

# **Control and Estimation of Piecewise Affine Systems**

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A thesis submitted to the Nanyang Technological University

in fulfillment of the requirements for the degree of

Doctor of Philosophy

**2006**

**Statement of Originality**

I hereby certify that the content of this thesis is the result of work done by me and has not been submitted for a higher degree to any other University or Institution.

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Date

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# Acknowledgments

First of all and most importantly, I am deeply indebted to my supervisor, Professor Xie Lihua, for his professional guidance, constant support, encouragement and more. His insightful comment and thoughtful discussion have been an inspiration for my work. What I appreciate most were the many opportunities and the freedom he has offered that allowed me to find the research topics I am most interested in. Without his great enthusiastic help, I would not be able to complete this thesis.

I would also like to thank Prof. Soh Yeng Cai, Prof. Feng Gang, Prof. Cao Xiaoli, Prof. Fu Minyue, Prof. Han Zhengzhong, Prof Zhang Huanshui and others, for their help, discussions and advice.

Appreciation also goes out to the members of Computer Control Laboratory for their general support. Computer Control Lab has been my home for the last three years and everybody around has contributed to my understanding of control systems in some way or another.

I must also acknowledge the research scholarship I received from the Nanyang Technological University when the research work was carried out in the school of Electrical and Electronic Engineering.

Yet, in the end, I am most grateful to my parents for the love and encouragement provided through my entire life, and to all my friends in the world who reminded me that there is a life beyond the lab.

# Abstract

Piecewise affine (PWA) system has wide applications in circuit and control systems. It is considered as a powerful tool to study nonlinear systems, as well as switched and hybrid systems. This thesis presents our research findings in control and estimation of PWA systems.

We begin with the examination of the stability of PWA systems. We introduce a piecewise homogeneous polynomial Lyapunov function approach to achieve less conservative stability conditions by using two types of power transformations. Both continuous-time and discrete-time PWA systems are considered. The approach leads to less conservative stability analysis than existing results. In addition to these stability results based on halfspace representation, we also find ways to address the stability of PWA systems based on vertex representation. Specifically, we incorporate vertex information of local partition and utilize parameterized Lyapunov functions to deduce less conservative conditions for stability.

Next, we address the controllability and reachability of a class of PWA systems. Based on a general classification method, explicit necessary and sufficient conditions in terms of system parameters for controllability and reachability are presented. We show that controllability and reachability can be asserted in finite steps in some situations. We also briefly discuss the general cases.

We then turn to the controller design problems for discrete-time PWA systems. We first consider the state feedback control for PWA systems. By incorporating partition information of the system and applying *S-procedure*, linear matrix inequality

(LMI) based design methods are proposed. To further reduce the conservatism, we give a method to convert the control problem into a bilinear matrix inequality (BMI) problem and propose several iterative procedures for solving the problem.  $H_\infty$  and generalized  $H_2$  control of PWA systems are also investigated.

Parallel to the control problem, we also consider  $H_\infty$  and generalized  $H_2$  estimation for both continuous-time and discrete-time PWA systems. The Luenberger type estimators with less conservatism are designed by introducing proper transformations embedded with the partition information.

Finally, we take the Takagi-Sugeno fuzzy system as an application of PWA systems and derive the corresponding stability conditions and controller design approaches.

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# Symbols and Acronyms

## Logic Operators

$A \implies B$  A implies B, i.e. if A is true then B is true

$A \iff B$  A implies B and B implies A, i.e. A is true if and only if B is true

$A \cup B$  A or B

$A \cap B$  A and B

## Algebraic Operators

$A^T$  Transpose of matrix A

$A^{-1}$  Inverse of matrix A

$A^+$  Moore-Penrose inverse of A

$A^\perp$  The kernel of A, i.e.,  $A^\perp A^T = 0$

$\det(A)$  Determinant of matrix A

$A > 0$  ( $A \geq 0$ ) A is positive definite (positive semi-definite)

$A < 0$  ( $A \leq 0$ ) A is negative definite (negative semi-definite)

$A \succeq 0$  All entries of A are non-negative

$A \sqsupseteq 0$  ( $A \sqsubset 0$ ) A is (strictly) copositive, i.e.,  $x^T A x \geq 0, \forall x \succeq 0$   
( $x^T A x > 0, \forall x \succeq 0, x \neq 0$ )

$\|y\|$  Euclidian norm of vector y

$\|y\|_1$   $\sum_{i=1}^n |y_i|, y = [y_1, \dots, y_n]^T$

$\|y\|_{\ell_\infty[0,N]}$   $\sup_{t \in [0,N]} \{y^T y\}^{\frac{1}{2}}$

$\|y\|_{\ell_2[0,N]}$   $\sum_{t=0}^N \{y^T y\}^{\frac{1}{2}}$

$\|y\|_{l_\infty(0,t_f)}$   $\sup_{t \in [0,t_f]} \{y^T y\}^{\frac{1}{2}}$

$\|y\|_{l_2(0,t_f)}$   $\int_0^{t_f} \{y^T y\}^{\frac{1}{2}} dt$

**Sets and Set Operators**

$\mathcal{R}(\mathcal{R}_+)$	Set of (non-negative) real numbers
$\mathcal{N}$	Set of non-negative integers
$\mathcal{R}^n$	Set of real vectors with $n$ elements
$\mathcal{R}^{n \times m}$	Set of real matrices with $n$ rows and $m$ columns
$\emptyset$	The empty set
$P \cap Q$	Set intersection $P \cap Q = \{x   x \in P \text{ and } x \in Q\}$
$P \cup Q$	Set union $P \cup Q = \{x   x \in P \text{ or } x \in Q\}$
$P \setminus Q$	Set difference $P \setminus Q = \{x   x \in P \text{ and } x \notin Q\}$
$P \subseteq Q$	The set P is a subset of Q, $x \in P \implies x \in Q$
$P \subset Q$	The set P is a strict subset of Q, $x \in P \implies x \in Q$ and $\exists x \in (Q \setminus P)$

**Others**

<b>I</b>	Identity matrix
<b>1</b>	Vector of ones, $\mathbf{1} = [1 \ 1 \ \dots \ 1]^T$
<b>0</b>	Vector of zeros, $\mathbf{0} = [0 \ 0 \ \dots \ 0]^T$
$\begin{bmatrix} A & * \\ B & C \end{bmatrix}$	Symmetric matrix $\begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$

**Acronyms**

BMI - Bilinear Matrix Inequality

CPS - Continuous-time Piecewise Affine Systems

DPS - Discrete-time Piecewise Affine Systems

LMI - Linear Matrix Inequality

LTI - Linear Time Invariant

MPC - Model Predictive Control

HPLF - Homogeneous Polynomial Lyapunov Functions

PHPLF - Piecewise Homogeneous Polynomial Lyapunov Functions

PQ - Piecewise Quadratic

PQLF - Piecewise Quadratic Lyapunov Function

PWA - Piecewise Affine

PWL - Piecewise Linear

PDLF - Partition-dependent Lyapunov Function

SDP - Semidefinite Programming

SOS - Sums of Squares

VDLF - Vertex-dependent Lyapunov Function

# Chapter 1

## Introduction

### 1.1 Motivations

Piecewise affine/linear (PWA or PWL)<sup>1</sup> systems have been studied for a long time [102, 119, 137]. The first study of PWA systems in the control literature may date back to Andronov's investigation on oscillations in nonlinear systems in the 1940's [4]. An early practical example appears in piecewise linear servomechanisms [152]. A concrete work on qualitative understanding of piecewise linear systems was made by Kalman, who considered a saturated system as a series of polyhedral regions in the state space, separated by switching boundaries in the 1950's [110].

However, it was not until the 1970's that PWA systems were perceived as a class of system models by the circuit community, where efficient simulations and analysis of large-scale circuits with diodes and other piecewise linear elements were hankeringly needed, which led to a considerable research effort on the efficient representation of PWA systems [41, 42, 155, 156]. As one of the pioneering works, Sontag analyzed the discrete-time PWA systems in the 1980's. His idea is still employed in some recent works, say, [112, 102, 145].

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<sup>1</sup>In many literatures, PWA and PWL are often used interchangeably, though PWA system has at least an affine term in the system model representation in general. In this thesis, we use PWA to include PWL.

In recent years, there have been increasing interests in PWA systems due to their applications in the following aspects.

Firstly, PWA maps have universal approximation properties [123, 124]. For example, in [123], the canonical representation of piecewise-linear functions is considered as a universal approximation scheme of multivariate functions. Meanwhile, two universal approximation schemes in terms of combinations of univariate canonical piecewise-linear functions are proposed. Hence PWA models are suitable for analysis and synthesis of classes of nonlinear systems [41, 102]. In the recent work [102], Johansson shows how approximation errors can be accounted for in the analysis to yield rigorous results for the underlying smooth nonlinear system, and develops a converse theorem which states that a piecewise quadratic Lyapunov function (PQLF) is enough to prove the exponential stability of a smooth system.

Secondly, PWA systems can be used to model many types of switched and hybrid systems. Heemels *et al.* [85] establish an equivalence between PWA systems and some classes of hybrid systems: mixed logical dynamical (MLD) systems, linear complementarity (LC) systems, extended linear complementarity (ELC) systems, and max-min-plusscaling (MMPS) systems [41, 29]. Thus PWA systems provide a powerful means for analysis and design of switched and hybrid systems.

Thirdly, PWA systems can be employed to formulate a large class of intelligent systems, especially, fuzzy systems [56, 62, 59]. The intelligent controllers are commonly rule-based. Linear controllers can be designed at the main operating points and switched when required. For instance, Johansson [102] analyzes continuous-time Takagi-Sugeno (T-S) fuzzy systems using a differential inclusion that involves all consequent dynamics via operating regime based models. Feng *et al.* use uncertain PWA system as the underlying model of T-S fuzzy system for stability analysis and controller design [56, 62, 59].

Fourthly, linear time-invariant (LTI) systems combined with PWL sector-bounded nonlinearity [70, 82] can be considered as a kind of PWA system. For example, the system consisting of the feedback interconnection of a linear system and a PWL

sector-bounded nonlinearity is a PWA system [82].

Fifthly, some special cases of sampled-data systems can be treated as PWA systems. In [93, 94], Imura proposes a so-called sampled-data PWA system, where the switching action of the discrete state is determined at each sampling time according to a condition on the continuous state.

Finally, besides the various applications listed above, some identification techniques for PWA systems have been developed. These include the statistical clustering technique [78], the K-means clustering-based procedure [66], the Bayesian procedure [109], the mixed-integer linear or quadratic programming [150], a bounded-error approach [15], the polynomial factorization (algebraic approach) [166] and so on. Thus many physical systems can be treated as PWA models, such as a fermentation process [53] and a pick-and-place machine [109].

There have been many works done on PWA systems, ranging from the modelling [102, 103, 146, 56, 13], characteristic analysis [92, 13, 95, 55] to controller [60, 59, 57, 56, 62, 54, 102] and estimator [1, 2, 108, 107, 172, 61] design. However, many properties of PWA systems have not been fully explored and their design approaches are generally conservative.

For example, the results for estimation problems of PWA systems [1, 2] are conservative because some important characteristics of PWA models, such as the partition information, are not taken into account. Recently, some researchers explored ways of integrating the partition information into observer design [108, 107]. However, the results are only for bimodal systems.

A similar problem also appears in control of PWA systems. In [43, 57, 102], controllers are designed for PWA systems without considering the partition information. In [56], Feng includes the partition information for a fuzzy controller design based on continuous-time PWA systems. Nevertheless, it is difficult to solve this problem for discrete-time systems due to the non-convex nature of optimization involved. In this thesis, we shall address this problem.

Moreover, many properties of PWA systems remain unknown and many existing results are generally conservative. For example, although many works have been done on the stability conditions [55, 102, 139, 64] based on piecewise Lyapunov functions, which is more suitable than common quadratic Lyapunov functions (CQLFs), they still cannot verify the stability of some simple examples [172]. The conditions for well-posedness of general PWA systems appear to be very difficult to be established [102]. Imura and Schaft [95] derive necessary and sufficient conditions for bimodal systems to be well-posed, based on the lexicographic inequalities and the smooth continuity of solutions. Extensions to multi-modal cases have also been reported [95, 92]. Other important properties, such as controllability and observability, also lack of investigation. In fact, due to the declaration of NP-hardness [19], most researches on the controllability of discrete-time PWA systems either focus on deducing efficient computational algorithms [114, 116], or on deriving the conditions for special/simplified models [168].

Many of these analysis and synthesis results above are related to the so-called Lyapunov functions, which also play a significant role in our study. Although CQLF has advantages in terms of simple structure and highly efficient computation, several facts make CQLF conservative. Firstly, CQLF does not allow affine terms in the dynamics so that simple systems such as those with saturation can not be analyzed [102, 43]. Secondly, it does not consider the partition information in the analysis, which leads to a problem that regional dynamics are forced into global dynamics. Thus several non-quadratic Lyapunov functions<sup>2</sup> are proposed [102, 55, 1, 2, 135, 38, 99]. PQLFs are widely employed in many recent researches [102, 55, 1, 2, 135], because they are relatively simple and highly efficient. There is no doubt that other types of non-quadratic Lyapunov functions are worth studying though they are more complex, such as bi-quadratic Lyapunov functions [10, 163], parameterized Lyapunov functions [129] and homogeneous Lyapunov functions [38, 99]. Exploring the advantages of these functions for PWA systems,

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<sup>2</sup>We use the term of non-quadratic Lyapunov functions to refer to all types of Lyapunov functions that are not commonly quadratic.

therefore, becomes one of the topics of this thesis.

In addition, due to the amazing advances in computer technologies, it is very promising to develop analysis and design methods by using numerical computation. In the past years, linear matrix inequality (LMI) [22] and bilinear matrix inequality (BMI) [72, 73] techniques have been widely applied in control engineering, which are also adopted in our research. A brief introduction to LMI and BMI is available in Appendix A.

This thesis studies control and estimation of PWA systems, as well as some fundamental properties of PWA systems.

## 1.2 Objective and Contributions

The aim of this thesis is to investigate the control and estimation problems of PWA systems, as well as properties of PWA systems, such as stability and controllability. Because PWA systems often exhibit very complex behaviors, say, limit cycles, multiple equilibrium points, chaos, and etc. [120, 137, 102], it is not expected that necessary and sufficient conditions can be formulated for general cases. However, it is possible that approximate analysis and synthesis methods can be developed for various practical problems and explicit conditions can be established for some special cases.

The main original contributions of the thesis are listed as follows.

- (1) We propose a construction method of piecewise homogeneous polynomial Lyapunov functions (PHPLFs) for PWA systems by applying the SOS theory [143]. Sufficient conditions for the existence of PHPLFs of a given degree are obtained in terms of LMIs for both continuous-time and discrete-time PWA systems. The results contain existing criteria based on PQLFs as special cases and provide a less conservative assertion of the stability of PWA systems.
- (2) We propose approaches to construct two kinds of Lyapunov functions by

including the partition information based on vertex representation, which are different from the existing stability conditions [102, 55, 43], which use halfspace representation. One is the partition-dependent Lyapunov function (PDLF) using copositive matrix formulation, which can be solved by the recently developed algorithms based on SDP. We prove that PDLF is equivalent to PQLF in some special cases, while in general PDLF is less conservative than PQLF. The other is vertex-dependent Lyapunov function (VDLF). We provide two LMI methods to calculate the corresponding Lyapunov matrices under the condition that the Lyapunov functions are not in a quadratic form. We also show that VDLF is more powerful than PQLF and PDLF. Furthermore, we prove that these two kinds of Lyapunov functions are necessary and sufficient for the piecewise quadratic stability in a sense.

- (3) We derive explicit necessary and sufficient conditions for the controllability and reachability of discrete-time planar bimodal piecewise linear systems based on a general classification method. The complexity of the controllability problem for piecewise linear systems is clearly demonstrated by the present study. We also briefly discuss the high-order and multi-model cases.
- (4) We provide some less conservative approaches to feedback controller design for discrete-time PWA systems. Using the less conservative analytical results of stability, our results arrive at some design approaches based on LMIs and BMIs. We propose some efficient iterative LMI algorithms to solve the BMI problems.  $H_\infty$  and generalized  $H_2$  control problems are considered too.
- (5) We present an LMI/BMI based approach to the design of non-synchronized  $H_\infty$  and generalized  $H_2$  estimators for PWA systems. Our approach employs *S-procedure* and partition-dependent slack variables to reduce the design conservatism, and it is applicable for both continuous-time and discrete-time PWA systems. Estimators of the Luenberger type are considered.
- (6) We give a new mathematical model for a class of Takagi-Sugeno (T-S) fuzzy

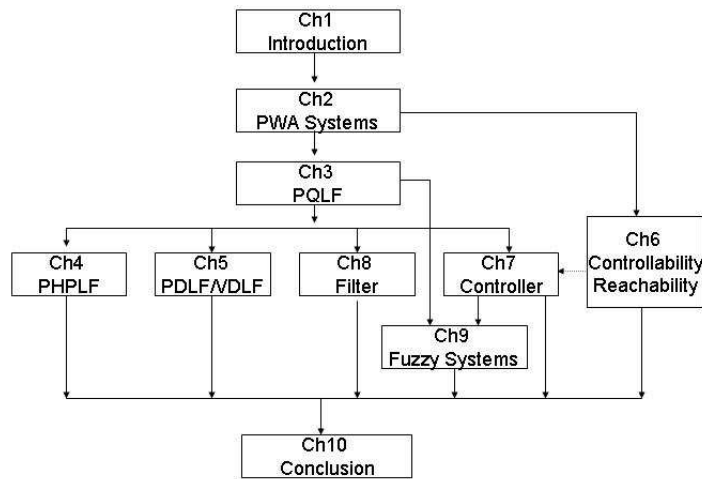


Figure 1.1: Relationship of all chapters

systems. This new model in a sense is a nested linear differential/difference inclusion. Sufficient conditions for exponential stability of discrete-time fuzzy systems are proposed. We also consider the stabilization problem via state feedback control for discrete-time cases. Less conservative analysis and synthesis results are derived by making use of the free variables appended via *S-procedure* and partition dependent slack variables.

### 1.3 Organization of the Thesis

In this thesis, Chapter 2 and Chapter 3 introduce some necessary backgrounds and preliminaries. Chapter 4-9 form the main body of the thesis, where a common structure of “Introduction - Main content - Conclusion” is adopted. Relationships among chapters are illustrated in Figure 1.1. Hence, these chapters are organized as follows.

In Chapter 2, we give a mathematical view of PWA systems. After introducing a PWA model, we analyze the structure of PWA systems with the aim to rule out degeneracies in PWA systems and distinguish it from other types of switched sys-

tems. Meanwhile it also provides critical engineering insights. Some applications are presented to illustrate the importance of PWA systems.

In Chapter 3, we first summarize some stability results of PWA systems. We then analyze these results based on PQLFs. We also derive some transformations, which provide a unified representation of discrete-time and continuous-time PWA systems. Furthermore, it also provides some useful lemmas for the design problems of PWA systems.

In Chapter 4, we address the construction of PHPLFs for PWA systems. Sufficient conditions for the existence of PHPLFs of a given degree for both continuous-time and discrete-time PWA systems are obtained in terms of LMIs. We also give a unified view of these stability criteria based on halfspace representation.

In Chapter 5, we continue to discuss the stability issue of PWA systems. Instead of halfspace representation, we propose less conservative stability conditions based on vertex representation. Firstly, we address the PDLFs and present a result based on copositive matrices. Secondly, we construct VDLFs by using the SOS theory [132] and Pólya's theorem [81, 138] via LMI solver. Examples are given to illustrate the advantages of PDLF and VDLF over PQLF.

In Chapter 6, we first review the literature on controllability and reachability (as well as the observability) for PWA systems and switched/hybrid dynamical systems. And then we discuss the controllability and reachability of a special class of PWA systems: discrete-time planar bimodal PWL systems. Sufficient and necessary conditions are given in an explicit form. Finally, we discuss some possible extensions for general cases.

In Chapter 7, we investigate the design of state feedback and dynamic output feedback stabilizing controllers, as well as  $H_\infty$  and generalized  $H_2$  state feedback controllers, for discrete-time PWA systems. The main objective is to derive design methods that incorporate the partition information of the PWA systems so as to reduce the design conservatism embedded in existing design methods. We first introduce some transformations that convert the feedback control design problem

into a BMI problem. Then, iterative algorithms are proposed to compute suitable feedback controllers. Several simulation examples are given to demonstrate the advantages of the proposed design.

In Chapter 8, we discuss the non-synchronized  $H_\infty$  and generalized  $H_2$  estimation problems for both continuous-time and discrete-time PWA systems. Due to the output noise and estimation errors, the transitions of the actual state and estimate state of a PWA system may not be synchronized. A type of Luenberger estimator is proposed to guarantee the non-synchronized  $H_\infty$  and generalized  $H_2$  performance by using arguments from the Lyapunov theory. Numerical examples show the good estimation performance of our approaches.

In Chapter 9, we first present a new model for a class of Takagi-Sugeno fuzzy dynamic systems based on a PWA system formulation. And then less conservative stability criteria for a class of discrete-time fuzzy systems are derived using recently developed dilated LMI characterizations. The stability analysis results lead to improved approaches to state feedback controller design based on LMI and BMI techniques. Some examples are presented to illustrate the modelling and synthesis of the proposed methods.

In the last chapter, some conclusions are given, together with possible future research directions.

# Chapter 2

## Piecewise Affine Systems

### 2.1 Introduction

Piecewise affine systems have wide applications in a range of engineering and science [102, 119, 15, 137]. Among these applications, some are naturally PWA, such as diodes and transistors, key components in even the simplest electronic circuits. However, many advanced controllers, notably gain-scheduled flight control systems, are indeed enlightened by piecewise linear ideas. From the basic analysis to the advanced design, PWA systems actually have undergone a long time development.

This chapter discusses the mathematical modelling of PWA systems. As we know, there are mainly three types of switching systems. The first one is arbitrary switching systems, whose switching sequences cannot be determined. The second one is time-dependent switching systems, whose switching signals are classified by the time-interval between two consecutive switching instances. The third one is state (or input/output vector)-dependent, whose switching signal is driven by the state-vector. For the second and third types, the switching signal may be predetermined before analysis and synthesis, or may be a design goal. Hence a clear definition and structural description is necessary.

## 2.2 System Model

Generally speaking, PWA models have rather loose definitions in literature, which vary from an interconnection of linear components [42] to polyhedral partition based model [102]. For clarity, we give the following definitions first.

**Definition 2.2.1** [137, 102] (*Polyhedron and Polytope; Halfspace and Vertex Representation*) *A polyhedron is a set of the form*

$$S = \{x \in \mathcal{R}^n | a_i^T x \geq b_i, i = 1, \dots, l\} = \{x \in \mathcal{R}^n | Ax \geq b\} \quad (2.1)$$

or, equivalently,

$$S = \{x = V\lambda + W\mu | \sum_i \lambda_i = 1, \lambda \succeq 0, \mu \succeq 0\} \quad (2.2)$$

where  $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_l]^T$  and  $\mu = [\mu_1, \mu_2, \dots, \mu_j]^T$ . (2.1) and (2.2) interpret the polyhedron in different ways, i.e., as the intersection of a finite number of halfspaces, and as the set addition (Minkowski sum) of the convex hull of the columns of the matrix  $V$  and the conic hull of the columns of the matrix  $W$ , respectively. The columns of  $V$  are called the vertices of the polyhedron, and the columns of  $W$  are called extreme rays.

A polytope is a bounded polyhedron. It can be represented as

$$S = \{x = V\lambda | \sum_i \lambda_i = 1, \lambda \succeq 0\} \quad (2.3)$$

Sometimes, the representation (2.1) is called the halfspaces representation, while (2.3) is referred as the vertex representation.

**Definition 2.2.2** [137, 76, 102] (*Polytopic/Polyhedral Partition*) *A collection of polytopic (polyhedral) sets  $\{S_i\}_{i=1}^N = \{S_1, \dots, S_N\}$  is a polytopic (polyhedral) partition of a polytopic (polyhedral) set  $\Theta \subseteq \mathcal{R}^n$  if (i)  $\bigcup_{i=1}^N S_i = \Theta$ , (ii)  $(S_p \setminus \partial S_p) \cap (S_q \setminus \partial S_q) = \emptyset, \forall q \neq p, p, q = 1, 2, \dots, N$ , where  $\partial$  denotes the boundary, (iii) if  $S_p \cap S_q \neq \emptyset$ , where  $q \neq p$ , then  $S_p \cap S_q$  is a common face of  $S_q$  and  $S_p$ .*

**Definition 2.2.3** [76, 102] (PWL, PWA and PWQ) Consider the function  $f$  over a polyhedral set  $S$ .  $f : S \rightarrow \mathcal{R}^d$  with  $d \in \mathcal{N}$  is piecewise affine (PWA), if a partition  $\{S_i\}_{i=1}^N$  of set  $S$  exists, such that  $f(x) = A_i x + a_i$  if  $x \in S_i$ . Specially, if  $a_i = 0$ , then the function is called piecewise linear (PWL).  $f : S \mapsto R$  is piecewise quadratic (PWQ), if a partition  $\{S_i\}_{i=1}^N$  of the set  $S$  exists, such that  $f(x) = x^T Q_i x + L_i x + C_i$  when  $x \in S_i$ .

Thus, roughly speaking, these systems whose dynamics are defined by PWA functions can be called as PWA systems. However, in this thesis, we only consider the PWA systems whose partitions are defined on polyhedrons/polytopes, though there also exist some other partitions, such as ellipses.

### 2.2.1 Model Representation

A PWA system can be represented as follows.

$$\begin{aligned} \delta x(t) &= A_i x(t) + B_{i2} u(t) + B_{i1} w(t) + a_i \\ y(t) &= C_i x(t) + D_{i2} u(t) + D_{i1} w(t) \end{aligned} \quad \text{for } x(t) \in S_i, i \in \mathcal{I} \quad (2.4)$$

where  $x(t) \in \mathcal{R}^n$  is the state vector,  $u(t) \in \mathcal{R}^m$  is the control input vector,  $w(t) \in \mathcal{R}^l$  is the exogenous noise signal and  $y(t) \in \mathcal{R}^r$  is the output vector. We assume  $m \leq n$ .  $\delta$  denotes the derivative operator, i.e.  $\delta x(t) = (d/dt)x(t)$ , for continuous-time systems and the shift forward operator, i.e.  $\delta x(t) = x(t+1)$ , for discrete-time systems. We say that the system dynamic locates in mode  $i$  if  $A_i x(t) + B_{i2} u(t) + B_{i1} w(t) + a_i$  in (2.4) is active.

$\mathcal{I}$  is the index set of these finite partitions/subspaces.  $|\mathcal{I}|$  denotes the size of the set  $\mathcal{I}$ .  $\{S_i\}_{i \in \mathcal{I}}$  denotes a set of polyhedral partitions of the state space,

$$S_i = \{x(t) | F_i x + f_i \geq 0\} \quad (2.5)$$

Let  $\tilde{\Omega}_c$  represent switch index pairs for the continuous-time case. The definition for  $\tilde{\Omega}_c$  will be slightly different in different situations. For example, if we assume that

the finite discrete state  $i$  is a piecewise constant function of time and piecewise continuous from the right, then we have

$$\tilde{\Omega}_c = \{(i, j) | x(t^-) \in S_i, x(t) \in S_j, i \neq j, i, j \in \mathcal{I}\} \quad (2.6)$$

The notation  $t^-$  indicates that the finite state is piecewise continuous from the right. Let  $\Omega_c = \tilde{\Omega}_c \cup \{(i, i) | i \in \mathcal{I}\}$ . If  $i, j \in \tilde{\Omega}_c$ , their boundary can be defined as<sup>1</sup>:

$$S_{ij} = \left\{ x_t | F_{ij} x(t) + f_{ij} = 0, (i, j) \in \tilde{\Omega}_c \right\} \quad (2.7)$$

The boundary defined in (2.7) is also called as a switching boundary.

Let  $\tilde{\Omega}_d$  represent index pairs for the discrete-time case:

$$\tilde{\Omega}_d = \{(i, j) | x(t) \in S_i, x(t+1) \in S_j, i \neq j, i, j \in \mathcal{I}\} \quad (2.8)$$

Let  $\Omega_d = \tilde{\Omega}_d \cup \{(i, i) | i \in \mathcal{I}\}$ . Similarly, we can define the partition  $S_i$  and switching boundary  $S_{ij}$ <sup>2</sup>.

**Remark 2.2.1** Cuzzola et al.[43] first propose a partition method based on both state space and control variable space, i.e.,  $\left\{ S_i = \left\{ \varsigma(t) \mid F_i^{(ux)} \varsigma(t) + f_i^{(ux)} \geq 0 \right\} \right\}_{i \in \mathcal{I}}$ , where  $\varsigma(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ , because the switching sequence may depend on the input signal under some situations, such as saturation effects or limitation on the control signal. Thus partition method is widely used in the Model predictive control (MPC)-based optimal control of PWA systems, see, for example, [13, 16].

