Type II codes over $\mathbb{F}_{2^m} + u\mathbb{F}_{2^m}$ ☆

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Abstract

We look at the special class of self-dual codes called Type II codes over the alphabet $R_m = \mathbb{F}_{2^m} + u\mathbb{F}_{2^m}$ where $u^2 = 0$. Properties of the Gray maps which take self-dual codes over R_m to self-dual codes over the subrings $\mathbb{F}_{2^m}$ and $R_r = \mathbb{F}_{2^r} + u\mathbb{F}_{2^r}$ for divisors r of m are studied. We give a mass formula for Type II codes over R_m and classify such codes of length $n \leqslant 8$ for $m = 2, 3, 4, 5, 6$ and 7.

Keywords: Type II codes; Mass formula; Gray map; Self-dual codes; Classification

1. Introduction

There has been much interest lately in a special class of self-dual codes, called Type II codes, over finite fields and rings of characteristic 2. One can define a Gray map from $\mathbb{F}_{2^r}^n$ to $\mathbb{F}_2^{rn}$ with respect to a fixed trace-orthogonal basis of $\mathbb{F}_{2^r}$ over $\mathbb{F}_2$ (cf. [5]). A Type II code over $\mathbb{F}_{2^r}$ is a self-dual code whose image under the Gray map is a doubly even self-dual binary code. This definition has been shown in [1] to be independent of the choice of trace-orthogonal basis.

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In [6], Type II codes were studied over the alphabet $\mathbb{F}_2+u\mathbb{F}_2$, where u is a nilpotent ring element of order 2. Among the motivations for these investigations were the similarities of such codes with quaternary Type II codes. Type II codes over $\mathbb{F}_4$ were studied in [4,8], while Type II codes over the ring $\mathbb{F}_4+u\mathbb{F}_4$ were introduced in [11]. Codes over this ring were shown to yield, under suitable Gray mappings, Type II codes over the subrings $\mathbb{F}_4$ and $\mathbb{F}_2+u\mathbb{F}_2$, and consequently, doubly even self-dual binary codes.

In this paper we extend these notions and results to the ring $\mathbb{F}_{2^m}+u\mathbb{F}_{2^m}$. We define Gray maps from $\mathbb{F}_{2^m}+u\mathbb{F}_{2^m}$ to the subrings $\mathbb{F}_{2^m}$, $\mathbb{F}_{2^r}$ and $\mathbb{F}_{2^r}+u\mathbb{F}_{2^r}$ for divisors r of m, and study the properties of Type II codes in these rings. We build on a result given in [6] and give a mass formula for Type II codes over $\mathbb{F}_{2^m}+u\mathbb{F}_{2^m}$. This resolves, in the general setting, the open problem posed in [11] on the existence of a mass formula for Type II codes over $\mathbb{F}_4+u\mathbb{F}_4$, and allows us to classify codes of modest lengths over the ring $\mathbb{F}_{2^m}+u\mathbb{F}_{2^m}$ for $m=2,3,4,5,6$ and 7.

2. Preliminaries

In this section, we give the necessary definitions and notations (cf. [10]).

Let q be a power of a prime number. We denote by $\mathbb{F}_q$ the finite field of order q. We recall some definitions on finite fields. If $E=\mathbb{F}_{q^m}$ is a finite extension of a finite field $F=\mathbb{F}_q$, then the *trace* $\mathrm{Tr}_{E/F}(\alpha)$ over F of an element $\alpha\in E$ is defined as

$$\mathrm{Tr}_{E/F}(\alpha)=\sum_{i=0}^{m-1}\alpha^{q^i}.$$

A basis $B=\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ of E over F is a *trace-orthogonal basis* (TOB) if

$$\mathrm{Tr}_{E/F}(\alpha_i\alpha_j)=\begin{cases}\text{non-zero} & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$$

In particular, a TOB is called a *self-dual basis* or *self-complementary basis* if $\mathrm{Tr}_{E/F}(\alpha_i^2)=1$ for $i=1,2,\ldots,m$. It is known (cf. [9]) that if q is even, there exists a self-dual basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ for all $m\geqslant 1$.

Let $B=\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ be a TOB of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$. Since the base field is the binary field, B is a self-dual basis. If $x\in\mathbb{F}_{2^m}$ is written as $x=\sum_{i=1}^m a_i\alpha_i$, with $a_i\in\mathbb{F}_2$, then the *Lee weight* $\mathrm{wt}_\mathrm{L}^B(x)$ *with respect to* B of the element x is the number of i's with $a_i=1$. (We shall simply call this the Lee weight if the reference to the basis B is clear.) The Lee weight $\mathrm{wt}_\mathrm{L}^B(v)$ of a vector $v\in\mathbb{F}_{2^m}^n$ is the sum of the Lee weights of its components.

A *code* of length n over $\mathbb{F}_{2^m}$ is an $\mathbb{F}_{2^m}$-subspace of $\mathbb{F}_{2^m}^n$. Elements of codes are called *codewords*. Two codewords $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ are *orthogonal* if their Euclidean inner product $x\cdot y=\sum_i x_iy_i$ is zero. (We shall use this notation for the Euclidean inner product for other settings as well.) The *dual* $C^\perp$ of a code C consists of all vectors of $\mathbb{F}_{2^m}^n$ which are orthogonal to every codeword in C. A code C is said to be *self-dual* (resp. *self-orthogonal*) if $C=C^\perp$ (resp. $C\subseteq C^\perp$).

Two codes are *permutation-equivalent* if one can be obtained from the other by permutation of coordinates. Note that this is not the usual definition of equivalence: *monomial-equivalence*. We have omitted multiplication by non-zero scalars because this does not, in general, preserve the self-duality and Lee weights generally.

A self-dual code C is said to be of *Type II* with respect to the basis B if the Lee weight of every codeword in C is a multiple of 4. If $m=1$, Type II codes are called doubly even self-dual binary codes. For $m=2$, Type II codes with respect to the unique TOB $\{\omega,\bar{\omega}\}$ were studied in [4,8]. In [1], it was shown that the definition of Type II codes over $\mathbb{F}_{2^m}$ is independent of the choice of TOB. Thus we may choose and fix what we deem as a suitable TOB of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$.

Define the Gray map

$$\psi'_{m,1} : (\mathbb{F}_{2^m})^n \to (\mathbb{F}_2)^{mn},$$

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m \mapsto (x_1,x_2,\ldots,x_m),$$

where $x_1,x_2,\ldots,x_m \in \mathbb{F}_2^n$. The Gray map $\psi'_{m,1}$ is an isometry which maps self-dual codes (resp. Type II codes) over $\mathbb{F}_{2^m}$ into self-dual binary codes (resp. doubly even self-dual binary codes) (cf. [5]). We point out that while this definition of the Gray map depends on the order of the basis elements α_i, a reordering of basis elements yields a Gray image which is permutation-equivalent.

Let R_m denote the commutative ring $\mathbb{F}_{2^m}+u\mathbb{F}_{2^m}=\mathbb{F}_{2^m}[u]/(u^2)$ of order 2^{2m} containing the nilpotent element u with $u^2=0$. A *code* C of length n over R_m is an R_m-submodule of R_m^n. An *additive code* is a subgroup of $(R_m^n,+)$. Elements of C are called *codewords*. Again, duality is defined with respect to the Euclidean inner product. A code C is said to be *self-dual* (resp. *self-orthogonal*) if $C = C^\perp$ (resp. $C \subseteq C^\perp$). Two codes over R_m are said to be *symmetry-equivalent* (or *equivalent*, whenever no confusion arises) if one is obtained from the other by permutation of coordinates and multiplication by $(1+u)$ in certain coordinates.

Following [6,11], we define the Gray map

$$\phi_m : R_m^n \to \mathbb{F}_{2^m}^{2n},$$

$$x + yu \mapsto (y, x+y),$$

where $x,y \in \mathbb{F}_{2^m}^n$. The map ϕ_m is an $\mathbb{F}_{2^m}$-linear isometry from (R_m^n, Lee distance) to ($\mathbb{F}_{2^m}^{2n}$, Lee distance).

Fix a TOB $B=\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$. We define the *Lee weight with respect to* B of an element $a+bu \in R_m$ (where $a,b \in \mathbb{F}_{2^m}$) to be the Lee weight with respect to B of its image $(b,a+b)$ under ϕ_m. Again, we omit reference to the basis B if the context is clear. The *Lee weight* $\mathrm{wt}_{\mathrm{L}}^{B}(v)$ *of a vector* $v \in R_m^n$ is the sum of the Lee weights of its components. A self-dual code over R_m is called *Type II* if the Lee weight of every codeword is a multiple of 4.

We can extend in the natural way the map $\psi'_{m,1}$ to the Gray map

$$\psi_{m,1} : R_m^n \to R_1^{mn},$$

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m \mapsto (x_1,x_2,\ldots,x_m),$$

where $x_1, x_2, \ldots, x_m \in R_1^n$. The Gray map $\psi_{m,1}$ is an R_1-linear isometry from (R_m^n, Lee distance) to (R_1^{mn}, Lee distance) where the *Lee distance between two codewords* c_1 and c_2 is defined to be the Lee weight of $c_1 - c_2$.

