

New developments in transport of intensity equation for phase retrieval and computational imaging

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ABSTRACT

The commonly known transport of intensity equation (TIE) was proposed by Teague as a method for the deterministic optical phase-retrieval problem. The aim is to deduce optical phase from only intensity measurements using non-interferometric techniques. Compared to many other phase retrieval algorithms, TIE is non-iterative, purely computational and no complicated auxiliary device needs to be introduced. In this paper, we introduce some recent developments in TIE phase retrieval and microscopy: including its fast numerical solution, treatment of boundary problem and the low-frequency artifacts, and two configurations for dynamic phase imaging. We also reexamine TIE in terms of phase-space optics, demonstrating the effect of partially coherent illumination on phase reconstruction, and connecting it to light field imaging at the geometry optics limit.

Keywords: transport-of-intensity equation; 3D measurement; phase retrieval;

1. INTRODUCTION

Phase is an important component of an optical wavefield bearing the information of the refractive index, optical thickness, or the topology of the specimen. Phase retrieval is a central problem in many areas of physics and optics since the phase of a wavefield is not accessible directly. The most well-established method for obtaining quantitative phase is through interferometry, such as digital holography. However, this class of methods relies on coherent illumination, therefore, plagued with problems of speckle that prevent the formation of high quality images. On a different note, quantitative phase can be retrieved by transport-of-intensity equation (TIE) [1] using only object field intensities at multiple axially displaced planes. TIE has been increasingly investigated during recent years due to its unique advantages over interferometric techniques: it is non-interferometric, works with partially coherent illumination, computationally simple, no need to phase unwrapping, and does not require a complicated optical system. In this paper, we will introduce some recent new developments in TIE phase retrieval: including its numerical solution, treatment of boundary problem and the low-frequency artifacts, and two configurations for dynamic phase imaging. We also reexamine TIE in terms of phase-space optics, demonstrating the effect of partially coherent illumination on phase reconstruction, and connecting it to light field imaging at the geometry optics limit.

2. TRANSPORT OF INTENSITY EQUATION

Assume that a paraxial beam is propagating along the z axis, and let the complex amplitude of the object be written as $\sqrt{I(\mathbf{r})} \exp[ik\phi(\mathbf{r})]$. The derivative of intensity in the light propagation direction z contains phase information that can be retrieved via TIE [1]:

$$-k \frac{\partial I(\mathbf{r})}{\partial z} = \nabla \cdot [I(\mathbf{r}) \nabla \phi(\mathbf{r})], \quad (1)$$

where k is the wave number $2\pi/\lambda$, \mathbf{r} is the position vector representing the spatial coordinates (x, y) . $I(\mathbf{r})$ is the in-focus image intensity. ∇ is the gradient operator over \mathbf{r} which is normal to the beam propagation direction. The TIE can

be derived from Helmholtz equation under paraxial approximation, or alternatively, using Fresnel wave propagation formulation with small propagation distance approximation. Expanding the RHS of Eq. (1), we obtain

$$-k \frac{\partial I}{\partial z} = \nabla I \cdot \nabla \phi + I \nabla^2 \phi, \quad (2)$$

In the above expression, the \mathbf{r} dependence has been omitted for notation simplicity and note that the Hamilton ∇ applies only to the lateral coordinates (x, y) . The first term on RHS of Eq. (2) is called prism term which stands for the longitude intensity variation due to the local wavefront slope. The second term is called lens terms which represent the intensity variation caused by the local wavefront curvature. It is seen that TIE links the longitudinal intensity derivative with the slope and curvature of the wavefront which produces the change in intensity as the wavefront propagates

3. SOLUTIONS TO TRANSPORT OF INTENSITY EQUATION

TIE is an elliptic partial differential equation for the phase function ϕ , which seems not hard to solve. With a specified boundary condition, one may find a unique solution provided that the intensity distribution is strictly positive [2]. As suggested by Teague [1], an auxiliary function ψ was introduced, which satisfies

$$\nabla \psi = I \nabla \phi \quad (3)$$

This assumption means the vector field $I \nabla \phi$, which is associated with the Poynting vector [3], can be characterized by the scalar potential ψ . Then TIE can be solved via the following Poisson equations [4]

