ON SOME DOUBLE-SUM FALSE THETA SERIES

SONG HENG CHAN AND BYUNGCHAN KIM

Abstract. We compute false theta series obtained from the double-sum

$$g_{a,b,c}(x,y,q) = \left(\sum_{r,s\geq 0} + \sum_{r,s<0} \right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}.$$

We investigate its number-theoretic and combinatorial implications using q-series and unimodal sequences.

1. Introduction

In the theory of q-series, false theta functions appear in lots of places. False theta functions are series that would be theta series if there are no alternation of the signs of some series terms. These series were first introduced by Rogers [12] and Ramanujan provided lots of identities involving false theta series in his notebooks [13] and lost notebook [14]. We do not know why Ramanujan studied these series, but his identities play important roles in recent studies on quantum modular forms.

In this paper, we compute false theta series obtained from the double-sum

$$g_{a,b,c}(x,y,q) = \left(\sum_{r,s\geq 0} + \sum_{r,s<0} \right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}. \tag{1.1}$$

Note that $g_{a,b,c}(x,y;q)$ differs from the traditional Hecke type double-sum

$$f_{a,b,c}(x,y,q) = \left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}, \tag{1.2}$$

in that we have a plus sign between two double-sums instead of a minus sign. The function $f_{a,b,c}(x,y,q)$ has played a very important roles in the theory of mock theta functions and satisfies many beautiful identities. For example, from [4, Example 1.1], we find that

$$f_{1,2,1}(q,-q,q) = (-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty \phi(q),$$

where $\phi(q)$ is the sixth order mock theta function

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q)_{2n}},$$

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and here and in the sequel, we use standard q-series notation:

$$(a;q)_n = (a)_n = \prod_{k=1}^{n}(1-aq^{k-1})$$

for $n \geq 0$. We say $f_{a,b,c}(x,y,q)$ is a Hecke-type double sum with type I symmetry while we refer $g_{a,b,c}(x,y,q)$ as a Hecke-type double-sum with type II symmetry. Hickerson and Mortenson [4] systemically study Hecke-type double sums with type I symmetry.

An interesting instant of a Hecke-type double-sum with type II symmetry appears in Andrews, Dyson, and Hickerson's beautiful paper [1, Theorem 1]:

$$g_{1,5,1}(-q,-q,q) - q^2 g_{1,5,1}(-q^4,-q^4,q) = \sigma(q), \tag{1.3}$$

where

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n}.$$

The function $\sigma(q)$ is deeply related with a real quadratic field [1, 3], and is an important example of quantum modular forms [17]. A number of researches have been conducted to understand or to find the identities similar to (1.3). (For example, see [2, 4, 8, 10, 11].)

Another instance of Hecke-type double-sums with type II symmetry appears in a recent work of the second author [5] while investigating the number of a certain type of unimodal sequences. Recall that a unimodal sequence is a sequence which is weakly increasing up to a point (called the peak), and then weakly decreasing thereafter. The weight of such a sequence is the sum of all of its terms. Let $u(n)$ be the number of unimodal sequences of weight n such that the largest part in the partition before the peak is larger than or equal to the largest part in the partition after the peak. Then we can rewrite [5, Theorem 6] as

$$\sum_{n\geq 0} u(n)q^n = \frac{1}{(q)_\infty^2} g_{1,2,4}(q,-q^4,q). \tag{1.4}$$

It turns out that $g_{1,2,4}(q,-q^4,q)$ is essentially a false theta function.

Theorem 1.1. *We have*

$$\left(\sum_{n,r\geq 0} + \sum_{n,r<0}\right) q^{n(n+1)/2+2nr+2r(r+1)} = g_{1,2,4}(-q,-q^4,q) = \frac{1}{1-q},$$

$$\left(\sum_{n,r\geq 0} + \sum_{n,r<0}\right) (-1)^n q^{n(n+1)/2+2nr+2r(r+1)} = g_{1,2,4}(q,-q^4,q) = \frac{1}{1-q}\sum_{n\geq 0}(-1)^n q^{n(n+1)/2}(1-q^{n+1}).$$

The following congruence is immediate.