Feng [57] partitions the output space, i.e.,  $\left\{ S_i = \left\{ y(t) \mid F_i^{(y)} y(t) + f_i^y \geq 0 \right\} \right\}_{i \in \mathcal{I}}$ . In fact, this partition method can be easily transferred into the partition based on state space:  $\left\{ S_i = \left\{ x(t) \mid F_i^{(y)} C_i x(t) + f_i^y \geq 0 \right\} \right\}_{i \in \mathcal{I}}$ .

<sup>1</sup> $S_i$  and  $S_j$  must be neighboring in the CPS.  $S_{ij}$  may also include inequality constraints. However, since we only use the equality information in most cases, we only provide the equality constraints, without further explanation.

<sup>2</sup>In DPS,  $S_i$  and  $S_j$  are not necessarily neighboring. Hence  $\Omega_d$  is much more complex than  $\Omega_c$ . See Section 2.3.2 and Section 8.2 for further discussion.

The commonly used partitions are only based on state space representation [102, 145, 55, 64, 135] as shown in (2.5). In our study, without further explanation, we only consider this type of space partition.

**Remark 2.2.2** Let  $\mathcal{I}_0 \subseteq \mathcal{I}$  be the set of indices for the partitions that contain the origin and  $\mathcal{I}_1 = \mathcal{I} \setminus \mathcal{I}_0 \subseteq \mathcal{I}$  be the set of indices for the partitions that do not contain the origin. We assume that for  $i \in \mathcal{I}_0$ ,  $a_i = 0$ . Under this assumption, 0 is one of the equilibria of the PWA systems. In fact, any  $x_{eq}$  satisfies  $A_i x_{eq} + a_i = 0$  is an equilibrium of the operating mode  $i$ . If  $A_i$  is full-rank, then  $x_{eq} = -A_i^{-1}a_i$  is the unique equilibrium of operating mode  $i$ . In general, the number of equilibria may not be the unique one. However, in this thesis, we are only interested in 0, as other equilibria can be easily treated in a similar way with a proper transformation.

In most discussions of this thesis, we only consider the case that all  $i \in \mathcal{I}_0$ . However, we can easily extend the corresponding results to general cases. In fact, we can transfer (2.4) to the following extended system [43, 55, 102]:

$$\delta\zeta(t) = \bar{A}_i\zeta(t) + \bar{B}_i u, \quad x(t) \in S_i, \quad i \in \mathcal{I} \quad (2.9)$$

where

$$\zeta(t) = \begin{bmatrix} x(t) \\ 1 \end{bmatrix}, \quad \bar{A}_i = \begin{cases} (CPS) : \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix} \\ (DPS) : \begin{bmatrix} A_i & a_i \\ 0 & 1 \end{bmatrix} \end{cases}, \quad \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix} \quad (2.10)$$

We also define <sup>3</sup>

$$\bar{F}_i = \begin{bmatrix} F_i & f_i \end{bmatrix}, \quad \bar{F}_{ij} = \begin{bmatrix} F_{ij} & f_{ij} \end{bmatrix} \quad (2.11)$$

Hence,  $S_i$  and  $S_{ij}$  can be represented as

$$S_i = \{x | \bar{F}_i \zeta \geq 0\}, \quad S_{ij} = \{x | \bar{F}_{ij} \zeta \geq 0\}$$

---

<sup>3</sup> $\bar{F}_i$  is called polyhedral cell identifier for  $S_i$ . A similar concept is the so-called polyhedral cell bounding  $\bar{E}_i$  [102], which satisfies the zero interpolation properties for  $i \in \mathcal{I}_0$ . It is easy to derive  $\bar{E}_i$  from  $\bar{F}_i$ . In the case when no confusion is caused, we use  $\bar{F}_i$  instead of  $\bar{E}_i$  for simplification.

These transformations will simplify our representation. In the sequel, in most cases, we only consider the PWL systems, i.e., system (2.4) with  $a_i = 0$  and  $f_i = 0$  without further notification.

Another important concept is the so-called continuity matrix  $\vec{\bar{F}}_i$  of  $\bar{F}_i$ . For  $(i, j) \in \tilde{\Omega}_c$ ,  $\vec{\bar{F}}_i$  is a continuity matrix of  $\bar{F}_i$  if  $\vec{\bar{F}}_i \zeta = \vec{\bar{F}}_j \zeta$  for  $\forall x \in S_{ij}$  [102].

## 2.2.2 Applications

As mentioned earlier, a wide range of industry systems can be modelled using piecewise affine approximations of their nonlinear elements. Especially, in mechanical and electrical systems, some of the most common nonlinearities, such as stiction, saturation, backlash and relay action, have distinct piecewise affine types of behavior [137]. Meanwhile, logic-based controllers, ranging from PLC (programmable logic controller) to complex rule-based controllers and gain schedulers, are widely employed in industry. The systems with these controllers can naturally fit into a piecewise affine framework. Three simple and typical examples will be given first, followed by two more complex examples.

**Example 2.2.1 (*Switched circuit*)** Consider a simple RC circuit with switch [180] as shown in Figure 2.1. We assume the switching is instantaneous based on the value of  $i_c$  or  $v_c$ . Now if we take voltage  $v_c$  as the state, the system dynamic is

$$\frac{dv_c}{dt} = \begin{cases} -\frac{1}{R_1 C} v_c + \frac{E_1}{R_1 C} & \text{Position 1} \\ -\frac{1}{R_2 C} v_c + \frac{E_2}{R_2 C} & \text{Position 2} \end{cases} \quad (2.12)$$

Note that  $v_c$  will be continuous at the switching instants. If current  $i_c$  through the capacitor is chosen as the continuous state, we have

$$\frac{di_c}{dt} = \begin{cases} -\frac{1}{R_1 C} i_c & \text{Position 1} \\ -\frac{1}{R_2 C} i_c & \text{Position 2} \end{cases} \quad (2.13)$$

In this case,  $i_c$  will be discontinuous at the switching instants whatever what the switching order is. This example illustrates that sometimes it is depending on the

choice of continuous state variables whether the system state is continuous during the switchings. Figure 2.2 shows a simulation result for (2.12) with the the system parameters  $R_1 = 1000(\text{ohm})$ ,  $R_2 = 1500(\text{ohm})$ ,  $E_1 = 1.1(\text{volt})$ ,  $E_2 = 2(\text{volt})$  and  $C = 0.001(\text{farad})$  at initial conditions  $v_c = 0.3, 0.8, 1.5, 2(\text{volt})$ , respectively. The switching condition is as follows. If  $v_c \geq 1$ , switch to position 1; otherwise switch to position 2.

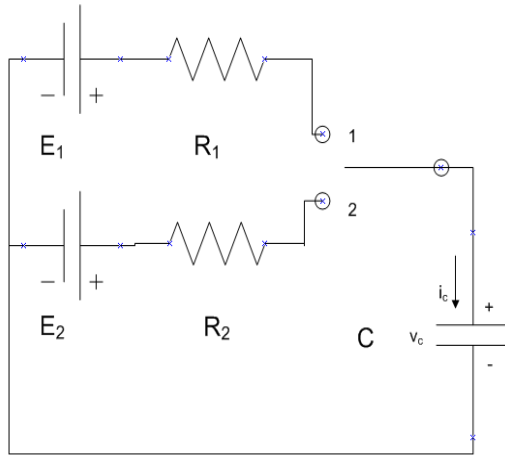


Figure 2.1: Switched circuit

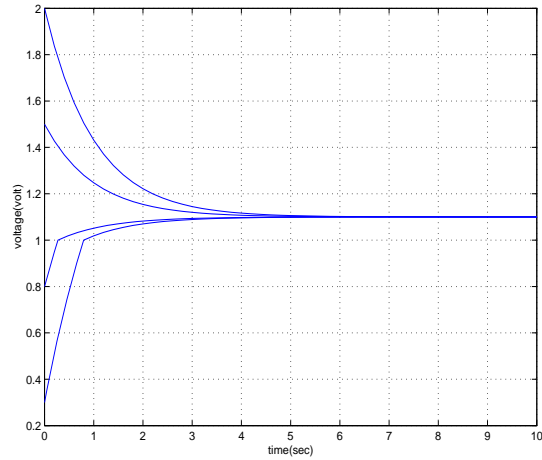


Figure 2.2: Simulation results

**Example 2.2.2 (Tunnel diode circuit)** This kind of circuits [42, 115, 146] is widely used in electrical engineering. Figure 2.3 presents such circuit with the inductor current in mili-Amps and the capacitor voltage in Volts. The circuit dynamics are described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -30 & -20 \\ 0.05 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 24 \\ -50g(x_2) \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} u \quad (2.14)$$

Following [115], the characteristic of the tunnel diode  $g(x_2)$  is modelled as a piecewise affine function as the over-plotted load line in Figure 2.4. The load line (The end points of the load line correspond to the steady-state current at the diode terminals with an open circuit between them and the steady-state current with a short circuit between them) intersects the diode characteristic in three locations, which generates three equilibrium points for the open-loop system. The measured output

is the current, i.e.,  $x_1$ . This example actually illustrates the way to analyze the smooth nonlinear systems using PWA model.

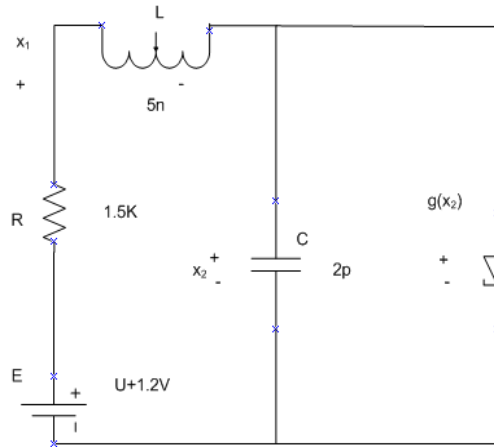


Figure 2.3: Tunnel diode circuit

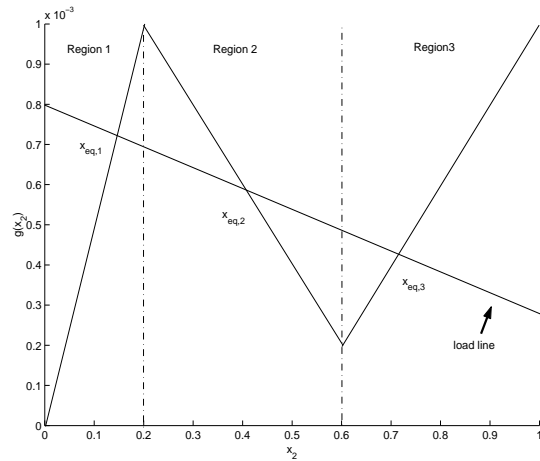


Figure 2.4: Tunnel diode characteristic

**Example 2.2.3 (LTI systems with state or input bound)** The system consists of the feedback interconnection of a linear systems and a PWL sector-bounded nonlinearity, and is therefore a PWA system [82]. A typical example is the so-called saturation systems, which often appear in circuit system and mechanical systems [127, 91, 88, 89, 90].

$$\delta x(t) = Ax + bu = Ax + b \text{ sat}(v), \quad v = k^T x \quad (2.15)$$

where  $k$  is the state feedback gain. As illustrated in Figure 2.5, (2.15) describes a class of single input saturation systems. The dynamics can be further expressed as [102]:

$$\delta x(t) = \begin{cases} Ax - b, & x \in S_1 \\ Ax + bk^T x, & x \in S_2 \\ Ax + b, & x \in S_3 \end{cases} \quad (2.16)$$

In Figure 2.6, we give an example with simulation result for a LTI system with an input saturation. It is borrowed from [137] with slight changes.

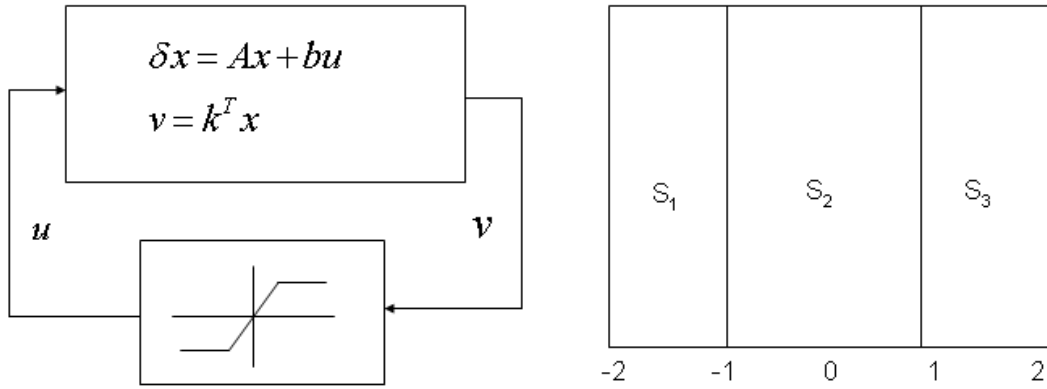
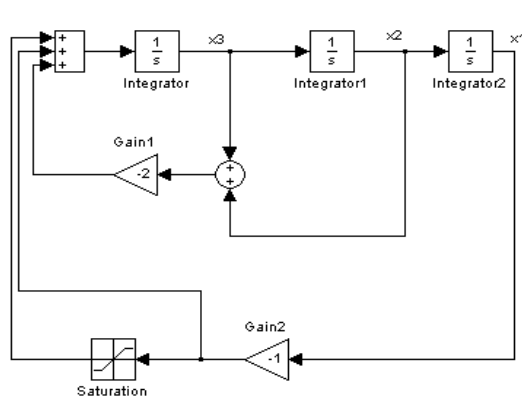
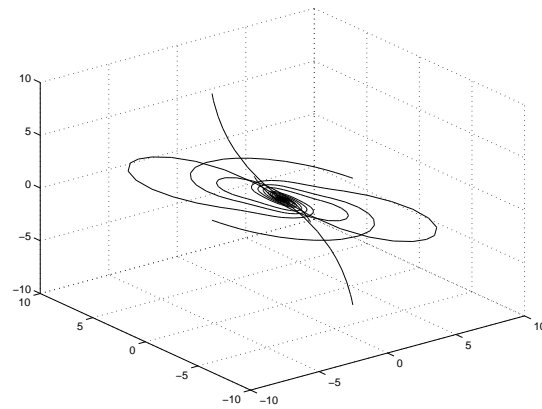


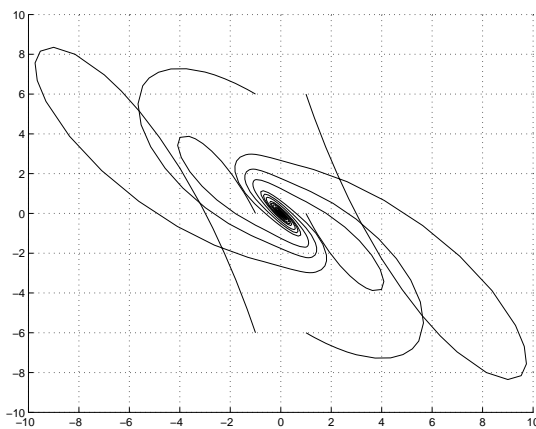
Figure 2.5: A saturation feedback systems with PWA representation



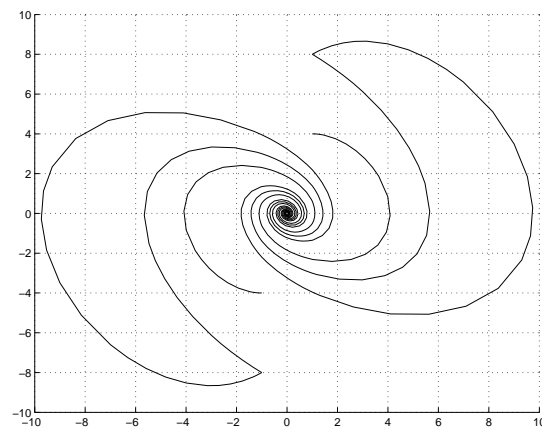
The block diagram



A 3D view of the system



A side view of the system



A top view of the system

Figure 2.6: A saturation example with simulation

Other examples include batch processes, cars with automatic gear-box and cruise controller [135], autonomous land vehicle and inverted pendulum on a cart [146]. Two more examples are given below to show how PWA systems can be used to model some complex systems.

**Example 2.2.4 (Hybrid systems)** Consider the following (continuous-time) controlled hybrid system [102, 135]

$$\dot{x}(t) = f(x(t), q(t), u(t), w(t)) \quad (2.17)$$

$$q(t) = g(x(t), q(t^-), u(t), w(t)) \quad (2.18)$$

where  $x(t) \in \mathcal{R}^n$ ,  $q(t) \in \mathcal{Q} \triangleq \{1, 2, \dots, N\}$ .  $f(\cdot) : \mathcal{R}^n \rightarrow \mathcal{R}^n$  is globally Lipschitz continuous.  $g(\cdot) : \mathcal{R}^n \times \mathcal{Q} \rightarrow \mathcal{Q}$  is the finite discrete dynamics of (2.17). The continuous dynamics of (2.17) generally depend on  $x(t)$  and the logical and finite dynamics  $q(t)$ .  $u(t) \in \mathcal{R}^m$  is the controlled vector.  $w(t)$  is the exogenous noise. The notation  $t^-$  indicates that the finite state is piecewise continuous from the right. Let  $[x^T, i]$  be the vector combined with the continuous state  $x$  and discrete dynamic  $i$ . Thus starting at  $[x_0, i]$ , the continuous state trajectory  $x(t)$  evolves according to (2.17) and (2.18). If  $x(t)$  hits some  $g(\cdot) = j \neq i$  at time  $t_1$ , then the state becomes  $[x(t_1), j]$ , from which the process continues. Thus, the hybrid system can be represented as a PWA system by linearization [135]:

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_{i1} w(t) + B_{i2} u(t) + a_i \\ y(t) &= C_i x(t) + D_{i1} w(t) + D_{i2} u(t) \end{aligned}, \quad i = q(t) \quad (2.19)$$

In terms of a hybrid model, the PWA system can be described as autonomous switching between a set of PWA partitions.

**Example 2.2.5 (Fuzzy systems)** Fuzzy systems have been applied in many fields, such as control systems, communication systems and signal processing. Johansson [105] and Feng [59] present different PWA analytical models for continuous-time

*Takagi-Sugeno (T-S) fuzzy systems. The corresponding discrete-time versions are considered in [167] and [60], respectively.*

*A typical example using fuzzy systems is an inverted pendulum on a cart [59], which is an abstract model in many applications, such as missile control and robotics control. The system model is as follows.*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g \sin(x_1) - amlx_2^2 \sin(2x_1)/2 - a \cos(x_1)u}{4l/3 - aml \cos^2(x_1)} \end{aligned} \quad (2.20)$$

*where  $x_1$  denotes the angle of the pendulum from the vertical axis, and  $x_2$  is the angular velocity.  $g = 9.8\text{m/s}^2$  is the gravity constant,  $m$  is the mass of the pendulum,  $a = 1/(n + M)$ ,  $M$  is the mass the cart,  $2l$  is the length of the pendulum, and  $u$  is the force applied to the cart. In the simulation, the pendulum parameters are chosen as  $m = 2\text{kg}$ ,  $M = 8\text{kg}$ , and  $2l = 1.0\text{m}$  [59].*

*The fuzzy model is obtained by linearizing the plant around the origin and  $x = [\pm 88^\circ \ 0]^T$  and has the form of (9.5), where  $L_1^i \in \{\text{about } 0, \text{ about } +88^\circ, \text{ about } -88^\circ\}$ , and  $m = 3$ , and*

$$\begin{aligned} A_1^{(c)} &= \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix}, A_2^{(c)} = A_3^{(c)} = \begin{bmatrix} 0 & 1 \\ 0.2395 & 0 \end{bmatrix}, a_1^{(c)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ a_2^{(c)} = -a_3^{(c)} &= \begin{bmatrix} 0 \\ -0.3675 \end{bmatrix}, B_1^{(c)} = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix}, B_2^{(c)} = B_3^{(c)} = \begin{bmatrix} 0 \\ -0.0052 \end{bmatrix} \end{aligned}$$

*The common membership function shown in Figure 2.7 is chosen as  $\mu_i = \frac{h_i}{\sum_1^3 h_i}$ ,  $i = 1, 2, 3$ , where*

$$\begin{aligned} h_1 &= \exp(-7x_1^2) \\ h_2 &= \exp(-7(x_1 + \pi/2)^2) \\ h_3 &= \exp(-7(x_1 - \pi/2)^2) \end{aligned} \quad (2.21)$$

*We consider the working range of state  $x_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  with the following partitions:*

$$S_1 = \left\{ x | x_1 \in \left[ \frac{-\pi}{2}, \frac{-\pi}{4} \right) \right\}, S_2 = \left\{ x | x_1 \in \left[ \frac{-\pi}{4}, 0 \right) \right\},$$

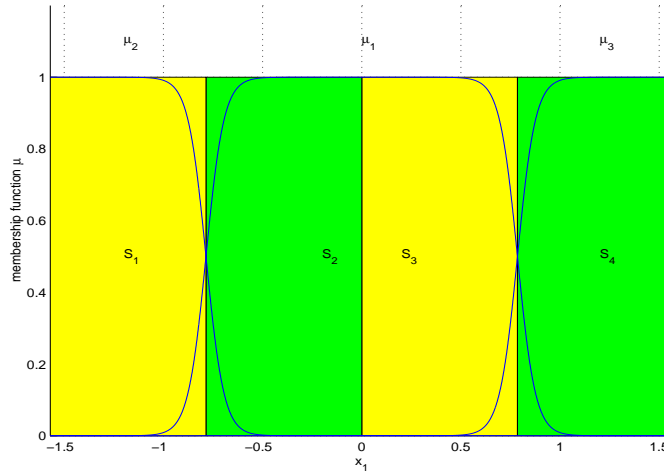


Figure 2.7: Membership functions of Example 2.2.5

$$S_3 = \left\{ x | x_1 \in \left[ 0, \frac{\pi}{4} \right) \right\}, \quad S_4 = \left\{ x | x_1 \in \left[ \frac{\pi}{4}, \frac{\pi}{2} \right] \right\}$$

Then the system parameters are as follows.

$$A_1 = A_3^{(c)}, \quad A_2 = A_3 = A_1^{(c)}, \quad A_4 = A_2^{(2)}, \quad a_1 = a_3^{(c)}, \quad a_4 = a_2^{(c)}$$

$$B_1 = B_3^{(c)}, \quad B_2 = B_3 = B_1^{(c)}, \quad B_4 = B_2^{(2)}$$

We will discuss fuzzy systems further in Chapter 9.

As for discrete-time examples, a typical instance is interconnection of discrete-time linear systems and automata [155]. They can also be often found in digital control of hybrid systems. Recently, some researchers also apply discrete-time PWA models to switched power converters [119], direct torque control of three phase induction motors [70], automotive robotized gearbox [12] and system biology [50].

## 2.3 Structural Analysis

This section recalls some basic concepts for analysis of PWA systems. As aforementioned, PWA systems can have very complex behavior, even for those with low

order. Multiple equilibrium points [135, 102], multi limit cycles [42, 135] and chaos [115] may exist. Hence, structural analysis shall be given to rule out degeneracies in PWA systems, providing critical engineering insight. Meanwhile, it would be a valuable complement for the analysis of other system properties, such as stability and controllability, and synthesis of PWA systems. In [102], equilibrium points, steady-state characteristic, verification of affine state constraints and detection of attractive sliding modes for continuous-time PWA systems are discussed. Here we recap some of important contents.

### 2.3.1 Trajectory, Sliding Modes and Well-posedness

For continuous-time PWA systems, if the right-hand side of the trajectory is discontinuous, it is possible that trajectory that enters a switching boundary cannot be continued into any of the neighboring regions, or that trajectory can be continued into several regions [102]. Hence Jahasson [102] presents the following definition in order to restrict the study to the case when the non-smooth dynamic does not lead to the above problems.

**Definition 2.3.1** (*Trajectory of CPS*)[102] *Let  $x(t) \in \bigcup_{i \in \mathcal{I}} S_i$  be an absolutely continuous function.  $X(t)$  is a trajectory of the continuous-time system (2.4) on  $[t_0, t_f]$  if, for almost all  $t \in [t_0, t_f]$ , the equation  $\dot{x}(t) = A_i x(t) + B_i u(t) + a_i$  holds for all  $i$  with  $x(t) \in S_i$ .*

However, as illustrated in [102], in the sense of Definition 2.3.1, trajectory might be non-unique or even not exist. In the view of analysis, the main difficulty lies in the cases when non continuation of a trajectory in the sense of Definition 2.3.1 is possible. Hence the so-called attractive sliding mode<sup>4</sup> is defined [67, 164, 102],

---

<sup>4</sup>Note that the trajectory cannot stay on the boundary  $S_{ij}$  unless it satisfies the differential equation for both  $i$  and  $j$  simultaneously. Hence most kinds of sliding modes are excluded [102]. See [102, 67] for illustrated examples. However, in this thesis, we further assume that only one mode is active at any time, if there is no explanation for sliding mode. More strictly, we can assume that the systems satisfy the well-posedness conditions [95], if necessary.

which states a phenomena for the CPS (2.4) at  $x_s$ , if there exists a trajectory with final state  $x_s$  but no trajectory with initial state. Nevertheless, in this thesis, we mainly deal with systems without sliding modes<sup>5</sup>, although the definition of meaningful solution concepts is possible<sup>6</sup>. Filippov [67] and Utkin [164] deliver their well-known solution concepts for the systems with attractive sliding mode when the function  $x(t)$  remains on the switching boundaries for some time interval.

**Definition 2.3.2** (*Filippov solution of CPS*)[67, 164, 102] *Let  $x(t) \in \bigcup_{i \in \mathcal{I}} S_i$  be an absolutely continuous function.  $x(t)$  is a Filippov solution of the continuous-time system (2.4) on  $[t_0, t_f]$  if, for almost all  $t \in [t_0, t_f]$ ,*

$$\dot{x}(t) = \mathbf{Co}_{i \in \mathcal{K}(t)} \{A_i x(t) + B_i u(t) + a_i\} \quad (2.22)$$

where  $\mathcal{K}(t)$  is the set of partition indices  $i$  such that  $x(t) \in S_i$ , and  $\mathbf{Co}$  denotes the convex hull.

Contrary to the non-uniqueness, it is also natural if the conditions exist such that the system produces unique trajectories from every initial state, i.e., the well-posedness property. The problem is partially solved by Imura *et al.* [95, 92, 93]. In [95], Imura and Var der Schaft discuss this problem under the definition of solution of Carathéodory. The necessary and sufficient condition for bimodal PWA systems is provided. Its extensions to multi-mode cases are also addressed. In [93], the well-posedness of switch-driven PWA systems is analyzed. There are also some results on particular classes of PWA systems, say, linear complementarity systems [84].

For DPS, in the sense of well-posedness, we make the following assumption [55]: the system dynamic is governed by the dynamic of the local mode of  $S_i$  when the dynamic jumps from  $S_i$  to  $S_j$ . Hence the unique solution of (2.4) is possible.

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<sup>5</sup>In most cases, we assume there is only one active mode at any time, if there is no further explanation.

<sup>6</sup>In [102], Johansson shows that even the silding mode occurs, the stability still can be guaranteed with their stability criteria. In [108], Juloski *et.al.* give a similar result for observer design. In [136], Pettersson provides conditions to check the silding mode, and also design controllers to avoid silding mode.

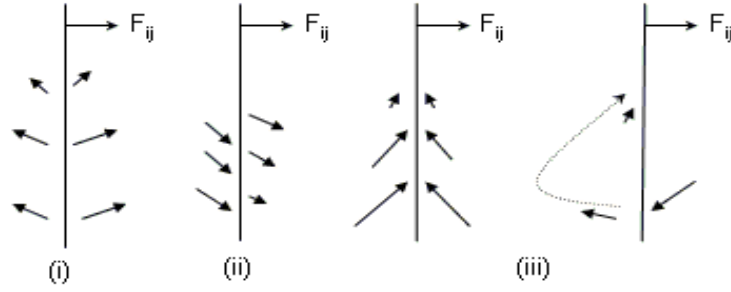


Figure 2.8: Several cases on the vector fields

### 2.3.2 Switching Sequences

We consider the CPS first. For the possible switching between two different partitions, there are three situations as illustrated in Figure 2.8: (i) no jump between two parts; (ii) jump only from one side to another side; (iii) jump from both sides. Define  $F_{ij}x + f_{ij} = 0$  for  $x \in S_{ij}$ . Then the situation (i) can be expressed as  $F_{ij}^T(A_i x + a_i) > 0$  and  $F_{ij}^T(A_j x + a_j) < 0$  for  $\forall x \in S_{ij}$ . Similarly, the situation (ii) is  $F_{ij}^T(A_i x + a_i) < 0$  and  $F_{ij}^T(A_j x + a_j) < 0$  for  $\forall x \in S_{ij}$ . For the case that  $S_{ij}$  are polytopic, we only need to check the vertices of  $S_{ij}$ . However, the inspection procedure is not so easy. One may also refer to [102, 80, 79] for further information.

