

# COMPLEX SYMMETRY OF COMPOSITION OPERATORS ON HILBERT SPACES OF ENTIRE DIRICHLET SERIES

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ABSTRACT. A criterion for boundedness of composition operators acting on a class of Hilbert spaces of entire Dirichlet series, namely the class  $\mathcal{H}(E, \beta_S)$ , was obtained in J. Math. Anal. Appl. (2013) for those spaces that do not contain non-zero constant functions, while other possibilities were not studied. In this paper, we first provide a complete characterization of boundedness of composition operators on any space  $\mathcal{H}(E, \beta_S)$ , which may or may not contain constant functions. We then study complex symmetry of composition operators on  $\mathcal{H}(E, \beta_S)$ , via analysis of composition conjugations.

## 1. INTRODUCTION

Complex symmetry has recently become an attractive topic in operator theory. Complex symmetric operators are generalizations of complex symmetric matrices and include several types of well-known operators such as normal operators, operators defined by Hankel matrices, Toeplitz matrices, etc. Complex symmetric operators were first studied in [5], and then have been widely discussed on numerous types of function spaces (see, e.g. [2, 9] and the references therein).

In the present paper, we are interested in complex symmetric composition operators on a class of Hilbert spaces of entire Dirichlet series. In the general context, let  $\mathcal{H}$  be a Banach space of holomorphic functions on a domain  $G \subseteq \mathbb{C}$ . A holomorphic self-mapping  $\varphi: G \rightarrow G$  induces a linear *composition operator*  $C_\varphi$  acting on  $\mathcal{H}$  by  $(C_\varphi f)(z) = (f \circ \varphi)(z)$ , for all  $z \in G$  and  $f \in \mathcal{H}$ .

Let  $\Lambda = (\lambda_n)_{n=1}^\infty$  be a sequence of real numbers satisfying  $0 \leq \lambda_n \uparrow +\infty$ . A *Dirichlet series with real frequencies* is a series of the form

$$(1.1) \quad f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad (a_n, z \in \mathbb{C}).$$

Associate to the sequence  $\Lambda$  the constant

$$(1.2) \quad L = \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n},$$

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and to the series (1.1) the quantity

$$D = \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n}.$$

It is well-known (see, e.g., [6, 12]) that in case  $L < \infty$ , the series (1.1) represents an entire function if and only if  $D = -\infty$ . This representation is unique, i.e., every representable entire function  $f$  corresponds to exactly one sequence  $(a_n)$  inducing a series of the form (1.1).

In [7], the authors studied several properties of composition operators on a specific class of Hilbert spaces of entire Dirichlet series, namely the spaces  $\mathcal{H}(E, \beta_S)$ , whose detailed construction is given in Section 2. We briefly describe these spaces here. Let  $\beta = (\beta_n)_{n=1}^{\infty}$  be a sequence of positive real numbers with the property that there exists some  $\alpha > 0$  such that,

$$\liminf_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n^{1+\alpha}} = \infty.$$

Then, the Hilbert space  $\mathcal{H}(E, \beta_S)$  induced by  $\beta$  consists of all Dirichlet series  $f$  of the form (1.1) such that

$$\|f\| := \left( \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < +\infty.$$

The paper [7] provides a criterion for boundedness of such composition operators. Thereafter other properties of these operators were investigated by many authors. In particular, [7] also studies their compactness and compact difference of two composition operators. The authors of [16] worked on invertibility, cyclicity, and computed the operator norms via reproducing kernels. Numerical range is investigated in [8], while topological structure is considered in [11].

We would like to emphasize that all the aforementioned results depend strongly on the following theorem of boundedness, which is the first and the most important property. Note that by the *invariance* of an operator  $T$  on  $\mathcal{H}(E, \beta_S)$ , we mean  $T$  maps  $\mathcal{H}(E, \beta_S)$  into itself.

**Theorem 1.1** ([7] Thms 4.2, 4.9; Prop 4.4). *Let  $\varphi(z)$  be an entire function on  $\mathbb{C}$ . The following are equivalent.*

- i) *The induced composition operator  $C_\varphi$  is invariant on  $\mathcal{H}(E, \beta_S)$ .*
- ii) *The induced composition operator  $C_\varphi$  is bounded on  $\mathcal{H}(E, \beta_S)$ .*
- iii)  *$\varphi(z) = z + b$  for some  $b \in \mathbb{C}$  with  $\Re(b) \geq 0$ .*

We remark that the proof of boundedness in [7, Prop 4.4] assumes  $\lambda_n > 0$  for all  $n \geq 1$ , i.e., when the Hilbert space  $\mathcal{H}(E, \beta_S)$  does not contain any non-zero constant function. If the Hilbert space contains the constant functions, i.e., when  $\lambda_1 = 0$ , then the proof is no longer valid. Moreover, while it is obvious that in this case any constant function  $\varphi$  also induces a bounded composition operator  $C_\varphi$ , this fact is not reflected in Theorem 1.1. This means that another criterion for boundedness different from that of [7] is required for the general case  $\lambda_1 \geq 0$ .

The aim of this paper is first to provide a complete criterion for boundedness of composition operators published in [7], which reduces to Theorem 1.1 for the particular case  $\lambda_1 > 0$ . We then proceed to investigate the complex

symmetry property of bounded composition operators on  $\mathcal{H}(E, \beta_S)$ . Using the composition conjugation form, we show that all composition operators induced by a non-constant  $\varphi$  are complex symmetric.

The paper is organized as follows. In Section 2, we discuss the construction of the Hilbert spaces of entire Dirichlet series  $\mathcal{H}(E, \beta_S)$  based on different notions of order of a Dirichlet series. In Section 3, we state and prove a complete criterion for bounded composition operators on *any* space  $\mathcal{H}(E, \beta_S)$ . This criterion applies for the general case  $\lambda_1 \geq 0$  and is a considerable refinement of the proof given in [7] for the case  $\lambda_1 > 0$ . With this we obtain a *complete characterization* of boundedness of composition operators on all Hilbert spaces  $\mathcal{H}(E, \beta_S)$ . Section 4 studies the complex symmetry of composition operators on  $\mathcal{H}(E, \beta_S)$ . First, we introduce the composition conjugation form  $J_\xi$  induced by an entire function  $\xi$ . Using the result of Section 3, we then establish a criterion for  $\xi$  such that  $J_\xi$  is a conjugation (Theorem 4.4). Finally, we prove that all bounded  $C_\varphi$  induced by a non-constant  $\varphi$  are complex symmetric with respect to composition conjugation  $J_\xi$  (Proposition 4.11). With an existing result on rank-one operators, we thus demonstrate that all bounded composition operators on the spaces  $\mathcal{H}(E, \beta_S)$  are complex symmetric (Theorem 4.14).

## 2. DIFFERENT NOTIONS OF ORDERS AND THE HILBERT SPACES $\mathcal{H}(E, \beta_S)$

**2.1. Orders of entire Dirichlet series.** Some of the most important properties of entire Dirichlet series are their growth orders. We shall primarily consider the ordinary order, the Ritt order, and the logarithmic orders, which are important in the construction of our intended spaces  $\mathcal{H}(E, \beta_S)$ .