Corollary 1.2. *We have*

$$\sum_{n\geq 0} u(n)q^n \equiv \frac{1}{(1-q)(q^2;q^2)_\infty} \pmod 2.$$

Moreover, the first identity in Theorem 1.1 means that all positive integers can be written in the unique form as

$$T_n + 2nr + 4T_r$$

for some integers n and r with $n, r \geq 0$ or $n, r < 0$, where T_n is a n-th triangular number $n(n+1)/2$.

The second identity in Theorem 1.1 implies that the q-expansion of $g_{1,2,4}(q, -q^4, q)$ has an interesting sign-pattern, namely,

$$+, --, +++, ----, 5+, 6-, 7+, \cdots$$

Since (1.4) arises from a Bailey Lemma method, it is natural to investigate whether there are similar Hecke-type double-sums with type II symmetry arising from Bailey lemmas and to examine their combinatorial and number-theoretic implications.

For example, Lovejoy and the second author [6] studied the number of unimodal sequences with double peaks. Let $dp(n)$ be the number of such unimodal sequences of weight n. Then, its generating function can be rewritten using the function $g_{a,b,c}(x, y, q)$.

Theorem 1.3. *We have*

$$\sum_{n\geq 0} dp(n)q^n = \sum_{n\geq 0} \frac{q^{2n}}{(q)_n^2} = \frac{1}{(q)_\infty^2} g_{1,2,2}(q, -q^3, q).$$

The rest of paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.3 and provide other combinatorial generating functions involving Hecke-type double-sums with type II symmetry. We also investigate their arithmetic properties.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. It is based on rearrangement of the summand, and we will use the same idea in the later sections.

Proof of Theorem 1.1. By setting $n = k - 2m$, we see that

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{n(n+1)/2+2mn+2m(m+1)} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{(n+2m)(n+2m+1)/2+m} \\ &= \sum_{m=0}^{\infty}\sum_{k=2m}^{\infty} q^{k(k+1)/2+m} \\ &= \sum_{k=0}^{\infty} q^{k(k+1)/2} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} q^m.\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{m=-\infty}^{-1}\sum_{n=-\infty}^{-1} q^{n(n+1)/2+2mn+2m(m+1)} &= \sum_{m=-\infty}^{-1}\sum_{k=-\infty}^{2m-1} q^{k(k+1)/2+m} \\
&= \sum_{m=1}^{\infty}\sum_{k=2m+1}^{\infty} q^{k(k-1)/2-m} \\
&= \sum_{m=1}^{\infty}\sum_{k=2m}^{\infty} q^{k(k+1)/2-m} \\
&= \sum_{k=0}^{\infty} q^{k(k+1)/2} \sum_{m=-\lfloor \frac{k}{2} \rfloor}^{-1} q^{m}.
\end{aligned}$$

Therefore, combining both sums, we obtain

$$\begin{aligned}
&\left(\sum_{m,n\geq 0} + \sum_{m,n<0}\right) q^{n(n+1)/2+2mn+2m(m+1)} \\
&= \sum_{k=0}^{\infty} q^{k(k+1)/2} \sum_{m=-\lfloor \frac{k}{2} \rfloor}^{\lfloor \frac{k}{2} \rfloor} q^{m} \\
&= \sum_{k=0}^{\infty} q^{k(k+1)/2} \frac{q^{-\lfloor \frac{k}{2} \rfloor}(1-q^{2\lfloor \frac{k}{2} \rfloor+1})}{1-q} \\
&= \sum_{k=0}^{\infty} q^{k(2k+1)} \frac{q^{-k}(1-q^{2k+1})}{1-q} + \sum_{k=0}^{\infty} q^{(k+1)(2k+1)} \frac{q^{-k}(1-q^{2k+1})}{1-q} \\
&= \frac{1}{1-q}\left(\sum_{k=0}^{\infty} q^{2k^2} - q^{2k^2+2k+1} + q^{2k^2+2k+1} - q^{2k^2+4k+2}\right) \\
&= \frac{1}{1-q}.
\end{aligned}$$

This proves the first identity.