Now we take a look at DPS, whose switching is much more complex than that of CPS, since the switching is not limited to the neighboring partitions. In [14], Bemporad *et al.* provide a numerical algorithm to calculate it using the MDL model. We may also consider this problem for polytope partitions in the context of the so-called one-step reachable set  $\vec{\mathcal{S}}_i$  [114]. By checking whether  $\vec{\mathcal{S}}_i \cap S_j$ ,  $\forall j \neq i, j \in \mathcal{I}$ , is empty, we may get the switching index set  $\Omega_d$ .

## 2.4 Conclusion

In this chapter, we have discussed the definition, model representation, structure and applications of PWA systems. One of the main purposes is to distinguish

PWA systems from other types of switching systems. From our aforementioned discussion in this chapter, we can see that PWA systems belong to the third type of switching systems. Furthermore, their state-dependent switching partition is preset and not a design objective.

# Chapter 3

## Stability Conditions Based on PQLFs

### 3.1 Introduction

In this chapter, we will consider the stability of autonomous (unforced) PWA system (2.4), i.e.,

$$\delta x(t) = A_i x(t) + a_i, \quad x \in S_i, \quad i \in \mathcal{I} \quad (3.1)$$

Note that the stability is one of the most important issues in the control system theory and also the fundamental for the design problems. We mainly focus on the so-called asymptotic stability and exponential stability, though there are some other concepts, such as absolute stability and practical stability [77]. We will review some existing results, including several types of indirect Lyapunov approaches. However, we put our emphasis on PQLFs and polynomial Lyapunov functions, because these functions give better analytical results than others. Furthermore, the PQLFs are the main tools for the controller and filter design in Chapter 7 and Chapter 8, respectively. The polynomial Lyapunov functions are the main tools for the improved stability criteria in Chapter 4 and Chapter 5. We

will also present a unified framework for the approach based on PQLFs in Section 3.2.2, which will show its advantages in the design late.

Unlike the LTI system, where the Hurwitz matrix (respectively, Schur matrix) guarantees the stability for the continuous-time case (respectively, discrete-time case), the situations in the PWA systems are very complex. Some simple examples in [48, 102] show that the eigenvalues constraint (Hurwitz or Schur condition) of the system matrix  $A_i$  may be neither sufficient nor necessary for its stability. In fact, the stability of PWA systems, as well as other kinds of switching systems, depends not only on the dynamics of each subsystems but also the properties of switching signals, as illustrated in Example 3.1.1. Furthermore, Blondel *et al.* [20] show that the globally asymptotical stability for general discrete-time PWA systems with  $n \geq 2$  is undecidable. The results in [154, 155] also imply that the problem of trajectory going to the origin is undecidable.

**Example 3.1.1** Consider the following two piecewise linear systems [102]:

$$\dot{x} = \begin{cases} \begin{bmatrix} -0.1 & 1 \\ -10 & -0.1 \end{bmatrix} & x_1 x_2 \geq 0 \\ \begin{bmatrix} -0.1 & 10 \\ -1 & -0.1 \end{bmatrix} & x_1 x_2 \leq 0 \end{cases} \quad (3.2)$$

$$\dot{x} = \begin{cases} \begin{bmatrix} -0.1 & 1 \\ -10 & -0.1 \end{bmatrix} & x_1 x_2 \leq 0 \\ \begin{bmatrix} -0.1 & 10 \\ -1 & -0.1 \end{bmatrix} & x_1 x_2 \geq 0 \end{cases} \quad (3.3)$$

The corresponding partitions  $S_i, i = 1, 2, 3, 4$  are four quadrants. We assume the system trajectory is continuous from right side. It is obvious that each system matrices are stable and have the same eigenvalue  $-0.1 \pm 3.1623i$ . Note that two systems are same except that the system matrices are exchanged in different modes. The phase portraits of two systems are shown in Figure 3.1, from which, we may

easily find the system trajectories transits are  $\dots \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \rightarrow S_1 \rightarrow \dots$ .

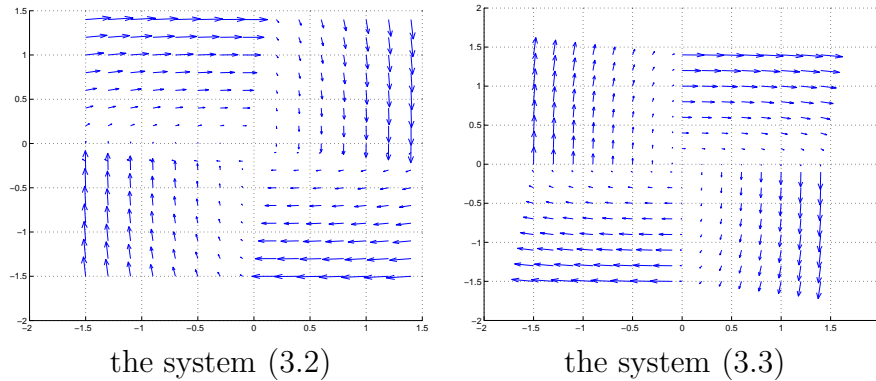


Figure 3.1: Phase portrait

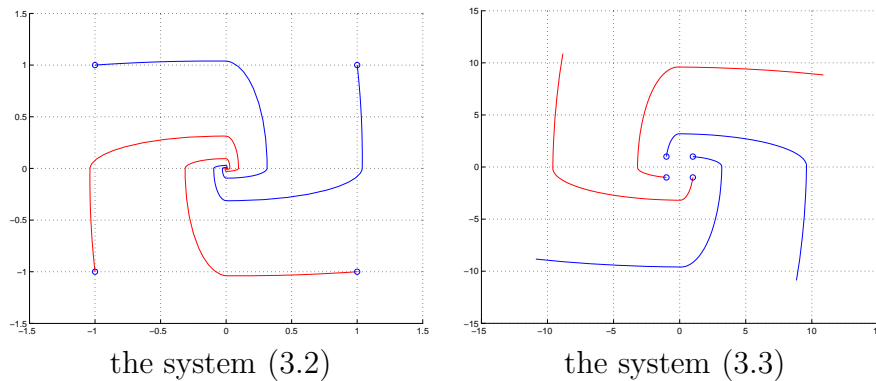


Figure 3.2: System trajectories

However, the system (3.2) is stable while the system (3.3) is unstable. (The proof will be given in next section.) Figure 3.2 shows some trajectories of two systems from initial conditions  $[\pm 1 \pm 1]$ .

Nevertheless, for some special cases, such as planar systems and bimodal systems, some criteria can be obtained based on some geometrical properties. Iwatani and Hara [96] provide two explicitly necessary and sufficient stability conditions for continuous-time planar bimodal PWL systems. These two conditions are given in terms of eigenvalue loci of subsystems and coefficients of characteristic polynomials, respectively. Further they discuss the stability conditions for planar multi-model PWL systems in [97], where the conditions are given in terms of poles and zeros

of subsystems, and they are computational tractable. Hu and Lin [89, 88] provide necessary and sufficient conditions for the globally asymptotical stability of planar saturation systems. The behaviors of the trajectories are examined in detail. In general, those analytical approaches are difficult or impossible to extend to general cases, although they deliver good performance or explicit conditions in some settings.

Hence, most of stability conditions for general PWA systems are only sufficient and gain benefits from the computational efforts of modern computers. One such approach is based on simulation. For example, Kantner and Dolye [113, 112] show that robust simulation which maps sets to sets may be an alternative approach. However, the most widely used approaches benefit from Lyapunov theory [115], especially, the Lyapunov indirect method.

Lyapunov indirect method provides a powerful tool to study stability. Roughly speaking, the stability established by Lyapunov functions can be called as Lyapunov stability. The following lemma gives a criterion based on Lyapunov function for asymptotic stability and exponential stability.

**Lemma 3.1.1** [115, 102] (*Asymptotic stability and exponential stability*) *Let  $x(t) : [0, \infty) \rightarrow R^n$  and let  $V(t) : [0, \infty) \rightarrow R$  be a nonincreasing and piecewise  $C^1$  function satisfying*

$$\frac{d}{dt}V(t) \leq -\gamma\|x(t)\|^p \quad (3.4)$$

*for some  $\gamma > 0$  and some  $p > 0$ , almost everywhere on  $[0, \infty)$ . If there exist  $\alpha > 0$  and  $\beta > 0$  such that*

$$\alpha\|x(t)\|^p \leq V(t) \leq \beta\|x(t)\|^p \quad (3.5)$$

*then  $\|x(t)\|$  tends to zero exponentially.*

However, the main problem with Lyapunov theory is that there is no general method for searching Lyapunov functions [115], although many types of Lyapunov

functions are suggested to solve this problem, such as linear functions, quadratic functions, or even general polynomials. A direct approach is to find a common quadratic Lyapunov function (CQLF). In this approach, only one Lyapunov matrix is needed which covers whole state space. Its advantages lie in its simplicity and efficiency, although it is very conservative. Even now, there are some researchers still using CQLF for PWA systems, say, [148, 147, 134, 133].

Due to the piecewise nature of the PWA systems, the corresponding piecewise Lyapunov functions are applied to achieve better results. PWL/PWA Lyapunov functions [102, 106] are constructed to establish the stability. A so-called polytopic Lyapunov function [183, 184, 128] is a special case of the PWL Lyapunov functions. However, the most widely used Lyapunov function is the piecewise quadratic Lyapunov function (PQLF) [101, 25, 82, 102, 55, 64]. Johansson and Rantzer [104, 102] give a sufficient stability condition with piecewise quadratic Lyapunov functions via a computational approach for continuous-time PWA systems. They systematically study the PQLF construction and ways to use partition information to relax conservatism in stability analysis. Similar ideas can be found in [82, 101]. The discrete-time counterpart has been studied by Feng [55] and Ferrari-Trecate *et al.* [64]. We will discuss PQLF and review some main results in the next sections, since PQLF is the one of main techniques used in our controller and estimator design.

There are some other approaches, such as bi-quadratic functions [10] and global surface functions [74]. Bean *et al.* [10] convert the PWA system with polytope partitions into a parameterized system in terms of some logical variables. Then by introducing some additional variables  $\pi$ , which are parallel to system dynamical state  $x$ , they solve the stability problem for the augmented state  $[\pi^T \ x^T]$ . The benefit is more freedom brought into LMI solver, while at the expense of more intensive computation. In addition, the construction of the auxiliary matrices is a burdensome job. Gonçalves [74] presents a constructive global analysis methodology, which infers global properties of PWA systems solely by studying the behavior

at switching surfaces associated. The main idea is to analyze impact maps, i.e., maps from one switching surface to the next switching surface. They mainly analyze the relay systems and saturation systems. This method actually is not easy to be used for analyzing the general PWA systems. In [101], Johansen presents a computational approach, which guarantees the uniform exponential stability, based on a gridding technique using a linear inequality defined pointwise.

The aforesaid Lyapunov functions actually are all in (piecewise) linear or quadratic forms. A promising technique based on higher order positive polynomials, or precisely, sums of squares (SOS) polynomials [99, 38, 39, 40], which also use SDP as their solver, becomes increasingly attractive. Hence, the corresponding piecewise SOS Lyapunov functions are used in PWA systems [139, 172, 99, 98, 11]. Nevertheless, the main idea is similar to PQLF, i.e., to find proper local Lyapunov functions for each partition. Using this technique, the methods for the computation of piecewise polynomial Lyapunov functions are exploited for analyzing stability of continuous-time hybrid and switched systems [139, 172]. Gonçalves *et al.* also extend their work [157] with the SOS method to construct polynomial surface Lyapunov functions of arbitrary order for the impact maps of limit cycles in PWA systems. The discrete-time results are discussed in [130, 11, 172]. In Chapter 4, we will consider how to use SOS polynomial to explicitly construct piecewise polynomial Lyapunov functions for both CPS and DPS. Meanwhile, we note that the existing results actually all use the partition information from halfspace representation (see Definition 2.2.1), is it possible to derive stability conditions using vertex representation? We will explore the vertex representation in Chapter 5.

## 3.2 Piecewise Quadratic Lyapunov Functions

The key ideas applied here are ambilateral. Firstly, quadratic Lyapunov functions are defined for each partition. Then they are combined to form a global Lyapunov

function. For CPS, we choose the Lyapunov function as

$$v(x) \triangleq \sum_{i=1}^{|\mathcal{I}|} \tau_i v_i(x), \quad \tau_i = \begin{cases} 1 & \text{mode } i \text{ is active} \\ 0 & \text{otherwise} \end{cases}, \quad v_i(x) = x^T P_i x, \quad P_i = P_i^T, \quad (3.6)$$

where  $|\mathcal{I}|$  is the size of  $\mathcal{I}$ , satisfying

$$v_i(x) > 0, \quad \forall x \in S_i, \quad \forall i \in \mathcal{I}, \quad x \neq 0; \quad (3.7)$$

$$v_i(x) \geq v_j(x), \quad \forall x \in S_{ij}, \quad (i, j) \in \tilde{\Omega}_c; \quad (3.8)$$

$$\dot{v}_i(x) < 0, \quad \forall x \in S_i, \quad i \in \mathcal{I}, \quad x \neq 0 \quad (3.9)$$

**Remark 3.2.1** Note that (3.7) guarantees the positiveness of the Lyapunov function. (3.9) means that the Lyapunov function is decreasing in  $S_i$ . (3.8) makes the Lyapunov function trajectory is non-increasing during mode switching. If “=” is chosen, then the Lyapunov function is continuous; otherwise, the Lyapunov function is discontinuous. (3.7) and (3.9) are not sufficient to guarantee the stability of PWA systems as we state early that the stability is partition-dependent (see [102] for examples). In fact, The functions derived from (3.7) and (3.9) may not absorb the energy increase caused by the mode switching, which leads to the unboundedness of trajectory.

For DPS, we choose the Lyapunov function of form (3.6) satisfying (3.7) and

$$\Delta v(x) = v_i(x(t+1)) - v_j(x(t)) < 0, \quad \forall (i, j) \in \Omega_d \quad (3.10)$$

Secondly, the *S-procedure* is employed to force (3.7)-(3.9) in a local region by outer elliptically approximating the polyhedra partition.

A detail proof procedure for continuous-time case with continuous Lyapunov functions can be found in [102] and discontinuous Lyapunov functions in [182].

**Remark 3.2.2** For CPS, if  $\tilde{\Omega}_{ij}$  is predetermined (see Section 2.3.2), then the continuity of Lyapunov function at the switching boundary is not necessary. At

time  $t$ , if  $x(t) \in S_{ij}$ , according to Definition 2.3.1 and inequalities (3.8) and (3.9), we have

$$v_i(x(t_-)) > v_i(x(t)) \geq v_j(x(t)) > v_j(x(t_+)) \quad (3.11)$$

where  $t_-$  denotes the time sequence just before  $t$  and  $t_+$  the time sequence just after  $t$ . Hence whatever the switching rule is, we guarantee the Lyapunov function is decreasing during the switching. For example, as defined in (2.6), if the switching rule is continuous from the right side, then  $v_i(x(t^-)) > v_i(x(t)) \geq v_j(x(t))$ .

Note that (3.8) is only required for  $x \in S_{ij}$ , that is, only at the switching boundary. Thus, it will not lead to  $v_i(x) = v_j(x)$  in general.

### 3.2.1 Stability Criteria

The stability result for the continuous-time PWA systems is stated as follows.

**Theorem 3.2.1** (CPS [102, 83]) *The system (3.1) is exponentially stable, if there exist some  $(T = T^T, P_i = \vec{F}_i^T T \vec{F}_i, \bar{P}_i = \vec{\bar{F}}_i^T T \vec{\bar{F}}_i, U_i = U_i^T \succeq 0, W_i = W_i^T \succeq 0)$ , where  $i \in \mathcal{I}$ , satisfying*

$$P_i - F_i^T W_i F_i > 0, \quad \forall i \in \mathcal{I}_0 \quad (3.12)$$

$$A_i^T P_i + P_i A_i + F_i^T U_i F_i < 0, \quad \forall i \in \mathcal{I}_0 \quad (3.13)$$

$$\bar{P}_i - \bar{F}_i^T W_i \bar{F}_i > 0, \quad \forall i \in \mathcal{I}_1 \quad (3.14)$$

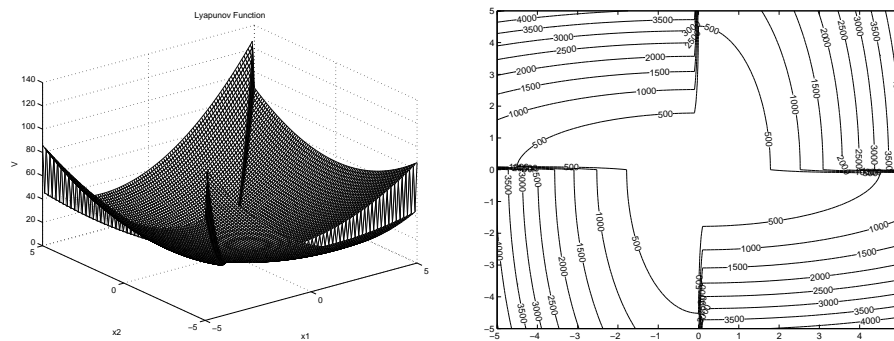
$$\bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{F}_i^T U_i \bar{F}_i < 0, \quad \forall i \in \mathcal{I}_1 \quad (3.15)$$

where the matrix  $\vec{F}_i$  and  $\vec{\bar{F}}_i$  are the continuity matrices<sup>1</sup> for  $F_i$  and  $\bar{F}_i$ , respectively.

**Remark 3.2.3** *In Theorem 3.2.1, an outer approximation is obtained by finding a matrix  $U_i = U_i^T \succeq 0$ , such that  $S_i \subseteq \{x | x^T F_i^T U_i F_i x \geq 0\}$ . The formulation*

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<sup>1</sup>See Remark 2.2.2 or Reference [102]. Here we assume that  $f_i = 0$  for  $i \in \mathcal{I}_0$ . Otherwise, we shall construct polyhedral cell bounding  $E_i$  of  $F_i$  to replace  $F_i$  in (3.12) and (3.13).



the value of Lyapunov function      the contour of Lyapunov function

Figure 3.3: The Lyapunov function of the system (3.2)

$P_i = \vec{F}_i^T T \vec{F}_i$  guarantees the continuity at the switching boundary. In fact, the continuity at the switching boundary is not necessary as stated in (3.8). A less conservative version of Theorem 3.2.1 [102] eliminates the continuity constraint by replacing  $P_i = \vec{F}_i^T T \vec{F}_i$  with <sup>2</sup>

$$P_i - P_j + F_{ij}^T T_{ij}^T + T_{ij} F_{ij} \geq 0, \quad \forall (i, j) \in \tilde{\Omega}_c \quad (3.16)$$

where  $T_{ij}$  is a free variable. (3.16) actually is deduced by Finsler's lemma (see Appendix A) by noting that  $F_{ij}x = 0$  when  $x \in S_{ij}$ .

**Example 3.2.1** Using the main idea of Theorem 3.2.1 and Remark 3.2.3, we can prove the stability of the system (3.2) by establishing the PQLF with following Lyapunov matrices:

$$P_1 = P_3 = \begin{bmatrix} 156.877 & 0.185236 \\ 0.185236 & 24.4707 \end{bmatrix}, \quad P_2 = P_4 = \begin{bmatrix} 24.4707 & -0.185236 \\ -0.185236 & 156.877 \end{bmatrix}$$

The corresponding Lyapunov function and its contour are shown in Figure 3.3. Figure 3.4 shows some trajectories of Lyapunov functions. Note that we cannot find a corresponding PQLF for the system (3.3).

The corresponding stability result for the discrete-time PWA systems is stated as follows.

<sup>2</sup>If  $S_{ij}$  contains inequality information, then the information can be further employed here by *S-procedure*.

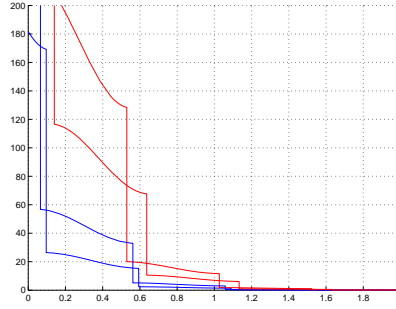


Figure 3.4: Some trajectories of the Lyapunov function of the system (3.2)

**Theorem 3.2.2** (DPS [55, 64]) *The system (3.1) is exponentially stable, if there exist some  $(P_i = P_i^T, \bar{P}_i = \bar{P}_i^T, W_i \succeq 0, U_{ij} \succeq 0)$  satisfying (3.12), (3.14) and*

$$A_i^T P_j A_i - P_i + F_i^T U_{ij} F_i < 0, \forall (i, j) \in \Omega_d, i \text{ and } j \in \mathcal{I}_0 \quad (3.17)$$

$$\bar{A}_i^T \bar{P}_j \bar{A}_i - \bar{P}_i + \bar{F}_i^T U_{ij} \bar{F}_i < 0, \forall (i, j) \in \Omega_d, i \text{ and/or } j \in \mathcal{I}_1 \quad (3.18)$$

where  $\bar{P}_i = [I \ 0]^T P_i [I \ 0]$  if  $i \in I_0$  and  $\bar{P}_j = [I \ 0]^T P_j [I \ 0]$  if  $j \in I_0$  in (3.18).

**Remark 3.2.4** *In Theorem 3.2.2, Feng [55] assumes that the system dynamic is governed by the dynamics of the local model  $S_i$  when the state of the system transits from the region  $S_i$  to  $S_j$ . If it is not the case, we may consider the following set*

$$\tilde{S}_{ij} = \{x \in \mathcal{R}^n | x(t) \in S_i, x(t+1) \in S_j\} \triangleq \{x \in \mathcal{R}^n | F_{ij}x + f_{ij} \geq 0\} \quad (3.19)$$

Then (3.17) can be rewritten as

$$A_i^T P_i A_i - P_i + F_i^T U_i F_i < 0, \forall i \in \mathcal{I}_0 \quad (3.20)$$

$$A_i^T P_j A_i - P_i + F_{ij}^T U_{ij} F_{ij} < 0, \forall (i, j) \in \Omega_d, i \text{ and } j \in \mathcal{I}_0 \quad (3.21)$$

(3.18) can be rewritten similarly.

**Remark 3.2.5** *If all Lyapunov matrices are forced to have global properties without looking into the partition information, i.e., all  $U_i, W_i, W_{ij}$  in Theorem 3.2.1*

and Theorem 3.2.2 are set to zeros, we can get compact but more conservative results, which have been illustrated in [102, 64], say, for example, (3.12) and (3.17) can be simplified as

$$\begin{cases} P_i > 0, & \forall i \in \mathcal{I}_0 \\ A_i^T P_j A_i - P_i < 0, & \forall (i, j) \in \Omega_d, i \text{ and } j \in \mathcal{I}_0 \end{cases} \quad (3.22)$$

### 3.2.2 A Unified Presentation

In this subsection, we show that the results of Theorem 3.2.1 and Theorem 3.2.2 can be obtained under a unified framework by a proper transformation. We only consider the case that all  $i \in \mathcal{I}_0$ , since the rest cases are similar.

$$p \triangleq \begin{bmatrix} x(t) \\ \delta x(t) \end{bmatrix}, \mathcal{P}_{ij} = \begin{cases} (CPS) : \begin{bmatrix} F_i^T U_i F_i & P_i \\ P_i & 0 \end{bmatrix} \\ (DPS) : \begin{bmatrix} F_i^T U_{ij} F_i - P_i & 0 \\ 0 & P_j \end{bmatrix} \end{cases}, \mathcal{A}_i = \begin{bmatrix} A_i & -I \end{bmatrix}$$

It is obvious that  $\mathcal{A}_i p = 0$  according to (3.1). And Theorem 3.2.2 and Theorem 3.2.2 can be unified as follows.

**Theorem 3.2.3** *The system (3.1) is exponentially stable, if there exists some  $(P_i = P_i^T, U_i \succeq 0, W_i \succeq 0, U_{ij} \succeq 0)$  such that (3.12) and*

$$p^T \mathcal{P}_{ij} p < 0, \forall \mathcal{A}_i p = 0 \quad (3.23)$$

hold, where for CPS,  $P_i = \vec{F}_i^T T \vec{F}_i$  and  $\forall i = j \in \mathcal{I}$ , and for DPS,  $\forall (i, j) \in \Omega_d$ .

**Lemma 3.2.4** *(DPS) (3.17) of Theorem 3.2.2 is equivalent to either one of the following inequalities*

$$\Phi_{ij}^{(1)}(A_i, P_i, U_{ij}) \triangleq \begin{bmatrix} F_i^T U_{ij} F_i - P_i - A_i^T A_i & A_i^T \\ A_i & P_j - I \end{bmatrix} < 0 \quad (3.24)$$

$$\Phi_{ij}^{(2)}(A_i, P_i, U_{ij}, \Upsilon_{ij}, \Psi_j) \triangleq \begin{bmatrix} -P_i + \mathcal{W}_{ij} & -\Upsilon_{ij}^T + A_i^T \Psi_j \\ -\Upsilon_{ij} + \Psi_j^T A_i & P_j - (\Psi_j + \Psi_j^T) \end{bmatrix} < 0 \quad (3.25)$$

for some  $(P_i = P_i^T, U_{ij} \succeq 0, \Upsilon_{ij}, \Psi_j)$  and  $\forall (i, j) \in \Omega_d$ , where  $\mathcal{W}_{ij} = F_i^T U_{ij} F_i + A_i^T \Upsilon_{ij} + \Upsilon_{ij}^T A_i$ .

**Proof:** Using the result from Theorem 3.2.3 and applying Finsler's lemma (see Appendix A), we can conclude that (3.17) is equivalent to either one of

$$\mathcal{P}_{ij} - \mu \mathcal{A}_i^T \mathcal{A}_i < 0, \text{ for some scalar } \mu \quad (3.26)$$

$$\mathcal{P}_{ij} + \mathcal{X}_{ij}^T \mathcal{A}_i + \mathcal{A}_i^T \mathcal{X}_{ij} < 0, \text{ for some matrix } \mathcal{X}_{ij} \quad (3.27)$$

where  $\mathcal{P}_{ij}$  is defined in (3.23). Now (3.26) leads to

$$\begin{bmatrix} F_i^T U_{ij} F_i - P_i - \mu A_i^T A_i & \mu A_i^T \\ \mu A_i & P_j - \mu I \end{bmatrix} < 0 \quad (3.28)$$

Since (3.26) only stands for  $\mu > 0$ , we can let  $P_i = \frac{1}{\mu} P_i$  and  $U_{ij} = \frac{1}{\mu} U_{ij}$ . Hence we have (3.24).

Now we expand (3.27) and observe that the entries of left side matrix of (3.27) are only related to the index  $j$  except for  $-P_i + F_i^T W_{ij} F_i$ , hence we may choose  $\mathcal{X}_{ij} = [\Upsilon_{ij} \ \Psi_j]$ . Thus the result follows. Alternatively, we give another proof for (3.25).

$$(3.25) \implies (3.17): \text{ Multiplying (3.25) from the left and the right by } \Theta_i^T = \begin{bmatrix} I & A_i^T \\ 0 & I \end{bmatrix}$$

and  $\Theta_i$ , respectively, we obtain

$$\begin{bmatrix} A_i^T P_j A_i - P_i + F_i^T U_{ij} F_i & -\Upsilon_{ij}^T + A_i^T P_j - A_i^T \Psi_j \\ * & P_j - (\Psi_j + \Psi_j^T) \end{bmatrix} < 0 \quad (3.29)$$

whose (1,1)-block implies (3.17).

(3.17)  $\implies$  (3.25): Given  $P_j$  that satisfies (3.12), let  $\Psi_j = \frac{1}{2}(P_j + I)$ ,  $\Upsilon_{ij}^T = A_i^T P_j - A_i^T \Psi_j$ . By substituting them into (3.29), we have:

$$\begin{bmatrix} A_i^T P_j A_i - P_i + F_i^T U_{ij} F_i & 0 \\ 0 & -I \end{bmatrix} < 0 \quad (3.30)$$

Noting that  $\Theta_i$  is invertible, we get the result.  $\blacksquare$

Now we consider a simplified version of Theorem 3.2.2 as stated in Remark 3.2.5 and have the following lemma.