$$-k \frac{\partial I}{\partial z} = \nabla^2 \psi, \quad (4)$$

and

$$\nabla \cdot (I^{-1} \nabla \psi) = \nabla^2 \phi. \quad (5)$$

Solving the above two equations is straightforward mathematically, however, implementing the solution has been proven to be difficult in practice. The first serious difficulty is how to define or supply appropriate boundary conditions, since no simple method for obtaining the needed boundary value has been found so far. The second difficulty lies in that only very few methods available of phase reconstruction can provide sufficiently high speed and acceptable memory requirement. The most well adopted method for TIE phase imaging utilizes fast Fourier transform (FFT) for speedy processing of the inverse Laplacian. However, the FFT-based method implies periodic boundary conditions, which is often in contradiction to the experimental setup, leading to significant boundary artifacts.

The Green's function solution to TIE which is first proposed by Teague [1], he represented the solution of Poisson equation in a general two dimensional domain Ω by [1]

$$\psi(\mathbf{r}) = -k \iint_{\Omega} \frac{\partial I(\mathbf{r}')}{\partial z} G(\mathbf{r}, \mathbf{r}') d\mathbf{r}' - \int_{\partial\Omega} G(\mathbf{r}, \mathbf{r}') \frac{\partial \psi(\mathbf{r}')}{\partial n} ds' + \int_{\partial\Omega} \psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} ds', \quad (6)$$

where $\partial\Omega$ is the region boundary, the $G(\mathbf{r}, \mathbf{r}')$ is the Green's function which satisfies $\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$. He assumed that the boundary values of the phase function can be obtained by other means. Imposing a Dirichlet boundary condition on $\partial\Omega$, ϕ is determined to within an additive constant. However, the phase boundary values $\partial\psi/\partial n$ are difficult to be measured directly or be known beforehand in practice.

To address the difficulty of obtaining the required phase boundary values, one need to introduce an aperture on the object plane or the intermediate infocus image plane. The aperture blocks off all the intensity outside the domain Ω , producing a strong transverse intensity derivative signal along the pupil boundary normal direction. For objects with smooth intensity and phase distribution, it is reasonable to assume that the intensity derivative parallel to the region boundary is negligible compared with the intensity derivative towards the boundary normal. Then the gradient of I at the region boundary is dominated by the component in pupil boundary normal direction, which equals the negative of the intensity boundary value, pointing outward from the domain:

$$\nabla I = \begin{cases} -I_{\partial\Omega} \delta_{\partial\Omega} \mathbf{n} & \mathbf{r} \in \partial\Omega \\ A_{\Omega} \nabla I, & \mathbf{r} \in \Omega \\ 0 & \text{others} \end{cases}, \quad (7)$$

where the A_Ω is the aperture function (=1 inside Ω , = 0 outside Ω). Note the gradient of I inside the domain is unaltered by the aperture. Similarly,

$$I = \begin{cases} A_\Omega I, & \mathbf{r} \in \Omega \\ 0 & \text{others} \end{cases} \quad (8)$$

Inserting Eqs. (7) and (8) into the TIE [Eq. (2)], we have

$$-k \frac{\partial I}{\partial z} = A_\Omega (I \nabla^2 \phi + \nabla I \cdot \nabla \phi) - I_{\partial\Omega} \frac{\partial \phi}{\partial n} \delta_{\partial\Omega}. \quad (9)$$

The first term on RHS is the longitudinal intensity variation inside the domain due to the phase slope and curvature *as if the aperture is not present*. The second term provides the Neumann condition of ψ at the region boundary. Since the two terms do not overlap, one can measure them separately thus having enough information to solve the Poisson equation [Eq. (4)] for ψ . The Green's function representation of the solution can be represented by

$$\psi(\mathbf{r}) = \iint_{\Omega} (I \nabla^2 \phi + \nabla I \cdot \nabla \phi) G(\mathbf{r}, \mathbf{r}') d\mathbf{r}' - \int_{\partial\Omega} G(\mathbf{r}, \mathbf{r}') I_{\partial\Omega}(\mathbf{r}') \frac{\partial \phi(\mathbf{r}')}{\partial n} ds', \quad (10)$$

where $G(\mathbf{r}, \mathbf{r}')$ is the Green's function with a Neumann boundary condition on $\partial\Omega$, which satisfies:

$$\nabla G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') - A^{-1} \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \Big|_{\Omega} = 0, \quad (11)$$

where A is the area of the region Ω . As we can see that the Green's function method represents the solution of the partial differential equation by one area and one curve integral of the Green's function. The two terms stands for the two contributory sources to the solution: the interior signal and the boundary signal of the region, which can be both obtained from the measured longitudinal intensity derivative. The Green's function which obeys Eq. (11), can be expanded in terms of eigen-functions of Laplacian, which is Fourier cosine harmonics, thus when the aperture is of rectangular shape, the numerical solution can be implemented with fast discrete cosine transforms (DCT) efficiently.