**Definition 2.1** (Ordinary Order). Let  $f$  be an entire function. If there exist  $\mu > 0$  and  $r_0 \geq 0$  such that the maximum modulus function  $M(r) := \max_{|z|=r \geq 0} |f(z)|$  satisfies

$$M(r) < e^{r^\mu}, \quad \forall r > r_0,$$

then  $f$  is said to be of *finite ordinary order* (or just *finite order*). The following finite limit

$$\rho(f) = \inf \mu = \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r} \geq 0,$$

where the infimum is taken over all  $\mu$  for which there exists such an  $r_0$ , is called the *ordinary order* (or just *order*) of  $f$ .

If no such  $\mu$  exists (i.e.,  $\rho(f) = +\infty$ ), then we say that  $f$  is of *infinite ordinary order* (or just *infinite order*).

Besides the ordinary order, the following notion of order introduced by Ritt [15] is only defined for entire Dirichlet series.

**Definition 2.2** (Ritt Order). Associate to an entire Dirichlet series (1.1) the function  $\mathcal{M}(\sigma) := \sup_{t \in \mathbb{R}} |f(\sigma + it)|$  where  $\sigma \in \mathbb{R}$ . If there exists  $\mu > 0$  such that  $\mathcal{M}(\sigma) \leq e^{e^{-\sigma\mu}}$  for all sufficiently large  $-\sigma$ , then the Dirichlet series is said to be of *finite Ritt order*, and we define the *Ritt order* to be

$$\rho_R(f) = \inf \mu = \limsup_{\sigma \rightarrow -\infty} \frac{\log \log \mathcal{M}(\sigma)}{-\sigma} \geq 0.$$

If no such  $\mu$  exists, then  $f$  has *infinite Ritt order*.

Finally, for entire Dirichlet series of Ritt order 0, Reddy defined in [14] the following logarithmic orders.

**Definition 2.3.** Let  $f$  be an entire Dirichlet series (1.1) with Ritt order 0, the *logarithmic orders* are given by

$$\rho_R(\mathcal{L})(f) = \limsup_{\sigma \rightarrow -\infty} \frac{\log \log \mathcal{M}(\sigma)}{\log(-\sigma)},$$

$$\rho_*(\mathcal{L})(f) = \limsup_{\sigma \rightarrow -\infty} \frac{\log \log \mu(\sigma)}{\log(-\sigma)},$$

where  $\mu(\sigma) = \max_n |a_n e^{-\lambda_n(\sigma+it)}|$ , and

$$\rho_c(\mathcal{L})(f) = \limsup_{n \rightarrow \infty} \frac{\log \lambda_n}{\log \left( \frac{1}{\lambda_n} \log \frac{1}{|a_n|} \right)}.$$

It is well-known that for any given  $f$ ,

$$\rho_R(\mathcal{L})(f) = \rho_*(\mathcal{L})(f) = \rho_c(\mathcal{L})(f) + 1 \geq 1.$$

Hence one of these orders is finite if and only if all of them are finite.

Note that logarithmic orders presuppose zero Ritt order, so without ambiguity we shall drop the notion ‘‘Ritt order 0’’ when we mention an entire Dirichlet series of finite logarithmic orders.

**2.2. The spaces  $\mathcal{H}(E, \beta_S)$ .** Let  $\beta = (\beta_n)$  be a sequence of positive real numbers. Associate to the sequence the *weighted sequence space*  $\ell_\beta^2$  with *weight*  $\beta$  defined as

$$\ell_\beta^2 = \left\{ a = (a_n)_{n=1}^\infty, a_n \in \mathbb{C} : \|a\|_{\ell_\beta^2} := \left( \sum_{n=1}^\infty |a_n|^2 \beta_n^2 \right)^{1/2} < \infty \right\}.$$

This is a Hilbert space with inner product

$$\langle a, b \rangle_{\ell_\beta^2} = \sum_{n=1}^\infty a_n \bar{b}_n \beta_n^2, \quad \forall a = (a_n), b = (b_n) \in \ell_\beta^2.$$

The spaces  $\ell_\beta^2$  are important in the study of many well-known Hilbert spaces of holomorphic functions, such as the classical Hardy spaces, Bergman spaces, and Dirichlet spaces. The interested reader can look for more details in the book [3].

We now summarize the use of spaces  $\ell_\beta^2$  in the construction of Hilbert spaces of Dirichlet series  $\mathcal{H}(E, \beta_S)$  introduced in [7]. We suppose for the remainder of this paper that the condition  $L < +\infty$  always holds, where  $L$  is defined as in (1.2). Consider the following sequence space

$$E = \left\{ (a_n)_{n=1}^\infty, a_n \in \mathbb{C} : \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = -\infty \right\}.$$

If  $(a_n) \in E$ , then by the remark below equation (1.2), the induced series (1.1) represents an entire function in  $z$ .

We set

$$\beta_* = \liminf_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n} \quad \text{and} \quad \beta^* = \limsup_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n}.$$

Then we have the following proposition.

**Proposition 2.4** ([7]). *Precisely one of the following statements must hold.*

- (1)  $\ell_\beta^2 \subsetneq E$ , which is equivalent to  $\beta_* = \infty$ .
- (2)  $E \subsetneq \ell_\beta^2$ , which is equivalent to  $\beta^* < \infty$ .
- (3)  $E \setminus \ell_\beta^2 \neq \emptyset$  and  $\ell_\beta^2 \setminus E \neq \emptyset$ , which is equivalent to  $\beta_* < \beta^* = \infty$ .

Thus, to obtain a Hilbert space of entire Dirichlet series, we consider the case where  $\beta_* = \infty$  holds, and give a name to this class of spaces.

**Definition 2.5.** Let  $\beta = (\beta_n)$  be a given sequence of positive numbers. We say that condition (E) holds, if

$$\beta_* = \liminf_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n} = \infty. \quad (E)$$

We define the corresponding inner product space  $\mathcal{H}(E, \beta)$  as

$$\mathcal{H}(E, \beta) = \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : (a_n) \in \ell_\beta^2 \right\},$$

with the inner product of  $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$  and  $g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$  in  $\mathcal{H}(E, \beta)$  given by

$$\langle f, g \rangle = \sum_{n=1}^{\infty} a_n \bar{b}_n \beta_n^2.$$

Obviously, by Proposition 2.4, every entire Dirichlet series  $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$  in  $\mathcal{H}(E, \beta)$  indeed represents an entire series and

$$\|f\| = \left( \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < +\infty.$$

We need two auxiliary results for the analysis of Ritt orders and logarithmic orders of elements in a space  $\mathcal{H}(E, \beta)$ .

**Lemma 2.6** ([7]). *Every series in a space  $\mathcal{H}(E, \beta)$  is an entire function of Ritt order zero if and only if the following condition (R) holds:*

$$\liminf_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n \log \lambda_n} = \infty. \quad (R)$$

It is shown that condition (R) is stronger than condition (E). Thus, if we denote by  $\mathcal{H}(E, \beta_R)$  a space constructed from a sequence  $\beta$  satisfying (R) in the same manner as Definition 2.5, then the space  $\mathcal{H}(E, \beta_R)$  is a special case of the class  $\mathcal{H}(E, \beta)$ , in which elements are entire Dirichlet series of Ritt order zero.

Note that we still have no information about the finiteness of the logarithmic orders of the functions in  $\mathcal{H}(E, \beta_R)$ . This leads us to the next result.