**Lemma 3.2.5** (DPS) [170, 46] *The following statement holds for  $P_i = P_i^T$ ,  $Q_i = P_i^{-1}$ ,  $G_i$ ,  $H_i$ ,  $i \in \mathcal{I}$  and  $(i, j) \in \Omega_d$ :*

$$\begin{aligned} (3.22) &\iff \begin{bmatrix} P_j & P_j A_i \\ A_i^T P_j & P_i \end{bmatrix} > 0 \iff \begin{bmatrix} -P_i & A_i^T G_j \\ G_j^T A_i & P_j - G_j - G_j^T \end{bmatrix} < 0 \\ &\iff \begin{bmatrix} P_j^{-1} & A_i \\ A_i^T & P_i \end{bmatrix} > 0 \iff \begin{bmatrix} -Q_j & A_i Q_i \\ Q_i A_i^T & -Q_i \end{bmatrix} < 0 \\ &\iff \begin{bmatrix} -P_i + A_i^T H_j + H_j^T A_i & -H_j^T + A_i^T G_j \\ -H_j + G_j^T A_i & P_j - G_j - G_j^T \end{bmatrix} < 0 \end{aligned} \quad (3.31)$$

For CPS, we can also have a similar result to Lemma 3.2.4. The proof is omitted.

**Lemma 3.2.6** (CPS) (3.13) of Theorem 3.2.1 is equivalent to either one of the following inequalities

$$\Phi_i^{(3)}(A_i, P_i, U_i) \triangleq \begin{bmatrix} F_i^T U_i F_i - A_i^T A_i & P_i + A_i^T \\ P_i + A_i & -I \end{bmatrix} < 0 \quad (3.32)$$

$$\Phi_i^{(4)}(A_i, P_i, U_i, \Upsilon_i, \Psi_i) \triangleq \begin{bmatrix} \mathcal{W}_i & P_i - \Upsilon_i^T + A_i^T \Psi_i \\ P_i - \Upsilon_i + \Psi_i^T A_i & -(\Psi_i + \Psi_i^T) \end{bmatrix} < 0 \quad (3.33)$$

for some  $(P_i = P_i^T, U_i = U_i^T \succeq 0, \Upsilon_i, \Psi_j)$  and  $\forall (i, j) \in \Omega_c$ , where  $\mathcal{W}_i = F_i^T U_i F_i + A_i^T \Upsilon_i + \Upsilon_i^T A_i$ .

The above new results stated in Theorem 3.2.3, Lemma 3.2.4 and Lemma 3.2.6 are equivalent in stability analysis, however, useful in investigating the problem of controller and estimator design for PWA systems. In Chapter 7 and Chapter 8, we can see such a small step may alleviate the design conservatism a lot.

### **3.3 Conclusion**

This chapter has mainly presented some stability conditions based on PQLFs for both continuous-time and discrete-time PWA systems. A unified representation of those conditions has been provided for PWA systems. We note that these stability conditions are only sufficient for PWA systems, while for standard linear systems, there exist necessary and sufficient conditions for stability in terms of a Lyapunov inequality. This motivates our study for better stability criteria.

# Chapter 4

## Stability Conditions Based on SOS Polynomials

### 4.1 Introduction

Recently, the positive polynomial theory [143, 86, 159], especially the polynomial in sums of squares (SOS) form, has attracted much attention from the control community [86]. There are mainly two types of works using the SOS polynomials. One gives explicit conditions based on Gram matrix, which seems to be first proposed by Bockett [26], and then extended by Barkin and Zelentsovsky [9, 185], and further studied recently by Chesi *et al.* [38, 39, 40, 35, 37, 86, 98]. These explicit conditions are directly given in terms of LMI. The other is implicit conditions directly using positive polynomial formulation [132, 99, 131, 139]. Some toolboxes, such as SOSTOOLS [140] and GloptiPoly [87], are used to solve the problems. However, both explicit conditions and implicit conditions are derived from SOS theory [132]. In this chapter, we will mainly discuss explicit conditions in Gram matrices for both continuous-time and discrete-time PWA systems. For clarity, we first recall some definitions here.

**Definition 4.1.1** [38, 26, 98] (*Power transformation of degree  $m$* ) Consider

the vector  $x \in \mathcal{R}^n$ ,  $x = [x_1, \dots, x_n]^T$ . The power transformation of degree  $m$  is a nonlinear change of coordinates that forms a new vector  $x^{[m]}$  of all integer powered monomials of degree  $m$  from the vector  $x$ ,

$$\begin{aligned} x_l^{[m]} &= c_l x_1^{m_{l1}} x_2^{m_{l2}} \cdots x_n^{m_{ln}}, \quad m_{lj} \in \{0, 1, 2, \dots, m\}, \\ \sum_{j=1}^n m_{lj} &= m, \quad l = 1, \dots, d_{n,m}, \quad d_{n,m} = \binom{n+m-1}{m} \end{aligned} \quad (4.1)$$

Usually we take  $c_l = 1$ .

**Definition 4.1.2** [38, 26, 98] (**Homogeneous polynomial**) Let  $P_{n,m}$  be the set of all polynomials in  $n$  variables with degree  $m$ , i.e. all items in  $P_{n,m}$  are in the form of (4.1). If a polynomial  $f_{n,m} \in P_{n,m}$  and  $f_{n,m}(\lambda x) = \lambda^m f_{n,m}(x)$ , then  $f_{n,m}$  is a homogeneous polynomial. We define the set of all homogeneous polynomials in  $n$  variables with degree  $m$  as  $H_{n,m}$ . The square matricial representation (SMR) of  $f_{n,2m} \in H_{n,2m}$  is defined as  $f_{n,2m} = x^{[m]T} C_f x^{[m]}$ , where  $x^{[m]} \in \mathcal{R}^{d_{n,m}}$  is the base vector of homogeneous forms of degree  $m$  in  $x$ , and  $C_f = C_f^T \in \mathcal{R}^{d_{n,m}} \times \mathcal{R}^{d_{n,m}}$ .

**Remark 4.1.1** An important property of the SMR is that the matrix  $C_f$  is not unique. In fact, we can identify some identities of the form:  $x_i^{[m]} x_j^{[m]} - x_k^{[m]} x_h^{[m]} = 0$ ,  $i, j, k, h \in \{1, 2, \dots, d_{n,m}\}$ , which is easily translated into a symmetric matrix  $C_\ell^{(0)} \in \mathcal{R}^{d_{n,m}} \times \mathcal{R}^{d_{n,m}}$  with its  $(i,j)$  and  $(j,i)$  entries being 1 (if  $i=j$ , then  $(i,i)=2$ ) and  $(k,h)$  and  $(h,k)$  entries being -1, or some scalar multiple thereof, and all other entries being zeros. Note that we can have a total of  $L_{n,m}$  such identities, i.e.  $\ell = 1, 2, \dots, L_{n,m}$ , where  $L_{n,m} = \frac{1}{2} d_{n,m}(d_{n,m} + 1) - d_{n,2m}$  [38]. Thus  $f_{n,2m}$  can be represented as  $f_{n,2m} = x^{[m]T} (C_f + \sum_{\ell=1}^{L_{n,m}} \gamma_\ell C_\ell^{(0)}) x^{[m]}$ , where  $\gamma_\ell \in \mathcal{R}$  is a scalar.

For further use, we also define<sup>1</sup>

$$M_{i,j} \in \mathcal{R}^{d_{n,i} \times d_{n,j}}, \quad \Sigma_{l,n,m} = \sum_{i=l}^m d_{n,i}$$

---

<sup>1</sup>We note that the positive semi-definite matrix set  $\bar{\mathcal{X}}_{n,m}$  is first introduced by Jarvis-Wloszek in [98].

$$\hat{x}^{[m]} = \begin{bmatrix} x \\ x^{[2]} \\ \dots \\ x^{[m]} \end{bmatrix}, \quad \hat{M}_{[m]} = \begin{bmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,m} \\ M_{1,2}^T & M_{2,2} & \dots & M_{2,m} \\ \vdots & & \ddots & \vdots \\ M_{1,m}^T & M_{2,m}^T & \dots & M_{p,p} \end{bmatrix} \in \mathcal{R}^{\Sigma_{1,n,m} \times \Sigma_{1,n,m}} \quad (4.2)$$

$$\begin{aligned} \bar{\mathcal{X}}_{n,m} = \{ & X \geq 0 : x^{[m]T} X x^{[m]} = \sum_{i=1}^n \epsilon_i x_i^{2m}, X \in \mathcal{R}^{d_{n,m} \times d_{n,m}}, \\ & \epsilon_i \geq 0, \epsilon = [\epsilon_1, \dots, \epsilon_n] \neq 0 \} \end{aligned} \quad (4.3)$$

## 4.2 Useful Lemmas

Firstly, we recall a transformation from matrix  $A$  to an extended matrix  $A^{[m]}$ . This type of transformation connects  $Ax$  to homogeneous polynomials of high degree.

**Lemma 4.2.1** [26, 98] *Given  $A \in \mathcal{R}^{r \times n}$ ,  $x \in \mathcal{R}^n$  and  $m \in \mathcal{N}$ , then under the power transformation, there exists a  $A^{[m]} \in \mathcal{R}^{d_{r,m} \times d_{n,m}}$  such that  $(Ax)^{[m]} = A^{[m]}x^{[m]}$ ,  $\forall x \in \mathcal{R}^n$ .*

**Remark 4.2.1** *Note that the transformation in Lemma 4.2.1 under some constraint is unique, which means that one  $A$  is corresponding to an unique  $A^{[m]}$ . For example, we force the sequence of  $x_i^{[m]}$  to satisfy the lexicographic ordering:*

$$\begin{aligned} x^{[m]} = [ & x_1^m \quad x_1^{m-1}x_2 \quad x_1^{m-2}x_2^2 \quad x_1^{m-2}x_2x_3 \quad \dots; \\ & x_2^m \quad x_2^{m-1}x_3 \quad x_2^{m-2}x_3^2 \quad \dots; \quad \dots; \quad x_n^m ]^T \end{aligned}$$

*However, the reverse is not true.*

**Lemma 4.2.2** [26, 98] *Given  $A, B \in \mathcal{R}^{n \times n}$ ,  $A^{[m]}$  and  $B^{[m]}$  satisfy*

$$(AB)^{[m]} = A^{[m]}B^{[m]} \quad (4.4)$$

$$(A^q)^{[m]} = (A^{[m]})^q, \text{ with } q \text{ integer and } A^q \text{ well defined} \quad (4.5)$$

$$(A^T)^{[m]} = (A^{[m]})^T \quad (4.6)$$

**Lemma 4.2.3** *If  $Ax \geq 0$ , then  $A^{[m]}x^{[m]} \geq 0$ . If  $Ax = 0$ , then  $A^{[m]}x^{[m]} = 0$ .*

**Remark 4.2.2** *Note that the transformation  $A^{[m]}$  does not possess linear properties, i.e.,  $[\alpha A + \beta B]^{[m]} \neq \alpha A^{[m]} + \beta B^{[m]}$ .*

Secondly, we present the transformation from  $A$  to  $A_{[m]}$ . This transformation is widely used in uncertain linear systems, say, [38, 39]. It expands a differential equation to a homogeneous polynomial of high degree.

**Lemma 4.2.4** [185, 38] *Given  $A \in \mathcal{R}^{n \times n}$  and  $m \in \mathcal{N}$ , with the relation  $\dot{x}(t) = A(t)x(t)$ , then there exists a matrix  $A_{[m]}(t) \in \mathcal{R}^{d_{n,m} \times d_{n,m}}$  such that  $\frac{d}{dt}(x^{[m]}) = A_{[m]}(t)x(t)^{[m]}$ .*

**Lemma 4.2.5** [185, 38] *Letting  $\alpha, \beta \in \mathcal{R}$  and  $A, B \in \mathcal{R}^{n \times n}$ ,*

$$[\alpha A + \beta B]_{[m]} = \alpha A_{[m]} + \beta B_{[m]} \quad (4.7)$$

**Lemma 4.2.6** [185, 38] *If  $A$  is Hurwitz, then  $A_{[m]}$  is Hurwitz as well.*

For more properties of these transformation, readers may refer to [185, 38, 9, 98].

### 4.3 SOS Based Stability Results

Due to the different transformations used for CPS and DPS, we discuss them separately. Based on the Lyapunov stability theory, if we can find a set of HPLFs of degree  $2m$ , denoted by  $v_i^{(2m)}(x)$ , such that

$$v_i^{(2m)}(x) > 0, \forall x \in S_i, i \in \mathcal{I}, x \neq 0; \quad (4.8)$$

$$v_i^{(2m)}(x) \geq v_j^{(2m)}(x), \forall x \in S_{ij}; \quad (4.9)$$

$$\dot{v}_i^{(2m)}(x) < 0, \forall x \in S_i, i \in \mathcal{I}, x \neq 0, \quad (4.10)$$

then we can define a PHPLF candidate as

$$v^{(2m)} = \sum_i \tau_i v_i^{(2m)}, \tau_i = \begin{cases} 1 & \text{mode } i \text{ is active} \\ 0 & \text{otherwise} \end{cases} \quad (4.11)$$

which implies the asymptotic stability of the corresponding system.

### 4.3.1 Continuous-time Case

Let us introduce the extended system corresponding to continuous-time setting of (3.1) defined by

$$\dot{x}^{[m]} = A_{i[m]}x^{[m]}, \quad x \in S_i, \quad i \in \mathcal{I} \quad (4.12)$$

The extended system (4.12) plays a key role in our stability analysis.

**Remark 4.3.1** *The partition properties still hold after the transformation. For  $\forall x \in S_i$ , we have  $F_i x \geq 0$ . Thus according to Lemma 4.2.1 and Lemma 4.2.3, we have  $F_i^{[m]}x^{[m]} \geq 0$ . Similarly, for  $\forall x \in S_{ij}$ , we can deduce  $x^{[m]} = F_{ij}^{[m]}z^{[m]}$  from  $x = F_{ij}z$ ,  $z \in \mathcal{R}^{n-1}$ .*

**Theorem 4.3.1** *The continuous-time PWA system (3.1) is asymptotically stable, if there exists a solution  $(P_i = P_i^T, U_i \succeq 0, V_i \succeq 0, \gamma_{il})$  such that*

$$P_i - F_i^{[m]T} U_i F_i^{[m]} - \bar{I}_{n,m} \geq 0, \quad \forall i \in \mathcal{I}_0 \quad (4.13)$$

$$F_{ij}^{[m]T} (P_i - P_j) F_{ij}^{[m]} = 0, \quad \forall i, j \in \Omega_c \quad (4.14)$$

$$P_i A_{i[m]} + A_{i[m]}^T P_i + \sum_{\ell=1}^{L_{n,m}} \gamma_{i\ell} P_{i\ell}^{(0)} + F_i^{[m]T} V_i F_i^{[m]} + \bar{I}_{n,m} \leq 0, \quad \forall i \in \mathcal{I}_0, \quad (4.15)$$

where  $P_{i\ell}^{(0)}$  is such that  $x^{[m]T} P_{i\ell}^{(0)} x^{[m]} = 0$ ,  $\bar{I}_{n,m} \in \bar{\mathcal{X}}_{n,m}$ ,  $\epsilon_i$  are sufficiently small non-negative scalars and  $F_i^{[m]}$ ,  $F_{ij}^{[m]}$  and  $A_{i[m]}$  are the corresponding transformations of  $F_i$ ,  $F_{ij}$  and  $A_i$ , respectively.

**Proof:** Define  $v_i^{(2m)}(x) = x^{[m]T} P_i x^{[m]}$ . Now we prove that  $v_i^{(2m)}$  is a desired Lyapunov function. For each region  $S_i$ , we have  $F_i x \geq 0, \forall x \in S_i$ . Thus  $F_i^{[m]} x^{[m]} \geq 0$ . From (4.13), for  $x \neq 0$ , we have

$$v_i^{(2m)} = x^{[m]T} P_i x^{[m]} \geq x^{[m]T} (F_i^{[m]T} U_i F_i^{[m]} + \bar{I}_{n,m}) x^{[m]} > 0$$

Since  $x = F_{ij} z, \forall x \in S_{ij}$ ,  $x^{[m]} = F_{ij}^{[m]} z^{[m]}$ . From (4.14), we have

$$\begin{aligned} v_i^{(2m)} &= x^{[m]T} P_i x^{[m]} = z^{[m]T} F_{ij}^{[m]T} P_i F_{ij}^{[m]} z^{[m]} \\ &= z^{[m]T} F_{ij}^{[m]T} P_j F_{ij}^{[m]} z^{[m]} = x^{[m]T} P_j x^{[m]} = v_j^{(2m)} \end{aligned}$$

Further, it follows from (4.15) that

$$\begin{aligned} \frac{d}{dt} v_i^{(2m)} &= x^{[m]T} (P_i A_{i[m]} + A_{i[m]}^T P_i) x^{[m]} \\ &\leq -x^{[m]T} \left( \sum_{\ell=1}^{L_{n,m}} \gamma_{i\ell} P_{i\ell}^{(0)} + F_i^{[m]T} V_i F_i^{[m]} + \bar{I}_{n,m} \right) x^{[m]} \\ &= -x^{[m]T} (F_i^{[m]T} V_i F_i^{[m]} + \bar{I}_{n,m}) x^{[m]} < 0, \quad x \neq 0 \end{aligned}$$

Observe that

$$\frac{d}{dt} v_i^{(2m)} \Big|_{\dot{x}=A_i x} = \frac{d}{dt} v_i^{(2m)} \Big|_{\dot{x}^{[m]}=A_{i[m]} x^{[m]}}$$

Thus we can conclude that  $v_i^{(2m)}$  is a PHPLF for system (3.1) and the system is asymptotically stable. ■

**Remark 4.3.2** *The equality constraint (4.14) can be eliminated. One approach is suggested by Johansson [102] via constructing a type of so-called continuity matrices to guarantee the continuity of a Lyapunov function as stated in Theorem 3.2.1 and Remark 3.2.3. Thus we have:  $\vec{F}_i^{[m]} x^{[m]} = \vec{F}_j^{[m]} x^{[m]}$ . An alternative approach to replace (4.14) is to choose  $P_i = \vec{F}_i^{[m]T} T \vec{F}_i^{[m]}$ , where  $T = T^T$ , such that*

$$\begin{aligned} v_i^{(2m)} &= x^{[m]T} P_i x^{[m]} = x^{[m]T} \vec{F}_i^{[m]T} T \vec{F}_i^{[m]} x^{[m]} = \\ &= x^{[m]T} \vec{F}_j^{[m]T} T \vec{F}_j^{[m]} x^{[m]} = x^{[m]T} P_j x^{[m]} = v_j^{(2m)} \end{aligned}$$

When  $m = 1$ , the result will reduce to these conditions suggested by Johansson and Rantzer [102] and Hassibi and Boyd [82].

Some SDP solvers such as SDPSOL handle equality constraints such as those given in (4.14). However, other SDP solvers, say, LMI Toolbox of Matlab, can not handle equality constraints directly, though we can use some transformation techniques to eliminate the equality constraints and rewrite the LMIs. In this case, we can use discontinuous Lyapunov functions similar to (3.16).

As stated in Remark 3.2.3, continuity of the Lyapunov function is not necessary here. According to (4.9), we have

$$x^{[m]T} P_i x^{[m]} \geq x^{[m]T} P_j x^{[m]}, \quad x \in S_{ij} \quad (4.16)$$

Note that the set  $S_{ij}$  can be represented by  $\{x | \bar{F}_{ij} x = 0\}$ . Thus  $\forall x \in S_{ij}$ ,  $\bar{F}_{ij}^{[m]} x^{[m]} = 0$ . (4.16) is equivalent to the following LMI in  $P_i, P_j, T_{ij}$  by Finsler's Lemma:

$$P_i - P_j + \bar{F}_{ij}^{[m]T} T_{ij} + T_{ij}^T \bar{F}_{ij}^{[m]} \geq 0 \quad (4.17)$$

Hence, (4.14) of Theorem 4.3.1 can be replaced by (4.17) for  $\forall (i, j) \in \Omega_c$ .

**Remark 4.3.3** As stated in Remark 2.2.2, for  $i \in \mathcal{I}_1$ , the system has an alternative representation. Note that if  $\bar{F}_i \xi \geq 0$ , then  $\bar{F}_i^{[m]} \xi^{[m]} \geq 0$ . Define  $\bar{A}_i$ ,  $\bar{F}_i$ ,  $\bar{F}_{ij}$  as (2.10)-(2.11), and  $\bar{z} = \begin{bmatrix} z^T & 1 \end{bmatrix}^T$ . If  $\xi = \bar{F}_{ij} \bar{z}$ , then  $\xi^{[m]} = \bar{F}_{ij}^{[m]} \bar{z}^{[m]}$ . Now we define  $\bar{v}_i^{(2m)} = \xi^{[m]T} \bar{P}_i \xi^{[m]}$ , which is no longer homogeneous. Fortunately, in (4.13)-(4.15), if we replace  $A_{i[m]}$  by  $\bar{A}_{i[m]}$ ,  $F_i^{[m]}$  by  $\bar{F}_i^{[m]}$  and  $F_{ij}^{[m]}$  by  $\bar{F}_{ij}^{[m]}$  and with the variables properly dimensioned, the inequalities about  $\bar{v}_i^{(2m)}$  (replace  $v_i^{(2m)}$  with  $\bar{v}_i^{(2m)}$ ) in (4.8)-(4.10) still hold. In fact, the last row of  $\bar{A}_i$  is zero and the derivative of constant 1 is zero, thus there must exist an  $\bar{A}_{i[m]}$ , such that  $\frac{d}{dt}((\xi)^{[m]}) = \bar{A}_{i[m]} \xi^{[m]}$ . We also note that for  $\bar{F}_i$  and  $\bar{F}_{ij}$ , there must exist  $\bar{F}_i^{[m]}$  and  $\bar{F}_{ij}^{[m]}$ , such that  $(\bar{F}_i \xi)^{[m]} = \bar{F}_i^{[m]} \xi^{[m]}$  and  $(\bar{F}_{ij} \xi)^{[m]} = \bar{F}_{ij}^{[m]} \xi^{[m]}$ . Hence, we have

$$\frac{d}{dt} \bar{v}_i^{(2m)} \Big|_{\dot{x}=A_i x + a_i} = \frac{d}{dt} \bar{v}_i^{(2m)} \Big|_{\xi=\bar{A}_i \xi} = \frac{d}{dt} \bar{v}_i^{(2m)} \Big|_{\xi^{[m]}=\bar{A}_{i[m]} \xi^{[m]}}$$

We can thus conclude that  $\bar{v}_i^{(2m)}$  is a Lyapunov function segment for the region  $S_i$ ,  $i \in \mathcal{I}_1$ .

Theorem 4.3.1 introduces a PHPLF for the continuous-time PWA system (3.1). The remaining questions are whether the more general functions: non-homogeneous polynomial Lyapunov functions (NHPLF) in the form of SOS can improve the stability analysis and why the increase of degree  $m$  can relax the conservatism of stability analysis. To answer these questions, we give the following theorems.

**Theorem 4.3.2** *The continuous-time PWA system (3.1) is asymptotically stable, if for  $\forall i \in \mathcal{I}$ ,  $\forall (i, j) \in \Omega_c$  and a sufficiently small  $\epsilon_i \geq 0$ ,*

$$\begin{cases} \hat{M}_i - \hat{F}_i^{[m]T} \hat{U}_i \hat{F}_i^{[m]} - \hat{I}_{n,m} \geq 0 \\ \hat{F}_{ij}^{[m]T} (\hat{M}_i - \hat{M}_j) \hat{F}_{ij}^{[m]} = 0 \\ \hat{M}_i \hat{A}_{i,[m]} + \hat{A}_{i,[m]}^T \hat{M}_i + \sum_l \hat{\gamma}_{il} \hat{P}_{il}^{(0)} + \hat{F}_i^{[m]T} \hat{V}_i \hat{F}_i^{[m]} + \hat{I}_{n,m} \leq 0 \end{cases} \quad (4.18)$$

where  $\hat{A}_{i,[m]} = \text{diag}\{A_{i[1]}, \dots, A_{i[m]}\}$ ,  $\hat{I}_{n,m} = \text{diag}\{0, \dots, 0, \bar{I}_{n,m}\}$ ,  $\bar{I}_{n,m} \in \bar{\mathcal{X}}_{n,m}$ ,  $\hat{F}_{i,[m]} = \text{diag}\{F_{i[1]}, \dots, F_{i[m]}\}$  and  $\hat{F}_{ij,[m]} = \text{diag}\{F_{ij[1]}, \dots, F_{ij[m]}\}$ , admit a feasible solution  $\hat{M}_i = \hat{M}_{i,[m]} = \hat{M}_{i,[m]}^T \in \mathcal{R}^{\Sigma_{1,n,m} \times \Sigma_{1,n,m}}$ ,<sup>2</sup>  $\hat{\gamma}_{il} \in \mathcal{R}$ ,  $l = 1, \dots, \sum_{i=1}^m (L_{n,i} + \beta_i)$ ,<sup>3</sup>  $\hat{U}_i \succeq 0$ ,  $\hat{V}_i \succeq 0$ ,  $i \in \mathcal{I}$ .

**Proof:** The proof is similar to that of Theorem 4.3.1. ■

Now we show that the NHPLF introduced in Theorem 4.3.2, in fact, cannot improve the analysis performance, though more free variables are added.

**Theorem 4.3.3** *If the LMIs (4.18) in Theorem 4.3.2 admit a feasible solution, then LMIs (4.13)-(4.15) in Theorem 4.3.1 also exist a solution, and vice versa.*

**Proof:** Assume  $\hat{M}$  satisfies (4.18) (For brevity, we omit the subscript  $i$  of  $\hat{M}_i$ ).

Partition

$$\hat{M} = \begin{bmatrix} \hat{M}_{[m-1]} & \bar{M} \\ \bar{M}^T & M_{[m]} \end{bmatrix}, \quad \sum_l \hat{\gamma}_{il} \hat{P}_{il}^{(0)} = \begin{bmatrix} P_{il}^{11}(\gamma_{il}) & P_{il}^{12}(\gamma_{il}) \\ P_{il}^{12}(\gamma_{il})^T & P_{il}^{22}(\gamma_{il}) \end{bmatrix}$$

<sup>2</sup>  $\hat{M}_{i,[m]}$  is defined in (4.2)

<sup>3</sup>  $\beta_i$  is an additional free integer number due to the non-homogeneous form. Note that  $\beta_m = 0$ .  $\hat{x}^{[m]T} \hat{P}_{il}^{(0)} \hat{x}^{[m]} = 0$ .

$$\hat{F}_i^{[m]T} \hat{V}_i \hat{F}_i^{[m]} = \begin{bmatrix} V_i^{11} & V_i^{12} \\ V_i^{12T} & V_i^{22} \end{bmatrix}, \quad \hat{F}_i^{[m]T} \hat{U}_i \hat{F}_i^{[m]} = \begin{bmatrix} U_i^{11} & U_i^{12} \\ U_i^{12T} & U_i^{22} \end{bmatrix}$$

It is clear that  $U_i^{22} = F_i^{[m]T} U_i F_i^{[m]}$  and  $V_i^{22} = F_i^{[m]T} V_i F_i^{[m]}$ . From the first inequality of (4.18), (4.13) holds. Also the equality of (4.18) implies (4.14).

Rewrite the third one of (4.18) as  $\begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{B}_i^T & \mathcal{C}_i \end{bmatrix} \leq 0$ , where

$$\begin{aligned} \mathcal{A}_i &= \hat{M}_{[m-1]} \tilde{A}_{i,[m-1]} + \tilde{A}_{i,[m-1]}^T \hat{M}_{[m-1]} + P_{il}^{11}(\gamma_{il}) + V_i^{11} \\ \mathcal{B}_i &= \bar{M} \tilde{A}_{i,[m-1]} + A_{i,[m]}^T \bar{M} + P_{il}^{12}(\gamma_{il}) + V_i^{12} \\ \mathcal{C}_i &= M_{[m]} A_{i,[m-1]} + A_{i,[m]}^T M_{[m]} + P_{il}^{22}(\gamma_{il}) + V_i^{22} + \bar{I}_{n,m} \end{aligned}$$

Now we shall prove that  $M_{[m]}$  is the solution  $P$  in (4.15). In fact, since  $\mathcal{C}_i \leq 0$ , we only need to prove that  $P_{il}^{22}(\gamma_{il})$  is equivalent to  $\sum_l \gamma_{il} P_{il}^{(0)}$  in (4.15). Considering the fact that all the terms of degree  $2m$  in  $\sum_l \hat{\gamma}_{il} \hat{P}_{il}^{(0)}$  come from  $P_{il}^{22}(\gamma_{il})$  and  $\begin{bmatrix} 0 & 0 \\ 0 & \sum_l \gamma_{il} P_{il}^{(0)} \end{bmatrix}$  is a special case of  $\sum_l \hat{\gamma}_{il} \hat{P}_{il}^{(0)}$ , the result follows.

