED FDFS

► **Lagrange interpolator** based FDFs are established favorites for their maximally flat magnitude response (cf. pg. 40, [1] and references therein). An N^{th} order FDF is obtained by setting $\vec{\eta} = ([\alpha_n]; [\gamma_m])$ where poles $\alpha_n = 0, n \in [1, N+1]$, zeros or $\gamma_{(2m-1)} = -\gamma_{(2m)}^*$, $m \in [1, \lfloor N/2 \rfloor]$ (Hermitian symmetric) and $\lfloor \cdot \rfloor$ denotes the floor function. Let $S_N^{(m)}$ be Stirling numbers of first kind. The zeros (or γ_m) are the corresponding roots of polynomial $\Lambda(\omega) = 1 + \sum_{m=1}^{\lfloor N/2 \rfloor} \lambda[m] \cdot \omega^{2m} = 0$ and the values of $\lambda[m]$ are given by [16], $\lambda[m] = (-1)^m \frac{(N-2m)!}{N!} \sum_{k=0}^{2m} S_N^{(N-k)} \cdot \binom{N-k}{N-2m} \left(\frac{N-1}{2}\right)^{2m-k}$.

The spectrum of the resultant Lagrange interpolator is $\hat{\beta}_{\vec{\eta}}(\omega) = \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{N+1} \left(\prod_{m=1}^{\lfloor N/2 \rfloor} (\omega - \gamma_{2m-1})(\omega + \gamma_{2m}^*)\right)$.

► **B-spline** based FDFs were known long before [10], [11], e.g. see [12]. The results were encouraging. They are the closest relatives of GenCESPs and they are obtained by setting $\vec{\eta} = ([\alpha_n])$ and the poles $\alpha_n = 0, n \in [1, N+1]$.

► **Matérn B-splines** [17] have never been used for FDF design. They are obtained by setting $\vec{\eta} = ([\alpha_n])$, $\alpha_n = \alpha$ with condition $\Re(\alpha) \leq 0$. Due to loss in diversity of α , the continued product reduces to a power function and Matérn B-splines are defined for any $N \in \mathbb{R}, N > -1$.

Besides these examples, FDFs based on **Fractional B-splines** [18] and some other typical spline families (e.g. hyperbolic, trigonometric and Tchebycheffian splines) can easily be constructed by tuning the parameters of GenCESP based FDF.

weights $c[k]$ can be obtained by inverse-filtering, $((\varphi)^{-1} * s)[k]$ [14], [15]. Back-substituting the coefficients into (3) provides

$$s(t) = \sum_{k=-\infty}^{+\infty} s[k] \underbrace{((\varphi)^{-1}[k] * \varphi(t-k))}_{\text{interpolating form of } \varphi(t)=\varphi_{\text{int}}(t)}$$

and, from now on, we will refer to the new interpolating function as $\varphi_{\text{int}}(t) = (\varphi)^{-1}[n] * \varphi(t-n)$. Remarkably, the spectrum $\hat{\varphi}_{\text{int}}(\omega) = (\hat{\varphi}(\omega)) / (\sum_{n=-\infty}^{+\infty} \hat{\varphi}(\omega + 2\pi n))$ conforms to the condition in (2). If $\hat{\varphi}(\omega)$ is an interpolant, then $\hat{\varphi}_{\text{int}}(\omega) = \hat{\varphi}(\omega)$ as $\sum_{n=-\infty}^{+\infty} \hat{\varphi}(\omega + 2\pi n) = 1$ in $\hat{\varphi}_{\text{int}}(\omega)$ (due to (2)).

In context of FDFs, generalized interpolation based design can be very advantageous. A delay of τ is simply a shift in the basis function, meaning

$$s_{\tau}[k] = \sum_{n=-\infty}^{+\infty} s[n] ((\varphi)^{-1}[n] * \varphi(k-n-\tau)). \quad (4)$$

Let $\hat{p}(z) = \sum_{k=-\infty}^{+\infty} p[k]z^{-k} = \mathcal{Z}\{p[k]\}$ denote the z -transform of sequence $p[k], k \in \mathbb{Z}$. By letting $\tau = (P/Q), P, Q \in \mathbb{Z}$, we obtain,

$$\mathcal{Z}\{s_{\tau}[k]\} = [z^{-P} \mathcal{Z}\{s[k/Q]\}]_{Q\downarrow}$$

where $Q \downarrow$ denotes the decimation operation. From (4), we have

$$\mathcal{Z}\{s_{\tau}[k]\} = \hat{s}(z) \underbrace{\hat{\varphi}^{-1}(z) [z^{-P} \mathcal{Z}\{\varphi(k/Q)\}]_{Q\downarrow}}_{\text{Fractional Delay Filter}=\hat{\varphi}_{\text{FDF}}(z)}$$