**Lemma 2.7** ([7]). *Every series in a space  $\mathcal{H}(E, \beta_R)$  is an entire function of finite logarithmic orders if and only if the following condition (S) holds:*

$$\exists \alpha > 0 : \quad \liminf_{n \rightarrow \infty} \frac{\log \beta_n}{\lambda_n^{1+\alpha}} = \infty. \quad (S)$$

Condition  $(S)$  is proven to be stronger than both conditions  $(R)$  and  $(E)$ . Thus by Lemma 2.7, a space  $\mathcal{H}(E, \beta)$  with weight  $\beta$  satisfying  $(S)$  consists of elements that are entire Dirichlet series of finite logarithmic orders. Finally, we introduce the following Hilbert spaces, which are the main concern of this paper.

**Definition 2.8.** Let  $\beta$  be a sequence of positive numbers satisfying  $(S)$ . The Hilbert space of entire Dirichlet series  $\mathcal{H}(E, \beta_S)$  with weight  $\beta$  is defined as

$$\mathcal{H}(E, \beta_S) = \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} : \|f\| := \left( \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < +\infty \right\}.$$

We have the following important remark.

*Remark 2.9.* If  $\lambda_1 = 0$ , then any space  $\mathcal{H}(E, \beta_S)$  contain all constant functions, since a constant  $c \in \mathbb{C}$  has the trivial representation

$$c = ce^{-0z} + \sum_{n=2}^{\infty} 0 \cdot e^{-\lambda_n z}.$$

This is not true when  $\lambda_1 > 0$ . In fact, no non-zero constant is representable in  $\mathcal{H}(E, \beta_S)$  due to the uniqueness of representations of entire functions [6].

The following remark summarizes what we have achieved so far.

*Remark 2.10.* We have the following proper inclusions:

$$\{\mathcal{H}(E, \beta_S)\} \subsetneq \{\mathcal{H}(E, \beta_R)\} \subsetneq \{\mathcal{H}(E, \beta)\},$$

where  $\{\mathcal{H}(E, \beta_S)\}$  denotes the class of all spaces  $\mathcal{H}(E, \beta_S)$ , and likewise for  $\{\mathcal{H}(E, \beta_R)\}$  and  $\{\mathcal{H}(E, \beta)\}$ .

A natural question raised is why we need to consider spaces  $\mathcal{H}(E, \beta_S)$ . In order to answer this, we have the following relation between finite logarithmic orders and finite ordinary orders of Dirichlet series.

**Lemma 2.11** ([7]). *Let  $\beta$  be a sequence of positive real numbers satisfying condition  $(S)$ . Then every element of the induced space  $\mathcal{H}(E, \beta_S)$  has finite ordinary order.*

Therefore, any space  $\mathcal{H}(E, \beta_S)$  contains only entire functions of finite order. The rationale for finite orders is contained in the lemma below, which shows the connection between compositions of entire functions and their growth orders. This result is crucial for the study of composition operators in [7] and in the present paper.

**Lemma 2.12** (Pólya [13]). *Let  $g$  and  $h$  be entire functions and suppose  $g \circ h$  is an entire function of finite ordinary order. Then, exactly one of the following is true.*

- (1)  $h$  is a polynomial and  $g$  is of non-zero finite ordinary order, or
- (2)  $h$  is not a polynomial but is a function of finite ordinary order, and  $g$  is of zero ordinary order.

We close this section with the following proposition.

**Proposition 2.13.** *Let  $\beta$  be a sequence of positive real numbers satisfying condition (S). Then the induced space  $\mathcal{H}(E, \beta_S)$  is a reproducing kernel Hilbert space with the reproducing kernel*

$$K(x, y) = \sum_{n=1}^{\infty} \frac{e^{-\lambda_n(\bar{x}+y)}}{\beta_n^2}.$$

We note that the general theory of reproducing kernel Hilbert spaces can be found in [1].

*Proof.* We need to show that for any  $c \in \mathbb{C}$ , the evaluation functional  $\delta_c: f \mapsto f(c)$  is bounded. Consider the (formal) Dirichlet series  $K_c(z) = \sum_{n=1}^{\infty} \beta_n^{-2} e^{-\lambda_n(z+\bar{c})}$ . A simple computation gives

$$\|K_c\|^2 = \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n \Re(c)}}{\beta_n^2}.$$

Condition  $L < \infty$  and condition (E) implies there exists a constant  $C > 0$  such that for  $n$  large,

$$-\frac{\log \beta_n}{\lambda_n} - \Re(c) < -C < -\frac{\log n}{\lambda_n} \implies \frac{e^{-\lambda_n \Re(c)}}{\beta_n} < \frac{1}{n},$$

so the summation  $\|K_c\|^2$  is absolutely convergent by comparison test. Hence,  $K_c(z)$  represents an entire function (Proposition 2.4). Now, by Cauchy–Schwarz inequality, we have

$$|f(c)|^2 = \left| \sum_{n=1}^{\infty} a_n e^{-\lambda_n c} \right|^2 = |\langle f, K_c \rangle|^2 \leq \|f\|^2 \|K_c\|^2 < \infty.$$

This implies  $\delta_c$  is bounded, and by Riesz representation theorem,  $K_c$  is the unique element of  $\mathcal{H}(E, \beta_S)$  such that  $\delta_c(f) = \langle f, K_c \rangle$  for all  $f \in \mathcal{H}(E, \beta_S)$ . Hence,  $\mathcal{H}(E, \beta_S)$  is a reproducing kernel Hilbert space, and the reproducing kernel is given by

$$K(x, y) = \langle K_x, K_y \rangle = K_x(y) = \sum_{n=1}^{\infty} \frac{e^{-\lambda_n(\bar{x}+y)}}{\beta_n^2}. \quad \square$$

*Remark 2.14.* The fact that a space  $\mathcal{H}(E, \beta_S)$  is a reproducing kernel Hilbert space implies, by a consequence of Closed Graph Theorem, that composition operators are automatically bounded whenever they are invariant.

### 3. A CRITERION FOR BOUNDEDNESS OF COMPOSITION OPERATORS ON $\mathcal{H}(E, \beta_S)$

Our first interest of this paper is to establish the following criterion for boundedness of composition operators on *any* space  $\mathcal{H}(E, \beta_S)$ .

**Theorem 3.1.** *Let  $\beta_S$  be a sequence of positive real numbers satisfying condition (S), and  $\varphi$  be an entire function. Consider the following statements.*

- (i)  $\varphi$  is a constant function, and
- (ii)  $\varphi(z) = z + b$  for some  $b \in \mathbb{C}$ ,  $\Re(b) \geq 0$ .

*The following are true about the boundedness of the composition operator  $C_\varphi$  acting on the Hilbert space  $\mathcal{H}(E, \beta_S)$ :*

- (1) If  $\lambda_1 = 0$ , then  $C_\varphi$  is bounded if and only if exactly one of conditions (i) or (ii) holds.  
(2) If  $\lambda_1 > 0$ , then  $C_\varphi$  is bounded if and only if (ii) holds.

This result is clearly different from that of [7], as constant functions  $\varphi$  are included in the case  $\lambda_1 = 0$ . We provide a new approach to Theorem 3.1 that works for both cases  $\lambda_1 = 0$  and  $\lambda_1 > 0$ . Our result covers all possibilities for the boundedness of composition operators on spaces  $\mathcal{H}(E, \beta_S)$  and hence provides a *complete characterization* for such boundedness.