If m has factorization $m = rs$, we choose bases for the tower of finite fields $\mathbb{F}_2 \subseteq \mathbb{F}_{2^r} \subseteq \mathbb{F}_{2^m}$ in the following manner. First we consider the more general setting.

Consider the tower $F = \mathbb{F}_q \subseteq E = \mathbb{F}_{q^r} \subseteq K = \mathbb{F}_{q^{rs}}$ and let $B_1 = \{\alpha_1, \ldots, \alpha_s\}$ and $B_2 = \{\beta_1, \ldots, \beta_r\}$ be TOBs for K/E and E/F, respectively. In general, it does not follow that $B = \{\alpha_i \beta_j | 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant r\}$ is a TOB for K/F. (Consider for example $\mathbb{F}_{16} = \mathbb{F}_2[\alpha]/(\alpha^4 + \alpha + 1)$. While $\{\alpha^5, \alpha^{10}\}$ and $\{\alpha^7, \alpha^{13}\}$ are TOBs for $\mathbb{F}_4/\mathbb{F}_2$ and $\mathbb{F}_{16}/\mathbb{F}_4$, respectively, $\{\alpha^2, \alpha^3, \alpha^8, \alpha^{12}\}$ is not a TOB of $\mathbb{F}_{16}$ over $\mathbb{F}_2$.) However, the following holds:

Proposition 1. *Let B, B_1 and B_2 be the bases defined above. If B_1 and B_2 are TOBs such that $\mathrm{Tr}_{K/E}(\alpha_i^2)$ is a non-zero element of F for every $i = 1, 2, \ldots, s$, then B is a TOB. In particular, if B_1 and B_2 are self-dual, then B is self-dual.*

Proof. We want to show that $\mathrm{Tr}_{K/F}(\alpha_i \beta_j \alpha_{i'} \beta_{j'})$ is non-zero if and only if $i = i'$ and $j = j'$.

If $\mathrm{Tr}_{K/E}(\alpha_i \alpha_j)$ is a non-zero element of F for $i = j$, we have

$$\begin{aligned}
\mathrm{Tr}_{K/F}(\alpha_i \beta_j \alpha_{i'} \beta_{j'}) &= \mathrm{Tr}_{E/F}(\mathrm{Tr}_{K/E}(\alpha_i \beta_j \alpha_{i'} \beta_{j'})) \\
&= \mathrm{Tr}_{E/F}(\beta_j \beta_{j'} \mathrm{Tr}_{K/E}(\alpha_i \alpha_{i'})) \\
&= \begin{cases} \mathrm{Tr}_{K/E}(\alpha_i^2)\mathrm{Tr}_{E/F}(\beta_j \beta_{j'}) & \text{if } i = i', \\ 0 & \text{if } i \neq i', \end{cases} \\
&= \begin{cases} \text{non-zero} & \text{if } i = i' \text{ and } j = j', \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

If B_1 and B_2 are self-dual bases, then

$$\mathrm{Tr}_{K/E}(\alpha_i \alpha_{i'}) = \begin{cases} 1 & \text{if } i = i', \\ 0 & \text{if } i \neq i' \end{cases}$$

and

$$\mathrm{Tr}_{E/F}(\beta_j \beta_{j'}) = \begin{cases} 1 & \text{if } j = j', \\ 0 & \text{if } j \neq j'. \end{cases}$$

Therefore we get

$$\mathrm{Tr}_{K/F}(\alpha_i \beta_j \alpha_{i'} \beta_{j'}) = \begin{cases} 1 & \text{if } i = i' \text{ and } j = j', \\ 0 & \text{otherwise,} \end{cases}$$

showing that B is self-dual. □

Thus, rather than arbitrary TOBs, we choose self-dual bases $B_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ of $\mathbb{F}_{2^m}$ over $\mathbb{F}_{2^r}$ and $B_2 = \{\beta_1, \beta_2, \ldots, \beta_r\}$ of $\mathbb{F}_{2^r}$ over $\mathbb{F}_2$. We may therefore define the intermediate Gray maps $\psi'_{m,r}$ and $\psi'_{r,1}$ as follows:

$$\psi'_{m,r} : \mathbb{F}_{2^m}^n \to \mathbb{F}_{2^r}^{sn},$$

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_s\alpha_s \mapsto (x_1, x_2, \ldots, x_s)$$

for $x_1, x_2, \ldots, x_s \in \mathbb{F}_{2^r}^n$ and

$$\psi'_{r,1} : \mathbb{F}_{2^r}^n \to \mathbb{F}_2^{rn},$$

$$x_1\beta_1 + x_2\beta_2 + \cdots + x_r\beta_r \mapsto (x_1, x_2, \ldots, x_r)$$

for $x_1, x_2, \ldots, x_r \in \mathbb{F}_2^n$. These maps have the following natural extensions:

$$\psi_{m,r} : R_m^n \to R_r^{sn},$$

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_s\alpha_s \mapsto (x_1, x_2, \ldots, x_s),$$

where $x_1, x_2, \ldots, x_s \in (\mathbb{F}_{2^r} + u\mathbb{F}_{2^r})^n$ and

$$\psi_{r,1} : R_r^n \to R_1^{rn},$$

$$x_1\beta_1 + x_2\beta_2 + \cdots + x_r\beta_r \mapsto (x_1, x_2, \ldots, x_r),$$

where $x_1, x_2, \ldots, x_r \in (\mathbb{F}_2 + u\mathbb{F}_2)^n$.

By the previous proposition, self-dual bases B_1 and B_2 also give us a self-dual basis $B = \{\alpha_i\beta_j | 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant r\}$ of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$ with the corresponding Gray maps

$$\psi'_{m,1} : \mathbb{F}_{2^m}^n \to \mathbb{F}_2^{mn}$$

and

$$\psi_{m,1} : R_m^n \to R_1^{mn}.$$

From a suitable ordering of the basis elements $\alpha_i\beta_j$, we get $\psi'_{m,1} = \psi'_{r,1} \circ \psi'_{m,r}$ and $\psi_{m,1} = \psi_{r,1} \circ \psi_{m,r}$. The choice of self-dual basis is necessary so that the corresponding Gray maps will preserve the self-dual property of a code.

3. Codes over $\mathbb{F}_{2^m} + u\mathbb{F}_{2^m}$

In this section, we study the properties of the Gray maps and Type II codes over R_m.

Proposition 2. *Let C be a code of length n over R_m. If C is self-orthogonal, so is $\phi_m(C)$. A code C is of Type II over R_m if and only if $\phi_m(C)$ is a Type II code over $\mathbb{F}_{2^m}$. The minimum Lee weight of C is equal to the minimum Lee weight of $\phi_m(C)$.*

Proof. Let $c = x + yu$ and $c' = x' + y'u$ be codewords in C, where $x, y, x', y' \in \mathbb{F}_{2^m}^n$. Then the Euclidean inner product of c and c' is $c \cdot c' = x \cdot x' + (x \cdot y' + x' \cdot y)u$. If C is self-orthogonal, $x \cdot x' = x \cdot y' + x' \cdot y = 0$. Thus in $\mathbb{F}_{2^m}$, we have $\phi_m(c) \cdot \phi_m(c') = x \cdot x' + x \cdot y' + x' \cdot y = 0$.

The last two statements hold because ϕ is an isometry. $\square$

Proposition 3. *Let $E=\mathbb{F}_{2^m}$ and $F=\mathbb{F}_{2^r}$ with $m=rs$, and ψ' the Gray map corresponding to a self-dual basis $B=\{\alpha_1,\alpha_2,\ldots,\alpha_s\}$ of E over F. Let C be a code of length n over E. If C is self-orthogonal, so is $\psi'_{m,r}(C)$. A code C is of Type II over E if and only if $\psi'_{m,r}(C)$ is a Type II code over F. The minimum Lee weight of C is equal to the minimum Lee weight of $\psi'_{m,r}(C)$.*

Proof. Let $c=\sum_{i=1}^{s} a_i\alpha_i$ and $c'=\sum_{i=1}^{s} a'_i\alpha_i$ be codewords in C, where $a_i, a'_i \in F^n$. If C is self-orthogonal, then

$$c\cdot c' = \sum_{i,j}(a_i\cdot a'_j)\alpha_i\alpha_j = 0.$$

Taking the trace over F, we get

$$0=\mathrm{Tr}_{E/F}\left(\sum_{i,j}(a_i\cdot a'_j)\alpha_i\alpha_j\right)=\sum_{i,j}\mathrm{Tr}_{E/F}((a_i\cdot a'_j)\alpha_i\alpha_j)$$
$$=\sum_{i,j}(a_i\cdot a'_j)\mathrm{Tr}_{E/F}(\alpha_i\alpha_j).$$

Since B is a self-dual basis, all the terms of the sum with $i\neq j$ vanish while $\mathrm{Tr}_{E/F}(\alpha_i^2)=1$ for all i, leaving $\sum_{i=j} a_i\cdot a'_j=0$. This latter sum is equal to $\psi'_{m,r}(c)\cdot\psi'_{m,r}(c')$. Hence $\psi'_{m,r}(C)$ is self-orthogonal.