The reverse is obvious. We may simply choose  $\hat{M} = \begin{bmatrix} 0 & 0 \\ 0 & P_i \end{bmatrix}$ ,  $\hat{U}_i = \begin{bmatrix} 0 & 0 \\ 0 & U_i \end{bmatrix}$ ,  $\hat{V}_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & V_{ij} \end{bmatrix}$ , where  $P_i$ ,  $U_i$  and  $V_{ij}$  are solutions of (4.13)-(4.15) in Theorem 4.3.1. ■

On the other hand, to relax the conservatism of stability analysis of the system (3.1) using the PHPLFs, we can in general increase the degree  $m$ . In fact, we can easily prove that if for  $m = 1$ , (4.15) has a solution, then for  $m > 1$ , it admits a solution as well. For a larger  $m$ , (4.15) involves more free parameters which shall allow a better (at least no worse) stability analysis result. We note that the Positivstellensatz [99, 37, 132] suggests that when  $m$  is sufficiently large, the relaxation is not conservative.

### 4.3.2 Discrete-time Case

Consider the discrete-time PWA system (3.1). We introduce the extended systems corresponding to (3.1) defined by<sup>4</sup>

$$x_{t+1}^{[m]} = A_i^{[m]} x_t^{[m]}, \quad x_t \in S_i, \quad i \in \mathcal{I} \quad (4.19)$$

Note that a different transformation for matrix  $A_i$  is applied here, compared with the continuous-time case. The partition properties of Remark 4.3.1 associated with the extended state also hold. We define a PHPLF similar to (4.11), and have the following result.

**Theorem 4.3.4** *The discrete-time PWA system (3.1) is asymptotically stable, if there exists a solution  $(P_i = P_i^T, U_i \succeq 0, V_{ij} \succeq 0, \gamma_{i\ell})$  such that*

$$P_i - F_i^{[m]T} U_i F_i^{[m]} - \bar{I}_{n,m} \geq 0, \quad \forall i \in \mathcal{I}_0 \quad (4.20)$$

$$\begin{aligned} & A_i^{[m]T} P_j A_i^{[m]} - P_i + \sum_{\ell=1}^{L_{n,m}} \gamma_{i\ell} P_{i\ell}^{(0)} + F_i^{[m]T} V_{ij} F_i^{[m]} \\ & + \bar{I}_{n,m} \leq 0, \quad \forall i, j \in \Omega_d, \quad i, j \in \mathcal{I}_0 \end{aligned} \quad (4.21)$$

where  $P_{i\ell}^{(0)}$  is such that  $x^{[m]T} P_{i\ell}^{(0)} x^{[m]} = 0$ ,  $\bar{I}_{n,m} \in \bar{\mathcal{X}}_{n,m}$ ,  $\epsilon_i \geq 0$ ,  $F_i^{[m]}$  and  $A_i^{[m]}$  are the corresponding transformations of  $F_i$  and  $A_i$ .

**Proof:** Define  $v_i^{(2m)}(t) = x_t^{[m]T} P_i x_t^{[m]}$ . For each region  $S_i$ , we have  $F_i x_t \geq 0, \forall x_t \in S_i$ . Thus  $F_i^{[m]} x_t^{[m]} \geq 0$ . From (4.20), we have

$$v_i^{(2m)}(t) = x_t^{[m]T} P_i x_t^{[m]} \geq x_t^{[m]T} (F_i^{[m]T} U_i F_i^{[m]} + \bar{I}_{n,m}) x_t^{[m]} > 0, \quad x_t^{[m]} > 0$$

Similar to [55], we assume that the dynamics of the system are governed by the dynamics of the local model of  $S_i$  when the state of the system transits from the

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<sup>4</sup>Here we use  $x_t$  instead of  $x(t)$  for clarity.

region  $S_i$  to  $S_j$ . From (4.21), we have

$$\begin{aligned} \Delta v(t) &= v^{(2m)}(t+1) - v^{(2m)}(t) = x_t^{[m]T} (A_i^{[m]T} P_j A_i^{[m]} - P_i) x_t^{[m]} \\ &\leq -x_t^{[m]T} \left( \sum_{\ell=1}^{L_{n,m}} \gamma_{i\ell} P_{i\ell}^{(0)} + F_i^{[m]T} V_i F_i^{[m]} + \bar{I}_{n,m} \right) x_t^{[m]} \\ &= -x_t^{[m]T} (F_i^{[m]T} V_i F_i^{[m]} + \bar{I}_{n,m}) x_t^{[m]} < 0, \quad x_t^{[m]} \neq 0 \end{aligned}$$

for  $(i, j) \in \Omega_d$ . Note that

$$\begin{aligned} \Delta v(t)|_{x_{t+1}=A_i x_t} &= x_{t+1}^{[m]T} P_j x_{t+1}^{[m]} - x_t^{[m]T} (t) P_i x_t^{[m]} \\ &= (A_i x_t)^{[m]T} P_j (A_i x_t)^{[m]} - x_t^{[m]T} P_i x_t^{[m]} \\ &= (A_i^{[m]} x_t^{[m]})^T P_j A_i^{[m]} x_t^{[m]} - x_t^{[m]T} P_i x_t^{[m]} \\ &= \Delta v(t)|_{x_{t+1}^{[m]}=A_i^{[m]} x_t^{[m]}} \end{aligned} \tag{4.22}$$

Thus we can conclude that  $\{v_i^{(2m)}\}$  is the right set of HPLFs for system (3.1). The proof is completed.  $\blacksquare$

**Remark 4.3.4** *It is obvious that when  $m = 1$ , the result will reduce to those conditions suggested by Feng [55] or Ferrari-Trecate et al. [64].*

**Remark 4.3.5** *For the case that  $i, j \in \mathcal{I}_1$ , we can easily check that for inequalities (4.20)-(4.21), when  $A_i^{[m]}$  is replaced by  $\bar{A}_i^{[m]}$  and  $F_i^{[m]}$  by  $\bar{F}_i^{[m]}$ , where  $\bar{A}_i$  and  $\bar{F}_i$  are defined in (2.10)-(2.11), the properties of  $\bar{v}_i^{(2m)}$  still hold for  $i, j \in I_1$  with the variables properly dimensioned.*

However, different from the continuous-time case, jumps of state from one partition to another are much more complex. In fact, we have to deal with the situation when the state jumps from the partition  $i \in I_0$  to the partition  $j \in I_1$  or vice versa.

*Case 1 ( $i \in \mathcal{I}_0, j \in \mathcal{I}_1$  and  $(i, j) \in \Omega_d$ ): We choose  $\bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 1 \end{bmatrix}$  and  $\bar{F}_i = \begin{bmatrix} F_i & 0 \end{bmatrix}$ . Now we have to extend the Lyapunov matrix  $P_i$  to a proper dimensioned matrix  $\bar{P}_i$ . The procedure is described as follows:*

*Given a variable matrix  $Z_i = (z_{i_1, i_2}^{(i)}) \in \mathcal{R}^{d_{n+1,m} \times d_{n+1,m}}$ . Thus  $\bar{W}_i = \xi^{[m]T} Z_i \xi^{[m]} = \sum_{j=1}^{d_{n+1,m} \times d_{n+1,m}} z_{i_1, i_2}^{(i)} x_1^{m_{j1}} \cdots x_n^{m_{jn}}$ , where  $\sum_{k=1}^n m_{jk}$  can take a value from the set  $\{0, 2m -$*

$2, 2m - 1, 2m\}$ . Compare  $\bar{W}_i$  with  $v_i = x^{[m]T} P_i x^{[m]}$  and determine the entries of  $Z_i$  such that  $z_{i_1, i_2}^{(i)}$  for  $x_i^{[2m]}$  are the same as the coefficient of  $x_i^{[2m]}$  in  $v_i$ , and let the rest entries of  $Z_i$  be zeros. Choose such  $Z_i$  as  $\bar{P}_i$ . In this situation, we can easily check that  $v_i = x^{[m]T} P_i x^{[m]} = \xi^{[m]T} \bar{P}_i \xi^{[m]} = \bar{v}_i$ .

Case 2 ( $i \in \mathcal{I}_1, j \in \mathcal{I}_0$  and  $(i, j) \in \Omega_d$ ): Simply let  $\bar{A}_j = \begin{bmatrix} A_j & 0 \\ 0 & 1 \end{bmatrix}$  and  $\bar{E}_j = \begin{bmatrix} E_j & 0 \end{bmatrix}$ , and choose the proper  $\bar{P}_j$  from  $P_j$  as Case 1.

Hence a new version of Theorem 4.3.4 shall also contain the following additional conditions :

$$\bar{P}_i - \bar{F}_i^{[m]T} \bar{U}_i \bar{F}_i^{[m]} - \bar{I}_{n,m} \geq 0, \quad \forall i \in \mathcal{I}_1 \quad (4.23)$$

$$\begin{aligned} \bar{A}_i^{[m]T} \bar{P}_j \bar{A}_i^{[m]} - \bar{P}_i + \sum_l \gamma_{il} \bar{P}_{il}^{(0)} + \bar{F}_i^{[m]T} \bar{V}_{ij} \bar{F}_i^{[m]} + \bar{I}_{n,m} &\leq 0, \\ \forall (i, j) \in \Omega_d; (i, j \in \mathcal{I}_1) \cup (i \in \mathcal{I}_0 \cap j \in \mathcal{I}_1) \cup (i \in \mathcal{I}_1 \cap j \in \mathcal{I}_0) \end{aligned} \quad (4.24)$$

where  $\bar{P}_{il}^{(0)}$  is such that  $\xi^{[m]T} \bar{P}_{il}^{(0)} \xi^{[m]} = 0$ ;  $\bar{F}_i^{[m]}$  and  $\bar{A}_{i[m]}$  are the corresponding transformations of  $\bar{F}_i$  and  $\bar{A}_i$ , respectively.

Thus, we have already checked all the four possible situations ( $i, j \in \mathcal{I}_0$ ;  $i, j \in \mathcal{I}_1$ ;  $i \in \mathcal{I}_0, j \in \mathcal{I}_1$ ;  $i \in \mathcal{I}_1, j \in \mathcal{I}_0$ ; for  $(i, j) \in \Omega_d$ ) that may happen in the state jump for discrete-time systems.

### 4.3.3 Examples

Now we give some examples to illustrate our results. We compare our results with existing works in two aspects: the decay rate of the system and the assertion of stability.

**Example 4.3.1** The following CPS example is from [102]. The system parameters are given by

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}$$

$$F_1 = -F_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad F_2 = -F_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

The stability has been verified in [102]. Figure 4.1 provides two stable trajectories from initial state  $[0 \ 1]^T$  and  $[1 \ 0]^T$ . Here we will compare the decay rate of the continuous-time PWA system (3.1). By setting (4.15) as

$$P_i A_{i[m]} + A_{i[m]}^T P_i - \rho P_i + \sum_{\ell} \gamma_{i\ell} P_{i\ell}^{(0)} + F_i^{[m]T} V_i F_i^{[m]} + \bar{I}_{n,m} \leq 0$$

where  $\rho$  is the corresponding decay ratio. We compare  $\rho$  by solving  $m = 1$  and  $m = 2$ , respectively. When  $m = 1$ , the optimal  $\rho = -0.21$ . When  $m = 2$ , the corresponding  $A_{i[2]}$  and  $F_i^{[2]}$  are

$$A_{1[2]} = A_{3[2]} = \begin{bmatrix} -0.2 & 10 & 0 \\ -1 & -0.2 & 5 \\ 0 & -2 & -0.2 \end{bmatrix}, \quad A_{2[2]} = A_{4[2]} = \begin{bmatrix} -0.2 & 2 & 0 \\ -5 & -0.2 & 1 \\ 0 & -10 & -0.2 \end{bmatrix},$$

$$F_1^{[2]} = F_3^{[2]} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}, \quad F_2^{[2]} = F_4^{[2]} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix},$$

and the optimal  $\rho = -0.40$ . It is obvious that when  $m = 2$ , the PHPLF gives a much better result than that of PQLF, or  $m = 1$ .

If we change the system parameters  $A_i$  to

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 1 \\ -5 & 0.1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}$$

When  $m = 1$ , we cannot get a feasible solution from (4.13)-(4.15), i.e., Johansson's [102] or Hassibi and Boyd [82]'s methods cannot confirm the stability of the system. In fact, if we choose  $m = 2$ , we can show that the system is stable and  $P_i$  for (4.13)-(4.15) are given by:

$$P_1 = \begin{bmatrix} 39.3319 & -8.47619 & -12597.9 \\ -8.47619 & 25590.7 & -41.6463 \\ -12597.9 & -41.6463 & 982.788 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 981.565 & -36.8693 & -8311.88 \\ -36.8693 & 17019.3 & -13.2532 \\ -8311.88 & -13.2532 & 39.9871 \end{bmatrix}$$

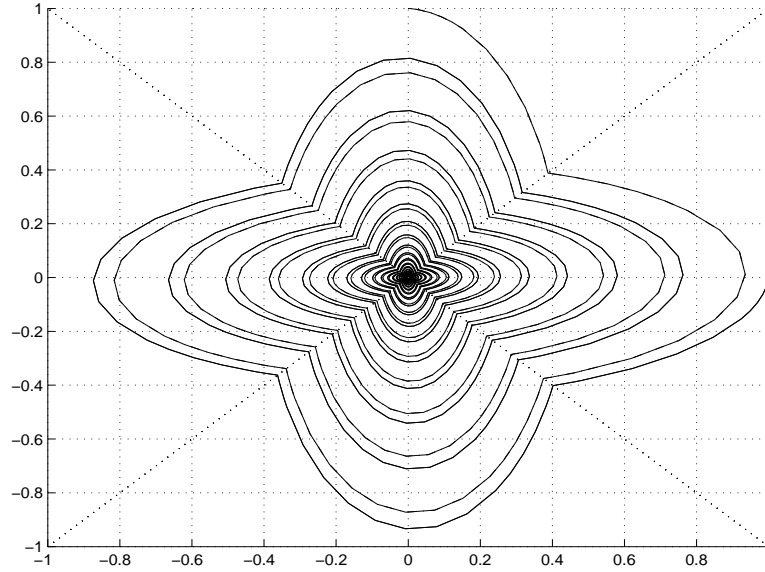


Figure 4.1: System state trajectory  $[0 \ 1]^T$  and  $[1 \ 0]^T$

$$P_3 = \begin{bmatrix} 39.3319 & -8.47619 & -11666.2 \\ -8.47619 & 23727.4 & -41.6463 \\ -11666.2 & -41.6463 & 982.788 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 981.565 & -36.8693 & 510.634 \\ -36.8693 & -625.727 & -13.2532 \\ 510.634 & -13.2532 & 39.9871 \end{bmatrix}$$

Note that in [102], the authors suggest a way to further divide partitions, in order to establish the stability of system. This method is valid here.

**Example 4.3.2** *In this example, we look at a DPS (3.1). The system parameters are given by*

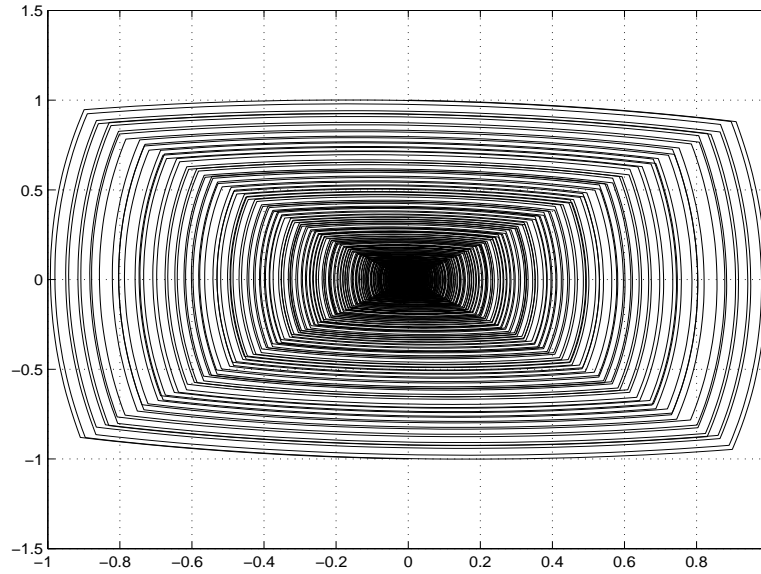
$$A_1 = A_3 = \begin{bmatrix} 1 & 0.01 \\ -0.05 & 0.897 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 1 & 0.05 \\ -0.01 & 0.897 \end{bmatrix}$$

$F_i$ ,  $i \in \mathcal{I}$  is the same as those of Example 4.3.1.

The stability can be verified by the methods in [43, 55]. Figure 4.2 shows two stable trajectories from initial state  $[0 \ 1]^T$  and  $[0 \ -1]^T$ . By setting (4.21) as

$$A_i^{[m]T} P_j A_i^{[m]} - (1 + \rho) P_i + \sum_{\ell} \gamma_{i\ell} P_{i\ell}^{(0)} + F_i^{[m]T} V_{ij} F_i^{[m]} + \bar{I}_{n,m} \leq 0$$

where  $\rho$  is the corresponding decay rate, we compare  $\rho$  by solving  $m = 1$  and  $m = 2$ , respectively. When  $m = 1$ , the optimal  $\rho = -0.01$ . When  $m = 2$ , the

Figure 4.2: System state trajectory from  $[0 \ 1]^T$  and  $[0 \ -1]^T$ 

optimal  $\rho = -0.02$ . It is obvious that  $m = 2$  gives a much better result. In fact, if we change the system parameters to:

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.01 \\ -0.05 & 0.997 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 1 & 0.05 \\ -0.01 & 0.998 \end{bmatrix}$$

we cannot get a feasible solution when  $m = 1$ . However, when  $m = 2$ , Theorem 4.3.4 proves the stability of the system with  $P_i, i \in \mathcal{I}$  shown as follows.

$$P_1 = \begin{bmatrix} 6.98249 & 0.122781 & 117286 \\ 0.122781 & -234570 & -0.0225782 \\ 117286 & -0.0225782 & 0.249983 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.261448 & 0.18896 & 186391 \\ 0.18896 & -372779 & 0.719387 \\ 186391 & 0.719387 & 7.01142 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 6.98249 & 0.122781 & 137957 \\ 0.122781 & -275911 & -0.0225782 \\ 137957 & -0.0225782 & 0.249983 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0.261448 & 0.18896 & 119429 \\ 0.18896 & -238855 & 0.719387 \\ 119429 & 0.719387 & 7.01142 \end{bmatrix}$$

#### 4.3.4 Results Based on SOSTOOLS

In the above section, we gave an explicit constructive method for high-order homogeneous Lyapunov functions based on the Gram matrices. As an alternative,

one may use the newly released Matlab Toolbox: SOSTOOLS [140] to obtain the following results similar to these in [11, 139].

**Theorem 4.3.5** (CPS) *The continuous-time PWA system (3.1) is asymptotically stable, if there exist polynomials  $v_i(x)$ ,  $c_{ij}(x)$ ,  $a_{ik}(x) \geq 0$ , and  $b_{ik}(x) \geq 0$ , such that*

$$v_i(0) = 0, \quad \forall i \in \mathcal{I}_0 \quad (4.25)$$

$$v_i(x) - \sum_{k=1}^{m_i} a_{ik}(x) F_i^{(k)}(x) - \bar{I}_{n,\ell} \geq 0, \quad \forall x \neq 0, \quad \forall i \in \mathcal{I} \quad (4.26)$$

$$\dot{v}_i(A_i x) + \sum_{k=1}^{m_i} b_{ik}(x) F_i^{(k)}(x) + \bar{I}_{n,\ell} \leq 0, \quad \forall x \neq 0, \quad \forall i \in \mathcal{I} \quad (4.27)$$

$$v_i(x) + c_{ij} F_{ij}^{(k)}(x) - v_j = 0, \quad \forall (i, j) \in \Omega_c \quad (4.28)$$

where  $F_i^{(k)}(x)$  is the  $k$ -th entry in  $F_i x \in \mathcal{R}^{m_i}$ ,  $F_{ij}^{(k)}(x)$  is the  $k$ -th entry in  $F_{ij} x \in \mathcal{R}^{m_{ij}}$ ,  $\bar{I}_{n,\ell} \in \bar{\mathcal{X}}_{n,\ell}$ ,  $2\ell$  is the degree of  $v_i(x)$ .

**Theorem 4.3.6** (DPS) *The discrete-time PWA system (3.1) is asymptotically stable, if there exist polynomials  $v_i(x)$ ,  $c_{ij}(x)$ ,  $a_{ik}(x) \geq 0$ , and  $b_{ik}(x) \geq 0$ , such that*

$$v_i(0) = 0, \quad \forall i \in \mathcal{I}_0 \quad (4.29)$$

$$v_i(x) - \sum_{k=1}^{m_i} a_{ik}(x) F_i^{(k)}(x) - \bar{I}_{n,\ell} \geq 0, \quad \forall x \neq 0, \quad \forall i \in \mathcal{I} \quad (4.30)$$

$$\Delta v_i(x) + \sum_{k=1}^{m_i} b_{ik}(x) F_i^{(k)}(x) + \bar{I}_{n,\ell} \leq 0, \quad \forall x \neq 0, \quad \forall i \in \mathcal{I} \quad (4.31)$$

where  $F_i^{(k)}(x)$ ,  $F_{ij}^{(k)}(x)$  and  $\bar{I}_{n,\ell}$  are defined in Theorem 4.3.5.

Compared the method using SOSTOOLS with our explicit method for analyzing PWA systems, our method has several advantages, one of which is that the explicit form of Gram matrices allows us to easily study the properties of this type of Lyapunov functions. For example, in Theorem 4.3.3, we show that the NHPLF

cannot improve the performance by carefully studying the Gram matrices in (4.18).

Another useful toolbox on SOS is GloptiPoly[87] based on the dual theory of polynomial moments [144]. In [131], Prajna *et.al.* use GloptiPoly to analyze nonlinear control systems, which may be extended to PWA systems.

As a promising technique, the SOS theory is still in development. In fact, since Hilbert presented his 17th problem in 1900, the SOS theory has been paid much attention. Although Artin [143] shows that the mere existence of SOS presentation of a positive polynomial, the solution offers a global characterization of positivity of polynomials. Some results on special issues, such as decomposition [45], computational complexity and simplification [132], approximation [117] have appeared in recent years. Using these new results, we may refine our analysis results for PWA systems.

Based on SOS technique, more generalized classes of switched and hybrid systems can be analyzed:

$$\delta x = f_i(x), \quad x \in S_i \quad (4.32)$$

where  $S_i = \{x \in \mathcal{R}^n | g_{ik}(x) \geq 0, k = 1, \dots, o_i\}$ ,  $o_i$  is the number of inequality constraints that form  $S_i$ ;  $i \in \mathcal{I}$  such that  $\bigcup_{i \in \mathcal{I}} S_i = \mathcal{R}^n$ .  $\mathcal{I}$  is the index set.  $f_i(x)$  and  $g_{ik}(x)$  are in the form of polynomials [139]. In [11], Bemporad *et al.* use the similar technique to [139] for passivity analysis:

$$\delta x(t) = f_i(x, u), \quad [x^T, u^T]^T \in S_i \quad (4.33)$$

where  $S_i = \{[x^T \ u^T]^T \in \mathcal{R}^{n+m} | g_{ik}(x, u) \geq 0, k = 1, \dots, o_i\}, i \in \mathcal{I}$ .

SOS technique can also be applied to analyze the  $L_2$  gain [142]. For the system with general polynomial dynamics such as (4.33), an improvement is expected. However, for the PWA systems described as (2.4), say,

$$\begin{aligned} \delta x &= A_i x(t) + B_i u(t) \\ y(t) &= C_i x(t) \end{aligned}, \quad x(t) \in S_i \quad (4.34)$$

A direct application of the technique introduced in this chapter may not improve the analysis performance of  $L_2$  gain compared with the piecewise quadratic functions if we simply use the new vector  $[x^T, u^T]^T$ . This fact can be easily proved by carefully studying the corresponding Gram matrices in each LMI.

**Remark 4.3.6** *In [102], Johansson provides a unified view of four classes of Lyapunov functions (PQLF, CQLF, PWA Lyapunov function (PWLF), polytopic Lyapunov functions (PTLF)). And he points out that PQLF can contain other three Lyapunov functions as special cases. Here we further indicate that PQLF is a special case of PHPLF. We note that all these methods actually draw the partition information based on the halfspace representation. However, as mentioned in Definition 2.2.1, the polytope or polyhedra also has vertex representation. The question is whether less conservative stability conditions can be found based on vertex representation. We will discuss this problem in the next chapter.*

## 4.4 Conclusion

This chapter has provided sufficient conditions for the stability of PWA systems based on piecewise homogeneous polynomial Lyapunov functions (PHPLFs). The conditions can be checked by solving a set of LMIs. With respect to previous work on PQLFs, improved stability analysis results can be obtained by increasing the degree of the PHPLF. Some numerical examples have demonstrated the comparative advantages of the PHPLFs. We have also briefly compared our results with the conditions based on SOSTOOLS.

# Chapter 5

## Stability Conditions Based on Vertex Representation

### 5.1 Introduction

We have reviewed the existing results and presented our results on stability analysis for PWA systems based on halfspace representation in Chapter 4. In this chapter we will investigate stability conditions for PWA systems based on vertex representation.

Instead of discussing both CPS and DPS as in Chapter 4, here we mainly consider CPS, since we can treat the DPS in a similar way. Consider the following continuous-time PWA system:

$$\dot{x} = A_i x + a_i, \quad x \in S_i, \quad i \in \mathcal{I} \quad (5.1)$$

where  $x \in \mathcal{R}^n$  is the system state vector,  $\{S_i = \{x | F_i x + f_i \geq 0\}\}_{i \in \mathcal{I}} \subseteq \mathcal{R}^n$  denotes a polyhedral partition  $S$  of the state space into a set of convex polytopes. For the sake of compactness, we mainly consider the case that all partitions contain the origin, i.e.  $\mathcal{I} = \mathcal{I}_0$ . An extension to  $\mathcal{I}_1$  will be discussed briefly.

Since  $S_i$  is convex, then  $S_{ij}$  is convex too. Thus the elements in  $S_i$  can be repre-

sented by their convex hull of vertices as defined in (2.3), i.e.

$$S_i = \mathbf{Co}\{x_{i1}, \dots, x_{iL_i}\} = \left\{ x = \sum_{l=1}^{L_i} \lambda_l x_{il} \mid \sum_{l=1}^{L_i} \lambda_l = 1, \lambda_l \geq 0 \right\} \quad (5.2)$$

where  $x_{il}$  is the vertex vector and  $L_i$  is the total number of vertices of  $S_i$ . (5.2)

can be rewritten as

$$\begin{aligned} S_i &= \{x = X_i \lambda \mid \lambda \in \Upsilon_{L_i}\}, \quad X_i = [x_{i1}, \dots, x_{iL_i}], \\ \Upsilon_{L_i} &= \{\lambda = [\lambda_1, \dots, \lambda_{L_i}]^T \in \mathcal{R}^{L_i} \mid \sum_{l=1}^{L_i} \lambda_l = 1, \lambda_l \geq 0\} \end{aligned} \quad (5.3)$$

Similarly,  $S_{ij} = \{x = X_{ij} \lambda \mid \lambda \in \Upsilon_{L_{ij}}\}$ , where  $X_{ij}$  is formed by the vertices of  $S_{ij}$ .

We call  $X_i$  and  $X_{ij}$  vertex matrices of  $S_i$  and  $S_{ij}$ , respectively. For the case that  $S_i$  is a full-dimensional simplex, we have  $L_i = n + 1$ , i.e., each  $S_i$  has  $n + 1$  affinely independent vertices, where  $\lambda$  is called the barycentric coordinates of  $S_i$  [23].

The term quadratic stability refers to stability that can be established using a quadratic Lyapunov function (QLF). For PWA systems, we define the following piecewise quadratic (PQ) stability.

**Definition 5.1.1** *The PWA system (5.1) is piecewise quadratically stable if there exists a positive piecewise (parameter-dependent) quadratic function  $v(x)$  satisfying*

$$D^+v(x) \leq -2\alpha v(x)$$

where  $D^+v(x) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [v(x+h) - v(x)]$ <sup>1</sup> and  $\alpha$  is a positive scalar. Note that PQ stability implies exponential stability.