In the sequel, we fix a particular sequence  $\beta$  that satisfies condition (S) and the induced space  $\mathcal{H}(E, \beta_S)$ . We first have a simple lemma.

**Lemma 3.2.** *If  $f \in \mathcal{H}(E, \beta_S)$  and  $a_m$  is the first non-zero coefficient of its representation, then*

$$\lim_{\Re(z) \rightarrow +\infty} e^{\lambda_m z} f(z) = a_m.$$

One can easily check the sufficiency for Theorem 3.1, which we provide here in Propositions 3.3 for the sake of completeness.

**Proposition 3.3.** *Suppose  $\varphi(z) = z + b$  with  $\Re(b) \geq 0$ . Then the induced composition operator  $C_\varphi$  is bounded on the space  $\mathcal{H}(E, \beta_S)$ .*

*Moreover, suppose  $\varphi$  is a constant function, then  $C_\varphi$  is bounded on  $\mathcal{H}(E, \beta_S)$  if  $\lambda_1 = 0$ , and is not invariant on  $\mathcal{H}(E, \beta_S)$  if  $\lambda_1 > 0$ .*

*Proof.* • We write  $\varphi(z) = z + b$  with  $\Re(b) \geq 0$ , then for any series  $f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}$  in the given space  $\mathcal{H}(E, \beta_S)$ , we have

$$\begin{aligned} \|C_\varphi f\|^2 &= \left\| \sum_{n=1}^{\infty} (a_n e^{-\lambda_n b}) e^{-\lambda_n z} \right\|^2 = \sum_{n=1}^{\infty} |a_n|^2 e^{-2\lambda_n \Re(b)} \beta_n^2 \\ &\leq e^{-2\lambda_1 \Re(b)} \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 = e^{-2\lambda_1 \Re(b)} \|f\|^2. \end{aligned}$$

The second last inequality comes from the assumption  $\lambda_n$  is increasing and  $\Re(b)$  is non-negative. Hence,  $\|C_\varphi f\| \leq e^{-\lambda_1 \Re(b)} \|f\|$ , so  $C_\varphi$  is bounded. This argument works for both cases  $\lambda_1 = 0$  and  $\lambda_1 > 0$ .

• Suppose  $\varphi(z) \equiv c$  for some  $c \in \mathbb{C}$ . If  $\lambda_1 > 0$ , then  $C_\varphi f = f(c)$  is not representable in  $\mathcal{H}(E, \beta_S)$  (Remark 2.9).

If  $\lambda_1 = 0$ , the boundedness of  $C_\varphi$  follows the boundedness of the evaluation functional  $\delta_c$  in Proposition 2.13 (with a note that  $\|C_\varphi\| = \beta_1 \|\delta_c\|$ ).  $\square$

Therefore, it suffices to prove the necessary condition. In fact, we only need to provide the proof for the case  $\lambda_1 = 0$ . The statement for the case  $\lambda_1 > 0$  follows as a corollary.

**Theorem 3.4.** *Suppose  $\lambda_1 = 0$  and that  $\varphi$  induces a bounded composition operator  $C_\varphi$  on  $\mathcal{H}(E, \beta_S)$ . Then exactly one of the following assertions holds:*

- (i)  $\varphi$  is a constant function, or  
(ii)  $\varphi(z) = z + b$ , where  $\Re(b) \geq 0$ .

*Remark 3.5.* Before proving Theorem 3.4, we have some comments for this proof with respect to the proof provided in [7].

- (1) We prove first that  $\varphi(z) = az + b$ , then  $a \geq 1$  and  $a \leq 1$  are shown. However, besides providing a different approach for  $a \geq 1$ , we give a refinement of the proof  $a \leq 1$ .
- (2) We still, however, employ Lemmas 2.12 and 3.2 in our proof.

*Proof of Theorem 3.4.* Suppose  $C_\varphi$  is bounded on  $\mathcal{H}(E, \beta_S)$ . Then it maps every element  $f \in \mathcal{H}(E, \beta_S)$  into itself. In particular, for  $f(z) = e^{-\lambda_2 z} \in \mathcal{H}(E, \beta_S)$ , we have  $C_\varphi f = f \circ \varphi \in \mathcal{H}(E, \beta_S)$  so it is representable as

$$(3.1) \quad C_\varphi f(z) = e^{-\lambda_2 \varphi(z)} = a_1 + a_2 e^{-\lambda_2 z} + a_3 e^{-\lambda_3 z} + \dots$$

Since  $f$  is an entire function of order 1, and by Lemma 2.11  $f \circ \varphi$  has finite order, Lemma 2.12 shows that  $\varphi$  is a polynomial. We write

$$\varphi(z) = b_0 + b_1 z + \dots + b_N z^N, \quad b_N \neq 0.$$

If  $\varphi$  is constant, there is nothing to prove. Hence suppose  $N \geq 1$ . We divide our proof into several steps.

- Step 1: Show that  $b_N$  is a positive real number and that  $a_1 = 0$ .  
 Step 2: Show that  $N = 1$ .  
 Step 3: Show that  $b_1 \geq 1$ .  
 Step 4: Show that  $b_1 = 1$ .  
 Step 5: Show that  $\Re(b_0) \geq 0$ .

Once all these steps are fulfilled, Theorem 3.4 is proved.

- *Proof for Step 1.* Letting  $\Re(z) \rightarrow +\infty$  in (3.1), by Lemma 3.2, we have

$$(3.2) \quad \lim_{\Re(z) \rightarrow +\infty} e^{-\lambda_2 \varphi(z)} = a_1 \quad \implies \quad \lim_{\Re(z) \rightarrow +\infty} e^{-\lambda_2 \Re(\varphi(z))} = |a_1|.$$

We also have

$$(3.3) \quad \Re(\varphi(z)) = \sum_{k=0}^N \Re(b_k z^k) \leq \Re(b_N z^N) + \sum_{k=0}^{N-1} |b_k| |z|^k.$$

Assume  $b_N$  is not positive real. There exists  $\theta_0 \in (0, 2\pi)$  such that  $b_N = |b_N| e^{i\theta_0}$ , so if we write  $z = r e^{i\theta}$ , then from (3.3) we have

$$\Re(\varphi(z)) \leq |b_N| \Re(z^N e^{i\theta_0}) + \sum_{k=0}^{N-1} |b_k| r^k = |b_N| r^N \cos(N\theta + \theta_0) + \sum_{k=0}^{N-1} |b_k| r^k.$$

Let  $\vartheta := \frac{1}{N+1}(\pi - \theta_0)$ . Since  $N \geq 1$ , we can see that

$$-\frac{\pi}{2} < \vartheta < \frac{\pi}{2} \implies \cos \vartheta > 0 \text{ and } \cos(N\vartheta + \theta_0) = \cos(\pi - \vartheta) < 0.$$

Hence, if  $r > 0$ , then

$$\Re(r e^{i\vartheta}) > 0 \text{ and } \Re(b_N z^N) = |b_N| r^N \cos(N\theta + \theta_0) < 0.$$

Thus along  $\theta = \vartheta$ , we have  $z = r e^{i\vartheta}$  and

$$\lim_{\Re(z) \rightarrow +\infty} \Re(\varphi(z)) = \lim_{r \rightarrow +\infty} \left( |b_N| r^N \cos(N\vartheta + \theta_0) + \sum_{k=0}^{N-1} |b_k| r^k \right) = -\infty,$$

which implies

$$\lim_{\Re(z) \rightarrow +\infty} e^{-\lambda_2 \Re(\varphi(z))} = +\infty.$$

This contradicts (3.2). Hence  $b_N \in \mathbb{R}^+$  (as  $b_N \neq 0$ ).