The half-sample delay case (or $P/Q = 0.5$) is interesting and is mildly connected with the dilation equation or the two-scale [2] relationship (which is the key to the multiresolution structure of the wavelet transform):

$$\frac{1}{2} \varphi\left(\frac{t}{2}\right) = \sum_{k=-\infty}^{+\infty} h[k] \varphi(t-k) \Leftrightarrow \hat{h}(\omega) = \frac{\hat{\varphi}(2\omega)}{\hat{\varphi}(\omega)}.$$

Functions satisfying such a condition can be good contenders for half-sample delay FDFs. For example, B-splines [9], [14] of odd degree satisfy this constraint. A B-spline of degree N or $\beta^N(t)$ has Fourier transform $\hat{\beta}(\omega) = ((\sin(\omega/2)/(\omega/2))^{N+1})$. It is straight forward to establish that for Q times upsampled splines, there exists a function $\hat{h}(\omega)$ such that

$$\hat{h}(\omega) = \frac{\hat{\beta}^N(Q\omega)}{\hat{\beta}^N(\omega)} = \frac{1}{Q^{N-1}} \left(\frac{\sin(Q\omega/2)}{\sin(\omega/2)}\right)^{N+1} \quad (Q - \text{scale relationship}).$$

In the case of discrete B-splines, $\hat{h}(z)|_{z=e^{j\omega}}$ takes the form of $(N+1)$ -times continued product of the Dirichlet kernel. This indicates that $\hat{h}(z)$ is a cascade of $(N+1)$ moving average filters, each having Q -coefficients (a finite impulse response or FIR filter). As shown in [10], the B-spline based FDF can be written as,

$$\hat{\varphi}_{\text{B-spline-FDF}}(z) = \frac{1}{\hat{\beta}^N(z)} \underbrace{[z^{-P} \hat{h}(z) \hat{\beta}^N(z)]_{Q\downarrow}}_{\text{FIR filter}}. \quad (5)$$

We will now design FDFs based on (4) using generalized splines [13]. Thanks to this generalization, it gives us more flexibility for an efficient filter design with computational complexity which is comparable to B-splines.

B. Generalized Cardinal Exponential Splines (GenCESP)

Definition: GenCESPs are a generalization of exponential splines [13]. A GenCESP of order N parametrized by $\vec{\eta} = (\vec{\alpha}; \vec{\gamma})$ with $\vec{\alpha} = [\alpha_1, \dots, \alpha_N]$ and $\vec{\gamma} = [\gamma_1, \dots, \gamma_M], M < N$, $\alpha_n \neq \gamma_m$ is defined as [13],

$$\beta_{\vec{\eta}}(t) \stackrel{\text{def}}{=} \Delta_{\vec{\eta}}\{\rho_{\vec{\eta}}(t)\} \quad (6)$$

where

► $\rho_{\vec{\eta}}(t) = \mathbf{1}_+(t)e^{\eta t}$ is a causal-exponential function satisfying the constraint set by the first-order operator $L_{\vec{\eta}} = \mathcal{D} - \eta \mathcal{I}$ and where \mathcal{D} denotes (Euler's) differential operator and $\mathcal{I} = \mathcal{D}^0$ is the identity operator. In mathematical literature, $\rho_{\vec{\eta}}(t)$ is called Green's function and is one that satisfies $L_{\vec{\eta}}\{\rho_{\vec{\eta}}(t)\} = \delta(t)$. In view of the following fact:

Cascade of first-order systems.

$$L_{\vec{\eta}} = \overbrace{(\mathcal{D} - \alpha_1 \mathcal{I}) * \dots * (\mathcal{D} - \alpha_N \mathcal{I})}^{\mathcal{L}} \leftarrow \mathcal{L} \rightarrow L_{\vec{\eta}}(s) = \frac{\prod_{n=1}^N (s - \alpha_n)}{\prod_{m=1}^M (s - \gamma_m)}$$

we conclude that $\rho_{\vec{\eta}}(t) = \mathcal{L}^{-1}\{1/L_{\vec{\eta}}(s)\}$ where \mathcal{L}^{-1} denotes the inverse of Laplace transform operator.