4. AXIAL INTENSITY DERIVATIVE ESTIMATION

The TIE requires to know the in-focus intensity I and axial intensity derivative $\partial I / \partial z$. Experimentally, I is easy to obtain. However, the intensity derivative along the optic axis cannot be directly measured. Conventionally, it is approximated by a finite difference between two out-of-focus images, recorded symmetrically about the in-focus plane with $\pm \Delta z$ defocus distance.

$$\frac{\partial I(\mathbf{r})}{\partial z} \approx \frac{I_{\Delta z}(\mathbf{r}) - I_{-\Delta z}(\mathbf{r})}{2\Delta z}. \quad (12)$$

Mathematically, this approximation is valid in the limit of small defocus distances, where the error is the second order of the focus distance if the data are noise-free. However, experimentally the derivative estimate will become quite unstable when the distance Δz is too small because of the noise and quantitation error [5]. On the other hand, increasing the two-plane separation Δz provides better signal-to-noise ratio (SNR) in the derivative estimate; but the breakdown of the linear approximation induces nonlinearity errors, which results in the loss of the phase lateral resolution [6]. Thus a *compromise* is made where Δz is chosen to balance the high-order (or non-linearity) error and the noise effect [7]. Specifically, the optimal Δz is dependent on both the maximum physically significant frequency of the object and the noise level [8]. A priori knowledge of these two aspects is difficult not be known in advance.

To overcome this difficulty, one may simply increase the number intensity measurements at multiple planes [6, 9]. With more intensity measurements $I_{j\Delta z}(\mathbf{r}), j = -n, \dots, 0, \dots, n$, the longitudinal intensity derivative can be represented by their linear combination:

$$\frac{\partial I(\mathbf{r})}{\partial z} \approx \sum_{j=-n}^n \frac{a_j I_{j\Delta z}(\mathbf{r})}{\Delta z}. \quad (13)$$

Thus, it offers more flexibility for improving the accuracy and noise resistance in derivative estimation. The only difference in these multiple planes derivative estimation methods lies in the coefficients a_i in Eq. (11), and it has been found that all these method can be unified into Savitzky-Golay differentiation filter (SGDF) with different degrees if the finite difference [Eq. (13)] is viewed from the viewpoint of the digital filter [5]. One can use the so-called optimal

frequency selection (OFS) scheme to improve the derivative estimation accuracy while suppress the noise. OSF is a three-step process with the basic idea that selecting the optimal components of the phase recovered from SGDFs with various degrees in spatial frequency domain to account for both phase SNR and nonlinearity error: First, estimate the intensity derivatives with SGDFs with different degrees through Eq. (11). Second, reconstruct the phase distributions with TIE by solving the two Poisson equations [Eqs. (4) and (5)] with the estimated intensity derivatives. Finally, extract the optimal frequency components from these phase distributions using a complementary filter bank in spatial frequency domain and then recombine them into a composite phase.

5. INSTANTANEOUS FOCUS STACK COLLECTION FOR DYNAMIC IMAGING

Another important issue is TIE requires a series of images captured at different focal depths, which is usually realized by translating the camera or the object manually or mechanically. This not only complicates the image acquisition process, but also prolongs the measurement time, precluding real-time observation of dynamic process. To this end, we introduce two schemes to extend transport of intensity quantitative phase imaging to dynamics applications. Both the two schemes are built upon a commercial microscope by introducing an additional image-relay system to the camera port of microscope. The first scheme (called tunable lens based TIE (TL-TIE) [10]) introduce an electronically tunable lens (ETL) in the Fourier plane of the image-relay system, which allows for rapid electronically-controlled re-focusing with constant magnification without altering the original high imaging quality of the microscope. The second scheme ((called single-shot quantitative phase microscopy (SQPM) [11]) introduces a Michelson-like architecture with a spatial light modulator between the image-relay system, so that two laterally separated images at different focus distance can be obtained simultaneously by a single camera exposure, enabling TIE phase recovery to be performed at frame rates that are only camera limited.