Now note that

$$\Re(\varphi(z)) = b_N \Re(z^N) + \sum_{k=0}^{N-1} \Re(b_k z^k) \geq b_N \Re(z^N) - \sum_{k=0}^{N-1} |b_k| r^k.$$

Letting  $\Re(z) \rightarrow +\infty$  along the real line, we have

$$(3.4) \quad \lim_{\Re(z) \rightarrow +\infty} \Re(\varphi(z)) = \lim_{r \rightarrow +\infty} \left( b_N r^N - \sum_{k=0}^{N-1} |b_k| r^k \right) = +\infty.$$

From (3.2) and (3.4), we have  $|a_1| = a_1 = 0$ .

- Proof for Step 2. From Step 1, we have seen that if  $a_m$  is the first non-zero coefficient of  $C_\varphi f$ , then  $m \geq 2$ . Fix  $m$ .

Assume  $N > 1$ . Consider  $\xi(z) = \varphi(z) - \frac{\lambda_m}{\lambda_2} z$ , which is a polynomial of degree at least 2. Again by Lemma 3.2,

$$\lim_{\Re(z) \rightarrow +\infty} e^{\lambda_m z} e^{-\lambda_2 \varphi(z)} = \lim_{\Re(z) \rightarrow +\infty} e^{-\lambda_2 \xi(z)} = a_m.$$

By applying the same proof of Step 1 from (3.2) to (3.4) for  $\xi$  instead of  $\varphi$ , we have  $a_m = 0$ , which contradicts our assumption that  $a_m \neq 0$ . Hence,  $N = 1$  and  $\varphi(z) = b_1 z + b_0$  with  $b_1 > 0$ .

- Proof for Step 3. From Step 1, we have  $a_1 = 0$ , so by Step 2, equation (3.1) becomes

$$e^{-\lambda_2(b_1 z + b_0)} = a_2 e^{-\lambda_2 z} + a_3 e^{-\lambda_3 z} + a_4 e^{-\lambda_4 z} + \dots$$

Multiply both sides by  $e^{\lambda_2 b_0} \neq 0$ , we have

$$(3.5) \quad e^{-\lambda_2 b_1 z} = c_2 e^{-\lambda_2 z} + c_3 e^{-\lambda_3 z} + c_4 e^{-\lambda_4 z} + \dots,$$

where  $c_n = a_n e^{\lambda_2 b_0}$ . Suppose  $c_m$  is the first non-zero coefficient. Then  $m \geq 2$ .

Note that for any  $\mu \in \mathbb{R}$ ,

$$(3.6) \quad \frac{1}{2ti} \int_{\sigma-ti}^{\sigma+ti} e^{\mu z} dz = \begin{cases} 1 & \text{if } \mu = 0, \\ \frac{e^{\mu\sigma}}{\mu t} \sin(\mu t) & \text{if } \mu \neq 0. \end{cases}$$

We then have

$$c_m = \lim_{t \rightarrow \infty} \frac{1}{2ti} \int_{-ti}^{ti} e^{(\lambda_m - \lambda_2 b_1)z} dz.$$

Since  $c_m \neq 0$ , equation (3.6) implies  $\lambda_m - \lambda_2 b_1 = 0$ , so  $b_1 = \frac{\lambda_m}{\lambda_2} \geq 1$ .

- Proof for Step 4. Assume  $b_1 > 1$ . Consider the probe functions  $q_k(z) = \beta_k^{-1} e^{-\lambda_k z}$ ,  $k \geq 2$ . Clearly  $\|q_k\| = 1$ . Let  $N(k)$  be the index of the first non-zero coefficient of  $C_\varphi q_k$ . Similarly to Step 3, we can prove that

$$(3.7) \quad b_1 = \frac{\lambda_{N(k)}}{\lambda_k} = \frac{\lambda_{N(2)}}{\lambda_2}.$$

Since  $b_1 > 1$ , it is true that  $\lambda_{N(k)} > \lambda_k$ , and so  $N(k) > k$ .

Let  $m_0 = 2$ ,  $m_1 = N(2)$  and  $m_{k+1} = N(m_k)$ . Then the sequence  $(m_k)$  is increasing. Equation (3.7) implies  $\lambda_{m_{k+1}} = b_1 \lambda_{m_k}$ , so

$$C_\varphi q_{m_k}(z) = \beta_{m_k}^{-1} e^{-\lambda_{m_k}(b_1 z + b_0)} = \frac{e^{-\lambda_{m_k} b_0}}{\beta_{m_k}} e^{-\lambda_{m_k} b_1 z} = \frac{e^{-\lambda_{m_k} b_0}}{\beta_{m_k}} e^{-\lambda_{m_{k+1}} z}.$$

Thus we can easily compute that

$$\|C_\varphi q_{m_k}\| = e^{-\lambda_{m_k} \Re(b_0)} \frac{\beta_{m_{k+1}}}{\beta_{m_k}}.$$

Since  $C_\varphi$  is bounded, the sequence  $(\|C_\varphi q_{m_k}\|)_k$  is bounded, so there exists a constant  $B > 0$  such that for all  $k \geq 1$ ,

$$\|C_\varphi q_{m_k}\| = e^{-\lambda_{m_k} \Re(b_0)} \frac{\beta_{m_{k+1}}}{\beta_{m_k}} < B \implies \frac{\beta_{m_{k+1}}}{\beta_{m_k}} < B e^{\lambda_{m_k} \Re(b_0)}.$$

Therefore,

$$\beta_{m_k} = \frac{\beta_{m_k}}{\beta_{m_{k-1}}} \cdot \frac{\beta_{m_{k-1}}}{\beta_{m_{k-2}}} \cdots \frac{\beta_{m_1}}{\beta_1} \cdot \beta_1 < \beta_1 B^k \exp\left(\Re(b_0) \sum_{j=0}^{k-1} \lambda_{m_j}\right).$$

Note that from (3.7),  $\lambda_{m_j} = \lambda_2 b_1^j$ , so

$$\sum_{j=0}^{k-1} \lambda_{m_j} = \lambda_2 \frac{b_1^k - 1}{b_1 - 1}.$$

Since  $b_1 > 1$ , we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{\log \beta_{m_k}}{\lambda_{m_k}} &\leq \limsup_{k \rightarrow \infty} \frac{\log\left(\beta_1 B^k \exp\left(\lambda_2 \Re(b_0) \frac{b_1^k - 1}{b_1 - 1}\right)\right)}{\lambda_2 b_1^k} \\ &= \limsup_{k \rightarrow \infty} \frac{k \log B + \lambda_2 \frac{b_1^k - 1}{b_1 - 1} \Re(b_0) + \log \beta_1}{\lambda_2 b_1^k} = \frac{\Re(b_0)}{b_1 - 1} < +\infty. \end{aligned}$$

This contradicts both conditions (E) and (S). Thus, we must have  $b_1 = 1$ .