► $\Delta_{\vec{\kappa}}\{f_{\vec{\kappa}}(\cdot)\}$ is a K th forward-(exponentially) weighted difference operator with z -transform

$$\hat{\Delta}_{\vec{\kappa}}(z) \stackrel{\text{def}}{=} \prod_{k=1}^K (1 - e^{\kappa_k} z^{-1}). \quad (7)$$

The Fourier transform of $\beta_{\vec{\eta}}(t)$ is given by

$$\beta_{\vec{\eta}}(t) \xrightarrow{\text{FT}} \hat{\beta}_{\vec{\eta}}(\omega) = \left(\prod_{n=1}^N \frac{1 - e^{\alpha_n - j\omega}}{j\omega - \alpha_n}\right) \prod_{m=1}^M (j\omega - \gamma_m)$$

(8)

which can also be written as $\hat{\beta}_{\vec{\eta}}(\omega) = (\hat{\Delta}_{\vec{\alpha}}(e^{j\omega})/L_{\vec{\eta}}(j\omega))$.

C. FDF Design Based on GenCESP

Stability of GenCESP for Practicable FDFs: In order to devise physically realizable FDFs using GenCESP, we need to assure that the space generated by integer-translates of the extended family of splines i.e., $V(\beta_{\vec{\eta}})$, forms a Riesz basis. This property has been proven in [13], provided that $\Im(\alpha_m) - \Im(\alpha_n) \neq j2\pi p, p \in \mathbb{Z}, \forall m \neq n$, is true for all pairs of distinct and imaginary poles. This will be a constraint on our design.

Q-Scale Relation: This relationship for GenCESP is not direct. From simple algebraic manipulation of (8), it can be inferred that there exists a function $\hat{h}(e^{j\omega})$ such that

$$\hat{h}(e^{j\omega}) = \frac{\hat{\beta}_{\vec{\eta}}(Q\omega)}{\hat{\beta}_{\vec{\eta}/Q}(\omega)} = \frac{1}{Q^{N-M-1}} \frac{\Delta_{\vec{\alpha}}(e^{jQ\omega})}{\Delta_{\vec{\alpha}/Q}(e^{j\omega})}$$

which shows that one GenCESP with parameter $(\vec{\eta})$ bears a ratio with another GenCESP with Q -scaled parameter $(\vec{\eta}/Q)$. In z -domain, $\hat{h}(e^{j\omega})$ (independent of γ_m) takes form of

$$\hat{h}(z) = \frac{1}{Q^{N-M-1}} \prod_{n=1}^N \sum_{q=0}^{Q-1} \left(e^{\frac{\alpha_n}{Q}} z^{-1} \right)^q \text{ (FIR - Filter)} \quad (9)$$

which happens to be a $QN - 1$ size FIR filter. For $\vec{\eta} = 0$, it corresponds to a Q -coefficient, N -point moving average filter.

FDF based on GenCESP: Using the formalism developed so far, the transfer function of an FDF based on the generalized family of splines is given by

$$\hat{\beta}_{\vec{\eta}, \text{FDF}}(z) = \underbrace{\hat{\beta}_{\vec{\eta}}^{-1}(z)}_{\text{IIR-Part}} \underbrace{[z^{-P} \hat{h}(z) \hat{\beta}_{\vec{\eta}/Q}(z)]_{Q\downarrow}}_{\text{FIR-Part}} \quad (10)$$

where $\hat{\beta}_{\vec{\eta}}^{-1}(z)$ is the transfer function of the inverse discrete spline filter (IDSF)—usually an infinite impulse response (IIR) filter. The structure of (10) is a generalization of the FDF proposed in [10]. The FDF in (10) can be efficiently implemented by direct z -domain filtering algorithm without having to carry out interpolation and decimation operations. The key advantage of using (10) is that we have a higher degree of freedom as one can readily access libraries of other splines and FDFs. We present some examples in Table I.

D. Design Example Based on GenCESP

GenCESP based FDF design is quite non-specific in general [a characteristic of (8)]. The performance of FDF depends on properties of (8). Depending upon the specific needs pertaining to FDF-design, one can tailor a GenCESP by selecting $\vec{\eta}$ which optimizes parameters such as regularity, order of approximation, even-symmetry of spline, order of spline etc. which have been studied in approximation theory/interpolant design literature (cf. [7], [15] and references therein).