By the same computation, we find that

$$\begin{aligned}
&\left(\sum_{m,n\geq 0}+\sum_{m,n<0}\right)(-1)^n q^{n(n+1)/2+2mn+2m(m+1)}\\
&=\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)/2}\left(\sum_{m=0}^{\lfloor \frac{k}{2}\rfloor} q^m-\sum_{m=-\lfloor \frac{k}{2}\rfloor}^{-1} q^m\right)\\
&=\sum_{k=0}^{\infty} q^{k(2k+1)}\left(\frac{1-q^{k+1}}{1-q}-\frac{q^{-k}(1-q^k)}{1-q}\right)-\sum_{k=0}^{\infty} q^{(k+1)(2k+1)}\left(\frac{1-q^{k+1}}{1-q}-\frac{q^{-k}(1-q^k)}{1-q}\right)\\
&=\frac{1}{1-q}\sum_{k=0}^{\infty} q^{2k^2}\left(q^k-q^{2k+1}-1+q^k-q^{3k+1}+q^{4k+2}+q^{2k+1}-q^{3k+1}\right)\\
&=\frac{1}{1-q}\sum_{k=0}^{\infty} q^{2k^2}\left(-1+2q^k-2q^{3k+1}+q^{4k+2}\right).
\end{aligned}$$

Since

$$\sum_{k=0}^{\infty} q^{2k^2}(1-q^{4k+2})=\sum_{k=0}^{\infty} q^{2k^2+k}(1-q^{4k+3})=1,$$

we conclude that

$$\begin{aligned}
&\left(\sum_{m,n\geq 0}+\sum_{m,n<0}\right)(-1)^n q^{n(n+1)/2+2mn+2m(m+1)}\\
&=\frac{1}{1-q}\sum_{k=0}^{\infty} q^{2k^2}\left(q^k-2q^{3k+1}+q^{5k+3}\right)\\
&=\frac{1}{1-q}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)/2}(1-q^{k+1}),
\end{aligned}$$

and this completes the proof of the second identity.

□

3. Other examples

We first prove Theorem 1.3.

Proof of Theorem 1.3. The first equality follows easily from an elementary combinatorial argument. Now we prove the second equality. From [9, (1) in Theorem 1.1], we find that

$$\sum_{n=0}^{\infty} q^n\beta_n=\frac{1}{(aq)_\infty(q)_\infty}\sum_{n,r\geq 0}(-a)^n q^{n(n+1)/2+(2n+1)r}\alpha_r, \tag{3.1}$$

where (α_n,β_n) is a Bailey pair relative to a.

From [15], we find that

$$\begin{aligned} \alpha_0 &= 1, \\ \alpha_n &= -q^{n(n-1)}(1-q^{2n}) \text{ for } n>0, \\ \beta_n &= \frac{q^n}{(q)_n^2} \end{aligned} \tag{3.2}$$

consists of a Bailey pair relative to 1. By plugging (3.2) to (3.1), we find that

$$\begin{aligned} \sum_{n\geq 0} \frac{q^{2n}}{(q)_n^2} &= \frac{1}{(q)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \left(1 - \sum_{r=1}^{\infty} q^{(2n+1)r+r(r-1)}(1-q^{2r})\right) \\ &= \frac{1}{(q)_\infty^2} \sum_{n,r\geq 0} (-1)^n q^{n(n+1)/2+(2n+1)r+r(r+1)} \\ &\quad - \frac{1}{(q)_\infty^2} \sum_{n\geq 0, r>0} (-1)^n q^{n(n+1)/2+(2n+1)r+r(r-1)} \\ &= \frac{1}{(q)_\infty^2} \left(\sum_{n,r\geq 0} + \sum_{n,r<0} \right) (-1)^n q^{n(n+1)/2+2nr+r(r+2)}, \end{aligned}$$

where we replace n and r by $-n-1$ and $-r$ in the second summation for the last equality. Therefore, we have obtained that

$$\sum_{n\geq 0} \frac{q^{2n}}{(q)_n^2} = \frac{1}{(q)_\infty^2} g_{1,2,2}(q, -q^3, q).$$