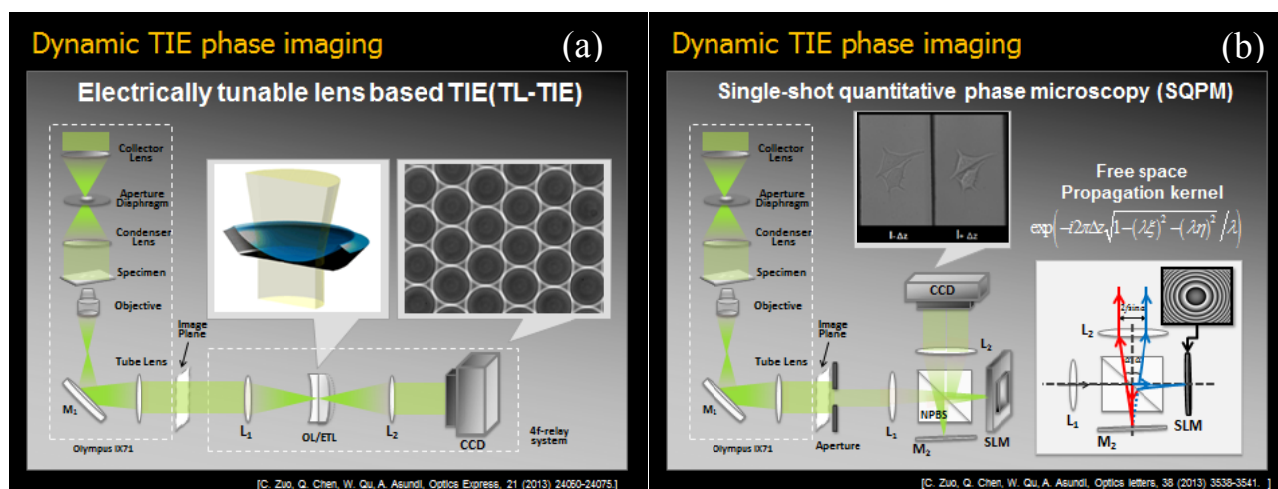


Fig. 1 Two system allowing dynamic TIE phase imaging. (a) Tunable lens based TIE (TL-TIE) system. (d) Single-shot quantitative phase microscopy (SQPM) system.

These schemes proposed here offer the possibility to extend TIE phase imaging to the study of fast moving objects and structural changes in dynamic processes. By using the numerical reconstruction implemented by the computer software, real-time transport of intensity imaging can be realized. We have successfully applied the technology in the characterization of microlens array, investigations of cellular dynamics and drug-induced morphology changes. Figure 2 shows the 3D rendering of phase map of one macrophage cell at different stages of phagocytosis.

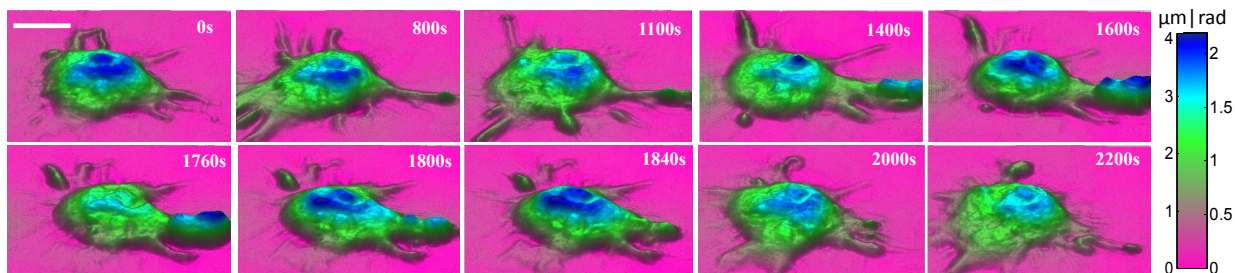


Fig. 2. Dynamic phase imaging of macrophage phagocytosis.