- *Proof for Step 5.* Consider once again functions  $q_k(z) = \beta_k^{-1} e^{-\lambda_k z}$ . Recall that the sequence  $(\|C_\varphi q_k\|)_k$  is bounded. From this, since  $0 \leq \lambda_k \uparrow +\infty$  and

$$\|C_\varphi q_k\| = e^{-\lambda_k \Re(b_0)},$$

it follows that  $\Re(b_0) \geq 0$ .

The proof is complete.  $\square$

**Corollary 3.6.** *Suppose  $\lambda_1 > 0$  and that  $\varphi$  induces a bounded composition operator  $C_\varphi$  on  $\mathcal{H}(E, \beta_S)$ . Then  $\varphi(z) = z + b$  with  $\Re(b) \geq 0$ .*

*Proof.* The technique used in the proof of Theorem 3.4 is still applicable here, except that terms  $\lambda_2$  in the proof are replaced by  $\lambda_1$ ,  $m \geq 2$  and  $k \geq 2$  are replaced by  $m \geq 1$  and  $k \geq 1$  respectively, and we do not need to prove  $a_1 = 0$  in Step 1 since that is obvious.  $\square$

*Proof of Theorem 3.1.* The result follows directly from Proposition 3.3, Theorem 3.4 and Corollary 3.6.  $\square$

#### 4. COMPLEX SYMMETRY OF COMPOSITION OPERATORS WITH RESPECT TO COMPOSITION CONJUGATIONS

The main interest for the rest of the paper is to develop a characterization for complex symmetry of composition operators that have the form of composition conjugation. Combined with prior established results in [4, 10], we thus show that all bounded composition operators on  $\mathcal{H}(E, \beta_S)$  are complex symmetric.

**4.1. The composition conjugation  $J_\xi$ .** First, we recall the definition of a composition conjugation.

**Definition 4.1.** Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{C}$ . A conjugation  $\mathcal{C}$  on  $\mathcal{H}$  is a function  $\mathcal{C} : \mathcal{H} \rightarrow \mathcal{H}$  which satisfies

- (a) anti-linearity, i.e.,  $\forall x, y \in \mathcal{H}$  and  $\lambda, \mu \in \mathbb{C}$ , we have  $\mathcal{C}(\lambda x + \mu y) = \bar{\lambda}\mathcal{C}x + \bar{\mu}\mathcal{C}y$ ,
- (b) involutivity, i.e.,  $\mathcal{C}^2 = I$  is the identity operator on  $\mathcal{H}$ , and
- (c) isometry, i.e.,  $\forall x \in \mathcal{H}$ , we have  $\|\mathcal{C}x\| = \|x\|$ .

**Definition 4.2.** Let  $\xi : \mathbb{C} \rightarrow \mathbb{C}$  be an entire function. A *composition conjugation*  $J_\xi$  acting on  $\mathcal{H}(E, \beta_S)$  is defined by

$$J_\xi f(z) = \overline{f(\overline{\xi(z)})}, \quad (f \in \mathcal{H}(E, \beta_S))$$

which satisfies the definition of a conjugation.

When  $\xi$  is the identity map on  $\mathbb{C}$ , then  $J_\xi$  is clearly a conjugation on  $\mathcal{H}(E, \beta_S)$ , which we denote simply by  $J$  and call the *standard conjugation*.

*Remark 4.3.* We will use the following three observations.

- (i) From [5], we see that checking for isometry is equivalent to verifying that  $\langle \mathcal{C}f, \mathcal{C}g \rangle = \langle g, f \rangle$ , for all  $f, g \in \mathcal{H}(E, \beta_S)$ .
- (ii) Notice that if  $J_\xi$  maps  $\mathcal{H}(E, \beta_S)$  to itself, then it is automatically anti-linear, so to test if  $J_\xi$  is a conjugation, it suffices to check for invariance, isometry, and involutivity.
- (iii) For any Dirichlet series  $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ , define the associated series

$$\tilde{f}(z) = \sum_{n=1}^{\infty} \bar{a}_n e^{-\lambda_n z}.$$

It is easy to see that  $f \in \mathcal{H}(E, \beta_S)$  if and only if  $\tilde{f} \in \mathcal{H}(E, \beta_S)$ . The proof is straightforward from the invariance of the standard conjugation  $J$ .

We now proceed to determine precisely those entire functions  $\xi$  such that  $J_\xi$  is a composition conjugation. Our goal is to establish the following criterion of  $\xi$  for  $J_\xi$  to be a composition conjugation.

**Theorem 4.4.** *The following statements are equivalent.*

- (i)  $\xi(z) = z + ci$ , for some  $c \in \mathbb{R}$ .
- (ii)  $J_\xi$  is a conjugation.

Before proving Theorem 4.4, we need to several additional results. The following lemma can be proved immediately.

**Lemma 4.5.** *Suppose  $J_\xi$  is invariant on  $\mathcal{H}(E, \beta_S)$ . Exactly one of the following is true:*

- (i) *If  $\lambda_1 > 0$ , then  $\xi(z) = z + b$  with  $\Re(b) \geq 0$ ,*
- (ii) *If  $\lambda_1 = 0$ , then  $\xi(z) = z + b$  with  $\Re(b) \geq 0$  or  $\xi(z) = c$  for some  $c \in \mathbb{C}$ .*

*Proof.* Given an arbitrary series  $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \in \mathcal{H}(E, \beta_S)$ , consider  $\tilde{f}(z) = \sum_{n=1}^{\infty} \overline{a_n} e^{-\lambda_n z}$  as in Remark 4.3. Then  $\tilde{f} \in \mathcal{H}(E, \beta_S)$ . Since  $J_\xi$  is invariant on  $\mathcal{H}(E, \beta_S)$ , we have

$$C_\xi f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n \xi(z)} = J_\xi \tilde{f}(z) \in \mathcal{H}(E, \beta_S).$$

This implies that  $C_\xi$  is an invariant composition operator on  $\mathcal{H}(E, \beta_S)$ . The two possible conclusions of the lemma follows directly from Theorem 3.1 and Remark 2.14.  $\square$

We determine the form of  $\xi$  by considering separately the properties of isometry and involutivity.

**Proposition 4.6.** *Suppose  $J_\xi$  is invariant on  $\mathcal{H}(E, \beta_S)$ . Then the isometry of  $J_\xi$  on  $\mathcal{H}(E, \beta_S)$  implies  $\xi(z) = z + ci$  for some  $c \in \mathbb{R}$ .*

*Proof.* By Lemma 4.5, we have two possibilities.

- *Case 1:  $\lambda_1 > 0$ .*

We have  $\xi(z) = z + b$  with  $\Re(b) \geq 0$  from Lemma 4.5. Then,

$$J_\xi f(z) = \sum_{n=1}^{\infty} \overline{a_n} e^{-\lambda_n b} e^{-\lambda_n z}.$$

Then by isometry, we have

$$\sum_{n=1}^{\infty} |\overline{a_n} e^{-\lambda_n b}|^2 \beta_n^2 = \|J_\xi f\|^2 = \|f\|^2 = \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2.$$

Set  $a_2 = 1$  and  $a_n = 0$  for  $n \neq 2$ , then  $e^{-2\lambda_2 \Re(b)} = 1$ . Hence  $\Re(b) = 0$ . So  $b = ci$  for some  $c \in \mathbb{R}$ .