□

From [6, Eqn (1.3)] with $x = 1$ (this is actually from Warnaar's partial theta function identity.), we find that

$$g_{1,2,2}(q, -q^3, q) = 1 + 2 \sum_{n\geq 1} (-1)^n q^{n(n+1)/2}.$$

We can also prove the above as follows. Proceeding as before in the proof of Theorem 1.1, we can find that

$$\begin{aligned}
g_{1,2,2}(q,-q^3,q) &= \sum_{k\geq 0}(-1)^k q^{k(k+1)/2}\left(\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor} q^{-m(m-1)} - \sum_{m=-\lfloor \frac{n}{2}\rfloor}^{-1} q^{-m(m-1)}\right) \\
&= \sum_{k\geq 0}(-1)^k q^{k(k+1)/2} + \sum_{k\geq 2}(-1)^k q^{k(k+1)/2}\sum_{m=1}^{\lfloor \frac{n}{2}\rfloor} q^{-m(m-1)}(1-q^{-2m}) \\
&= \sum_{k\geq 0}(-1)^k q^{k(k+1)/2} + \sum_{k\geq 1} q^{k(2k+1)}\sum_{m=1}^{k} q^{-m(m-1)}(1-q^{-2m}) \\
&\quad - \sum_{k\geq 1} q^{(k+1)(2k+1)}\sum_{m=1}^{k} q^{-m(m-1)}(1-q^{-2m}) \\
&= \sum_{k\geq 0}(-1)^k q^{k(k+1)/2} + \sum_{k\geq 1}\left(-q^{k^2} + q^{2k^2+k} - q^{2k^2+3k+1} + q^{(k+1)^2}\right) \\
&= 1 + 2\sum_{k\geq 0}(-1)^k q^{k(k+1)/2}.
\end{aligned}$$

As another example, we derive the following.

Theorem 3.1. *We have*

$$2\sum_{n\geq 0}\frac{q^n}{(q)_n^2(1+q^n)} = \frac{1}{(q)_\infty^2} g_{1,2,1}(q,-q,q).$$

We note that the left hand side is twice of the generating function for the number of unimodal sequences of weight n such that if the size of the peak is p, then the number of parts of size p before the peak is larger than or equal to the number of parts of size p after the peak. To see this, we first observe that the summand in the left hand side is

$$\frac{q^n}{(1-q^{2n})(q)_{n-1}(q)_n}.$$

We may think q^n corresponds to the peak of size n. For the part generated by $1/(1-q^{2n})$, we split them into two equal sizes, i.e. n and n, then put each of them before and after the peak. We put parts from $1/(q)_{n-1}$ after the peak and put the parts from $1/(q)_n$ before the peak.

Proof of Theorem 3.1. From [15], we find that

$$\begin{aligned}
\alpha_0 &= 1, \\
\alpha_n &= q^{n(n-1)/2}(1-q^n) \text{ for } n>0, \\
\beta_n &= \frac{2}{(q)_n^2(1+q^n)}
\end{aligned} \tag{3.3}$$

consists of a Bailey pair relative to 1. By plugging (3.3) to (3.1), we find that

$$\begin{aligned}\sum_{n\geq 0}\frac{2q^n}{(q)_n^2(1+q^n)} &= \frac{1}{(q)_\infty^2}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\left(1+\sum_{r=1}^{\infty}q^{(2n+1)r+r(r-1)/2}(1-q^r)\right)\\ &= \frac{1}{(q)_\infty^2}\sum_{n,r\geq 0}(-1)^n q^{n(n+1)/2+(2n+1)r+r(r-1)/2}\\ &\quad -\frac{1}{(q)_\infty^2}\sum_{n\geq 0,r>0}(-1)^n q^{n(n+1)/2+(2n+1)r+r(r+1)/2}\\ &= \frac{1}{(q)_\infty^2}\left(\sum_{n,r\geq 0}+\sum_{n,r<0}\right)(-1)^n q^{n(n+1)/2+2nr+r(r+1)/2},\end{aligned}$$

where we replace n and r by $-n-1$ and $-r$ in the second summation for the last equality. Therefore, we obtain that

$$2\sum_{n\geq 0}\frac{q^n}{(q)_n^2(1+q^n)} = \frac{1}{(q)_\infty^2}g_{1,2,1}(q,-q,q).$$

□

Corollary 3.2. *We have*

$$g_{1,2,1}(q,-q,q)\equiv (q^2;q^2)_\infty \pmod 2.$$

Proof. Since

$$2\sum_{n\geq 0}\frac{q^n}{(q)_n^2(1+q^n)}\equiv 1 \pmod 2,$$

we obtain that

$$g_{1,2,1}(q,-q,q)\equiv (q)_\infty^2\equiv (q^2;q^2)_\infty \pmod 2.$$

from Theorem 3.1. □

Theorem 3.3. *We have*

$$\sum_{n\geq 0}\frac{q^{2n}}{(q^2;q^2)_n}\begin{bmatrix}2n\\ n\end{bmatrix} = \frac{1}{(q)_\infty(q^2;q^2)_\infty}g_{2,2,2}(q^2,-q^3,q).$$