Figure 5.1 gives an example of PQLF for PQ stability, where  $v$  is composed of 3 quadratic functions  $v_1, v_2, v_3$ . In the switching boundaries,  $v$  is continuous at  $x_2$  and discontinuous at  $x_1$ . Note that the conditions in Theorem 3.2.1 and Theorem 3.2.2 also guarantee the PQ stability.

---

<sup>1</sup>If  $v : T \rightarrow \mathcal{R}$  is Lipschitz on the interval  $T$ , then the Dini-derivative  $D^+v(x)$  exists everywhere on  $T$ . The Dini-derivative is equal to the derivative if the derivative exists; the Dini-derivative of a Lipschitz function  $v(x)$  is defined everywhere on  $T$ ; and the behavior of  $\dot{v}(x)$  on the measure zero set does not affect the integral.

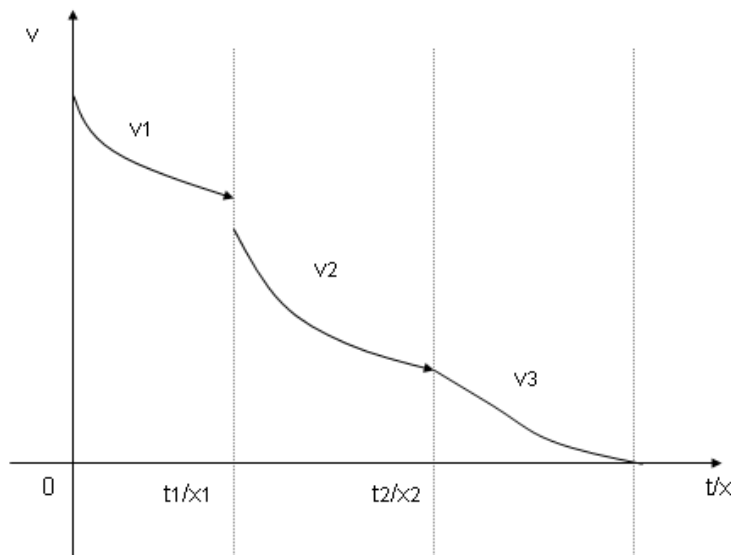


Figure 5.1: A PQLF along the state trajectory of a PQ stable system

For future use, we define  $\mathcal{L}(d)$  as the set of  $L$ -tuples obtained from all possible combinations of  $(\alpha_1, \alpha_2, \dots, \alpha_L)$ ,  $\alpha_i \in \mathcal{N}$ , such that  $\sum_{i=1}^L \alpha_i = d$ .  $\mathcal{L}_r(d)$  is the  $r$ th  $L$ -tuples of  $\mathcal{L}(d)$  which is lexically ordered,  $r = 1, \dots, J(d) = \frac{(L+d-1)!}{d!(L-1)!}$ .

$$\chi^r(d) = \frac{d!}{\alpha_1! \dots \alpha_L!}, \quad \chi_k^r(d, a) = \begin{cases} \frac{d!}{\alpha_1! \dots (\alpha_k - a)! \dots \alpha_L!} & \text{if } \alpha_k - a \in \mathcal{N} \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_{kl}^r(d, a, b) = \begin{cases} \frac{d!}{\alpha_1! \dots (\alpha_k - a)! \dots (\alpha_l - b)! \dots \alpha_L!} & \text{if } \alpha_k - a, \alpha_l - b \in \mathcal{N} \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_{klm}^r(d, a, b, c) = \begin{cases} \frac{d!}{\alpha_1! \dots (\alpha_k - a)! \dots (\alpha_l - b)! \dots (\alpha_m - c)! \dots \alpha_L!} & \text{if } \alpha_k - a, \alpha_l - b, \alpha_m - c \in \mathcal{N} \\ 0 & \text{otherwise} \end{cases}$$

where  $a, b, c \in \mathcal{N}$ .

## 5.2 Useful Lemmas

Before presenting our main results, we provide some useful lemmas first.

**Definition 5.2.1** A matrix  $M \in \mathcal{R}^{n \times n}$  is copositive, if  $x^T M x \geq 0$  for  $\forall x \in \mathcal{R}_+^n$ .  
 $M \in \mathcal{R}^{n \times n}$  is strictly copositive, if  $x^T M x > 0$  for  $\forall x \in \mathcal{R}_+^n, x \neq 0$ .

**Lemma 5.2.1**  $x^T M x \geq 0, \forall x \in \mathcal{R}_+^n$  iff  $\lambda^T M \lambda \geq 0, \forall \lambda \in \Upsilon_n$ . It also holds similarly for the strict inequality version.

**Proof:** For any  $x \in \mathcal{R}_+^n$ , there exists  $\lambda = [\lambda_1, \dots, \lambda_n]^T \in \Upsilon_n$ , such that  $x = \rho \lambda$ , where  $\rho = \|x\|_1 > 0$ . The result follows. ■

**Lemma 5.2.2** Let  $A_n = A_n^T = \{a_{ij}\}_{n \times n} = \begin{bmatrix} A_{n-1} & a_1 \\ a_1^T & a_{nn} \end{bmatrix}$ ,  $\lambda = [\lambda_1, \dots, \lambda_n]^T = [\bar{\lambda}, \lambda_n]^T \in \Upsilon_n$ , then  $\lambda^T A \lambda = \bar{\lambda}^T \bar{A}_{n-1} \bar{\lambda}$ , where  $\bar{A}_{n-1} = A_{n-1} + I_0 a_1^T + a_1 I_0^T + a_{nn} I_0 I_0^T$ ,  $I_0^T = [0, \dots, 0, 1]$ . Furthermore, if  $a_1 = 0$  and  $a_{nn} = 0$ , then  $\lambda^T A_n \lambda \geq 0$  iff  $\bar{\lambda}^T A_{n-1} \bar{\lambda} \geq 0$ , i.e.,  $A_n \supseteq 0$  iff  $A_{n-1} \supseteq 0$ . It also holds similarly for the strict inequality version.

**Lemma 5.2.3** [102]  $M \in \mathcal{R}^{n \times n}$  is (strictly) copositive, if it can be written as the sum of a positive semidefinite (definite)  $P \in \mathcal{R}^{n \times n}$  and an elementwise nonnegative matrix  $Q \in \mathcal{R}^{n \times n}$ :

$$M = P + Q \quad (5.4)$$

If  $n \leq 4$ , this condition is also necessary.

**Lemma 5.2.4** (Pólya's theorem) [81, 138] Let

$$F(\lambda) = F(\lambda_1, \lambda_2, \dots, \lambda_L) = \sum_{r=1}^{J(g)} \chi^r(g) \beta_r \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \dots \lambda_L^{\alpha_L} \quad (5.5)$$

where  $\alpha_1 \alpha_1 \dots \alpha_L = \mathcal{L}_r(g)$ , be a real homogeneous polynomial with degree  $d \in \mathcal{N}$ , which is positive for  $\forall \lambda \in \Psi_L$ . Then for a sufficiently large  $d \in \mathcal{N}$ , the product

$$\left( \sum_{i=1}^L \lambda_i \right)^d F(\lambda) \quad (5.6)$$

has all its coefficients strictly positive. Furthermore, we can choose

$$\begin{aligned} d &> \frac{g(g-1)\mu}{2\kappa} - g \\ \mu &= \max\{|\beta_r|, r = 1, \dots, J(g)\}, \quad \kappa = \min_{\lambda \in \Upsilon_L} F(\lambda) \end{aligned} \quad (5.7)$$

### 5.3 Partition-dependent Lyapunov Function

Define a partition-dependent Lyapunov function (PDLF) similar to (3.6) as follows

$$v(x) = \sum_{i=1}^{\|\mathcal{I}\|} \tau_i v_i(x), \quad \tau_i = \begin{cases} 1 & \text{mode } i \text{ is active} \\ 0 & \text{otherwise} \end{cases}, \quad v_i(x) = x^T P_i x \quad (5.8)$$

where  $P_i = P_i^T$  and  $v_i$  satisfies

$$v_i(x) > 0, \quad \forall x \in S_i, \quad \forall i \in \mathcal{I}, \quad x \neq 0; \quad (5.9)$$

$$v_i(x) \geq v_j(x), \quad \forall x \in S_{ij}, \quad i, j \in \tilde{\Omega}_c; \quad (5.10)$$

$$-\dot{v}_i(x) > 0, \quad \forall x \in S_i, \quad i \in \mathcal{I}, \quad x \neq 0 \quad (5.11)$$

**Theorem 5.3.1** *A PDLF given by (5.8) assures the PQ stability of the PWA system (5.1), iff there exists  $P_i = P_i^T$ , such that*

$$\lambda^T X_i^T P_i X_i \lambda > 0, \quad X_i \lambda \neq 0, \quad i \in \mathcal{I}, \quad \forall \lambda \in \Upsilon_{L_i}; \quad (5.12)$$

$$\lambda^T X_{ij}^T (P_i - P_j) X_{ij} \lambda \geq 0, \quad (i, j) \in \tilde{\Omega}_c, \quad \forall \lambda \in \Upsilon_{L_{ij}}; \quad (5.13)$$

$$-\lambda^T X_i^T (A_i^T P_i + P_i A_i) X_i \lambda > 0, \quad X_i \lambda \neq 0, \quad i \in \mathcal{I}, \quad \forall \lambda \in \Upsilon_{L_i} \quad (5.14)$$

**Proof:** Based on the definition of PQLF (5.8)-(5.11),  $v(x)$  is obviously Lipschitz, then the Dini-derivative  $D^+v(x)$  exists everywhere. Since  $S_i$  and  $S_{ij}$  are polytope, (5.9)-(5.11) are equivalent to (5.12)-(5.14). The necessary condition is obvious. And we only prove the sufficient condition. In fact, (5.11) implies  $x^T (A_i^T P_i + P_i A_i) x < -\epsilon_i x^T x$ , where  $\epsilon_i > 0$ ,  $x \in S_i$ . Choose  $\alpha_i = \frac{\epsilon_i}{2 \max_{\text{eig}(P_i)}$ , then  $x^T (A_i^T P_i + P_i A_i) x < -2\alpha_i x^T P_i x$ . In the switching boundary  $S_{ij}$ , we may choose  $\alpha_{ij} =$

$\min(\alpha_i, \alpha_j)$ , then  $D^+v(x) < -2\alpha_{ij}v(x)$ . (If  $v(i)$  is discontinuous at the switching boundary, then  $D^+v(x) = -\infty$ .) Finally, we choose  $\alpha = \min(\alpha_i)$ . ■

Noting that  $X_i\lambda_i \neq 0$  in (5.12) - (5.14), we may eliminate zero vector from  $X_i$  and  $X_{ij}$  if the origin is one of the vertices of  $S_i$  and  $S_{ij}$ . Based on Lemma 5.2.1 and Lemma 5.2.2, we can see that the elimination will not affect the inequalities of (5.12)-(5.14).

Now the inequalities (5.12)-(5.14) are equivalent to:

$$X_i^T P_i X_i \sqsupset 0, \quad i \in \mathcal{I}; \quad (5.15)$$

$$X_{ij}^T (P_i - P_j) X_{ij} \supseteq 0, \quad (i, j) \in \tilde{\Omega}_c; \quad (5.16)$$

$$-X_i^T (A_i^T P_i + P_i A_i) X_i \sqsupset 0, \quad i \in \mathcal{I} \quad (5.17)$$

Copositive problem is a NP-hard problem. A well-known sufficient result for matrix  $M$  to be copositive is shown in Lemma 5.2.3, which leads to the following theorem.

**Theorem 5.3.2** *A PDLF given by (5.8) assures the PQ stability of the PWA system (5.1), iff there exist  $P_i = P_i^T$ , such that (5.15)-(5.17) are satisfied. Furthermore, if  $\max L_i \leq 4$ , (5.15)-(5.17) are equivalent to the following LMI conditions*

$$X_i^T P_i X_i - U_i > 0, \quad i \in \mathcal{I}; \quad (5.18)$$

$$X_{ij}^T (P_i - P_j) X_{ij} - V_{ij} \geq 0, \quad (i, j) \in \tilde{\Omega}_c; \quad (5.19)$$

$$-X_i^T (A_i^T P_i + P_i A_i) X_i - W_i > 0, \quad i \in \mathcal{I} \quad (5.20)$$

where  $P_i = P_i^T$ ,  $U_i \succeq 0$ ,  $V_{ij} \succeq 0$ ,  $W_i \succeq 0$ . if  $\max L_i \geq 5$ , (5.18)-(5.20) are only sufficient.

**Remark 5.3.1** *In our formulation (5.12)-(5.14), the partition description is exact since the vertex representation of polytope is exact. However, in [102], ellipsoid is used to approximate the polytope. Nevertheless, our method has strong connection*

with the results in [102]. By a proper transformation, we can obtain the following form similar to that in [102].

$$P_i - Y_i^T U_i Y_i > 0, \quad i \in \mathcal{I}; \quad (5.21)$$

$$(P_i - P_j) - Y_{ij}^T V_{ij} Y_{ij} \geq 0, \quad (i, j) \in \tilde{\Omega}_c; \quad (5.22)$$

$$-(A_i^T P_i + P_i A_i) - Y_i^T W_i Y_i > 0, \quad i \in \mathcal{I} \quad (5.23)$$

where  $Y_i = X_i^+$  and  $Y_{ij} = X_{ij}^+$ .<sup>2</sup> For some special cases, we can even prove that the results in [103] are the same as ours under the assumption that Lyapunov function is continuous.<sup>3</sup>

Hence, considering the computational issue, if the sufficient conditions (5.18)-(5.20) are used, then our method has similar computational complexity compared with the one in [103]. In fact, if we note the conclusion in Lemma 5.2.2, we can further reduce the computational complexity by decreasing the dimension of  $S_i$  to  $L_i - 1$  and  $S_{ij}$  to  $L_{ij} - 1$ . For  $\max L_i \geq 5$ , the formulations (5.18)-(5.20) are only sufficient. There exist some other less conservative algorithms for computing copositive matrix [132, 21]. However, the computation of these algorithms is much more complex than that of (5.18)-(5.20), although they are also SDP-based.

**Example 5.3.1** Consider an example similar to Example 4.3 in [102] with simplex partitions:

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix},$$

$$F_1 = -F_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad F_2 = -F_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad f_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad i \in \mathcal{I}$$

<sup>2</sup>The Moore-Penrose inverse of  $X_i$  and  $Y_{ij}$  always exist. If  $S_i$  is full-dimensional simplex and  $X_i$  contains origin as its one of its column vector, after eliminate origin for  $X_i$ ,  $X_i$  is invertible, since  $\text{span}\{x_{1i}, \dots, x_{ni}\} = \mathcal{R}^n$ .

<sup>3</sup>For the case that  $i \in \mathcal{I}_0$  and  $S_i$  are full-dimensional simplex,  $X_i$  (with origin eliminated) is invertible, hence we can easily check that the cell bounding are exactly  $X_i^{-1}$ .

This example has already been proved to be stable [102]. We consider the decay rate  $2\alpha$  defined in Definition 5.1.1. In fact, using the results from Theorem 4.1 and Theorem 8.1 in [102] the non-trivial solutions give the almost same decay ratio  $2\alpha = 0.19$  as Theorem 5.3.2. This actually illustrates the conclusion in Remark 5.3.1.

However, our method shows advantage in cases when the partitions are not full-dimensional simplexes.

**Example 5.3.2** Consider the following example related to selector control, which is a common strategy for constraint handling in the process industry [24, 102].

$$\dot{x} = \begin{cases} Ax & K^T x \geq 0 \\ Ax + BK^T x & K^T x \leq 0 \end{cases} \quad |x_1| \leq 2, |x_2| \leq 4$$

$$A = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 21 \end{bmatrix}, \quad K = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In this example, each partition is a rectangle with four vertices. We shall choose  $U_i, W_i \in R^{4 \times 4}$  and  $V_{ij} \in R^{2 \times 2}$  for Theorem 5.3.2. Using the algorithms in [102], we can obtain maximum  $2\alpha = 2.0$ , while Theorem 5.3.2 gives  $2\alpha = 3.5$ . Note that in our LMI formulation, more variables are used: Theorem 5.3.2 uses 52 variables, Theorem 4.1 in [102] 18 variables and Theorem 8.1 in [102] 22 variables.

This example is with 4 vertices, so the equation (5.4) in Lemma 5.2.3 is both necessary and sufficient. In the next example, we will take a look at the case with more vertices.

**Example 5.3.3** Consider the bimodal system with the following parameters

$$A_1 = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0.5 \\ -1 & 0 & -0.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 2 & 0 \\ -2 & -2 & 0.5 \\ -1 & 0 & -0.1 \end{bmatrix},$$

$$F_1 = [1 \ 0 \ 0], \quad F_2 = [-1 \ 0 \ 0]$$

under the additional constraint:  $\max\{x_1, x_2, x_3\} = 2, \quad \min\{x_1, x_2, x_3\} = -2$ .

It is obvious that each partition is a cuboid with 8 vertices. Using the algorithms in [102], we can obtain maximum  $2\alpha = 0.90$ , while Theorem 5.3.2 gives  $2\alpha = 1.55$ .

**Remark 5.3.2** *In [102], it is shown that the partition refinement can improve the analysis performance, and an automated algorithm is given to subdivide the partitions into simplexes. This refinement procedure on non full-dimensional simplex may lead to better results. For example, we may connect the origin to other vertices in order to divide two subspaces into 6 simplex sub-partitions in Example 5.3.2, and then we can get almost the same results using these aforementioned algorithms, like Example 5.3.1. However, if we count the extra computation effects on the partition refinement, then our algorithm shall be more efficient than those in [102].*

**Remark 5.3.3** *The extension to the case  $i \in \mathcal{I}_1$  is straightforward if we choose the corresponding Lyapunov matrix as  $\bar{P}_i = \begin{bmatrix} P_i & p_i \\ p_i^T & r_i \end{bmatrix} \in \mathcal{R}^{(n+1) \times (n+1)}$  and  $\bar{X}_i = \begin{bmatrix} X_i \\ \mathbf{1} \end{bmatrix} \in \mathcal{R}^{(n+1) \times (L_i)}$ <sup>4</sup>. In fact, using the transformations introduced in Remark 2.10 (see also [102, 172]), we can easily obtain a similar result to Theorem 5.3.2 for general cases.*

## 5.4 Vertex-dependent Lyapunov Function

Due to the difficulty in partition refinement, we may consider the so-called parameter-dependent Lyapunov function. Actually, in a sense, the Lyapunov function defined by (5.8) is also a PDLF with respect to  $\tau_i$ . Here we use a more general kind

<sup>4</sup>For the case that  $S_i$  is full-dimensional simplex,  $\bar{X}_i$  is invertible, which can be proved by noting that  $x_{2i} - x_{1i}, \dots, x_{L_i i} - x_{(L_i-1)i}$  are linearly independent.

of parameter-dependent Lyapunov function, which can be considered as vertex-dependent Lyapunov function (VDLF):<sup>5</sup>

$$v(x) = \sum_{i=1}^{\|\mathcal{I}\|} \tau_i v_i(x), \quad \tau_i = \begin{cases} 1 & \text{mode } i \text{ is active} \\ 0 & \text{otherwise} \end{cases}, \quad v_i(x) = x^T \sum_{k=1}^{L_i} \lambda_k P_{ik} x \quad (5.24)$$

The new Lyapunov function candidate should satisfy (5.9)-(5.11), i.e.,

$$\Gamma_{0i}^{[1]} = \lambda^T X_i^T \left( \sum_{k=1}^{L_i} \lambda_k P_{ik} \right) X_i \lambda > 0, \quad X_i \lambda_i \neq 0, \quad i \in \mathcal{I}, \quad \lambda \in \Upsilon_{L_i}; \quad (5.25)$$

$$\Gamma_{0ij}^{[3]} = \lambda^T X_{ij}^T \left( \sum_{k=1}^{L_{ij}} \lambda_k P_{ik} - \sum_{k=1}^{L_{ij}} \lambda_k P_{jk} \right) X_{ij} \lambda \geq 0, \quad (i, j) \in \tilde{\Omega}_c, \quad \lambda \in \Upsilon_{L_{ij}}; \quad (5.26)$$

$$\Gamma_{0i}^{[2]} = -\lambda^T X_i^T \left( A_i^T \sum_{k=1}^{L_i} \lambda_k P_{ik} + \sum_{k=1}^{L_i} \lambda_k P_{ik} A_i + \Lambda_i \right) X_i \lambda > 0, \quad X_i \lambda \neq 0, \quad i \in \mathcal{I}, \quad \lambda \in \Upsilon_{L_i} \quad (5.27)$$

where  $\Lambda_i = \sum_{k=1}^{L_i} \lambda_k \sum_{l=1}^{L_i} \dot{x}_{ilk} P_{ik}$ ,  $X_i^+ A_i X_i := (\dot{x}_{ilk})_{L_i \times L_i}$ .<sup>6</sup>

Similar to Theorem 5.3.1, we have the following conclusion

**Theorem 5.4.1** *A VDLF given by (5.24) assures the PQ stability of the PWA system (5.1), iff there exists  $P_{ik} = P_{ik}^T$ , such that (5.25)-(5.27) are satisfied.*

However, (5.25)-(5.27) cannot be directly transferred into quadratic form with respect to  $\lambda$ , hence a new technique is necessary. In the following, we introduce two

<sup>5</sup>We consider the case that  $i \in \mathcal{I}_0$  first.

<sup>6</sup>Here we explain the derivation of (5.27). Note that

$$\dot{v}_i(x) = x^T \left( \sum_{k=1}^{L_i} \lambda_k P_{ik} \right) A_i x + x^T A_i^T \left( \sum_{k=1}^{L_i} \lambda_k P_{ik} \right) x + x^T \sum_{k=1}^{L_i} \dot{\lambda}_k P_{ik} x$$

Since  $x = X_i \lambda$ , a set of solution of  $\lambda$  with respect to  $x$  is  $\lambda = X_i^+ x + (I - X_i^+ X_i) y$ , where  $y \in R^{L_i}$  is any vector. Hence  $\dot{\lambda} = X_i^+ \dot{x} = X_i^+ A_i X_i \lambda$ . Now we have

$$\sum_{k=1}^{L_i} \dot{\lambda}_k P_{ik} = \sum_{k=1}^{L_i} \sum_{l=1}^{L_i} \dot{x}_{ikl} \lambda_l P_{ik} = \sum_{k=1}^{L_i} \lambda_k \sum_{l=1}^{L_i} \dot{x}_{ilk} P_{ik}$$

Hence we get (5.27).

new LMI formulations, which are inspired by the recent work on SOS presentation [132, 99, 172] and Pólya's theorem [129, 81, 138], to calculate the corresponding Lyapunov matrices.

Noting that  $\sum_{i=1}^L \lambda_i = 1$  for any  $\lambda \in \Upsilon_L$ , multiplying (5.25) -(5.27) on the left side with  $(\sum_{i=1}^{L^{(h)}} \lambda_i)^d$ , where  $L^{(h)}$  for (5.25) -(5.27) are  $L_i$ ,  $L_{ij}$  and  $L_i$ , respectively, and  $d \in \mathcal{N}$ , does not change the inequality. Hence, the inequalities (5.25) -(5.27) with multiplication  $(\sum_{i=1}^{L^{(h)}} \lambda_i)^d$  on the left side can be expressed as

$$\begin{aligned} (\sum_{i=1}^{L_i} \lambda_i)^d \Gamma_{0i}^{[1]} - \epsilon (\sum_{i=1}^{L_i} \lambda_i^\iota) &\geq 0 \\ (\sum_{i=1}^{L_{ij}} \lambda_{ij})^d \Gamma_{0i}^{[2]} &\geq 0 \\ (\sum_{i=1}^{L_i} \lambda_i)^d \Gamma_{0i}^{[3]} - \epsilon (\sum_{i=1}^{L_i} \lambda_i^\iota) &\geq 0 \end{aligned} \quad (5.28)$$

where  $\epsilon$  is a sufficiently small positive scalar and  $\iota$  is a proper even number, say,  $\iota = 4$ . Further, if  $d$  is an odd number, then (5.28) can be possibly reformulated as SOS polynomials with  $\iota = d + 3$ . If such SOS polynomials can be found, then (5.28) holds. Using SOSTOOLS [140] developed recently, the problem becomes simple.

It is known that when  $(L^{(h)}, d + 3) \in F = \{(\gamma, 2), (2, 2\delta), (3, 4) | \gamma, \delta \in \mathcal{N}\}$  [38], we can always find the SOS presentation for (5.28) if (5.28) holds. To illustrate, we consider the following example.

**Example 5.4.1** Consider a similar example as Example 5.3.1 with

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 1 \\ -5 & 0.1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}$$

and keeps other parameters the same.

Each partition has 3 vertices, and each switching boundary has 2 vertices. Now if we choose  $d = 1$ , then  $(L^{(h)}, d + 3) = (3, 4) \in F$  for the first and third inequalities in (5.28) and  $(L^{(h)}, d + 3) = (2, 2 * 2) \in F$  in the second inequality in (5.28). It means that (5.25)-(5.27) have a solution, iff (5.28) has one too. Using the methods

in [102] or Theorem 5.3.2, we cannot establish the stability. However, we can get the corresponding VDLP in Theorem 5.4.1 using formulation (5.28) with  $d = 1$ .

Nevertheless, for other situations that  $(L^{(h)}, d + 3) \notin F$ , we cannot make such conclusion. In fact, Chesi characterizes the positive polynomials which are not SOS in [36]. Based on our experience, the result using higher degree  $d > 1$  has no significant difference from the one using  $d = 1$  by this approach. Therefore, in the rest of this section, we introduce another approach. It is motivated by the recent work [129], where Oliveira and Peres provide a new condition for the stability of LTI polytopic uncertain systems using Pólya's theorem [81, 138] (see Lemma 5.2.4).