- *Case 2:  $\lambda_1 = 0$ .*

We have either  $\xi(z) = z + b$  with  $\Re(b) \geq 0$ , or  $\xi(z) = b$  with  $b \in \mathbb{C}$ . The first possibility follows Case 1, and we also obtain  $b = ci$  for some  $c \in \mathbb{R}$ . Assume  $\xi(z) = b$  where  $b \in \mathbb{C}$ , then

$$J_\xi f(z) = \sum_{n=1}^{\infty} \overline{a_n} e^{-\lambda_n b}, \text{ so } \|J_\xi f\|^2 = \beta_1^2 \left| \sum_{n=1}^{\infty} \overline{a_n} e^{-\lambda_n b} \right|^2.$$

Consider the probe functions  $q_n(z) = e^{-\lambda_n z}$ . By isometry, we have

$$\beta_1^2 e^{-2\lambda_n \Re(b)} = \|J_\xi q_n\|^2 = \|q_n\|^2 = \beta_n^2.$$

Thus  $e^{-2\lambda_n \Re(b)} = \frac{\beta_n^2}{\beta_1^2}$  for each  $n$ , which leads to

$$-\Re(b) = \frac{\log \beta_n}{\lambda_n} - \frac{\log \beta_1}{\lambda_n}.$$

Invoking Condition (E) and taking the limit as  $n \rightarrow \infty$ , we have  $-\Re(b) = \infty$ , a clear contradiction.  $\square$

This finishes the case for isometry. We move on to the involutivity of  $J_\xi$ .

**Proposition 4.7.** *Suppose  $J_\xi$  is invariant on  $\mathcal{H}(E, \beta_S)$ . Then the involutivity  $J_\xi$  on  $\mathcal{H}(E, \beta_S)$  implies  $\xi(z) = z + ci$  for some  $c \in \mathbb{R}$ .*

*Proof.* The involutivity implies

$$f(z) = J_\xi J_\xi f(z) = \overline{J_\xi f(\overline{\xi(z)})} = \overline{\overline{\overline{f(\xi(\overline{\xi(z)})})}} = \overline{f(\xi(\overline{\xi(z)})}).$$

With probe functions  $q_n(z) = e^{-\lambda_n z}$ , we obtain  $e^{-\lambda_n \overline{\xi(\overline{\xi(z)})}} = e^{-\lambda_n z}$ , for each  $n \geq 2$ . Hence,

$$(4.1) \quad \overline{\xi(\overline{\xi(z)})} = z + 2 \frac{k_n}{\lambda_n} \pi i$$

for some  $k_n \in \mathbb{Z}$  for each  $n \geq 2$ .

Since  $J_\xi$  is invariant on  $\mathcal{H}(E, \beta_S)$ , we again have two cases according to Lemma 4.5.

- *Case 1:*  $\lambda_1 > 0$ .

The invariance of  $J_\xi$  implies  $\xi(z) = z + b$  where  $\Re(b) \geq 0$ . Write  $b = s + ic$  ( $s, c \in \mathbb{R}$ ). Then, equation (4.1) implies

$$\overline{\xi(\overline{\xi(z)})} = \overline{\xi(z + s + ic)} = \overline{z + s - ic + s + ic} = z + 2s.$$

Then, we get

$$2s = 2 \frac{k_n}{\lambda_n} \pi i \in \mathbb{R},$$

so  $k_n = 0$  and  $s = 0$ . Hence  $\xi(z) = z + ci$ , where  $c \in \mathbb{R}$ .

- *Case 2:*  $\lambda_1 = 0$ .

The invariance of  $J_\xi$  implies  $\xi(z) = z + b$  with  $\Re(b) \geq 0$  or  $\xi(z) = b$ , where  $b \in \mathbb{C}$  is a constant. The first possibility is similar to Case 1. Now, assume  $\xi(z) = b$ . Obviously the relation (4.1) cannot hold, a contradiction.  $\square$

*Proof of Theorem 4.4.* The proof of (i)  $\rightarrow$  (ii) is a trivial verification of the definition, so we omit the details. The proof of (ii)  $\rightarrow$  (i) follows Lemma 4.5, Propositions 4.6 and 4.7, which we just have established.  $\square$

**4.2. Complex symmetry with respect to conjugations  $J_\xi$ .** Finally, we consider the complex symmetry property of bounded composition operators on  $\mathcal{H}(E, \beta_S)$  with respect to conjugations  $J_\xi$ . We first remind the reader of the definition of complex symmetry.

**Definition 4.8.** Let  $T$  be a continuous linear operator mapping a Hilbert space  $\mathcal{H}$  to itself. Given that  $\mathcal{C}$  is a conjugation, we say that  $T$  is  *$\mathcal{C}$ -symmetric*, or *complex symmetric with respect to  $\mathcal{C}$* , if

$$\mathcal{C} \circ T \circ \mathcal{C} = T^*.$$

If such a  $\mathcal{C}$  exists, we say that  $T$  is *complex symmetric*.

Here,  $T^*$  is the adjoint operator of  $T$ . Where the context is clear, we omit the composition symbol henceforth. It is worthwhile to note that the above definition is equivalent to both  $\mathcal{C}T = T^*\mathcal{C}$  and  $T\mathcal{C} = \mathcal{C}T^*$ , by involutivity of  $\mathcal{C}$ .

First, we retrieve the following result about the adjoint of a bounded composition operator in  $\mathcal{H}(E, \beta_S)$ .

**Lemma 4.9** ([8]). *Let  $\varphi(z) = z + b$ , where  $\Re(b) \geq 0$ , and let  $\psi(z) = z + \bar{b}$ . Then on  $\mathcal{H}(E, \beta_S)$ ,  $C_\varphi^* = C_\psi$ .*

*Remark 4.10.* Note that when  $\lambda_1 = 0$ , if  $\varphi(z) = b$ , and  $\psi(z) = c$ , where  $b, c \in \mathbb{C}$ , then on  $\mathcal{H}(E, \beta_S)$ , we always have  $C_\varphi^* \neq C_\psi$ .

Indeed, assume there exist  $b, c$  such that  $C_\psi = C_\varphi^*$ . Let  $f, g \in \mathcal{H}(E, \beta_S)$ , where  $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$  and  $g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$ . Then by definition,  $\langle C_\varphi f, g \rangle = \langle f, C_\psi g \rangle$ , which is equivalent to

$$\left\langle \sum_{n=1}^{\infty} a_n e^{-\lambda_n b}, \sum_{n=1}^{\infty} b_n e^{-\lambda_n z} \right\rangle = \left\langle \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \sum_{n=1}^{\infty} b_n e^{-\lambda_n c} \right\rangle.$$

Pick  $f(z) = e^{-\lambda_2 z}$  and  $g(z) = 1 + e^{-\lambda_2 z}$ . From the equality above, we get

$$0 \neq e^{-\lambda_2 b} \beta_1^2 = 0(1 + e^{-\lambda_2 c}) \beta_1^2 = 0,$$

which is a contradiction.

With Lemma 4.9 and Remark 4.10, we can prove that every self-mapping composition operator  $C_\varphi$  induced by a non-constant  $\varphi(z)$  is complex symmetric with respect to any conjugation  $J_\xi$ .