Here, $\begin{bmatrix}2n\\ n\end{bmatrix}$ is a q-binomial coefficient defined by

$$\begin{bmatrix}2n\\ n\end{bmatrix} := \frac{(q)_{2n}}{(q)_n^2},$$

which is a generating function for the number of partitions with at most n parts $\leq n$. Using this fact, we can think of the left hand side of the identity in Theorem 3.3 as a generating function for the number of unimodal sequences of weight n such that there is a double-peak of size p, and before the peak there are at most p parts $\leq p$, and after the peak, every part $\leq p$ appears even number of times.

Proof of Theorem 3.3. From [9, (5) in Theorem 1.1], if (α_n, β_n) is a Bailey pair relative to 1, then

$$\sum_{n\geq 0}(q;q^2)_n q^n \beta_n = \frac{1}{(q)_\infty (q^2;q^2)_\infty} \sum_{n,r\geq 0}(-1)^n q^{n^2+n+(2n+1)r}\alpha_r. \tag{3.4}$$

By plugging (3.2) to (3.4), we find that

$$\begin{aligned}\sum_{n\geq 0}\frac{(q;q^2)_n q^{2n}}{(q)_n^2} &= \frac{1}{(q)_\infty (q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\left(1-\sum_{r=1}^{\infty} q^{(2n+1)r+r(r-1)}(1-q^{2r})\right)\\ &= \frac{1}{(q)_\infty^2}\sum_{n,r\geq 0}(-1)^n q^{n(n+1)+(2n+1)r+r(r+1)}\\ &\quad - \frac{1}{(q)_\infty (q^2;q^2)_\infty}\sum_{n\geq 0, r>0}(-1)^n q^{n(n+1)+(2n+1)r+r(r-1)}\\ &= \frac{1}{(q)_\infty (q^2;q^2)_\infty}\left(\sum_{n,r\geq 0}+\sum_{n,r<0}\right)(-1)^n q^{n(n+1)+2nr+r(r+2)},\end{aligned}$$

where we replace n and r by $-n-1$ and $-r$ in the second summation for the last equality. Therefore, we obtain that

$$\sum_{n\geq 0}\frac{(q;q^2)_n q^{2n}}{(q)_n^2} = \sum_{n\geq 0}\frac{q^{2n}}{(q^2;q^2)_n}\begin{bmatrix}2n\\ n\end{bmatrix} = \frac{1}{(q)_\infty (q^2;q^2)_\infty} g_{2,2,2}(q^2,-q^3,q).$$

□

The double-sum false theta series $g_{2,2,2}(\pm q^2, -q^3, q)$ can be simplified.

Proposition 3.4. *We have*

$$g_{2,2,2}(-q^2,-q^3,q) = \frac{1}{1-q}$$

while

$$g_{2,2,2}(q^2,-q^3,q) = \frac{1}{1+q}\left(1+2\sum_{k\geq 1}(-1)^k q^{k(k+1)}\right).$$

Proof. By proceeding as before in the proof of Theorem 1.1, we can see that

$$g_{2,2,2}(-q^2,-q^3,q)=\sum_{k\geq 0}\sum_{|m|\leq k}q^{k(k+1)+m}=\frac{1}{1-q}$$

$$\begin{aligned}g_{2,2,2}(q^2,-q^3,q)&=\sum_{k\geq 0}\sum_{m=0}^{k}(-1)^{m+k}q^{k(k+1)+m}-\sum_{k\geq 0}\sum_{m=-k}^{-1}(-1)^{m+k}q^{k(k+1)+m}\\&=\sum_{k\geq 0}(-1)^k q^{k(k+1)}\left(\frac{(-1)^k q^{k+1}+1}{1+q}-\frac{(-1)^k q^{-k}-1}{1+q}\right)\\&=\frac{1}{1+q}\sum_{k\geq 0}q^{k^2}(q^{2k+1}+2(-1)^k q^k-1)\\&=\frac{1}{1+q}\left(1+2\sum_{k\geq 1}(-1)^k q^{k(k+1)}\right).\end{aligned}$$