Note that (5.25) and (5.27) can be expanded as

$$\begin{aligned} & \sum_{k=1}^{L_i} \lambda_k^3 Q_{1i}^{(h)} + \sum_{k=1}^{L_i} \sum_{l=1, l \neq k}^{L_i} \lambda_k^2 \lambda_l Q_{2i}^{(h)} \\ & + \sum_{k=1}^{L_i-2} \sum_{l=k+1}^{L_i-1} \sum_{m=l+1}^{L_i} \lambda_k \lambda_l \lambda_m Q_{3i}^{(h)} > 0 \end{aligned} \quad (5.29)$$

where  $h$  for (5.25) and (5.27) is 1 and 2, respectively, and  $Q_{1i}^{(h)}$ ,  $Q_{2i}^{(h)}$  and  $Q_{3i}^{(h)}$  are defined as follows.

$$\begin{aligned} Q_{1i}^{(h)} &= x_{ik}^T \bar{Q}_{ik}^{(h)} x_{ik}, \\ Q_{2i}^{(h)} &= x_{ik}^T \bar{Q}_{ik}^{(h)} x_{il} + x_{il}^T \bar{Q}_{ik}^{(h)} x_{ik} + x_{ik}^T \bar{Q}_{il}^{(h)} x_{ik}, \\ Q_{3i}^{(h)} &= x_{il}^T \bar{Q}_{ik}^{(h)} x_{im} + x_{im}^T \bar{Q}_{ik}^{(h)} x_{il} + x_{ik}^T \bar{Q}_{il}^{(h)} x_{im} \\ & \quad + x_{im}^T \bar{Q}_{il}^{(h)} x_{ik} + x_{ik}^T \bar{Q}_{im}^{(h)} x_{il} + x_{il}^T \bar{Q}_{im}^{(h)} x_{ik}, \\ \bar{Q}_{ik}^{(1)} &= P_{ik}, \quad \bar{Q}_{ik}^{(2)} = -(A_i^T P_{ik} + P_{ik} A_i + \sum_{l=1}^{L_i} \dot{x}_{ikl} P_{ik}), \quad h = 1, 2 \end{aligned} \quad (5.30)$$

(5.26) can be expanded as

$$\begin{aligned} & \sum_{k=1}^{L_{ij}} \lambda_k^3 Q_{1ij} + \sum_{k=1}^{L_{ij}} \sum_{l=1, l \neq k}^{L_{ij}} \lambda_k^2 \lambda_l Q_{2ij} \\ & + \sum_{k=1}^{L_{ij}-2} \sum_{l=k+1}^{L_{ij}-1} \sum_{m=l+1}^{L_{ij}} \lambda_k \lambda_l \lambda_m Q_{3ij} \geq 0 \end{aligned} \quad (5.31)$$

and  $Q_{1ij}$ ,  $Q_{2ij}$  and  $Q_{3ij}$  are defined as follows.

$$\begin{aligned}
Q_{1ij} &= x_{ijk}^T \bar{Q}_{ijk} x_{ijk}, \\
Q_{2ij} &= x_{ijk}^T \bar{Q}_{ijk} x_{ijl} + x_{ijl}^T \bar{Q}_{ijk} x_{ijk} + x_{ijk}^T \bar{Q}_{ijl} x_{ijk}, \\
Q_{3ij} &= x_{ijl}^T \bar{Q}_{ijk} x_{ijm} + x_{ijm}^T \bar{Q}_{ijk} x_{ijl} + x_{ijk}^T \bar{Q}_{ijl} x_{ijm} \\
&\quad + x_{ijm}^T \bar{Q}_{ijl} x_{ijk} + x_{ijk}^T \bar{Q}_{ijm} x_{ijl} + x_{ijl}^T \bar{Q}_{ijm} x_{ijk}, \\
\bar{Q}_{ijk} &= P_{ik} - P_{jk}
\end{aligned} \tag{5.32}$$

In the rest of this section, we consider (5.28) with  $\epsilon = 0$ . (5.28) can be further expressed as (5.33) and (5.34) as follows.

$$\sum_{r=1}^{J(d+3)} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_{L_i}^{\alpha_{L_i}} \Gamma_{ri}^{(h)} > 0 \tag{5.33}$$

where  $h = 1, 2$ ,  $\alpha_1 \alpha_1 \cdots \alpha_{L_i} = \mathcal{L}_r(d+3)$  and

$$\begin{aligned}
\Gamma_{ri}^{(h)} &= \sum_{k=1}^{L_i} \chi_k^r(d, 3) Q_{1i}^{(h)} + \sum_{k=1}^{L_i} \sum_{l=1, l \neq k}^{L_i} \chi_{kl}^r(d, 2, 1) Q_{2i}^{(h)} \\
&\quad + \sum_{k=1}^{L_i-2} \sum_{l=k+1}^{L_i-1} \sum_{m=l+1}^{L_i} \chi_{klm}^r(d, 1, 1, 1) Q_{3i}^{(h)} \\
\sum_{r=1}^{J(d+3)} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_{L_{ij}}^{\alpha_{L_{ij}}} \Gamma_{rij}^{(3)} &\geq 0
\end{aligned} \tag{5.34}$$

where  $\alpha_1 \alpha_1 \cdots \alpha_{L_{ij}} = \mathcal{L}_r(d+3)$  and

$$\begin{aligned}
\Gamma_{rij}^{(3)} &= \sum_{k=1}^{L_{ij}} \chi_k^r(d, 3) Q_{1ij} + \sum_{k=1}^{L_{ij}} \sum_{l=1, l \neq k}^{L_{ij}} \chi_{kl}^r(d, 2, 1) Q_{2ij} \\
&\quad + \sum_{k=1}^{L_{ij}-2} \sum_{l=k+1}^{L_{ij}-1} \sum_{m=l+1}^{L_{ij}} \chi_{klm}^r(d, 1, 1, 1) Q_{3ij}
\end{aligned}$$

However, Pólya's theorem [81, 138] cannot be directly applied here, because it implies that  $d$  would be rather large in order to solve (5.28), due to  $\kappa = \min \Gamma_{0i}^{[1]} = 0$  for  $i \in \mathcal{I}_0$  (see Lemma 5.2.4). Hence, we shall make some changes in order to overcome this obstacle. Here we first consider the so-called conewise linear systems [31], and then extend to general cases. The state space of a conewise linear system is covered by a finite number of conical regions, and that on each of these regions separately we have linear dynamics, with continuous transitions between different partitions. Here we further assume that these conical regions are closed and bounded.

To illustrate our idea, we consider the following example. Assume that the system has four partitions as shown in Figure 5.2. For  $S_1$ , it is expressed as  $co\{x_1, x_2, x_3\}$ . We introduce two new vertices  $x'_1$  and  $x'_2$ , such that  $\frac{x'_1}{x_1} = \frac{x'_2}{x_2} = \varrho$ , where  $\varrho$  is a real scalar, such that  $0 < \varrho < 1$ . Define  $S_1^\varepsilon = co\{\varepsilon x_1, \varepsilon x_2, \varepsilon x'_1, \varepsilon x'_2\}$ , where  $0 \leq \varepsilon \leq 1$ . Then  $S_1 = \bigcup_{0 \leq \varepsilon \leq 1} S_1^\varepsilon$ . Now we consider the first inequality in (5.28). We use the vertices from  $S_1^\varepsilon$  instead of these from  $S_1$ . Since the inequality is homogeneous, we may only consider  $S_1^1$ . If the inequality holds for  $S_1^1$ , it is also satisfied for any other  $S_1^\varepsilon$  because of scaling. Hence, the first inequality of (5.28) holds for  $S_1$ . We can consider the third inequality similarly. For other partitions, we may also choose some new vertices with the rate  $\varrho$  and have the similar conclusion.

Now we consider the second inequality in (5.28). This inequality also has homogeneous properties. Hence, we only need to consider some special cases. As shown in Figure 5.2, for the case that the state trajectory happens to jump from  $S_1^1$  to  $S_2 \setminus S_2^1$ , in fact, this situation is included by certain  $\varepsilon < 1$ , see the internal partitions by dashed lines. Based on the above discussion, we can see that for conewise linear systems with the scaling technique, we may use the new partitions instead of the original partitions for stability analysis.

**Theorem 5.4.2** *A VDLF given by (5.24) assures the PQ stability of the conewise linear system (5.1), if there exist  $P_{ik} = P_{ik}^T$  and a sufficiently large  $d$  such that for  $r = 1, \dots, J(d+3)$ , the following LMIs hold*

$$\Gamma_{ri}^{(h)} > 0, \quad h = 1, 2, \quad \alpha_1 \alpha_1 \cdots \alpha_{L_i} = \mathcal{L}_r(d+3), \quad \forall i \in \mathcal{I} \quad (5.35)$$

$$\Gamma_{rij}^{(3)} \geq 0, \quad \alpha_1 \alpha_1 \cdots \alpha_{L_{ij}} = \mathcal{L}_r(d+3), \quad \forall (i, j) \in \tilde{\Omega}_c \quad (5.36)$$

**Proof:** Using the scaling technique, we can avoid the partition including zero. (5.35) and (5.36) imply (5.33) and (5.34), which in turn imply (5.25)-(5.27). ■

**Remark 5.4.1** *From Lemma 5.2.4, we can see the necessary and sufficient condition of Pólya's Theorem holds for the strictly positive polynomial. However, the*

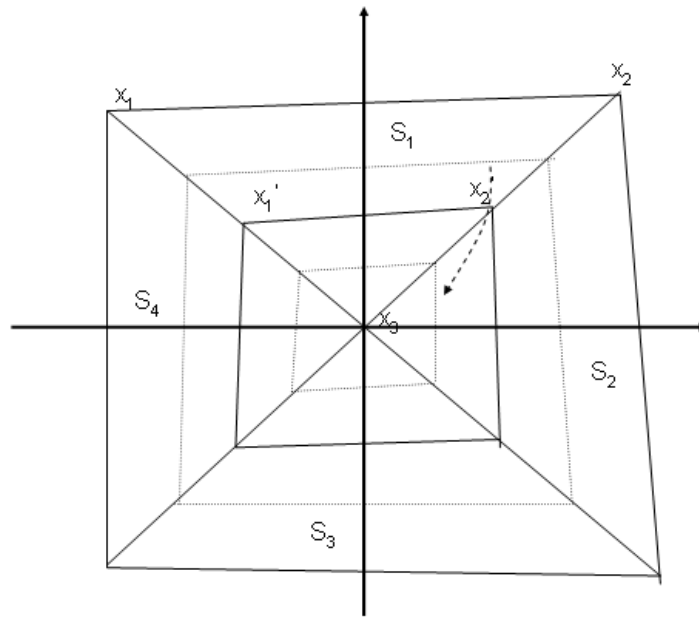


Figure 5.2: A conewise linear system with four partitions

*non-strict version of Pólya's Theorem does not hold generally. Since (5.26) is non-strict, Theorem 5.4.2 is only sufficient. Nevertheless, by increasing  $d$  up to the bound introduced in Lemma 5.2.4, we may expect better performance analytical result.*

**Example 5.4.2** *We still consider Example 5.4.1. This example has similar partition structure as shown in Figure 5.2. We choose  $\rho = 0.5$ , i.e.,*

$$X_1 = \begin{bmatrix} -1 & 1 & -0.5 & 0.5 \\ 1 & 1 & 0.5 & 0.5 \end{bmatrix},$$

*and so on. Using Theorem 5.4.2, with  $d = 0$ , we already get a feasible solution. This example illustrates the potential advantage of VDLP.*

**Remark 5.4.2** *For other situation rather than conewise linear systems, Theorem 5.4.2 is still applicable by noting two facts: 1) for those partitions with indices  $i \in \mathcal{I}_0$ , they can be divided as cone-like partitions; 2) for those partitions with indices  $i \in \mathcal{I}_1$ , we can directly apply the formulation in Theorem 5.4.2 without scaling.*

*It is also worthy of mentioning that Theorem 5.3.2 and Theorem 5.4.2 can be extended to a general polyhedron  $S$  defined by (2.2).*

## 5.5 Conclusion

This chapter has provided sufficient and necessary conditions for the PQ stability of PWA systems based on PDLF and VDLF with the aid of LMIs. Compared with the previous work on PQLF, PDLF gives an alternative interpretation to PQLF for some special cases, and better results in general. For VDLF, although it is not in a quadratic form, we have proposed two LMI approaches to overcome this obstacle. Some numerical examples have demonstrated the advantages of our methods. Moreover, our method can be easily extended to discrete-time PWA systems.

# Chapter 6

## Controllability and Reachability

### 6.1 Introduction

The notion of controllability and reachability, as well as observability, has played a central role throughout the history of modern control theory. In this section, we will first give a brief literature review on the controllability and reachability, as well as the observability of PWA systems and switched/hybrid dynamical systems. These properties have some direct applications. For example, the reachability asks if it is possible to steer the system from a given point or set of initial states to a given final state or a given set of final states. In fact, when we discuss the problem of safety verification, where one has to guarantee that some undesirable (unsafe) states are never reached while the systems is in operation. Hence we have to face the reachable set problem. Also, we can find that the reachability has implication in the study of the reduction of a realization [114, 14].

Controllability, reachability and observability of PWA systems generally cannot be simply deduced from system models, because the system is not always controllable even if the subsystem in every mode is controllable in common sense. Bemporad *et al.* [14] show that these properties of PWA systems can be very complex and present a number of counter-examples that rule out obvious conjectures about

inheriting observability and controllability from the composing linear subsystems. In fact, Blondel and Tsitsiklis [19] have already proved that the controllability problem of discrete-time PWA systems is a NP-hard problem. Sontag [156] shows that observability of general PWA systems is undecidable, but that N-observability<sup>1</sup> is decidable.

The curious fact to note here is that these early negative results have not prevented researchers from studying these properties of PWA systems [31]. Actually, along with the recognition of PWA systems as an important area of future research in recent years, renewed effort in analysis for PWA systems has been made, especially on the computational front. The researches have been mainly focusing on developing efficient computational algorithms, or deriving conditions for special/simplified models. For example, Kerrigan [114] suggest a computational approach using standard geometric concepts (Pontryagin/Minkowski difference, Minkowski sum, projection, and so on) to solve the robust reachability problem of disturbed discrete-time PWA systems, where the disturbance constraints are independent of state and input. And this approach is of orders of magnitude faster than the quantifier elimination approach used by Koutsoukos and Antsaklis [116] and Vidal *et al.* [165].

It is also meaningful to study some equivalent systems instead of PWA systems. It seems that the main line of work in this regard has emerged from the so-called mixed logical dynamical (MLD) systems introduced by Bemporad *et al.* [14]. In [14], a mixed integer linear programming (MILP)<sup>2</sup> based algorithm is introduced to check “incremental observability”, which is a stronger version of observability requiring a minimum amount of distinguishability between states, and controllability.

Although there are many issues to be explored with these general systems, we pay

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<sup>1</sup>Given an integer  $N$ , termed the observation horizon, the PWA system is N-observable if the sequence of measurements  $(y_0, y_1, \dots, y_N)$  determines the initial state  $x_0$  uniquely for all  $x_0 \in R^n$ .

<sup>2</sup>MILP is a kind of linear programming (LP), where the system model has some variables to be required to take on integer values. [17, 14]

our attention to some kinds of simplified systems. There are mainly three types.

The first type has simplification in dimension and model number, say, planar systems and bimodal systems. In this case, the conditions usually are given in explicit forms based on classification methods. For example, Çamlıbel *et al.* give a necessary and sufficient condition for the controllability of continuous-time planar bimodal linear complementarity systems in [29], and an extension for planar bimodal linear systems with continuous dynamics along the hyperplane boundary in [30]. Brogliato [28, 27] obtains necessary and sufficient conditions for global controllability of a class of continuous-time linear complementarity systems with planar evolution variational inequalities. Xie *et al.* [169] provide an explicitly necessary and sufficient condition for null controllability of a special class of discrete-time planar bimodal PWL systems, where two models have some same parameters. In [80], the reachability of PWA systems on polytope in the plane (two-dimension) is analyzed. Specially, the null controllability and (asymptotically) null controllable region of both continuous-time and discrete-time saturation systems are studied by Hu and Lin [90].

The second type has simplification in the region description. A typical example is conewise linear system. Necessary and sufficient conditions for observability and controllability are derived in [31, 32]. Another early example is the so-called hypersurface system<sup>3</sup>, a controllability characterization of which has been provided in [118].

The third type has simplification in switching sequences or models. As we know, once the switching sequences or models are known as “a priori”, the so-called controllability matrix or observability matrix can be used to analyze the corresponding properties directly [151]. There are several such system models. The first example is piecewise constant system (PWCS)<sup>4</sup>. An early work by Goshen-

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<sup>3</sup>It is assumed that the number of inputs in each subsystem is equal to the number of states minus one [118].

<sup>4</sup>PWCS's time-line of measurements is decided by finite number of time segments, i.e, the system matrices are piecewise constant functions of time. There are cases where the time varying system can be expressed by a PWCS with little loss of accuracy, and with no loss of the

Meshin and Bar-Itzhack [75] develop an observability matrix for PWCS in both continuous-time and discrete-time representations. In [52], Ezzine and Haddada discuss both observability and controllability. The second example is sampled-data PWA system. Azuma and Imura [6, 7] propose an algebraic characterization and a probabilistic approach to solve the controllability and reachability problem for such systems. The third example is the PWA system whose partitions are based on the output space [57]. In addition, some works are under the assumption that mode sequence is arbitrary and independent of the initial state and input sequence in the switched systems. Under this assumption, the so-called pathwise observability and controllability are decidable [8].

Although the conditions for simplified systems are difficult to be extended to general cases, we may still use them as reference. For example, the criteria of reachability for discrete-time arbitrary switching systems are sufficient conditions for DPS. Hence they are still worth investigating.

In this chapter we mainly discuss a type of bimodal PWA systems. The examples of bimodal PWA systems may exist in temperature control systems [14], digital control of autonomous cars with a two-value gear-box [135], chaotic systems [137, 34, 61] and digital control of ideal diodes with two modes: the blocking mode and the conducting mode. A simple switched circuit may also lead to a bimodal PWA systems [181].

The planar discrete-time system considered is defined as follows.

$$x(k+1) = \begin{cases} A_1x(k) + B_1u(k), & \text{if } C^T x(k) \geq 0 \\ A_2x(k) + B_2u(k), & \text{if } C^T x(k) < 0 \end{cases} \quad (6.1)$$

where  $x = [x_1 \ x_2]^T \in \mathcal{R}^2$  is the state vector,  $u \in \mathcal{R}$  is the input.  $A_1, A_2 \in \mathcal{R}^{2 \times 2}$  and  $B_1, B_2, C \in \mathcal{R}^2$  are constant matrices. We also denote  $\mathcal{S}_1 = \{x | C^T x \geq 0\}$  and  $\mathcal{S}_2 = \{x | C^T x < 0\}$ .

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characteristic behavior of the system [75].

Note that there are many different concepts on controllability, reachability and observability among these analysis [69, 114]. The corresponding (null) controllability and reachability adopted in this chapter are defined as follows.

**Definition 6.1.1** (*Null Controllability*) A nonzero state of (6.1) is (null) controllable, if there exist inputs such that the initial state is driven to zero in finite steps. The system (6.1) is null controllable (NC), if all nonzero states are NC.

**Definition 6.1.2** (*Reachability*) A given state of (6.1) is reachable, if there exist inputs such that the given state can be reached from zero in finite steps. The system (6.1) is reachable, if all nonzero states are reachable.

We also define the NC set and reachable set as follows.

**Definition 6.1.3** (*Null Controllable set*) The NC set  $\mathcal{X}_k$  in  $k$  steps is defined as  $\{x_0 \in \mathcal{R}^2 | \exists u(0), \dots, u(k-1), \text{ such that } x(0) = x_0 \text{ and } x(k) = 0\}$ .

**Definition 6.1.4** (*Reachable set*) The reachable set  $\mathcal{Y}_k$  in  $k$  steps is defined as  $\{x_f \in \mathcal{R}^2 | \exists u(0), \dots, u(k-1), \text{ such that } x(0) = 0 \text{ and } x(k) = x_f\}$ .

**Lemma 6.1.1**  $\mathcal{X}_k \subseteq \mathcal{X}_{k+1}$ , for any integer  $k > 0$ . Thus the system (6.1) is NC, if and only if there exists  $k < \infty$  such that  $\mathcal{X}_k = \mathcal{R}^2$ .

**Lemma 6.1.2**  $\mathcal{Y}_k \subseteq \mathcal{Y}_{k+1}$ , for any integer  $k > 0$ . Thus the system (6.1) is reachable, if and only if there exists  $k < \infty$  such that  $\mathcal{Y}_k = \mathcal{R}^2$ .

Throughout the rest of chapter, we denote  $u(k) = u_k$ .

## 6.2 Controllability

In order to simplify the analysis, we divide the discussion into two cases based on whether  $[B_1, B_2]$  is nonsingular. We discuss the nonsingular case in subsection 6.2.1, while leave the singular case in subsection 6.2.2.

### 6.2.1 $[B_1, B_2]$ is nonsingular

In this section, we consider the system (6.1) where the matrix  $[B_1, B_2]$  is nonsingular. Firstly, some preliminaries on the null controllability are given. We then discuss the null controllability problem under several cases of system parameters.

Under the non-singularity assumption, the system (6.1) can be simplified. Define  $P = [B_1, B_2]^{-1}$ ,  $\bar{x} = Px$ . (6.1) can be rewritten as

$$\bar{x}(k+1) = \begin{cases} \bar{A}_1 \bar{x}(k) + \bar{B}_1 u(k) & \text{if } \bar{C}^T \bar{x}(k) \leq 0 \\ \bar{A}_2 \bar{x}(k) + \bar{B}_2 u(k) & \text{if } \bar{C}^T \bar{x}(k) > 0 \end{cases} \quad (6.2)$$

where  $\bar{B}_1 = [1 \ 0]^T$ ,  $\bar{B}_2 = [0 \ 1]^T$ ,  $\bar{A}_i = PA_iP^{-1}$ ,  $i = 1, 2$  and  $\bar{C} = P^{-T}C$ .

It is obvious that for a non-singular matrix  $P$ , the transformation  $\bar{x} = Px$  will not change the property of null controllability. For this reason, we can assume, without loss of generality, that

$$A_i = \begin{bmatrix} A_{i1}^T \\ A_{i2}^T \end{bmatrix} = \begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix}, \quad i = 1, 2, \quad (6.3)$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

For future use, we denote  $\phi$  as the empty set,  $\lambda_1 = C^T \begin{bmatrix} \delta_1 \\ -\gamma_1 \end{bmatrix}$ ,  $\lambda_2 = C^T \begin{bmatrix} \beta_2 \\ -\alpha_2 \end{bmatrix}$ ,

and

$$\mathcal{S}_+ = \begin{cases} \{s \in \mathcal{R} | \lambda s \geq 0\} & \text{if } \lambda \neq 0 \\ \mathcal{R} & \text{if } \lambda = 0 \end{cases}, \quad \mathcal{S}_- = \begin{cases} \{s \in \mathcal{R} | \lambda s < 0\} & \text{if } \lambda \neq 0 \\ \emptyset & \text{if } \lambda = 0. \end{cases} \quad (6.4)$$

**Lemma 6.2.1** *The following propositions hold for the system (6.1) with parameters of (6.3):*

- (1) Assume that  $c_i \neq 0$ ,  $i = 1, 2$ . If there exists  $k$  such that  $\{x|C^T x \geq 0\} \subseteq \mathcal{X}_k$  or  $\{x|C^T x < 0\} \subseteq \mathcal{X}_k$  or  $\{x|C^T x = 0\} \subseteq \mathcal{X}_k$ , then  $\mathcal{X}_{k+1} = \mathcal{R}^2$  and system (6.1) is NC.
- (2) Assume that  $c_1 \neq 0$  and  $c_2 = 0$ .
- (a) If there exists  $k$  such that  $\{x|C^T x < 0\} \subseteq \mathcal{X}_k$ , then  $\mathcal{X}_{k+1} = \mathcal{R}^2$  and system (6.1) is NC.
- (b) When  $\beta_2 \neq 0$ , if there exists  $k$  such that  $\{x|C^T x < 0, A_{21}^T x \geq 0\} \cup \{x|C^T x \geq 0\} \subseteq \mathcal{X}_k$  (or  $\{x|C^T x < 0, A_{21}^T x = 0\} \cup \{x|C^T x \geq 0\} \subseteq \mathcal{X}_k$ ), then  $\mathcal{X}_{k+1} = \mathcal{R}^2$  and system (6.1) is NC.
- (c) When  $\delta_1 \neq 0$ , if there exists  $k$  such that  $\{x|C^T x < 0, A_{12}^T x \geq 0\} \cup \{x|C^T x \geq 0\} \subseteq \mathcal{X}_k$ , then  $\mathcal{X}_{k+1} = \mathcal{R}^2$  and system (6.1) is NC.
- (3) Assume that  $c_1 = 0$  and  $c_2 \neq 0$ .
- (a) If there exists  $k$  such that  $\{x|C^T x \geq 0\} \subseteq \mathcal{X}_k$ , then  $\mathcal{X}_{k+1} = \mathcal{R}^2$  and system (6.1) is NC.
- (b) When  $\gamma_1 \neq 0$ , if there exists  $k$  such that  $\{x|C^T x \geq 0, A_{12}^T x \geq 0\} \cup \{x|C^T x < 0\} \subseteq \mathcal{X}_k$ , then  $\mathcal{X}_{k+1} = \mathcal{R}^2$  and system (6.1) is NC.
- (4) If  $\mathcal{X}_{k+1} = \mathcal{X}_k$ , then  $\mathcal{X}_{k+l} = \mathcal{X}_k$ ,  $\forall l \geq 0$ .

**Proof:** The proposition (1) and proposition (4) have been considered in [169]. We only prove proposition (2) since proposition (3) is similar.

To show (2)(a), note that for any  $x \in \mathcal{R}^2$ , there are two situations: either  $x \in \{x|C^T x < 0\} \subseteq \mathcal{X}_k$  or  $x \in \{x|C^T x \geq 0\}$ . For the former, it is already NC. For the latter, considering that  $C^T(A_1 x + B_1 u) = c_1[\alpha_1, \beta_1]x + c_1 u$ , we may choose a proper  $u$ , such that  $C^T(A_1 x + B_1 u) < 0$ , which means after one step, the state will lie in  $\mathcal{X}_k$ . Thus  $\mathcal{X}_{k+1} = \mathcal{R}^2$ .

As for (2)(b), we also consider such situations. If  $x \in \{x|C^T x \geq 0\}$ , it is already

NC. If  $x \in \{x|C^T x < 0\}$  and the next (one-step) state lies in  $\mathcal{X}_k$ , then

$$C^T(A_2x + B_2u) \geq 0 \text{ or } \begin{cases} C^T(A_2x + B_2u) < 0 \\ A_{21}^T(A_2x + B_2u) \geq 0. \end{cases} \quad (6.5)$$

That is

$$c_1 A_{21}^T x \geq 0 \text{ or } \begin{cases} c_1 A_{21}^T x < 0 \\ A_{21}^T(A_2x + B_2u) \geq 0. \end{cases} \quad (6.6)$$

Since we can always find a proper  $u$  to satisfy  $A_{21}^T(A_2x + B_2u) \geq 0$  because  $\beta_2 \neq 0$ , so  $\{x|C^T x < 0\} \subseteq \mathcal{X}_{k+1}$ . Hence,  $\mathcal{X}_{k+1} = \mathcal{R}^2$ . From the proof, we may observe that  $\{x|C^T x < 0, A_{21}^T x = 0\} \cup \{x|C^T x \geq 0\} \subseteq \mathcal{X}_k$  is also a sufficient condition. The proof of (2)(c) is similar. ■

Let us first consider the following two special cases:

$$(1) \ A_{12} = 0$$

$$(2) \ A_{21} = 0$$

under which we can easily have the following results.

**Lemma 6.2.2** *If  $A_{12} = 0$  and  $c_2 \neq 0$  (or  $A_{21} = 0$  and  $c_1 \neq 0$ ), then the system (6.1) with parameters of (6.3) is NC and  $\mathcal{X}_2 = \mathcal{R}^2$ .*

**Proof:** If  $A_{12} = 0$ , the system dynamic corresponding to the first mode is determined by  $A_1x + B_1u = \begin{bmatrix} A_{11}^T x + u \\ 0 \end{bmatrix}$ . Thus  $\{x|C^T x \geq 0\}$  is NC. According to Lemma 6.2.1 (3)(a),  $\mathcal{X}_2 = \mathcal{R}^2$ . Similarly, if  $A_{21} = 0$ ,  $\{x|C^T x < 0\} \subseteq \mathcal{X}_1$ , thus the result follows. ■

**Lemma 6.2.3** *Consider the system (6.1) with parameters of (6.3) in the following two cases.*

(1)  $A_{12} = 0$ ,  $c_2 = 0$ ,  $\beta_2 = 0$  and  $A_{21} \neq 0$

(2)  $A_{21} = 0$ ,  $c_1 = 0$ ,  $\gamma_1 = 0$  and  $A_{12} \neq 0$

For case (1) (respectively, case (2)), the system is not NC if and only if  $\alpha_2 > 0$  (respectively,  $\delta_1 > 0$ ); the system is NC if and only if  $\alpha_2 < 0$  (respectively,  $\delta_1 < 0$ ).

**Proof:** For Case (1), it is easy to deduce that  $\mathcal{X}_1 = \{x | C^T x \geq 0\}$  and  $\mathcal{X}_2 = \{x | C^T x \geq 0\} \cup \{x | C^T x < 0, c_1 A_{21}^T x \geq 0\}$ . Hence, we can prove that if  $\alpha_2 > 0$ ,  $\mathcal{X}_3 = \mathcal{X}_2$ ; otherwise,  $\mathcal{X}_3 = \mathcal{R}^2$ .

For Case (2), it is easy to deduce that  $\mathcal{X}_1 = \{x | x_2 = 0\} \cup \{x | C^T x < 0\}$ . For any  $x \in \mathcal{X}_2$ , if  $C^T x \geq 0$ , we have:  $x_2 = 0$  or  $c_2 \delta_1 x_2 < 0$ . Thus we can see that if  $\delta_1 > 0$ ,  $\mathcal{X}_3 = \mathcal{X}_2$ ; otherwise,  $\mathcal{X}_3 = \mathcal{R}^2$ . ■

Lemmas 6.2.1, 6.2.2 and 6.2.3 show that if  $A_{12} = 0$  (respectively,  $A_{21} = 0$ ), the system is not NC if and only if  $c_2 = 0$ ,  $\beta_2 = 0$ ,  $\alpha_2 > 0$  and  $A_{21} \neq 0$  (respectively,  $c_1 = 0$ ,  $\gamma_1 = 0$ ,  $\delta_1 > 0$  and  $A_{12} \neq 0$ ). Hence in the following three subsections, we only need to consider the cases with  $A_{12} \neq 0$  and  $A_{21} \neq 0$ .

(I)  $\alpha_2 \delta_1 - \gamma_1 \beta_2 \neq 0$  and  $c_i \neq 0$ ,  $i = 1, 2$

Now we are ready to present one of the main results in this section.