**Proposition 4.11.** *Consider the following statements.*

- (i) *Let  $\varphi(z) = z + b$ , where  $\Re(b) \geq 0$ . The composition operator  $C_\varphi$  on  $\mathcal{H}(E, \beta_S)$  is  $J_\xi$ -symmetric where  $J_\xi$  is any conjugation induced by  $\xi(z) = z + ci$ , for any  $c \in \mathbb{R}$ .*
- (ii) *Let  $\varphi(z) = b$ ,  $b \in \mathbb{C}$ . There is no composition conjugation  $J_\xi$  such that  $C_\varphi$  on  $\mathcal{H}(E, \beta_S)$  is  $J_\xi$ -symmetric.*

*The following are true.*

- (1) *If  $\lambda_1 > 0$ , then (i) is true.*
- (2) *If  $\lambda_1 = 0$ , then both (i) and (ii) are true.*

*Proof.* Suppose  $\varphi(z) = z + b$  with  $\Re(b) \geq 0$ . Then, given  $\xi(z) = z + ci$  with  $c \in \mathbb{R}$ , we have

$$\begin{aligned} J_\xi C_\varphi J_\xi f(z) &= J_\xi C_\varphi \sum_{n=1}^{\infty} \bar{a}_n e^{-\lambda_n ci} e^{-\lambda_n z} = J_\xi \sum_{n=1}^{\infty} \bar{a}_n e^{-\lambda_n ci} e^{-\lambda_n b} e^{-\lambda_n z} \\ &= \overline{\sum_{n=1}^{\infty} \bar{a}_n e^{-\lambda_n ci} e^{-\lambda_n b} e^{-\lambda_n \bar{\xi}(z)}} = \sum_{n=1}^{\infty} a_n e^{\lambda_n ci} e^{-\lambda_n \bar{b}} e^{-\lambda_n \xi(z)} \\ &= \sum_{n=1}^{\infty} a_n e^{\lambda_n ci} e^{-\lambda_n ci} e^{-\lambda_n \bar{b}} e^{-\lambda_n z} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n(z+\bar{b})} = C_\varphi^* f(z). \end{aligned}$$

Thus  $C_\varphi$  is  $J_\xi$ -symmetric, so it is complex symmetric. This argument is valid for  $\lambda_1 \geq 0$ .

Now, suppose  $\lambda_1 = 0$  and  $\varphi(z) = b$  for some  $b \in \mathbb{C}$ , then

$$\begin{aligned} J_\xi C_\varphi J_\xi f(z) &= J_\xi C_\varphi \sum_{n=1}^{\infty} \bar{a}_n e^{-\lambda_n c i} e^{-\lambda_n z} = J_\xi \sum_{n=1}^{\infty} \bar{a}_n e^{-\lambda_n c i} e^{-\lambda_n b} \\ &= \overline{\left( e^{-\lambda_1 (\overline{z+ci})} \sum_{n=1}^{\infty} \bar{a}_n e^{-\lambda_n c i} e^{-\lambda_n b} \right)} = \sum_{n=1}^{\infty} \bar{a}_n e^{-\lambda_n c i} e^{-\lambda_n b} \\ &= \sum_{n=1}^{\infty} a_n e^{\lambda_n c i} e^{-\lambda_n \bar{b}} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n (\bar{b}-ci)}. \end{aligned}$$

Assume  $J_\xi C_\varphi J_\xi = C_\varphi^*$ . Then  $C_\varphi^* f(z) = f(\bar{b} - ci) = C_\psi f(z)$ , where  $C_\psi$  the bounded composition operator induced by the constant function  $\psi(z) = \bar{b} - ci$ . This contradicts Remark 4.10.  $\square$

Note that Proposition 4.11 does *not* state that composition operators  $C_\varphi$  induced by constant functions  $\varphi$  are not complex symmetric. These operators are not complex symmetric only with respect to composition conjugations  $J_\xi$ .

**4.3. Complex symmetry of composition operators induced by a constant function.** Recall that an operator  $T$  acting on a Hilbert space  $\mathcal{H}$  is of *rank one* if and only if there exist  $u, v \in \mathcal{H}$  such that  $Tf = \langle f, u \rangle v$  for all  $f \in \mathcal{H}$ . Our goal is to prove the following proposition for composition operators induced by constant functions in case  $\lambda_1 = 0$ .

**Proposition 4.12.** *Suppose  $\lambda_1 = 0$ . Then any composition operator  $C_\varphi$  induced by a constant function  $\varphi$  is complex symmetric.*

*Proof.* From Theorem 3.1,  $C_\varphi$  is bounded. Observe that  $C_\varphi$  is of rank one, since

$$C_\varphi f = f(b) = \langle f, K_b \rangle 1, \quad \forall f \in \mathcal{H}(E, \beta_S),$$

where  $K_b$  is as in the proof of Proposition 2.13.

Let  $W = \text{span}\{K_b, 1\}$ . We may decompose  $C_\varphi$  as a direct sum of two operators

$$C_\varphi = T_W \oplus 0_{W^\perp},$$

where  $T_W$  is an operator on the two-dimensional subspace  $W$  and  $0_{W^\perp}$  is the zero operator on  $W^\perp$ . By [4], any operator on a 2-dimensional Hilbert space is complex symmetric, thus  $T_W$  is complex symmetric on  $W$ . The zero operator is trivially complex symmetric. Since the direct sum of complex symmetric operators is complex symmetric,  $C_\varphi$  is also complex symmetric.  $\square$

*Remark 4.13.* We note that another approach via binormal operators to demonstrate that any rank one bounded linear operator on any Hilbert space is complex symmetric is given in [10].

We conclude with the following theorem, which summarizes the results presented in this paper regarding complex symmetry of composition operators on  $\mathcal{H}(E, \beta_S)$ .

**Theorem 4.14.** *Consider the following statements.*

- (i) Let  $\varphi(z) = z + b$ , where  $\Re(b) \geq 0$ . The composition operator  $C_\varphi$  on  $\mathcal{H}(E, \beta_S)$  is  $J_\xi$ -symmetric where  $J_\xi$  is any conjugation induced by  $\xi(z) = z + ci$ , for any  $c \in \mathbb{R}$ .
- (ii) Let  $\varphi(z) = b$ ,  $b \in \mathbb{C}$ . Then  $C_\varphi$  is complex symmetric, but not  $J_\xi$ -symmetric for any composition conjugation  $J_\xi$ .

The following are true.

- (1) If  $\lambda_1 > 0$ , then (i) is true.  
 (2) If  $\lambda_1 = 0$ , then both (i) and (ii) are true.

Consequently, all bounded composition operators on  $\mathcal{H}(E, \beta_S)$  are complex symmetric.

This paper focuses on the bounded composition operators on Hilbert spaces  $\mathcal{H}(E, \beta_S)$  and studies the complex symmetry of these operators. In particular, we introduced the composition conjugations  $J_\xi$  and study the aforementioned complex symmetry with respect to  $J_\xi$ . Although all bounded composition operators are complex symmetric, as we see from Theorem 4.14, the composition conjugations does not characterize the symmetry of composition operators induced by constant functions. Therefore, an open question for the future work is to find other forms of conjugations that characterize the complex symmetry of *all* bounded composition operators on more general classes of Hilbert spaces of entire Dirichlet series, such as  $\mathcal{H}(E, \beta_R)$  or  $\mathcal{H}(E, \beta)$ .

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