□

In [6], Lovejoy and the second author studied the number of unimodal sequences of weight n such that the parts before the peak are at most $c-k$, where k is the size of the Durfee square of the partition after the peak. Here we study a double-peak analogue $dv(n)$, where $dv(n)$ is the number of unimodal sequences of weight n such that there is a double-peak and such that the parts before the peak are at most $c-k$, where k is the size of the Durfee square of the partition after the peak. Then [6, Prop 3.1] implies that

$$\sum_{n\geq 0}dv(n)q^n=\sum_{n\geq 0}\frac{(q^{n+1})_n q^{2n}}{(q)_n^2}.$$

We can also express the generating function for $dv(n)$ by using two Hecke-type double sums with type II symmetry, while the generating function for the single peak version involves partial theta functions.

Theorem 3.5. *We have*

$$\sum_{n\geq 0}dv(n)q^n=\frac{1}{(q)_\infty^2}\left(g_{6,3,2}(-q^5,-q^3,q)-qg_{6,3,2}(-q^7,-q^4,q)\right).$$

Proof. Again from [9, Eqn (6) in Theorem 1.1], for a Bailey pair (α_n,β_n) relative to 1,

$$\sum_{n\geq 0}(q^{n+1})_n q^n\beta_n=\frac{1}{(q)_\infty^2}\sum_{n,r\geq 0}q^{3n^2+2n+3rn+r}(1-q^{2n+r+1})\alpha_r. \tag{3.5}$$

Again plugging (3.2) into (3.5), we find that

$$\begin{aligned}\sum_{n\geq 0}\frac{(q^{n+1})_n q^{2n}}{(q)_n^2} &= \frac{1}{(q)_\infty^2}\sum_{n,r\geq 0} q^{n(3n+2)+(3n+1)r}\left(q^{r(r+1)}-q^{2n+r^2+2r+1}\right)\\ &\quad+\frac{1}{(q)_\infty^2}\sum_{n\geq 0,r>0} q^{n(3n+2)+(3n+1)r}\left(q^{2n+1+r^2}-q^{r(r-1)}\right)\\ &=\frac{1}{(q)_\infty^2}\left(\sum_{n,r\geq 0}+\sum_{n,r<0}\right)q^{n(3n+2)+3nr+r^2+2r}\left(1-q^{2n+r+1}\right),\end{aligned}$$

where we replace n and r by $-n-1$ and $-r$ for the last identity. Therefore, we obtain that

$$\sum_{n\geq 0}\frac{(q^{n+1})_n q^{2n}}{(q)_n^2}=\frac{1}{(q)_\infty^2}\left(g_{6,3,2}(-q^5,-q^3,q)-qg_{6,3,2}(-q^7,-q^4,q)\right).$$

□

Surprisingly, the linear combination of two double-sum false theta series appeared in Theorem 3.5 is a sum of a twisted theta function and a false theta function.

Theorem 3.6. *Let*

$$G(q):=g_{6,3,2}(-q^5,-q^3,q)-qg_{6,3,2}(-q^7,-q^4,q).$$

Then,

$$q^4G(q^3)=q-\sum_{n\geq 1}\left(\frac{n}{3}\right)q^{n^2}-2\sum_{n\geq 0}q^{9n^2+15n+7}\left(1-q^{6n+6}\right),$$

where $\left(\frac{n}{3}\right)$ *is the Legendre symbol.*

The following congruence is immediate from Theorem 3.6.

Corollary 3.7. *We have*

$$\sum_{n\geq 0}dv(n)q^{3n+4}\equiv\frac{1}{(q^6;q^6)_\infty}\left(\sum_{n>1}q^{n^2}+\sum_{n\geq 1}q^{9n^2}\right)\pmod 2.$$

Proof of Theorem 3.6. First we note that

$$q^4G(q^3)=\left(\sum_{n,r\geq 0}+\sum_{n,r<0}\right)q^{3n(3n+2)+9nr+3r^2+6r+4}\left(1-q^{6n+3r+3}\right)$$

and

$$\begin{aligned}3n(3n+2)+9nr+3r^2+6r+4&=(3n+3r/2+1)^2+3(r/2+1)^2,\\ 3n(3n+4)+9nr+3r^2+9r+7&=(3n+3r/2+2)^2+3(r/2+1)^2.\end{aligned}$$