The last two statements hold because $\psi'_{m,r}$ is an isometry. □

Proposition 4. *Let $m=rs$, and $\psi_{m,r}$ the Gray map from R_m to R_r corresponding to a self-dual basis $B=\{\alpha_1,\alpha_2,\ldots,\alpha_s\}$ of $E=\mathbb{F}_{2^m}$ over $F=\mathbb{F}_{2^r}$. Let C be a code of length n over R_m. If C is self-orthogonal, so is $\psi_{m,r}(C)$. A code C is of Type II over R_m if and only if $\psi_{m,r}(C)$ is a Type II code over R_r. The minimum Lee weight of C is equal to the minimum Lee weight of $\psi_{m,r}(C)$.*

Proof. Let $c=\sum_{i=1}^{s} a_i\alpha_i$ and $c'=\sum_{i=1}^{s} a'_i\alpha_i$ be codewords in C and write $a_i, a'_i\in R_r$ as $a_i=a_{i1}+ua_{i2}$ and $a'_i=a'_{i1}+ua'_{i2}$, where $a_{i1},a_{i2},a'_{i1},a'_{i2}\in\mathbb{F}_{2^r}$. Then the inner product $c\cdot c'$ is equal to

$$c\cdot c'=\sum_{i,j}(a_{i1}\cdot a'_{j1})\alpha_i\alpha_j+u\sum_{i,j}(a_{i1}\cdot a'_{j2}+a'_{j1}\cdot a_{i2})\alpha_i\alpha_j.$$

Define the function $t: R_m\to R_r$, given by $t(a+bu):=\mathrm{Tr}_{E/F}(a)+u\mathrm{Tr}_{E/F}(b)$. We then have

$$t(c\cdot c')=\sum_{i,j}(a_{i1}\cdot a'_{j1})\mathrm{Tr}_{E/F}(\alpha_i\alpha_j)+u\sum_{i,j}(a_{i1}\cdot a'_{j2}+a'_{j1}\cdot a_{i2})\mathrm{Tr}_{E/F}(\alpha_i\alpha_j)$$
$$=\sum_{i=1}^{s}(a_{i1}\cdot a'_{i1})+u\sum_{i=1}^{s}(a_{i1}\cdot a'_{i2}+a'_{i1}\cdot a_{i2})$$

$$= \sum_{i=1}^{s} (a_i \cdot a_i')$$

$$= \psi_{m,r}(c) \cdot \psi_{m,r}(c').$$

Thus if $c \cdot c' = 0$, then $\psi_{m,r}(c) \cdot \psi_{m,r}(c') = 0$.

The map $\psi_{m,r}$ is an isometry, thus the last two assertions hold. □

Corollary 5. *Let $d(\mathbb{F}_{2^m}, n)$ and $d(R_m, n)$ be the highest minimum Lee weights of a Type II code of length n over $\mathbb{F}_{2^m}$ and R_m, respectively. Then*

(1) $d(\mathbb{F}_{2^m}, n) \leqslant 4\lfloor \frac{mn}{24} \rfloor + 4$,
(2) $d(R_m, n) \leqslant 4\lfloor \frac{mn}{12} \rfloor + 4$.

Proof. The upper bound $4\lfloor n/24 \rfloor + 4$ on the Hamming weight of a Type II binary code of length n is given in [13,14]. □

Lemma 6. *If C and C' are equivalent self-dual codes over R_m, then their images under the Gray maps ϕ_m and $\psi_{m,r}$ are equivalent over $\mathbb{F}_{2^m}$ and R_r, respectively.*

Proof. This follows immediately from the definition of the Gray maps.

Proposition 7. *A self-dual code over R_m contains the all-u vector.*

Proof. If C is a self-dual code over R_m, then its Gray image $\psi_{m,1}(C)$ is also self-dual and contains the all-u vector (cf. [6, Proposition 3.5]). The proposition then follows from the definition of the Gray map. □

Corollary 8. *The minimum Lee weight of a self-dual R_m-code C of length n does not exceed $2n$.*

Proof. Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a self-dual basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$. Since the Gray map ϕ_m preserves weight, $\mathrm{wt}_{\mathrm{L}}^B(\alpha_i u) = \mathrm{wt}_{\mathrm{L}}^B(\alpha_i) + \mathrm{wt}_{\mathrm{L}}^B(\alpha_i) = 2$ for $i = 1, 2, \ldots, m$. Hence the element $\alpha_i(u, u, \ldots, u) \in C$ has Lee weight $2n$. □

Let $m = rs$ ($m > r \geqslant 1$). We define the automorphism σ_r on $\mathbb{F}_{2^r}^{2sn}$ as follows. A vector

$$c = (x_1, x_2, \ldots, x_{2sn}) \in \mathbb{F}_{2^r}^{2sn},$$

can be written as $c = (X_1, X_2, \ldots, X_{2s})$ where

$$X_i = (x_{(i-1)n+1}, x_{(i-1)n+2}, \ldots, x_{in})$$

for $i = 1, 2, \ldots, s$. Let

$$\sigma_r(c) := (X_1, X_{s+1}, X_2, X_{s+2}, \ldots, X_s, X_{2s}).$$

Proposition 9. *There is a commutative diagram*

$$\begin{array}{ccc} R_m^n & \xrightarrow[\psi_{m,r}]{} & R_r^{sn} \\ \phi_m \downarrow & & \downarrow \sigma_r\circ\phi_r \\ \mathbb{F}_{2^m}^{2n} & \xrightarrow[\psi'_{m,r}]{} & \mathbb{F}_{2^r}^{2sn}. \end{array}$$

Proof. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ be a self-dual basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_{2^r}$. If $c = \sum_{i=1}^{s} a_i\alpha_i + \sum_{i=1}^{s} b_i\alpha_i u \in R_m^n$, where $a_i, b_i \in \mathbb{F}_{2^r}^n$, we have

$$\psi'_{m,r} \circ \phi_m(c) = (b_1, a_1 + b_1, b_2, a_2 + b_2, \ldots, b_s, a_s + b_s) \in \mathbb{F}_{2^r}^{2sn}$$

and

$$\phi_r \circ \psi_{m,r}(c) = (b_1, b_2, \ldots, b_s, a_1 + b_1, a_2 + b_2, \ldots, a_s + b_s) \in \mathbb{F}_{2^r}^{2sn}.$$

Therefore $\psi'_{m,r} \circ \phi_m(c) = \sigma_r \circ \phi_r \circ \psi_{m,r}(c)$. □

Corollary 10. *If m has non-trivial factorization $m = rs$ $(r, s > 1)$ then there is a commutative diagram*

$$\begin{array}{ccccc} R_m^n & \xrightarrow[\psi_{m,r}]{} & R_r^{sn} & \xrightarrow[\psi_{r,1}]{} & R_1^{rsn} \\ \phi_m \downarrow & & \downarrow \sigma_r\circ\phi_r & & \downarrow \sigma_1\circ\phi_1 \\ \mathbb{F}_{2^m}^{2n} & \xrightarrow[\psi'_{m,r}]{} & \mathbb{F}_{2^r}^{2sn} & \xrightarrow[\psi'_{r,1}]{} & \mathbb{F}_{2}^{2rsn}. \end{array}$$

Proof. This follows immediately from the previous proposition. □

Proposition 11. *Let $m = rs$ and consider the commutative diagram*

$$\begin{array}{ccc} R_m^n & \xrightarrow[\psi_{m,r}]{} & R_r^{sn} \\ \phi_m \downarrow & & \downarrow \sigma_r\circ\phi_r \\ \mathbb{F}_{2^m}^{2n} & \xrightarrow[\psi'_{m,r}]{} & \mathbb{F}_{2^r}^{2sn}. \end{array}$$

Let C' be a code over $\mathbb{F}_{2^r}$ of length $2sn$. If there are codes C_1 over $\mathbb{F}_{2^m}$ and C_2 over R_r, of lengths $2n$ and sn, respectively, such that $\psi'_{m,r}(C_1) = \sigma_r \circ \phi_r(C_2) = C'$, then there is a code C over R_m of length n such that $\phi_m(C) = C_1$ and $\psi_{m,r}(C) = C_2$.

Proof. The maps in the commutative diagram above are bijections and hence, the pre-images $\phi_m^{-1}(C_1)$ and $\psi_{m,r}^{-1}(C_2)$ coincide; let it be $C \subseteq R_m^n$. The Gray maps are linear, thus C is an additive code over R_m. We need to show that C is an R_m-submodule of R_m^n. Let

$$\boldsymbol{v} = \sum_{i=1}^{s} v_i\alpha_i + \left(\sum_{i=1}^{s} w_i\alpha_i\right) u$$

be a codeword in C, where $B = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ is the usual self-dual basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_{2^r}$, and $v_i, w_i \in \mathbb{F}_{2^r}^n$. We shall prove that $\lambda\boldsymbol{v} \in C$ for any scalar $\lambda = \lambda_1 + \lambda_2 u \in R_m$, where

$\lambda_1, \lambda_2 \in \mathbb{F}_{2^m}$, by showing that $\phi_m(\lambda \boldsymbol{v}) \in C_1$. Since $\phi_m(\lambda \boldsymbol{v}) = \phi_m(\lambda_1 \boldsymbol{v} + \lambda_2 u\boldsymbol{v}) = \phi_m(\lambda_1 \boldsymbol{v}) + \phi_m(\lambda_2 u\boldsymbol{v})$, it suffices that we show $\phi_m(\lambda_1 \boldsymbol{v}) \in C_1$ and $\phi_m(\lambda_2 u\boldsymbol{v}) \in C_1$. The Gray map ϕ_m is $\mathbb{F}_{2^m}$-linear and C_1 is an $\mathbb{F}_{2^m}$-vector space, hence $\phi_m(\lambda_1 \boldsymbol{v}) = \lambda_1 \phi_m(\boldsymbol{v}) \in C_1$. Without losing generality, it remains to show that $\phi_m(u\boldsymbol{v}) \in C_1$.