6. DIRECT CONTINUOUS RECOVERY FOR DIGITAL HOLOGRAPHY

One of the great advantages of TIE is able to directly reconstruct the continuous phase of the wave, without the need for phase unwrapping [12, 13]. Taking advantage of this appealing feature, we can combine it with digital holography. Numerical focusing is a unique capability of digital holography, where a single hologram is used to calculate the optical field at any number of image planes, emulating the focusing control of a conventional imaging system. This inspires us to properly combine these two techniques so that merits of both methods can be accrued. One can directly recover the continuous phase in digital holography based on solving the TIE from the normalized wavefield obtained during the numerical reconstruction of the hologram, and meanwhile, eliminate the tilt and quadratic phase aberration inherent in digital holography without cumbersome physical or numerical compensation procedure [14].

7. PHASE SPACE REPRESENTATION OF TRANSPORT OF INTENSITY

The fact that TIE is not restricted to the purely coherent regime but works also with a partially coherent source seems to be very well known. However, partially coherent field does not have a well-defined phase since the field experiences statistical fluctuations over time, which brings up an important question about what “phase” is being measured in such scenarios. Phase-space optics provides an elegant description of TIE under partially coherent illumination. The phase-space representation of TIE for a paraxial partially coherent field is given by

$$\frac{\partial I(\mathbf{x})}{\partial z} = -\lambda \nabla_{\mathbf{x}} \cdot \int \mathbf{u} W(\mathbf{x}, \mathbf{u}) d\mathbf{u}, \quad (14)$$

where $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$ are the two-dimensional spatial and spatial frequency vectors, respectively. $W(\mathbf{x}, \mathbf{u})$ is the Wigner distribution function (WDF) of the paraxial, temporally stationary, and quasi-monochromatic partially coherent stochastic field $u(\mathbf{x})$. $\nabla_{\mathbf{x}}$ is the 2D gradient operator over \mathbf{x} . Equation (14) reveals that the transport of intensity along optical axis gives the spatial divergence of first-order local moment of WDF with respect to the frequency variable. In the limit of the completely coherent wave, the field $u(\mathbf{x})$ becomes deterministic and can be fully described by the complex amplitude function. Then Eq. (14) reduced Teague’s TIE for coherent field [Eq. (1)].

Consider a thin sample illuminated by the mutual intensity $\Gamma_{in}(\mathbf{x}_1, \mathbf{x}_2)$, immediately after the sample, the WDF of the wave field can be viewed as the windowed WDF of the object transmittance, which has the form of a mere multiplication in the \mathbf{x} -direction and a convolution in the \mathbf{u} -direction of the WDFs of the incidence illumination and the object transmittance

$$W_{obj}(\mathbf{x}, \mathbf{u}) = W_T(\mathbf{x}, \mathbf{u}) \otimes_{\mathbf{u}} W_{in}(\mathbf{x}, \mathbf{u}) = \int W_T(\mathbf{x}, \xi) W_{in}(\mathbf{x}, \mathbf{u} - \xi) d\xi. \quad (15)$$

The above equation reveals that the partially coherent incident illumination blurs the object WDF along the frequency dimension. If the object is illuminated with a spatial stationary field, corresponding to Köhler illumination for an optical microscope, the WDF has the form

$$W_{in}(\mathbf{x}, \mathbf{u}) = \bar{s}(\mathbf{u}) = |P_{cond}(\mathbf{u})|^2. \quad (16)$$

Note it only depends on the frequency variable \mathbf{u} . The Fourier transform $\bar{s}(\mathbf{u})$ is a nonnegative function, which can be characterized using an effective source (condenser aperture), which is incoherent and produce the mutual intensity $s(\mathbf{x}')$ at the far-zone. Equation (16) is in fact the Van Cittert-Zernike theorem, wherein $|P_{cond}(\mathbf{u})|^2$ is the intensity distribution over the effective source (condenser aperture) plane.

The Wigner distribution function describes the signal in the space and the spatial frequency domain, simultaneously, which closely resembles the light field in geometrical optics, where the position and direction of a ray are also presented simultaneously. Utilizing this correspondence, light field imaging for a thin specimen can be computationally synthesized based on solving the partially coherent TIE. Apply the approximation $L(\mathbf{x}, \boldsymbol{\theta}) \approx W(\mathbf{x}, \lambda \mathbf{u})$ to Eq. (14), we obtain

$$\frac{\partial I(\mathbf{x})}{\partial z} \approx -\nabla_{\mathbf{x}} \cdot \int \boldsymbol{\theta} L(\mathbf{x}, \boldsymbol{\theta}) d\boldsymbol{\theta}. \quad (17)$$