By splitting r according to the parity, we can find that

$$\begin{aligned}\sum_{n,r\geq 0} q^{3n(3n+2)+9nr+3r^2+6r+4} &= \sum_{n,r\geq 0} q^{(3n+3r/2+1)^2+3(r/2+1)^2}\\ &= \sum_{n,r\geq 0} \left(q^{(3n+3r+1)^2+3(r+1)^2} + q^{(3n+3r+5/2)^2+3(r+3/2)^2}\right)\\ &= \sum_{\substack{n\geq 0\\ 0\leq r\leq n}} \left(q^{(3n+1)^2+3(r+1)^2} + q^{(3n+5/2)^2+3(r+3/2)^2}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{n,r\geq 0} q^{3n(3n+4)+9nr+3r^2+9r+7} &= \sum_{n,r\geq 0} q^{(3n+3r/2+2)^2+3(r/2+1)^2}\\ &= \sum_{n,r\geq 0} \left(q^{(3n+3r+2)^2+3(r+1)^2} + q^{(3n+3r+7/2)^2+3(r+3/2)^2}\right)\\ &= \sum_{\substack{n\geq 0\\ 0\leq r\leq n}} \left(q^{(3n+2)^2+3(r+1)^2} + q^{(3n+7/2)^2+3(r+3/2)^2}\right).\end{aligned}$$

Similarly, we can also observe that

$$\begin{aligned}\sum_{n,r<0} q^{3n(3n+2)+9nr+3r^2+6r+4} &= \sum_{\substack{n\geq 0\\ 0\leq r\leq n}} \left(q^{(3n+5)^2+3r^2} + q^{(3n+7/2)^2+3(r-1/2)^2}\right),\\ \sum_{n,r<0} q^{3n(3n+4)+9nr+3r^2+9r+7} &= \sum_{\substack{n\geq 0\\ 0\leq r\leq n}} \left(q^{(3n+4)^2+3r^2} + q^{(3n+5/2)^2+3(r-1/2)^2}\right).\end{aligned}$$

Now we find that

$$\begin{aligned}
\sum_{\substack{n\geq 0\\ 0\leq r\leq n}} q^{(3n+1)^2+3(r+1)^2} - \sum_{\substack{n\geq 0\\ 0\leq r\leq n}} q^{(3n+4)^2+3r^2} &\\
&= q^4 - \sum_{n\geq 1} q^{(3n+1)^2}\left(1-q^{3n^2}-q^{3(n+1)^2}\right),\\
\sum_{\substack{n\geq 0\\ 0\leq r\leq n}} q^{(3n+5)^2+3r^2} - \sum_{\substack{n\geq 0\\ 0\leq r\leq n}} q^{(3n+2)^2+3(r+1)^2} &\\
&= -q^7 + \sum_{n\geq 1} q^{(3n+2)^2}\left(1-q^{3n^2}-q^{3(n+1)^2}\right),\\
\sum_{\substack{n\geq 0\\ 0\leq r\leq n}} q^{(3n+5/2)^2}\left(q^{3(r+3/2)^2}-q^{3(r-1/2)^2}\right) &\\
&= -q^7 - 2\sum_{n\geq 1} q^{(3n+5/2)^2+3/4} + q^{13} + \sum_{n\geq 1} q^{(3n+5/2)^2}\left(q^{3(n+1/2)^2}+q^{3(n+3/2)^2}\right),\\
\sum_{\substack{n\geq 0\\ 0\leq r\leq n}} q^{(3n+7/2)^2}\left(q^{3(r-1/2)^2}-q^{3(r+3/2)^2}\right) &\\
&= q^{13} + 2\sum_{n\geq 1} q^{(3n+7/2)^2+3/4} - q^{19} - \sum_{n\geq 1} q^{(3n+7/2)^2}\left(q^{3(n+1/2)^2}+q^{3(n+3/2)^2}\right).
\end{aligned}$$