The image of $u\boldsymbol{v} = (\sum_{i=1}^{s} v_i \alpha_i)u$ under ϕ_m is $\phi_m(u\boldsymbol{v}) = \sum_{i=1}^{s} (v_i, v_i)\alpha_i$. We claim that

$$\psi'_{m,r} \circ \phi_m(u\boldsymbol{v}) = (v_1, v_1, v_2, v_2, \ldots, v_s, v_s) \in C'.$$

The image $\psi_{m,r}(\boldsymbol{v}) = (v_1 + w_1 u, v_2 + w_2 u, \ldots, v_s + w_s u)$ is an element of C_2, and hence so is $u\psi_{m,r}(\boldsymbol{v}) = (v_1, v_2, \ldots, v_s)u$. From this we get

$$\sigma_r \circ \phi_r(u\psi_{m,r}(\boldsymbol{v})) = (v_1, v_1, v_2, v_2, \ldots, v_s, v_s) \in C'.$$

This proves that $\phi_m(u\boldsymbol{v}) \in C_1$. □

We now recall some results needed for the next proposition and the following section on the mass formula.

Let C be a self-dual code over R_m of length n. As shown in [6], C is equivalent to a code with generator matrix

$$\begin{pmatrix} I_k + uM & D \\ 0 & uE \end{pmatrix}, \tag{1}$$

where M, D and E are matrices over $\mathbb{F}_{2^m}$. We associate with C two codes over $\mathbb{F}_{2^m}$: the residue code C_1 and the torsion code C_2 which are defined as follows:

$$C_1 = \{x \in \mathbb{F}_{2^m}^n \mid x + uy \in C \text{ for some } y \in \mathbb{F}_{2^m}^n\},$$

$$C_2 = \{x \in \mathbb{F}_{2^m}^n \mid ux \in C\}.$$

C_1 has generator matrix

$$(I_k \quad D)$$

and C_2 has generator matrix

$$\begin{pmatrix} I_k & D \\ 0 & E \end{pmatrix}.$$

Define a map f by

$$\begin{aligned} f : C_1 &\rightarrow \mathbb{F}_{2^m}^n / C_2, \\ x &\mapsto \{y \in \mathbb{F}_{2^m}^n \mid x + uy \in C\}, \end{aligned}$$

such that $C = \{x + uy \mid x \in C_1, y \in f(x)\}$.

Following [7,6], it can be shown that the set of self-dual codes C over R_m is in one-to-one correspondence with the set of triplets (C_1, C_2, f), where C_1 is a self-orthogonal code over $\mathbb{F}_{2^m}$ containing the all-one vector $\mathbf{1}$, $C_2 = C_1^{\perp}$ and f is a linear map from C_1 to $\mathbb{F}_{2^m}^n / C_2$, represented by the symmetric matrix M given above.

Proposition 12. *Let C_1 be a self-orthogonal code over $\mathbb{F}_{2^m}$ containing the all-one vector $\mathbf{1}$. If the Lee weights of all the codewords of C_1 are multiples of 4, then $C_1 + uC_1^{\perp}$ is a Type II code over R_m.*

Proof. After a suitable permutation, C_1 has a generator matrix of the form $(I_k\ \ D)$, where D is a matrix over $\mathbb{F}_{2^m}$. Since C_1 is self-orthogonal, $C_1^{\perp}$ has generator matrix $\left(\begin{smallmatrix} I_k & D \\ 0 & E \end{smallmatrix}\right)$.

Therefore $C = C_1 + uC_1^{\perp}$ has generator matrix

$$\begin{pmatrix} I_k & D \\ 0 & uE \end{pmatrix}.$$

Any two rows of $(0\ \ uE)$ are clearly orthogonal. Furthermore, every row of $(0\ \ uE)$ is orthogonal to every row of $(I_k\ \ D)$ because every row of $(0\ \ E)$ is in the dual of C_1. Thus, C is self-orthogonal, and by considering cardinality, it follows that C is self-dual over R_m. Since every row of $(0\ \ E)$ is orthogonal to $\mathbf{1} \in C_1$, the rows of $(0\ \ E)$ are of even Lee weight. Thus the rows of $(0\ \ uE)$ have weights divisible by 4, and by assumption, so do the rows of $(I_k\ \ D)$. Therefore C is of Type II over R_m. □

Define the *swap* map s on $\mathbb{F}_{2^m}^{2n}$ as

$$s((x_1, x_2, \ldots, x_{2n})) = (x_{n+1}, x_{n+2}, \ldots, x_{2n}, x_1, x_2, \ldots, x_n).$$

The following proposition and corollary characterize codes over $\mathbb{F}_{2^m}$ which come from codes over R_m via the Gray map ϕ_m. The proofs of these assertions are identical to those for R_1 and R_2 in [6,11].

Proposition 13. *A code C over $\mathbb{F}_{2^m}$ of length $2n$ is the image of a code over R_m of length n under the Gray map ϕ_m if and only if C is invariant under the swap map. In general, C is the image of some R_m-code under some Gray map if and only if it admits a fixed-point free involution in its automorphism group.*

Corollary 14. *A code C over $\mathbb{F}_{2^m}$ is the image of a self-dual code over R_m under ϕ_m if and only if C is self-dual and invariant under the swap map. In general, a self-dual code C over $\mathbb{F}_{2^m}$ is the image of a self-dual R_m-code under some Gray map if and only if it admits a fixed-point free involution in its automorphism group.*

4. Mass formula for Type II codes over $\mathbb{F}_{2^m} + u\mathbb{F}_{2^m}$

In this section, we prove the following theorem:

Theorem 15 (Mass formula). *Let $N_{2^m}(n)$ be the number of distinct Type II codes of length n over R_m. Then*

$$N_{2^m}(n) = \begin{cases} \displaystyle\sum_{k=1}^{n/2} \sigma_1^{(2^m)}(n,k) 2^{1+\frac{mk(k-1)}{2}} & if\ \ n \equiv 0 \pmod 2\ \ and\ \ mn \equiv 0 \pmod 4 \\ 0 & otherwise, \end{cases}$$

where $\sigma_1^{(2^m)}(n,k)$ is the number of self-orthogonal $[n,k]$-codes over $\mathbb{F}_{2^m}$ containing the all-one vector $\mathbf{1}$.

Remark. The formula for $\sigma_1^{(2^m)}(n,k)$ is

$$\sigma_1^{(2^m)}(n,k) = \prod_{i=1}^{k-1} \frac{2^{m(n-2k+2i)} - 1}{2^{mi} - 1},$$

given in [12, Chap. 19]. (Note that for self-dual codes, this formula reduces to $\prod_{i=1}^{k-1} 2^{mi} + 1$.)

Before we prove this theorem, we introduce some notions and lemmas.

We define the map $\imath : \mathbb{F}_2 \to \mathbb{Z}$ to be given by $\imath(0) = 0$, $\imath(1) = 1$. Throughout this section, we denote by Tr the absolute trace function $\mathrm{Tr}_{\mathbb{F}_{2^m}/\mathbb{F}_2}$.

A vector $(a_1, a_2, \ldots, a_n) \in R_m^n$ is *doubly even* (resp. *even*) if the Lee weight of $(xa_1, xa_2, \ldots, xa_n)$ is a multiple of 4 (resp. 2) for all $x \in R_m$. Note that while the Lee weight depends on the choice of the self-dual basis, the doubly even property of a vector does not (cf. [1]).

Lemma 16. *Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a self-dual basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$. Then*

$$\mathrm{wt}_{\mathrm{L}}^B(\alpha_j(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m)) \equiv c_j \pmod 2,$$

where $c_j \in \mathbb{F}_2$.

Proof. Recall that for any $x \in \mathbb{F}_{2^m}$,

$$x = \mathrm{Tr}(\alpha_1 x)\alpha_1 + \mathrm{Tr}(\alpha_2 x)\alpha_2 + \cdots + \mathrm{Tr}(\alpha_m x)\alpha_m. \tag{2}$$

Since $\mathrm{Tr}(\alpha) = \mathrm{Tr}(\alpha^2)$ for every $\alpha \in \mathbb{F}_{2^m}$, it follows that $\mathrm{Tr}(\alpha_i) = 1$ for $i = 1, 2, \ldots, m$. By letting $x = 1$ in (2), we see that $\sum_{i=1}^m \alpha_i = 1$.