Equation (30) is regarded as TIE at geometry limit which reveals that the transport of intensity along optical axis gives the spatial divergence of first-order local angular moment of the light field. For a thin sample under Köhler illumination of an optical microscope, the light field can be computed by

$$L(\mathbf{x}, \boldsymbol{\theta}) = I(\mathbf{x}) \left| P_{cond} \left[\boldsymbol{\theta} - \frac{1}{k} \nabla_{\mathbf{x}} \phi(\mathbf{x}) \right] \right|^2. \quad (18)$$

It can be seen that the angular distribution of the light field is determined by the intensity distribution over the source plane $|P_{cond}(\mathbf{u})|^2$. With a specified $|P_{cond}(\mathbf{u})|^2$, the $L(\mathbf{x}, \boldsymbol{\theta})$ can be calculated, thus perspective of the specimen from an arbitrary viewing angle can be obtained.

8. CONCLUSIONS

In this paper, we have introduced some recent new developments in TIE computational imaging. Due to its following unique advantages

- a) It is non-interferometric, works with partially coherent illumination.
- b) It does not require a complicated optical system, can well compatible with the conventional bright field microscope.
- c) It is a single beam method and less stringent to mechanical and environmental instability.
- d) It directly recovers the continuous phase without 2π discontinuity, so there is no need to phase unwrapping.
- e) It can provide high-quality phase measurement with high resolution in lateral (x and y) and vertical (z) and temporal (t) dimensions.

We firmly believed that the TIE will demonstrate even larger potentials in diverse fields in further development.

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REFERENCES

- [1] M. Reed Teague, "Deterministic phase retrieval: a Green's function solution," *J. Opt. Soc. Am.*, 73(11), 1434-1441 (1983).
- [2] T. E. Gureyev, A. Roberts, and K. A. Nugent, "Partially coherent fields, the transport-of-intensity equation, and phase uniqueness," *J. Opt. Soc. Am. A*, 12(9), 1942-1946 (1995).
- [3] D. Paganin, and K. A. Nugent, "Noninterferometric Phase Imaging with Partially Coherent Light," *Physical Review Letters*, 80(12), 2586-2589 (1998).
- [4] L. J. Allen, and M. P. Oxley, "Phase retrieval from series of images obtained by defocus variation," *Optics Communications*, 199(1-4), 65-75 (2001).
- [5] C. Zuo, Q. Chen, Y. Yu et al., "Transport-of-intensity phase imaging using Savitzky-Golay differentiation filter - theory and applications," *Opt. Express*, 21(5), 5346-5362 (2013).
- [6] L. Waller, L. Tian, and G. Barbastathis, "Transport of Intensity phase-amplitude imaging with higher order intensity derivatives," *Opt. Express*, 18(12), 12552-12561 (2010).
- [7] D. Paganin, A. Barty, P. J. McMahon et al., "Quantitative phase-amplitude microscopy. III. The effects of noise," *Journal of Microscopy*, 214(1), 51-61 (2004).

- [8] A. V. Martin, F. R. Chen, W. K. Hsieh et al., "Spatial incoherence in phase retrieval based on focus variation," *Ultramicroscopy*, 106(10), 914-924 (2006).
- [9] M. Soto, and E. Acosta, "Improved phase imaging from intensity measurements in multiple planes," *Appl. Opt.*, 46(33), 7978-7981 (2007).
- [10] C. Zuo, Q. Chen, W. Qu et al., "High-speed transport-of-intensity phase microscopy with an electrically tunable lens," *Optics Express*, 21(20), 24060-24075 (2013).
- [11] C. Zuo, Q. Chen, W. Qu et al., "Noninterferometric single-shot quantitative phase microscopy," *Optics letters*, 38(18), 3538-3541 (2013).
- [12] A. Barty, K. A. Nugent, D. Paganin et al., "Quantitative optical phase microscopy," *Opt. Lett.*, 23(11), 817-819 (1998).
- [13] C. J. Bellair, C. L. Curl, B. E. Allman et al., "Quantitative phase amplitude microscopy IV: imaging thick specimens," *Journal of Microscopy*, 214(1), 62-69 (2004).
- [14] C. Zuo, Q. Chen, W. Qu et al., "Direct continuous phase demodulation in digital holography with use of the transport-of-intensity equation," *Optics Communications*, 309(0), 221-226 (2013).