Therefore, we have arrive at

$$\begin{aligned}
q^4G(q^3) &= q - \sum_{n\geq 0} q^{(3n+1)^2} + \sum_{n\geq 0} q^{(3n+2)^2} - 2\sum_{n\geq 0} q^{9n^2+15n+7}\left(1-q^{6n+6}\right)\\
&= q - \sum_{n\geq 1}\left(\frac{n}{3}\right) q^{n^2} - 2\sum_{n\geq 0} q^{9n^2+15n+7}\left(1-q^{6n+6}\right).
\end{aligned}$$

□

4. Possible generalizations

Recall (see [16], for example) that if (α_n, β_n) is a Bailey pair relative to a, then

$$\begin{aligned}
\alpha'_n &= a^n q^{n^2}\alpha_n,\\
\beta'_n &= \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q)_{n-k}}\beta_k
\end{aligned}$$

is also a Bailey pair relative to a. By iterating (3.2), we could obtain an infinite series of identities involving $g_{a,b,c}$.

Beside iterations, we can find a generalized identity using [7, Theorem 1.1]: For $k \geq 1$ and $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$, and suppose that (α_n, β_n) is a Bailey pair relative to a^2. Then,

$$\sum_{n\geq 0}(a^2q)_{2n}q^n\beta_n \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(aq)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}$$
$$= \sum_{i=1}^{2k} \frac{(-1)^{i+1}a^{i-1}q^{\binom{i}{2}}(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q,q,aq)_\infty} \sum_{r,n\geq 0} a^{(2k+1)n}q^{kn((2k+1)n+2i)+(2k+1)rn+ri}\alpha_r. \tag{4.1}$$

By plugging (3.2) into (4.1), we find that

$$\sum_{n\geq 0} q^{2n}\begin{bmatrix}2n\\n\end{bmatrix} \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(q)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}$$
$$= \sum_{i=1}^{2k} \frac{(-1)^{i+1}q^{\binom{i}{2}}(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty^3}$$
$$\times \left(\sum_{r,n\geq 0} q^{kn((2k+1)n+2i)+(2k+1)rn+ri+r^2+r} - \sum_{n\geq 0, r>0} q^{kn((2k+1)n+2i)+(2k+1)rn+ri+r^2-r}\right).$$

By pairing i-th summand and $2k+1-i$-th summand, we derive that the above equals to

$$\sum_{i=1}^{k} \frac{(-1)^{i+1}(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty^3} \left(\sum_{r,n\geq 0} q^{A_i(n,r)} - \sum_{n\geq 0, r>0} q^{B_i(n,r)}\right.$$
$$\left. - \sum_{r,n\geq 0} q^{A_{2k+1-i}(n,r)} + \sum_{n\geq 0, r>0} q^{B_{2k+1-i}(n,r)}\right),$$

where $A_i(n,r)$ and $B_i(n,r)$ are defined by

$$A_i(n,r) := kn((2k+1)n+2i) + (2k+1)rn + ri + r^2 + r + \binom{i}{2}$$
$$B_i(n,r) := kn((2k+1)n+2i) + (2k+1)rn + ri + r^2 - r + \binom{i}{2}$$

By noting that

$$A_i(n,r) = B_{2k+1-i}(-n-1,-r) \quad \text{and} \quad A_{2k+1-i}(n,r) = B_i(-n-1,-r),$$

we derive that

$$\sum_{n\geq 0} q^{2n}\begin{bmatrix}2n\\n\end{bmatrix} \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_1+N_2+\cdots+N_{k-1}}}{(q)_{n-N_1}(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}$$
$$= \sum_{i=1}^{k} \frac{(-1)^{i+1}(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty^3} \left(\sum_{r,n\geq 0} + \sum_{r,n<0}\right) q^{A_i(n,r)} - q^{A_{2k+1-i}(n,r)}$$
$$= \sum_{i=1}^{2k} \frac{(-1)^{i+1}(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty^3} q^{\binom{i}{2}} g_{2k(2k+1),2k+1,2}(-q^{k(2k+2i+1)}, -q^{i+2}, q).$$

In particular, when $k = 1$, we can recover Theorem 3.5.

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Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang link, Singapore, 637371, Republic of Singapore

E-mail address: ChanSH@ntu.edu.sg

School of Liberal Arts, Seoul National University of Science and Technology, 232 Gongneung-ro, Nowon-gu, Seoul 01811, Korea

E-mail address: bkim4@seoultech.ac.kr