In order to facilitate analysis, we exemplify a particular case [15] which will also serve as a design example. Consider the configuration $\vec{\eta} = ([0, 0, 0, 0]; [-j\sqrt{42}, j\sqrt{42}])$. The resultant GenCESP is

$$\begin{aligned} \hat{\beta}_{\vec{\eta}}(\omega) &= \left(\frac{1 - e^{-j\omega}}{j\omega} \right)^4 \cdot (j(\omega - \sqrt{42}))(j(\omega + \sqrt{42})) \\ &\Rightarrow \beta_{\vec{\eta}}(t) = \beta^3(t) + \frac{1}{42} \mathcal{D}^2 \beta^3(t). \end{aligned}$$

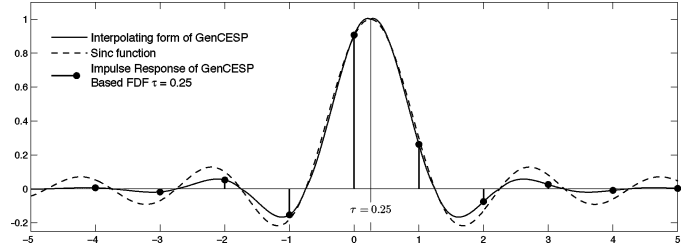


Fig. 1. Impulse response of GenCESP based FDF of order 4 with parameter $\vec{\eta} = ([0, 0, 0, 0]; [-j\sqrt{42}, j\sqrt{42}])$. The plot compares $\varphi_{\text{int}}(t - \tau)$, $\varphi = \beta_{\vec{\eta}}$ and $\text{sinc}(t - \tau)$ with $\tau = 1/4$. The impulse response of $\beta_{\vec{\eta}, \text{FDF}}[n]$ has an exponential decay rate—a desirable property which makes the FDF robust to distortion caused due to truncation.

Using the derivative property of B-splines [14], $\mathcal{D}^2 \beta^n(t) = (\delta(t - 1) - 2\delta(t) + \delta(t + 1)) * \beta^{n-2}(t)$, we have, $\beta_{\vec{\eta}}(k) = (8/42)\delta(k+1) + (26/42)\delta(k) + (8/42)\delta(k-1)$ and its z -transform $\mathcal{Z}\{\beta_{\vec{\eta}}(k)\} = \hat{\beta}_{\vec{\eta}}(z) = (8z + 26 + 8z^{-1})/42$. Similarly, depending upon Q , $\hat{\beta}_{\vec{\eta}/Q}(z)$ can be easily derived. Using (9), we have $\hat{h}(z) = (1 + z^{-1})^4$. This contributes to the FIR-part in (10). The resulting IDSF, $\hat{\beta}_{\vec{\eta}}^{-1}(z) = 42/(8z + 26 + 8z^{-1})$ is an all-pole, IIR filter. Following [14], it can be factorized into a first-order causal and anticausal filter

$$\begin{aligned} \hat{\beta}_{\vec{\eta}}^{-1}(z) &= \frac{-42\mu}{(1 - \mu^2)} \left(\frac{1}{1 - \mu z^{-1}} + \frac{1}{1 - \mu z} - 1 \right) \quad (\text{IIR-Part}) \\ &\Rightarrow z^{-1} \left\{ \hat{\beta}_{\vec{\eta}}^{-1}(z) \right\} = \frac{-42\mu}{(1 - \mu^2)} \mu^{|k|} \end{aligned}$$

where $\mu = (-13 + \sqrt{105})/8$ is the smallest (in magnitude) root of the transfer function of $\hat{\beta}_{\vec{\eta}}^{-1}(z)$ —an even symmetric filter with exponentially decreasing coefficients, thus characterizing $\beta_{\vec{\eta}, \text{FDF}}[n]$ with an exponential decay. Fig. 1 compares the impulse response of $\text{sinc}(t - \tau)$, interpolating version of $\beta_{\vec{\eta}}(t - \tau)$ and the FDF, $\beta_{\vec{\eta}, \text{FDF}}[n]$ for $\tau = P/Q = 1/4$. The coefficients of FDF decay exponentially making it robust against truncation.

Fig. 2 depicts the frequency response [Fig. 2(a)] and phase delay characteristics [Fig. 2(b)] of the (cubic) GenCESP and cubic B-spline [10] based FDF. The plot is obtained by substituting $z = e^{j\omega}$ in (10) with $\tau = (P/Q)$, $P = 1$ to 5 and $Q = 10$. The phase delay characteristics for the two FDFs are comparable. As $\tau \rightarrow 0.5$, both the FDFs approach an ideal half-sample delay response, however, as seen in Fig. 2(c), GenCESP is marginally slower.