From (2) and the linearity of the trace function, we have

$$\begin{aligned} \mathrm{wt}_{\mathrm{L}}^B(x) &\equiv \mathrm{Tr}(\alpha_1 x) + \mathrm{Tr}(\alpha_2 x) + \cdots + \mathrm{Tr}(\alpha_m x) \pmod 2 \\ &\equiv \mathrm{Tr}((\alpha_1 + \alpha_2 + \cdots + \alpha_m)x) \pmod 2 \\ &\equiv \mathrm{Tr}(x) \pmod 2. \end{aligned}$$

Since $\mathrm{Tr}(\alpha_j(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m)) = c_j$, the result follows. □

Lemma 17. *Let $(a_2, a_3, \ldots, a_n)$ be an even vector in R_m^{n-1}. Then there exists a unique element $a_1 \in \mathbb{F}_{2^m}$ such that $(ua_1, a_2, a_3, \ldots, a_n) \in R_m^n$ is doubly even.*

Proof. Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a self-dual basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$, and let $a_1 = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m$ for $c_i \in \mathbb{F}_2$. Then for $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} \mathrm{wt}_{\mathrm{L}}^B((u\alpha_i a_1, \alpha_i a_2, \alpha_i a_3, \ldots, \alpha_i a_n)) &= \mathrm{wt}_{\mathrm{L}}^B((u\alpha_i a_1)) + \mathrm{wt}_{\mathrm{L}}^B((\alpha_i a_2, \alpha_i a_3, \ldots, \alpha_i a_n)) \\ &= 2\imath(c_i) + \mathrm{wt}_{\mathrm{L}}^B((\alpha_i a_2, \alpha_i a_3, \ldots, \alpha_i a_n)) \end{aligned}$$

by Lemma 16. It means that $(ua_1, a_2, \ldots, a_n)$ is doubly even if and only if

$$c_i = \mathrm{wt}_{\mathrm{L}}^B((\alpha_i a_2, \alpha_i a_3, \ldots, \alpha_i a_n))/2 \pmod 2.$$

Thus a_1 exists and is uniquely determined. □

Lemma 18. *Let $B=\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ be a self-dual basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$. For any $a\in\mathbb{F}_{2^m}$,*

$$\mathrm{wt}_{\mathrm{L}}^B(\alpha_i(1+ua)) = 2\mathrm{wt}_{\mathrm{L}}^B(\alpha_i a) + 1 - 2\imath(\mathrm{Tr}(\alpha_i^2 a)).$$

Proof. Since the Gray map ϕ_m preserves the Lee weight, we have

$$\mathrm{wt}_{\mathrm{L}}^B(\alpha_i(1+ua)) = \mathrm{wt}_{\mathrm{L}}^B(\alpha_i a) + \mathrm{wt}_{\mathrm{L}}^B(\alpha_i + \alpha_i a).$$

Note that the image under $\psi_{m,1}$ of α_i is an m-tuple with 1 in the ith position and 0's elsewhere, while that of $\alpha_i a$ has $\mathrm{Tr}(\alpha_i(\alpha_i a))$ as its ith component. Thus the image under $\psi_{m,1}$ of $\alpha_i+\alpha_i a$ differs from the image of $\alpha_i a$ only in the ith position, which is determined by $\mathrm{Tr}(\alpha_i(\alpha_i a))$. Thus

$$\mathrm{wt}_{\mathrm{L}}^B(\alpha_i + \alpha_i a) = \begin{cases} \mathrm{wt}_{\mathrm{L}}^B(\alpha_i a) - 1 & \text{if } \mathrm{Tr}(\alpha_i(\alpha_i a)) = 1, \\ \mathrm{wt}_{\mathrm{L}}^B(\alpha_i a) + 1 & \text{if } \mathrm{Tr}(\alpha_i(\alpha_i a)) = 0. \end{cases}$$

The result follows. □

Lemma 19. *Let n be an even positive integer such that $nm \equiv 0 \pmod 4$, and $(a_1,a_2,\ldots,a_n)$ a vector in $\mathbb{F}_{2^m}^n$. Let $b=\sum_{j=1}^n a_j$. Then the vector $(1+ua_1,1+ua_2,\ldots,1+ua_n)$ is doubly even if and only if b is 0 or 1 when $n\equiv 0 \pmod 4$, or $b^2+b+1=0$ in $\mathbb{F}_{2^m}$ if $n\equiv 2 \pmod 4$. In particular, for any $(a_2,a_3,\ldots,a_n)\in\mathbb{F}_{2^m}^{n-1}$, there are exactly two values for a_1 such that $(1+ua_1,1+ua_2,\ldots,1+ua_n)$ is doubly even.*

Proof. First, we consider the case $n\equiv 0 \pmod 4$. Let $B=\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ be a self-dual basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$. Let $\mathbf{1}=(1,1,\ldots,1)$ and $\boldsymbol{a}=(a_1,a_2,\ldots,a_n)$.

By Lemma 18,

$$\begin{aligned}\mathrm{wt}_{\mathrm{L}}^B(\alpha_i(\mathbf{1}+u\boldsymbol{a})) &= \sum_{j=1}^n (2\mathrm{wt}_{\mathrm{L}}^B(\alpha_i a_j) - 2\imath(\mathrm{Tr}(\alpha_i^2 a_j))) + n \\ &\equiv \sum_{j=1}^n (2\imath(\mathrm{Tr}(\alpha_i a_j)) + 2\imath(\mathrm{Tr}(\alpha_i^2 a_j))) \pmod 4 \\ &\equiv 2\imath\left(\sum_{j=1}^n (\mathrm{Tr}(\alpha_i a_j) + \mathrm{Tr}(\alpha_i^2 a_j))\right) \pmod 4.\end{aligned}$$

Since $\mathrm{Tr}(x^2)=\mathrm{Tr}(x)$, we have

$$\begin{aligned}\mathrm{wt}_{\mathrm{L}}^B(\alpha_i(\mathbf{1}+u\boldsymbol{a})) &\equiv 2\imath\left(\sum_{j=1}^n \mathrm{Tr}(\alpha_i^2(a_j^2+a_j))\right) \pmod 4 \\ &\equiv 2\imath\left(\mathrm{Tr}\left(\alpha_i^2 \sum_{j=1}^n (a_j^2+a_j)\right)\right) \pmod 4.\end{aligned}$$

Therefore, as $x = \mathrm{Tr}(\alpha_1^2 x)\alpha_1^2 + \mathrm{Tr}(\alpha_2^2 x)\alpha_2^2 + \cdots + \mathrm{Tr}(\alpha_m^2 x)\alpha_m^2$ for every $x \in \mathbb{F}_{2^m}$ and $B^2 = \{\alpha_1^2, \alpha_2^2, \ldots, \alpha_m^2\}$ is also a self-dual basis, $\mathbf{1} + u\boldsymbol{a}$ is doubly even if and only if

$$\sum_{j=1}^{n} (a_j^2 + a_j) = 0,$$

i.e., if and only if

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} a_j + 1\right) = 0.$$

Now consider the case when $n \equiv 2 \pmod 4$. Note since $nm \equiv 0 \pmod 4$, m is even and there exists an $x \in \mathbb{F}_{2^m}$ such that $x^2 + x + 1 = 0$. As in the previous case, we have

$$\mathrm{wt}_{\mathrm{L}}^{B}(\alpha_i(\mathbf{1} + u\boldsymbol{a})) \equiv 2\imath\left(\mathrm{Tr}\left(\alpha_i^2 \sum_{j=1}^{n} (a_j^2 + a_j)\right)\right) + 2 \pmod 4. \tag{3}$$

Similarly, $\mathbf{1} + u\boldsymbol{a}$ is doubly even if and only if

$$\sum_{j=1}^{n} (a_j^2 + a_j) = 1,$$

i.e., if and only if

$$\left(\sum_{j=1}^{n} a_j\right)^2 + \sum_{j=1}^{n} a_j + 1 = 0.$$

The result follows. □

We can now prove Theorem 4.1. Recall the discussion in the previous section on the structure of self-dual codes over R_m.

To count Type II codes over R_m, we only have to count the number of self-orthogonal codes C_1 which contain the all-one vector and for each C_1, the number of functions f, represented by the matrix M in (1), with the condition that each row of the matrix $(I_k + uM\ D)$ is doubly even. We claim that there are $2^{1+mk(k-1)/2}$ possibilities for the matrix M.

We determine the symmetric matrix $M = \{m_{ij}\}$ over $\mathbb{F}_{2^m}$ in the following manner: First we can arbitrarily choose entries m_{ij} for $i \geqslant j, i \geqslant 2$. These determine the entries m_{ij} for $i < j, i \geqslant 2$, because $m_{ij} = m_{ji}$. By Lemma 17, the entries m_{i1} for $i \geqslant 2$ are uniquely determined so that each ith row of $(I_k + uM\ D)$, for $i \geqslant 2$, is doubly even. By symmetry, the remaining entries of M are thus determined, except for the entry m_{11}. The sum of all the rows of $(I_k + uM\ D)$ is $\mathbf{1} + u\boldsymbol{a}$ with some $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{F}_{2^m}^n$. Since $a_2, \ldots, a_n$ have been determined, there are exactly two possibilities for a_1, by Lemma 19. Each choice for a_1 determines m_{11}. Therefore there exist exactly $2(2^m)^{k(k-1)/2}$ such matrices M. □

5. Classification and examples

In this section, we give a classification of Type II codes over R_m of lengths up to 8, 6, 6, 6, 4 and 6 for $m = 2$, 3, 4, 5, 6 and 7, respectively. Throughout this section, by equivalence we shall mean symmetry-equivalence. In order to confirm completeness of the classifications, we use the mass formula in Theorem 15 as

$$N_{2^m}(n) = \sum_{C \in \mathscr{C}^{[n]}_{2^m}} \frac{|B_n|}{|\mathrm{Aut}(C)|},$$

where B_n is the signed symmetric group of order $2^n \cdot n!$ and $\mathrm{Aut}(C)$ is the group of automorphisms of the code C as a subgroup of B_n. Throughout this section, $\mathrm{Aut}(C)$ denotes the group of automorphisms of the code C. We have obtained the groups of automorphisms which appear below by brute force, i.e., collecting all elements of B_n preserving C.

We first give some remarks on the minimum Lee weight of a code, which by definition, depends on the choice of the self-dual basis. In fact, there is an example of a Type II code over $\mathbb{F}_{32}$ which has different minimum Lee weights when taken with respect to two distinct self-dual bases of $\mathbb{F}_{32}$ over $\mathbb{F}_2$ (cf. [3]). However, our computations on the Type II codes covered in this paper have yielded the same minimum Lee weights irrespective of the choice of basis. Therefore, in this section we shall refer to the minimum Lee weight of a code without reference to the self-dual basis, and we denote it by d_{L} in the tables.

5.1. Type II codes of length 2

We begin by considering R_m-codes of length 2. By the mass formula, there is no Type II code if m is odd. If m is even, we have the following result, which is independent of m.

Proposition 20. *If m is even, then there exists a unique Type II* [2,1]*-code C over R_m. Moreover, the minimum Lee weight of C is* 4.

Proof. The codes generated by $(1 + u\omega, 1)$ and $(1 + u\omega^2, 1)$ where $\omega^2 + \omega + 1 = 0$ in $\mathbb{F}_{2^m}$ are of Type II. The mass formula implies that there are no other Type II codes. Moreover, it is easy to see that they are equivalent.

By Proposition 7, the code contains the codeword $(\alpha_1 u, \alpha_1 u)$, which has Lee weight 4. □

5.2. Dimension of the residue codes

Recall that we can associate to any Type II code C of length n over R_m its residue code C_1 and torsion code C_2. In order to classify the codes, it is enough to consider the subclasses defined by the dimension of C_1, which ranges from 1 to $n/2$.

We first discuss the subclass of codes C where C_1 is of dimension 1.

Proposition 21. *There exists a unique Type II code over R_m such that the dimension of its residue code C_1 is 1 whenever n is even and nm is multiple of 4. The minimum Lee weight of C is 4.*

Proof. The codes generated by

$$\begin{pmatrix} 1+ux & 1 & 1 & \cdots & 1 & 1 \\ 0 & u & 0 & \cdots & 0 & u \\ 0 & 0 & u & \cdots & 0 & u \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & u & u \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} 1+u(1+x) & 1 & 1 & \cdots & 1 & 1 \\ 0 & u & 0 & \cdots & 0 & u \\ 0 & 0 & u & \cdots & 0 & u \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & u & u \end{pmatrix},$$

are of Type II, where $x=0$ if $n \equiv 0 \pmod 4$, and $x \in \mathbb{F}_{2^m}$ such that $x^2+x+1=0$ if $n \equiv 2 \pmod 4$. The mass formula implies that there are no other such codes. It is readily seen that the two codes above are equivalent.

By Proposition 7, the second row of each of the generator matrices above multiplied by α_1 is of Lee weight 4. This proves the second statement. □

5.3. Type II codes over R_2

Type II codes over $R_2 = \mathbb{F}_4 + u\mathbb{F}_4$ were studied in [11]. The unique self-dual basis of $\mathbb{F}_4 = \mathbb{F}_2[\omega]/(\omega^2+\omega+1)$ over $\mathbb{F}_2$ is $\{\omega, \omega^2\}$. We give a classification of Type II codes for lengths $n = 2, 4, 6$ and 8.

As discussed in Section 3, a Type II code over R_m has a generator matrix of the form

$$\begin{pmatrix} I_k + uM & D \\ 0 & uE \end{pmatrix}, \tag{4}$$

where I_k is the $k \times k$ identity matrix, and D, E and M are matrices over $\mathbb{F}_{2^m}$. The matrix E is determined by D, because C_2 is the dual code of C_1. In the following subsections, we describe the generator matrices of Type II codes by giving D and M.

5.3.1. $n = 4$

For the case $n = 4$, it is enough to consider the case where the dimension of C_1 is 2.

Table 1
All Type II R_2-codes of length 4

$C \in \mathscr{C}^{[4]_4}$	D	M	$\lvert\mathrm{Aut}(C)\rvert$	d_{L}	$\psi_{2,1}(C)$
$C_1^{[4,1]_4}$			192	4	$[8,2]\_2d_4$
$C_1^{[4,2]_4}$	$D_1^{[4,2]_4}$	$M_1^{[4,2]_4}$	32	4	$[8,4]\_4d_2b$
$C_2^{[4,2]_4}$	$D_1^{[4,2]_4}$	$M_2^{[4,2]_4}$	32	4	$[8,4]\_4d_3b$
$C_3^{[4,2]_4}$	$D_2^{[4,2]_4}$	$M_3^{[4,2]_4}$	24	4	$[8,4]\_e_8a$

Let $D_1^{[4,2]_4}$ and $D_2^{[4,2]_4}$ be the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \omega & \omega^2 \\ \omega^2 & \omega \end{pmatrix}$, respectively. Then $(I_2\ D_1^{[4,2]_4})$ and $(I_2\ D_2^{[4,2]_4})$ generate inequivalent self-orthogonal $[4,2]$-codes over $\mathbb{F}_4$. Let $M_1^{[4,2]_4}$, $M_2^{[4,2]_4}$ and $M_3^{[4,2]_4}$ be the matrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, respectively.

Let $C_1^{[4,1]_4}$ be the code characterized in Proposition 21 and let $C_1^{[4,2]_4}$ $C_2^{[4,2]_4}$ and $C_3^{[4,2]_4}$ be Type II codes, each of which is generated by a matrix of the form (4) where the relevant D and M in each case are given in Table 1. We are able to confirm that $\mathscr{C}^{[4]_4} := \{C_1^{[4,1]_4}, C_1^{[4,2]_4}, C_2^{[4,2]_4}, C_3^{[4,2]_4}\}$ consists of all the Type II $[4,2]$-codes over R_2, using the mass formula as follows:

$$N_4(4) = 2 + 40 = \sum_{C \in \mathscr{C}^{[4]_4}} \frac{2^4 \cdot 4!}{|\mathrm{Aut}(C)|}.$$

The last column of Table 1 gives the image of C under the Gray map $\psi_{2,1}$, using the notation in [6, Table II].

The code $C_1^{[4,1]_4}$ is the code indicated in Proposition 21. The code $C_3^{[4,2]_4}$ was given as an example in [11].

5.3.2. $n = 6$

For the case $n = 6$, it is enough to consider the cases where the dimension of C_1 is 2 or 3. We define the following matrices:

$$D_1^{[6,2]_4} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_2^{[6,2]_4} = \begin{pmatrix} 1 & 1 & \omega & \omega^2 \\ 0 & 0 & \omega^2 & \omega \end{pmatrix},$$

$$D_3^{[6,2]_4} = \begin{pmatrix} 1 & \omega & \omega & 0 \\ 0 & \omega^2 & \omega^2 & 1 \end{pmatrix},$$

$$E_1^{[6,2]_4} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad E_2^{[6,2]_4} = \begin{pmatrix} 1 & 0 & \omega & \omega^2 \\ 0 & 1 & \omega & \omega^2 \end{pmatrix},$$

$$E_3^{[6,2]_4} = \begin{pmatrix} 1 & 0 & \omega^2 & \omega \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$M_1^{[6,2]_4} = \begin{pmatrix} \omega^2 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_2^{[6,2]_4} = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix},$$

$$M_3^{[6,2]_4} = \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_4^{[6,2]_4} = \begin{pmatrix} \omega^2 & \omega \\ \omega & 1 \end{pmatrix},$$

$$D_1^{[6,3]_4} = \begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & 1 & \omega^2 \\ \omega^2 & \omega & 0 \end{pmatrix}, \quad D_2^{[6,3]_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D_3^{[6,3]_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{pmatrix},$$

$$M_1^{[6,3]_4} = \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_2^{[6,3]_4} = \begin{pmatrix} \omega & \omega & 0 \\ \omega & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_3^{[6,3]_4} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_4^{[6,3]_4} = \begin{pmatrix} \omega & \omega^2 & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix},$$

$$M_5^{[6,3]_4} = \begin{pmatrix} \omega & \omega & \omega \\ \omega & 1 & \omega^2 \\ \omega & \omega^2 & 1 \end{pmatrix}, \quad M_6^{[6,3]_4} = \begin{pmatrix} \omega & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$M_7^{[6,3]_4} = \begin{pmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \end{pmatrix}, \quad M_8^{[6,3]_4} = \begin{pmatrix} \omega & \omega & \omega \\ \omega & \omega & \omega \\ \omega & \omega & \omega \end{pmatrix},$$

$$M_9^{[6,3]_4} = \begin{pmatrix} \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{pmatrix}, \quad M_{10}^{[6,3]_4} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix},$$