Finally, we compare the interpolation error for the worst case approximation i.e., half-sample delay, using the same setup as in [10]: The test signal is $s[n] = \sin(n)$ ($\omega = 1$) and the absolute interpolation error is given by $|s_\tau[n] - \tilde{s}_\tau[n]|$, with $\tau = 0.5$ and where $\tilde{s}_\tau[n]$ is obtained by using FDF in (10). As seen in Fig. 3, GenCESP based FDF outperforms the scheme presented in [10] and its competitors, therein.

III. DISCUSSION AND CONCLUSION

Our aim here was to emphasize that GenCESP—a richer class of splines—can be used to design FDFs which perform better than (most) existing solutions e.g., Lagrange interpolator, B-splines, cubic convolution kernel [19], sinc, etc. For the same order of spline, the computational complexity is comparable to

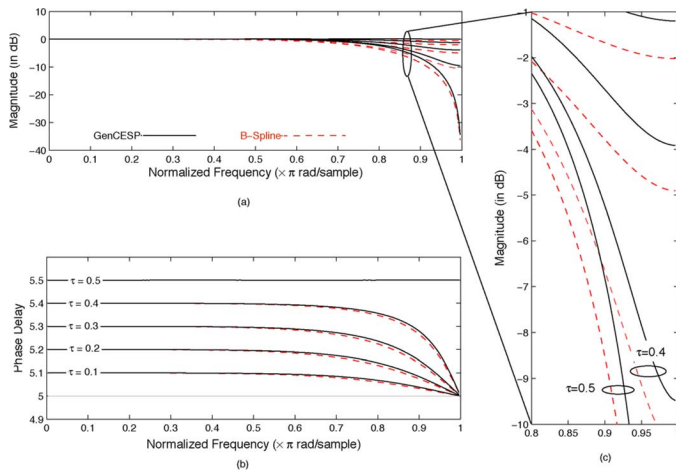


Fig. 2. (a) and (c) Magnitude response of GenCESP and B-spline based FDF for $\tau = P/Q$, $P = 1$ to 5 and $Q = 10$. (b) Phase delay characteristics.

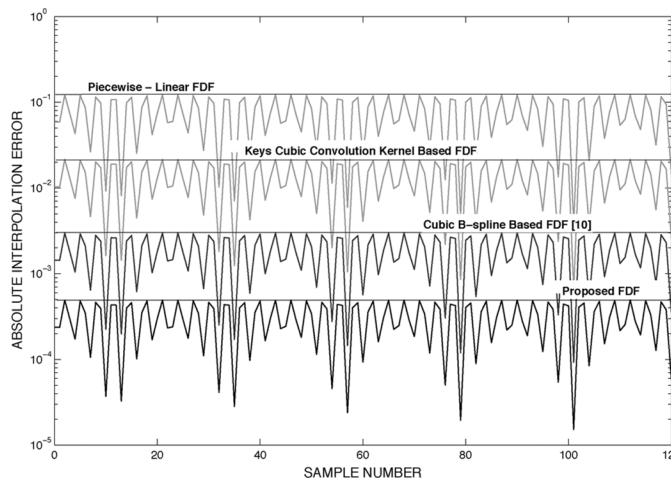


Fig. 3. Comparison of absolute interpolation errors incurred using FDFs based on GenCESP, cubic B-spline, Keys' cubic convolution kernel [19] and piecewise linear kernel (or GenCESP with $\vec{\eta} = [0, 0]$).

any other spline based technique [13]. The key advantages of using GenCESP for FDF design include:

- flexibility in design as GenCESP is a library of many other well known classes of splines;
- precise fractional delays for half-sample delay case due to linear phase characteristics and for other cases, the phase response is linear for frequency range $|\omega| \leq (\pi/2)$. Furthermore, we do not rely on windowing or Taylor series based methods. While the former suffers with phase-shift problem [1], the later leads to inaccurate designs as one has to truncate the series to n -terms;
- ease of implementation as the filters can be decomposed into recursive, causal and anticausal filters which considerably reduce the computational complexity [14];

- lesser interpolation errors, depending on the choice of $\vec{\eta}$.

The comparisons carried out in this paper are associated with a particular configuration of GenCESP. One can always construct an FDF which is better than the proposed example and extend designs for arbitrary shifts as in [11]. Systematic design and optimization of GenCESPs that are tailored for fractional delay filtering applications is relatively unexplored and it can be addressed in future studies.

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