Table 2
All Type II R_2-codes of length 6

$C \in \mathscr{C}^{[6]_4}$	D	M	$\lvert\mathrm{Aut}(C)\rvert$	d_L	$\phi_2(C)$
$C_1^{[6,1]_4}$			23040	4	$C_{12,5}$
$C_1^{[6,2]_4}$	$D_1^{[6,2]_4}$	$M_1^{[6,2]_4}$	768	4	$C_{12,5}$
$C_2^{[6,2]_4}$	$D_1^{[6,2]_4}$	$M_2^{[6,2]_4}$	768	4	$C_{12,2}$
$C_3^{[6,2]_4}$	$D_2^{[6,2]_4}$	$M_3^{[6,2]_4}$	144	4	$C_{12,4}$
$C_4^{[6,2]_4}$	$D_3^{[6,2]_4}$	$M_4^{[6,2]_4}$	192	4	$C_{12,6}$
$C_1^{[6,3]_4}$	$D_1^{[6,3]_4}$	$M_1^{[6,3]_4}$	12	8	$C_{12,7}$
$C_2^{[6,3]_4}$	$D_1^{[6,3]_4}$	$M_2^{[6,3]_4}$	36	4	$C_{12,3}$
$C_3^{[6,3]_4}$	$D_2^{[6,3]_4}$	$M_3^{[6,3]_4}$	128	4	$C_{12,2}$
$C_4^{[6,3]_4}$	$D_2^{[6,3]_4}$	$M_4^{[6,3]_4}$	128	4	$C_{12,6}$
$C_5^{[6,3]_4}$	$D_2^{[6,3]_4}$	$M_5^{[6,3]_4}$	128	4	$C_{12,6}$
$D_6^{[6,3]_4}$	$D_2^{[6,3]_4}$	$M_6^{[6,3]_4}$	128	4	$C_{12,5}$
$C_7^{[6,3]_4}$	$D_2^{[6,3]_4}$	$M_7^{[6,3]_4}$	384	4	$C_{12,6}$
$C_8^{[6,3]_4}$	$D_2^{[6,3]_4}$	$M_8^{[6,3]_4}$	384	4	$C_{12,5}$
$C_9^{[6,3]_4}$	$D_2^{[6,3]_4}$	$M_9^{[6,3]_4}$	384	4	$C_{12,5}$
$C_{10}^{[6,3]_4}$	$D_2^{[6,3]_4}$	$M_{10}^{[6,3]_4}$	384	4	$C_{12,1}$
$C_{11}^{[6,3]_4}$	$D_3^{[6,3]_4}$	$M_{11}^{[6,3]_4}$	32	4	$C_{12,6}$
$C_{12}^{[6,3]_4}$	$D_3^{[6,3]_4}$	$M_{12}^{[6,3]_4}$	96	4	$C_{12,1}$
$C_{13}^{[6,3]_4}$	$D_3^{[6,3]_4}$	$M_{13}^{[6,3]_4}$	24	4	$C_{12,4}$

$$M_{11}^{[6,3]_4} = \begin{pmatrix} \omega & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_{12}^{[6,3]_4} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_{13}^{[6,3]_4} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & \omega \end{pmatrix}.$$

We are able to complete the classification in a way similar to that for the case $n = 4$. Additively, we use a brute forced way to confirm the inequivalency between $C_4^{[6,3]_4}$ and $C_5^{[6,3]_4}$ and between $C_8^{[6,3]_4}$ and $C_9^{[6,3]_4}$. It suffices to confirm these cases directly because we can see the inequivalence among other combinations by a difference of invariants of the symmetry-equivalence: the minimum Lee weight, the order of automorphism group or the Gray map image. Therefore, Table 2 gives the complete classification, which we confirm using the mass formula

$$N_4(6) = 2 + 680 + 10880 = \sum_{C \in \mathscr{C}^{[6]_4}} \frac{2^6 \cdot 6!}{\lvert\mathrm{Aut}(C)\rvert}.$$

The last column of Table 2 gives the image of C under the Gray map ϕ_2, using the notation in [4, Table II].

Table 3
Type II R_2-codes of length 8 which have the highest minimum Lee weight

$C \in \mathscr{C}^{[4]_8}$	D	M	$\lvert\mathrm{Aut}(C)\rvert$	d_L	$\phi_2(C)$
$C_{12}^{[8,3]_4}$	$D_5^{[8,3]_4}$	$C_{12}^{[8,3]_4}$	384	8	D_{43}
$C_{34}^{[8,3]_4}$	$D_9^{[8,3]_4}$	$C_{34}^{[8,3]_4}$	48	8	D_{43}
$C_1^{[8,4]_4}$	$D_1^{[8,4]_4}$	$C_1^{[8,4]_4}$	96	8	D_{42}
$C_5^{[8,4]_4}$	$D_2^{[8,4]_4}$	$C_5^{[8,4]_4}$	16	8	D_{42}
$C_{10}^{[8,4]_4}$	$D_2^{[8,4]_4}$	$C_{10}^{[8,4]_4}$	96	8	D_{42}
$C_{11}^{[8,4]_4}$	$D_2^{[8,4]_4}$	$C_{11}^{[8,4]_4}$	96	8	D_{42}
$C_{12}^{[8,4]_4}$	$D_3^{[8,4]_4}$	$C_{12}^{[8,4]_4}$	8	8	D_{46}
$C_{13}^{[8,4]_4}$	$D_3^{[8,4]_4}$	$C_{13}^{[8,4]_4}$	32	8	D_{42}
$C_{14}^{[8,4]_4}$	$D_3^{[8,4]_4}$	$C_{14}^{[8,4]_4}$	4	8	D_{44}
$C_{15}^{[8,4]_4}$	$D_3^{[8,4]_4}$	$C_{15}^{[8,4]_4}$	16	8	D_{42}
$C_{17}^{[8,4]_4}$	$D_3^{[8,4]_4}$	$C_{17}^{[8,4]_4}$	16	8	D_{44}
$C_{18}^{[8,4]_4}$	$D_3^{[8,4]_4}$	$C_{18}^{[8,4]_4}$	16	8	D_{43}
$C_{19}^{[8,4]_4}$	$D_3^{[8,4]_4}$	$C_{19}^{[8,4]_4}$	32	8	D_{42}
$C_{21}^{[8,4]_4}$	$D_3^{[8,4]_4}$	$C_{21}^{[8,4]_4}$	16	8	D_{42}
$C_{22}^{[8,4]_4}$	$D_3^{[8,4]_4}$	$C_{22}^{[8,4]_4}$	16	8	D_{45}
$C_{25}^{[8,4]_4}$	$D_3^{[8,4]_4}$	$C_{25}^{[8,4]_4}$	32	8	D_{42}
$C_{27}^{[8,4]_4}$	$D_4^{[8,4]_4}$	$C_{27}^{[8,4]_4}$	2	8	D_{47}
$C_{28}^{[8,4]_4}$	$D_4^{[8,4]_4}$	$C_{28}^{[8,4]_4}$	4	8	D_{46}
$C_{29}^{[8,4]_4}$	$D_4^{[8,4]_4}$	$C_{29}^{[8,4]_4}$	4	8	D_{44}
$C_{30}^{[8,4]_4}$	$D_4^{[8,4]_4}$	$C_{30}^{[8,4]_4}$	4	8	D_{46}
$C_{31}^{[8,4]_4}$	$D_4^{[8,4]_4}$	$C_{31}^{[8,4]_4}$	4	8	D_{48}
$C_{33}^{[8,4]_4}$	$D_4^{[8,4]_4}$	$C_{33}^{[8,4]_4}$	12	8	D_{45}
$C_{130}^{[8,4]_4}$	$D_8^{[8,4]_4}$	$C_{130}^{[8,4]_4}$	24	8	D_{48}
$C_{132}^{[8,4]_4}$	$D_8^{[8,4]_4}$	$C_{132}^{[8,4]_4}$	32	8	D_{42}

5.3.3. $n=8$

Similarly, we can also complete the classification for the case $n=8$, and verify this using the mass formula as follows:

$$N_4(8) = 2 + 10920 + 2970240 + 45260800 = \sum_{C \in \mathscr{C}^{[8]_4}} \frac{2^8 \cdot 8!}{|\mathrm{Aut}(C)|}.$$

We have 189 inequivalent Type II $[8,4]$-codes. Due to space constraints, we give only the list of 24 Type II $[8,4]$-codes with minimum Lee weight 8 in Table 3. These are the extremal Type II codes.

The last column of Table 3 gives the image of C under the Gray map ϕ_2, using the notation in [2, Table 1].

5.4. Type II codes over R_3

In this section, we look at Type II codes over $R_3 = \mathbb{F}_8 + u\mathbb{F}_8$. The unique self-dual basis of $\mathbb{F}_8 = \mathbb{F}_2[\beta]/(\beta^3+\beta+1)$ over $\mathbb{F}_2$ is $\{\beta^3, \beta^5, \beta^6\}$.

Table 4
Type II R_3-codes of length 4

	D			
$D_1^{[4,2]_8}$	1	0	0	1
$D_2^{[4,2]_8}$	β	β^3	β^3	β

$C \in \mathscr{C}^{[4]_8}$	M				D	$\lvert\mathrm{Aut}(C)\rvert$	d_L	$\phi_3(C)$
$C_1^{[4,1]_8}$						192	4	$C_{8,1}$
$C_1^{[4,2]_8}$	1	1	1	1	$D_1^{[4,2]_8}$	32	4	$C_{8,2}$
$C_2^{[4,2]_8}$	β^6	β	β	β^6	$D_1^{[4,2]_8}$	32	4	$C_{8,2}$
$C_3^{[4,2]_8}$	β^5	β^2	β^2	β^5	$D_1^{[4,2]_8}$	32	4	$C_{8,2}$
$C_4^{[4,2]_8}$	β^3	β^4	β^4	β^3	$D_1^{[4,2]_8}$	32	4	$C_{8,2}$
$C_5^{[4,2]_8}$	β^6	1	1	β^6	$D_2^{[4,2]_8}$	8	8	$C_{8,3}$
$C_6^{[4,2]_8}$	1	β	β	1	$D_2^{[4,2]_8}$	8	4	$C_{8,2}$

5.4.1. $n=4$

We give the classification of Type II $[4,2]$-codes over R_3 in Table 4. D and M are 2×2 matrices whose first and second rows are given in the table. The mass formula yields the following:

$$N_8(4) = 2 + 144 = \sum_{C \in \mathscr{C}^{[4]_8}} \frac{2^8 \cdot 8!}{|\mathrm{Aut}(C)|}.$$

In the last column of Table 4, the image of C under the Gray map ϕ_3 is given using the notation in [5, Table 1].

5.5. Codes over R_4

In this section, we consider the case $\mathbb{F}_{16} = \mathbb{F}[\alpha]/(\alpha^4+\alpha+1)$. From the self-dual bases $B_1 = \{\alpha^2, \alpha^8\}$ of $\mathbb{F}_{16}$ over $\mathbb{F}_4$ and $B_2 = \{\alpha^5, \alpha^{10}\} = \{\omega, \omega^2\}$ of $\mathbb{F}_4$ over $\mathbb{F}_2$, we choose the self-dual $\mathbb{F}_2$-basis $B = \{\alpha^7, \alpha^{13}, \alpha^{12}, \alpha^3\}$ of $\mathbb{F}_{16}$ over $\mathbb{F}_2$.

5.5.1. $n=4$

We give the classification of Type II $[4,2]$-codes over R_4 in Table 5. D and M are 2×2 matrices whose first and second rows are given in the table. The mass formula yields the following:

$$N_{16}(4) = 2 + 544 = \sum_{C \in \mathscr{C}^{[16]_4}} \frac{2^4 \cdot 4!}{|\mathrm{Aut}(C)|}.$$

The codes $C_{13}^{[4,2]_{16}}$, $C_{16}^{[4,2]_{16}}$ and $C_{15}^{[4,2]_{16}}$ are the images of $C_9^{[4,2]_{16}}$, $C_{11}^{[4,2]_{16}}$ and $C_{12}^{[4,2]_{16}}$, respectively, under the Frobenius automorphism on R_4 given by $x+uy \mapsto x^2+uy^2$.

The sixth and seventh columns of the bigger table in Table 5 use the notation in Table 3 to denote the images of C under the Gray maps $\psi_{4,2}^{\{\alpha,\alpha^4\}}$ and $\psi_{4,2}^{\{\alpha^2,\alpha^8\}}$, defined

Table 5
Type II R_4-codes of length 4

	D			
$D_1^{[4,2]_{16}}$	1	0	0	1
$D_2^{[4,2]_{16}}$	α	α^4	α^4	α
$D_3^{[4,2]_{16}}$	α^2	α^8	α^8	α^2
$D_4^{[4,2]_{16}}$	α^5	α^{10}	α^{10}	α^5

$C \in \mathscr{C}^{[4]_{16}}$	M				D	$\lvert\mathrm{Aut}(C)\rvert$	d_{L}	$\psi_{4,2}^{\{\alpha,\alpha^4\}}(C)$	$\psi_{4,2}^{\{\alpha^2,\alpha^8\}}(C)$
$C_1^{[4,1]_{16}}$						192	4		
$C_1^{[4,2]_{16}}$	1	1	1	1	$D_1^{[4,2]_{16}}$	32	4		
$C_2^{[4,2]_{16}}$	α^5	α	α	α^5	$D_1^{[4,2]_{16}}$	32	4		
$C_3^{[4,2]_{16}}$	α^{10}	α^2	α^2	α^{10}	$D_1^{[4,2]_{16}}$	32	4		
$C_4^{[4,2]_{16}}$	α^4	α^3	α^3	α^4	$D_1^{[4,2]_{16}}$	32	4		
$C_5^{[4,2]_{16}}$	0	α^5	α^5	0	$D_1^{[4,2]_{16}}$	32	4		
$C_6^{[4,2]_{16}}$	α^8	α^6	α^6	α^8	$D_1^{[4,2]_{16}}$	32	4		
$C_7^{[4,2]_{16}}$	α^2	α^7	α^7	α^2	$D_1^{[4,2]_{16}}$	32	4		
$C_8^{[4,2]_{16}}$	α	α^{11}	α^{11}	α	$D_1^{[4,2]_{16}}$	32	4		
$C_9^{[4,2]_{16}}$	α^5	1	1	α^5	$D_2^{[4,2]_{16}}$	8	8	$C_{10}^{[8,4]}$	$C_{19}^{[8,4]}$
$C_{10}^{[4,2]_{16}}$	1	α	α	1	$D_2^{[4,2]_{16}}$	8	4		
$C_{11}^{[4,2]_{16}}$	α^2	α^3	α^3	α^2	$D_2^{[4,2]_{16}}$	8	8	$C_5^{[8,4]}$	$C_{21}^{[8,4]}$
$C_{12}^{[4,2]_{16}}$	α^4	α^7	α^7	α^4	$D_2^{[4,2]_{16}}$	8	8	$C_5^{[8,4]}$	$C_{21}^{[8,4]}$
$C_{13}^{[4,2]_{16}}$	α^{10}	1	1	α^{10}	$D_3^{[4,2]_{16}}$	8	8	$C_{13}^{[8,4]}$	$C_{11}^{[8,4]}$
$C_{14}^{[4,2]_{16}}$	1	α^2	α^2	1	$D_3^{[4,2]_{16}}$	8	4		
$C_{15}^{[4,2]_{16}}$	α^8	α^3	α^3	α^8	$D_3^{[4,2]_{16}}$	8	8	$C_{15}^{[8,4]}$	$C_5^{[8,4]}$
$C_{16}^{[4,2]_{16}}$	α^4	α^6	α^6	α^4	$D_3^{[4,2]_{16}}$	8	8	$C_{15}^{[8,4]}$	$C_5^{[8,4]}$
$C_{17}^{[4,2]_{16}}$	0	1	1	0	$D_4^{[4,2]_{16}}$	24	4		
$C_{18}^{[4,2]_{16}}$	α^{10}	α	α	α^{10}	$D_4^{[4,2]_{16}}$	8	8	$C_{132}^{[8,4]}$	$C_{132}^{[8,4]}$

with respect to the self-dual bases $\{\alpha, \alpha^4\}$ and $\{\alpha^2, \alpha^8\}$, respectively. The blank entries are also of Type II but references to these codes are not available.

5.5.2. $n = 6$

There are 2495 inequivalent Type II $[6, 3]$-codes over R_4. We verify this figure using the mass formula:

$$N_{16}(6) = 2 + 139808 + 35790848 = \sum_{C \in \mathscr{C}^{[16]_6}} \frac{2^6 \cdot 6!}{|\mathrm{Aut}(C)|}.$$

We omit the table of the codes because it is too large.

Table 6
The number of inequivalent Type II codes

	d_L	$n=2$	$n=4$	$n=6$	$n=8$
$R_2=\mathbb{F}_4+u\mathbb{F}_4$	4	1	$1+3$	$1+4+12$	$1+8+44+112$
	8	0	$0+0$	$0+0+1$	$0+0+2+22$
$R_3=\mathbb{F}_8+u\mathbb{F}_8$	4	0	$1+5$	0	
	8	0	$0+1$	0	
$R_4=\mathbb{F}_{16}+u\mathbb{F}_{16}$	4	1	$1+11$	$1+18+666$	
	8	0	$0+7$	$0+28+1782$	
$R_5=\mathbb{F}_{32}+u\mathbb{F}_{32}$	4	0	$1+35$	0	
	8	0	$0+21$	0	
$R_6=\mathbb{F}_{64}+u\mathbb{F}_{64}$	4	1	$1+155$		
	8	0	$0+43$		
$R_7=\mathbb{F}_{128}+u\mathbb{F}_{128}$	4	0	$1+651$	0	
	8	0	$0+85$	0	

5.6. Type II codes over larger base rings

There exist 57, 199 and 737 inequivalent Type II codes of length 4 over R_5, R_6 and R_7, respectively. Similarly, these figures can be verified using the mass formula.

5.7. Summary

We summarize the numbers of inequivalent Type II codes obtained above in Table 6. The generator matrices for all these codes are available on the World Wide Web at the following URL: `http://www.math.nagoya-u.ac.jp/~koichi/data/`.

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