**Theorem 6.2.4** Consider the system defined by (6.1) with parameters of (6.3) satisfying  $\alpha_2 \delta_1 - \gamma_1 \beta_2 \neq 0$  and  $c_1 c_2 \neq 0$ . The system is not NC if and only if one of the following conditions holds:

(1)  $\lambda_2 = 0$  and either one of the following conditions is satisfied:

(a)  $\gamma_1 = 0$ ,  $\delta_1 \lambda_1 \eta \geq 0$ ,  $\lambda_1 \neq 0$

(b)  $\gamma_1 \lambda_1 < 0$ ,  $\delta_1 c_2 \leq 0$ ,  $c_1 \lambda_1 > 0$ ,  $\delta_1 \lambda_1 \eta \geq 0$

(c)  $\gamma_1 \lambda_1 > 0$ ,  $\delta_1 c_2 \geq 0$ ,  $c_1 \lambda_1 > 0$ ,  $\delta_1 \lambda_1 \eta \geq 0$

(2)  $\lambda_2 \neq 0$  and either one of the following conditions is satisfied:

$$(a) \lambda_1 \lambda_2 > 0, \gamma_1 \lambda_1 \leq 0, \delta_1 \lambda_1 > 0, \alpha_2 \lambda_2 > 0, \beta_2 \lambda_2 \leq 0, c_2 \lambda_2 < 0$$

$$(b) \lambda_1 \lambda_2 > 0, \gamma_1 \lambda_1 > 0, \delta_1 \lambda_1 \leq 0, \alpha_2 \lambda_2 \leq 0, \beta_2 \lambda_2 > 0, c_2 \lambda_2 < 0$$

$$(c) \lambda_1 \lambda_2 < 0, \gamma_1 \lambda_1 \leq 0, \delta_1 \lambda_1 \leq 0, \alpha_2 \lambda_2 > 0, \beta_2 \lambda_2 > 0, c_2 \lambda_2 > 0$$

$$(d) \lambda_1 \lambda_2 < 0, \gamma_1 \lambda_1 > 0, \delta_1 \lambda_1 > 0, \alpha_2 \lambda_2 \leq 0, \beta_2 \lambda_2 \leq 0, c_2 \lambda_2 > 0$$

**Proof:** The main ideas used in the proof lie in the following observation:

- (1) The number of classification is finite with respect to the finite system parameters;
- (2) Based on the set mapping, the NC set  $\mathcal{X}_k$  can be determined as follows. First, we can easily deduce the set  $\mathcal{X}_1$ . Once  $\mathcal{X}_k$  is determined, for any  $x \in \mathcal{R}^2$ , considering  $\bar{x} = A_1 x + B_1 u$  when  $\{x | C^T x \geq 0\}$  and/or  $\bar{x} = A_2 x + B_2 u$  when  $\{x | C^T x < 0\}$ , we may determine  $\mathcal{X}_{k+1}$ . Thus we may find for some  $k$ , either  $\mathcal{X}_k = \mathcal{R}^2$ , which leads the system to be NC, or  $\mathcal{X}_{k+1} = \mathcal{X}_k \neq \mathcal{R}^2$ , which leads the system to be not NC. Fortunately, only finite number of such steps suffices.

The detail is shown in Appendix B.1 using a similar framework of [169]. ■

(II)  $\alpha_2 \delta_1 = \beta_2 \gamma_1$  &  $c_i \neq 0, i = 1, 2$

We have the following result.

**Theorem 6.2.5** Consider the system defined by (6.1) with parameters of (6.3) satisfying  $\alpha_2 \delta_1 = \beta_2 \gamma_1$  and  $c_i \neq 0, i = 1, 2$ . Then, the system (6.1) is NC.

**Proof:** See Appendix B.2. ■

(III)  $c_i = 0, i = 1$  or  $2$

Under this condition, the system (6.1) with the parameters of (6.3) has its modes determined by the  $x_2$ -axis or  $x_1$ -axis. We have the following result.

**Theorem 6.2.6** *Consider the system defined by (6.1) with parameters of (6.3) satisfying  $A_{12} \neq 0$  and  $A_{21} \neq 0$ . Then the system is not NC if and only if one of the following conditions holds:*

- (1)  $c_1 \neq 0, c_2 = 0, \delta_1 = 0, \beta_2 = 0, \alpha_2 > 0$
- (2)  $c_1 \neq 0, c_2 = 0, \delta_1 > 0, \beta_2 = 0, \gamma_1 \neq 0, \alpha_2 > 0$
- (3)  $c_1 = 0, c_2 \neq 0, \gamma_1 = 0, \delta_1 \geq 0$
- (4)  $c_1 = 0, c_2 \neq 0, \alpha_2 = 0, \delta_1 < 0, \gamma_1 \beta_2 > 0$

**Proof:** See Appendix B.3. ■

**Remark 6.2.1** *Consider the system defined by (6.1) with parameters of (6.3). From Theorems 6.2.4, 6.2.5 and 6.2.6 and their proofs, the following properties can be observed:*

- (1) *The null controllability is not relevant to the parameters  $\alpha_1, \beta_1, \gamma_2$  and  $\delta_2$ .*
- (2) *If it is NC, then a non-zero initial state in  $\mathcal{R}^2$  can be driven to zero within at most four steps with proper inputs.*
- (3) *If it is not NC, then  $\mathcal{X}_3$  is at most an angular region and for a non-zero initial state in any NC set (even for  $\mathcal{X}_k, k > 2$ ), zero can be reached within at most three steps with proper inputs.*

**Remark 6.2.2** *As pointed out in Remark B.1.3 of Appendix B.1, the conditions for NC are related to the relative position of lines  $C^T x = 0, A_{12}^T x = 0$  and  $A_{21}^T x = 0$ .*

This partially explains why the conditions in Theorem 6.2.5 and Theorem 6.2.6 are less complex than Theorem 6.2.4. For example, in the case that  $\lambda_1 \neq 0$  and  $\lambda_2 \neq 0$ , the situations in Theorem 6.2.4 are rather complex, whereas in Theorem 6.2.6, since  $\delta_1 \lambda_1 \beta_2 \lambda_2 > 0$ , it is relatively easier to deduce the NC set.

### 6.2.2 $[B_1, B_2]$ is singular

In this section, we briefly discuss the case that  $[B_1, B_2]$  is singular. For a non-zero  $[B_1, B_2]$ , it is obvious that there are two cases for  $\det[B_1, B_2] = 0$ .

- (1)  $B_1 \neq 0$  and  $B_2 \neq 0$
- (2)  $B_1 = 0$  or  $B_2 = 0$

For the first case, we can always find a nonsingular matrix  $Q$  that transforms  $B_i$ ,  $i = 1, 2$  into  $[1 \ 0]^T$ . Thus, without loss of generality, we can assume that the system (6.1) is of the form:

$$A_i = \begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix}, \quad i = 1, 2, \quad B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \quad (6.7)$$

In this case, Propositions (1), (2)(a) and (4) of Lemma 6.2.1 still hold. If we change the condition  $\beta_2 \neq 0$  in (2)(b) as  $\alpha_2 \neq 0$  and  $\delta_1 \neq 0$  in (2)(c) as  $\gamma_1 \neq 0$ , then (2)(b) and (2)(c) stand. Meanwhile, we may find that the analytic procedure in Section 6.2.1 can be applied here too. The detail is omitted.

The second case involves a partition where the control input is zero. This situation will be more complicated and no general result has been found yet. However, it is possible to deduce the null controllability. For example, we consider the case

$$A_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ 0 & \delta_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \alpha_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ 0 \end{bmatrix},$$

where  $\alpha_2 < 0$  and  $c_1 > 0$ . In this case, any  $x \in \{x|C^T x < 0\}$  can jump to the set  $\{x|x_1 > 0\}$  in one step and any  $x \in \{x|C^T x \geq 0\}$  can be driven to  $\{x|x_1 > 0\}$  by a proper  $u$ . Since  $\{x|x_1 > 0\}$  is NC, then the system is NC.

## 6.3 Reachability

Note that the full rank of the reachability matrix  $R_i(k)$  [151] of each of the two modes may not guarantee the reachability of the system (6.1). For example,  $A_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ 1 & 1 \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} 1 & 1 \\ \gamma_2 & \beta_2 \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , where  $\beta_2 \neq 0$ . Thus we have  $R_1(2) = \begin{bmatrix} 1 & \alpha_1 \\ 0 & 1 \end{bmatrix}$  and  $R_2(2) = \begin{bmatrix} 1 & 1 \\ 0 & \beta_2 \end{bmatrix}$ . It is obvious that  $R_i(2)$ ,  $i = 1, 2$ , is of full rank. But the system is not reachable. In fact,  $\mathcal{Y}_3 = \mathcal{Y}_2 = \{x|x_1 < 0\} \cup \{x|x_2 \geq 0\} \neq \mathcal{R}^2$ .

Nonetheless, along a similar line of the discussions on controllability, we can obtain the explicit conditions for reachability. Different from the NC set, where  $\mathcal{X}_k = \{x|A_1 x + B_1 u \in \mathcal{X}_{k-1}, x \in S_1\} \cup \{x|A_2 x + B_2 u \in \mathcal{X}_{k-1}, x \in S_2\}$ , the reachable set can be obtained by  $\mathcal{Y}_k = \{A_1 x + B_1 u | x \in (S_1 \cap \mathcal{Y}_{k-1})\} \cup \{A_2 x + B_2 u | x \in (S_2 \cap \mathcal{Y}_{k-1})\}$ .

### 6.3.1 $[B_1, B_2]$ is nonsingular

Since  $x(1) = B_1 u_1$ ,  $\mathcal{Y}_1 = \{x|x_2 = 0\}$ , then all states in  $\mathcal{S}_1$  by one-step transit remains in  $\mathcal{S}_1$  if and only if  $c_1 = 0$  and  $c_2 \neq 0$ . Firstly, we consider the case that  $c_1 = 0$  and  $c_2 \neq 0$ .  $x(2) = A_1 x(1) + B_1 u_2 = \begin{bmatrix} \alpha_1 u_1 + u_2 \\ \gamma_1 u_1 \end{bmatrix}$ . It is obvious that when  $\gamma_1 = 0$ ,  $\mathcal{Y}_2 = \mathcal{Y}_1$ , the system is not reachable. Otherwise,  $\mathcal{Y}_2 = \mathcal{R}^2$ .

Secondly, we consider the case that  $c_1 \neq 0$ . For any  $x_1 \in \mathcal{S}_1$ , i.e.,  $c_1 u_1 \geq 0$ ,

$$x(2) = A_1 x(1) + B_1 u_2 = \begin{bmatrix} \alpha_1 u_1 + u_2 \\ \gamma_1 u_1 \end{bmatrix} \quad (6.8)$$

and for any  $x_1 \in \mathcal{S}_2$ , i.e.,  $c_1 u_1 < 0$ ,

$$x(2) = A_2 x(1) + B_2 u_2 = \begin{bmatrix} \alpha_2 u_1 \\ \gamma_2 u_1 + u_2 \end{bmatrix} \quad (6.9)$$

Thus we have the results for  $\mathcal{Y}_2$  shown in Table 6.1.

Case	$\gamma_1$	$\alpha_2$	$\mathcal{Y}_2$
1	=0	=0	$\{x x_1 = 0\} \cup \{x x_2 = 0\}$
2	=0	$\neq 0$	$\{x x_2 = 0\} \cup \{x x_1 < 0\}$ , if $c_1 \alpha_2 > 0$
			$\{x x_2 = 0\} \cup \{x x_1 > 0\}$ , if $c_1 \alpha_2 < 0$
3	$\neq 0$	=0	$\{x x_2 \geq 0\} \cup \{x x_1 = 0\}$ , if $c_1 \gamma_1 > 0$
			$\{x x_2 \leq 0\} \cup \{x x_1 = 0\}$ , if $c_1 \gamma_1 < 0$
4	$\neq 0$	$\neq 0$	$\{x x_2 \geq 0\} \cup \{x x_1 > 0\}$ or $\{x x_2 \leq 0\} \cup \{x x_1 < 0\}$ , if $\gamma_1 \alpha_2 < 0$
			$\{x x_2 \geq 0\} \cup \{x x_1 < 0\}$ or $\{x x_2 \leq 0\} \cup \{x x_1 > 0\}$ , if $\gamma_1 \alpha_2 > 0$

Table 6.1: Four cases of  $\mathcal{Y}_2$

To consider  $\mathcal{Y}_k$ ,  $k \geq 3$ , similar to the analysis on controllability we can discuss the reachability under several situations. One of main steps is to identify the value set of  $f = A_{12}^T x$  or  $g = A_{21}^T x$  under the constraint  $x_1 \geq 0$  (or  $x_1 \leq 0$  or  $x_2 \geq 0$  or  $x_2 \leq 0$ , or strict version) and  $C^T x \geq 0$  (or  $C^T x < 0$ ). For example, we have:

**Lemma 6.3.1** Consider the value set of  $f$  in  $x_2 \geq 0$  and  $C^T x \geq 0$ , where  $c_1 \neq 0$ .

- (1)  $f \geq 0$  if and only if a)  $c_1 < 0$ ,  $\gamma_1 \leq 0$ ,  $\lambda_1 \leq 0$  or b)  $c_1 > 0$ ,  $\gamma_1 \geq 0$ ,  $\lambda_1 \geq 0$ .
- (2)  $f \leq 0$  if and only if a)  $c_1 < 0$ ,  $\gamma_1 \geq 0$ ,  $\lambda_1 \geq 0$  or b)  $c_1 > 0$ ,  $\gamma_1 \leq 0$ ,  $\lambda_1 \leq 0$ .
- (3)  $\{f\} = \mathcal{R}$  if and only if  $\gamma_1 \lambda_1 < 0$ .

We now present the reachability results for the case that  $A_{12} \neq 0$  and  $A_{21} \neq 0$ .

**Theorem 6.3.2** Consider the system defined by (6.1) with parameters of (6.3) satisfying  $c_1 \neq 0$ . If  $A_{12} \neq 0$  and  $A_{21} \neq 0$ , then the system is not reachable if and only if one of the following conditions holds:

$$(1) (\gamma_1 = 0) \cap (\Phi_{81} \cup \Phi_{82});$$

$$(2) (\alpha_2 = 0) \cap (\Phi_{51} \cup \Phi_{52} \cup \Phi_{53} \cup \Phi_{54});$$

$$(3) \Phi_{71} \cup \Phi_{72} \cup \Phi_{73} \cup \Phi_{74}$$

where  $\Phi_{ij}$  is defined in Appendix B.4.

**Proof:** See Appendix B.4. ■

For the case that  $A_{12} = 0$  and/or  $A_{21} = 0$ , we have the following result.

**Theorem 6.3.3** Consider the system defined by (6.1) with parameters of (6.3) satisfying  $c_1 \neq 0$ . If  $A_{12} = 0$  and/or  $A_{21} = 0$ , then the system is not reachable if and only if one of the following conditions holds:

$$(1) A_{12} = 0, A_{21} = 0;$$

$$(2) A_{12} = 0, A_{21} \neq 0 \text{ and either one of the following conditions:}$$

$$(a) \alpha_2 = 0, c_2 = 0;$$

$$(b) (\alpha_2 = 0, \beta_2 > 0, c_2 > 0, \lambda_1 \leq 0) \cup (\alpha_2 = 0, \beta_2 < 0, c_2 < 0, \lambda_1 \geq 0);$$

$$(c) (\alpha_2 = 0, \beta_2 > 0, c_2 < 0, \lambda_1 \geq 0) \cup (\alpha_2 = 0, \beta_2 < 0, c_2 > 0, \lambda_1 \leq 0);$$

$$(d) (c_1 \alpha_2 > 0) \cap (\Phi_{14} \cup (c_2 = 0, c_1 < 0));$$

$$(e) (c_1 \alpha_2 < 0) \cap (\Phi_{33} \cup (c_2 = 0, c_1 > 0));$$

$$(3) A_{12} \neq 0, A_{21} = 0 \text{ and either one of the following conditions:}$$

$$(a) (\gamma_1 = 0, \delta_1 > 0, c_2 > 0);$$

$$(b) (\gamma_1 = 0, \delta_1 > 0, c_2 < 0);$$

$$(c) (c_1 \gamma_1 > 0) \cap \Phi_{42} \cap (c_2 \delta_1 > 0);$$

$$(d) (c_1 \gamma_1 < 0) \cap \Phi_{43} \cap (c_2 \delta_1 < 0).$$

where  $\Phi_{ij}$  is defined in Appendix B.4.

**Proof:** The proof is similar to that of Theorem 6.3.2, thus we omit the detail.

■

**Remark 6.3.1** From Theorem 6.3.2 and Theorem 6.3.3, we can see that the system defined by (6.1) with parameters of (6.3) has the following properties:

- (1) The reachability is independent of the parameters:  $\alpha_1, \beta_1, \gamma_2, \delta_2$ .
- (2) If it is reachable, then a given final state in  $\mathcal{R}^2$  can be reached within at most four steps with proper inputs.
- (3) If it is not reachable, then  $\mathcal{Y}_3$  is at most a region containing three quadrants and for a given state in any reachable set, the state can be reached within at most three steps with proper inputs.

### 6.3.2 $[B_1, B_2]$ is singular

Similar to the discussion in Section 6.2.2, we may also analyze this situation by two different cases. The first case of  $B_1 \neq 0$  and  $B_2 \neq 0$  can be addressed in an analogous means of Section 6.3.1. However, for the case that  $B_1 = 0$  and  $B_2 = 0$ , it is also not easy to reduce the corresponding explicit conditions. Nevertheless, the reachability is still possible. For example,

$$A_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 \\ c_2 \end{bmatrix}, \gamma_1 \neq 0$$

**Remark 6.3.2** Although controllability and reachability share some common properties, for example,  $\mathcal{X}_{k-1} \subseteq \mathcal{X}_k$  ( $\mathcal{Y}_{k-1} \subseteq \mathcal{Y}_k$ ) and those in Remarks 6.2.1 and 6.3.1, the difference is obvious. For example, when  $A_{12} = 0$  and  $A_{21} = 0$ , the system is NC but not reachable. This is because matrices  $A_i$ ,  $i = 1, 2$  are not invertible.

**Remark 6.3.3** In general, the presented method of general classification can be extended to high order and multi-modal PWA systems. However, with the increased

system order and/or number of modes, the classification tends to be complex, which makes it hard to obtain explicit conditions for controllability in terms of system parameters. Nevertheless, the classification method can be implemented numerically. We assume that the corresponding region  $\mathcal{S}_i$  of each mode is polyhedra [102]. According to the previous analysis, we can see that once  $\mathcal{X}_{k-1}$  is known, we can deduce  $\mathcal{X}_k$  from  $\mathcal{X}_{k-1}$ . In fact,  $\mathcal{X}_i = \bigcup_{j=1}^m \{x | A_j x + B_j u \in \mathcal{X}_{i-1}, x \in \mathcal{S}_j\}$ . It is obvious that  $\mathcal{X}_0 = \{0\}$ . Hence, a recursive algorithm can be used to calculate the  $k$ -step region  $\mathcal{X}_k$  with the following two stop conditions: 1) if for some  $i$ ,  $\mathcal{X}_i = \mathcal{R}^n$ , then the system is NC; 2) if for some  $i$ ,  $\mathcal{X}_i = \mathcal{X}_{i-1} \neq \mathcal{R}^n$ , then the system is not NC.

Note that, in the worst case, this direct approach of calculating  $\mathcal{X}_k$  has the complexity growing exponentially with respect to  $k$ . That is because at any time step  $k$ ,  $\mathcal{X}_k$  may contain up to  $m^k$  sets. So the key issue is to reduce the computational complexity, i.e., to limit the number of regions in  $\mathcal{X}_k$  in each step. The unions of polyhedron [70] and the fact  $\mathcal{X}_{k-1} \subseteq \mathcal{X}_k$  may be helpful for this purpose. The detailed analysis is beyond our scope.

## 6.4 Conclusion

This chapter has discussed the null controllability and reachability of planar bimodal PWA systems. Some necessary and sufficient conditions have been presented in terms of the system parameters. We have also shown that in some situations, controllability and reachability can be asserted in finite steps. These analytical procedures indicate the complexity of PWA systems. For high order or multi-modal systems, we have suggested a recursive procedure to evaluate the controllability/reachability. We point out that the so-called set-controllability and set-reachability [14, 17] can be discussed using the approach introduced in this chapter.

# Chapter 7

## $H_\infty$ and Generalized $H_2$ Controller Design for Discrete-time PWA Systems

### 7.1 Introduction

In Chapter 4 and Chapter 5, we have provided several improved stability criteria for PWA systems using the SOS theory and vertex representation compared with existing results. Although the SOS theory is a powerful tool in analysis, it is not easy to be applied to design problems. As stated in [99], the design problem for CPS may be solved by SOS via a rather complex iterative procedure, while it seems to be arduous for DPS. Some other approaches also impose too many constraints [141, 51]. Using the criteria based on vertex representation for CPS, the design problem is naturally transferred to a BMI problem. However, it seems impossible to give a direct LMI-based solution. Hence, in this chapter, we mainly deal with the controller design problem with the aid of PQLF, which has been summarized in Chapter 3.

Though the approaches to linear differential/difference inclusions for the state feed-

back control problem can be employed directly, where the design problem can be transferred into an LMI problem using Lemma 3.2.5, it is inheritably conservative due to no partition information as stated in Remark 3.2.5. See [43, 126] for examples.

Theorem 3.2.1 and Theorem 3.2.2 provide less conservative criteria, however, a direct application into state feedback controller design may bring some problems partially due to the bias terms introduced by *S-procedure* to include the partition information. For CPS, the design problem becomes a BMI problem. For example, Feng [56, 54] provides some formulations based on BMIs. Hassibi [82] considers the same problem by using a pre-designed ellipsoid and gives an LMI formation with scalars. However, for DPS, it seems that there is no design method based on Theorem 3.2.2 before our attempt, as the design problem is neither LMI nor BMI.

The similar problems also appear in robust state feedback control for PWA systems [43, 126, 64, 63, 167, 58, 56, 54]. In [58], Feng presents an  $H_\infty$  controller design method for fuzzy dynamic systems based on a continuous-time PWA model. And in [167], Wang and Feng extend the result for discrete-time Fuzzy systems. In [43, 126], Cuzzola *et al.* consider the  $H_\infty$  control problem, generalized  $H_2$  problem and robust  $H_\infty$  control. Similar obstacles as in the stabilization problem also exist here.

In the case of output feedback control for PWA systems (2.4), it is rather difficult to consider static output feedback control, because besides the unknown states, we may not know the exact mode. Hence, dynamical output feedback controller, which estimates both the mode and system state, may be suitable for this case. A recent result by Rodrigues [149, 146, 148] provides an observer-based output feedback controller synthesis approach for CPS, where sliding mode is taken into account in detail. As for DPS, Feng in [57] proposes a method to synthesize an observer-based output feedback controller based on the so-called separation principle. However, his result is based on the output space partition and hence is not applicable to the system (2.4).

In this chapter, we mainly consider the state feedback controller design for DPS, since the corresponding part for CPS has been discussed [82, 56, 54, 149, 146, 148]. We aim to give a less conservative design, where partition-dependent slack variables are employed with the aid of *S-procedure*. In particular, required properties of Lyapunov function for each partition are only satisfied locally. However, since the Lyapunov matrices are no longer required to be positive definite, the standard Schur complement cannot be applied to convert the design problem into an LMI or even a BMI one. In this chapter, by introducing a proper transformation, we convert the design problem into a BMI problem. Iterative algorithms are then proposed to solve the BMI problem.

Consider the discrete-time PWL system (2.4), i.e.,

$$\begin{aligned} x(t+1) &= A_i x(t) + B_{2i} u(t) + B_{1i} w & \text{for } x(t) \in S_i, i \in \mathcal{I} \\ y(t) &= C_i x(t) + D_{2i} u(t) + D_{1i} w \end{aligned} \quad (7.1)$$

Note that here we only consider the PWL cases for  $i \in \mathcal{I}_0$ , since the case  $i \in \mathcal{I}_1$  can be discussed similarly.

## 7.2 State Feedback Control

In this section, we consider the following static state feedback controller for the system (7.1):

$$u(t) = K_i x(t), \quad x(t) \in S_i \quad (7.2)$$

Thus the closed-loop system of (7.1) and (7.2) is given by

$$\begin{aligned} x(t+1) &= \bar{A}_i x(t) + B_{1i} w & \text{for } x(t) \in S_i, i \in \mathcal{I} \\ y(t) &= \bar{C}_i x(t) + D_{1i} w \end{aligned} \quad (7.3)$$

where  $\bar{A}_i = A_i + B_{2i} K_i$  and  $\bar{C}_i = C_i + D_{2i} K_i$ .

### 7.2.1 Stabilization Problem

In this subsection, we consider the stabilization problem, i.e., finding a state feedback controller (7.2) such that the system (7.3) without noise (i.e.  $w \equiv 0$ ) is stable. In this case, the closed-loop system of (7.1) and (7.2) is given by

$$x(t+1) = \bar{A}_i x(t) \quad \text{for } x(t) \in S_i, i \in \mathcal{I} \quad (7.4)$$

Most existing feedback control design methods for discrete-time PWA systems are based on Lemma 3.2.5.

**Lemma 7.2.1** [43, 57, 126] *The PWA system defined by (7.4) is stabilizable via static state feedback (7.2), if there exists a solution  $(Q_i > 0, Y_i)$ , such that*

$$\begin{bmatrix} Q_j & A_i Q_i + B_i Y_i \\ * & Q_i \end{bmatrix} > 0, \quad (i, j) \in \Omega_d \quad (7.5)$$

*In this situation, a suitable state controller can be given by  $K_i = Y_i Q_i^{-1}$ ,  $i \in \mathcal{I}$ .*

Lemma 7.2.1 is a direct consequence of (3.31) in Lemma 3.2.5 by letting  $Y_i = K_i Q_i$ . This approach is simple and efficient. However, the results are generally conservative due to the inheritable conservatism of Lemma 3.2.5. On the other hand, although Theorem 3.2.2 [55, 64] gives a less conservative analytical result by taking into account the partition information of the system, it is difficult to be applied into controller design. Recall from Theorem 3.2.2 that for a given controller defined by (7.2), the stabilization of (7.4) is implied by (3.12) and

$$(A_i + B_i K_i)^T P_j (A_i + B_i K_i) - P_i + F_i^T U_{ij} F_i < 0, \quad \text{for } (i, j) \in \Omega_d \quad (7.6)$$

for some  $P_i = P_i^T$ ,  $W_i \succeq 0$  and  $U_{ij} \succeq 0$ . Note that for undetermined  $K_i$ , (7.6) cannot be directly transformed to an LMI feasibility problem, not even a BMI problem since the matrices  $P_j$  are not necessarily positive definite and the well known Schur complement cannot be applied. In the following, the useful transformations from

Lemma 3.2.4 and Lemma A.0.3 and Lemma A.0.4, will be applied to address this problem.

**Theorem 7.2.2** *The PWA system defined by (7.4) is stabilizable via static state feedback (7.2), if there exists a solution  $(K_i, P_i = P_i^T, \Upsilon_{ij}, \Psi_j, W_i \succeq 0, U_{ij} \succeq 0, X_{ij})$ , such that*

$$P_i - F_i^T W_i F_i > 0, \quad i \in \mathcal{I} \quad (7.7)$$

and either one of the following BMIs are satisfied,

$$\Phi_{ij}^{(1)}(\bar{A}_i, P_i, U_{ij}) < 0 \quad (7.8)$$

$$\Phi_{ij}^{(2)}(\bar{A}_i, P_i, U_{ij}, \Upsilon_{ij}, \Psi_i) < 0 \quad (7.9)$$

$$\begin{bmatrix} B_i^T P_j B_i & B_i^T P_j A_i \\ * & \mathcal{U}_{ij} \end{bmatrix} + X_{ij} \begin{bmatrix} -I & K_i \end{bmatrix} + \begin{bmatrix} -I \\ K_i \end{bmatrix} X_{ij}^T < 0 \quad (7.10)$$

$$\begin{bmatrix} B_i^T P_j B_i - I & B_i^T P_j A_i + K_i \\ * & \mathcal{U}_{ij} - K_i^T K_i \end{bmatrix} < 0 \quad (7.11)$$

where  $\Phi_{ij}^{(1)}$  and  $\Phi_{ij}^{(2)}$  are defined in (3.24)-(3.25), and  $\mathcal{U}_{ij} = A_i^T P_j A_i - P_i + F_i^T U_{ij} F_i$ , for  $\forall (i, j) \in \Omega_d$ .

**Proof:** (7.8) and (7.9) are directly from Lemma 3.2.4. Now we only prove the equivalence among (7.6), (7.11) and (7.10). (7.6) can be rewritten as

$$\begin{bmatrix} K_i \\ I \end{bmatrix}^T \mathcal{P}_{ij} \begin{bmatrix} K_i \\ I \end{bmatrix} < 0, \quad \mathcal{P}_{ij} = \begin{bmatrix} B_i^T P_j B_i & B_i^T P_j A_i \\ * & A_i^T P_j A_i - P_i + F_i^T U_{ij} F_i \end{bmatrix} \quad (7.12)$$

It is obvious that  $\begin{bmatrix} K_i \\ I \end{bmatrix}^T \begin{bmatrix} -I \\ K_i^T \end{bmatrix} = 0$ . Thus we have  $\begin{bmatrix} -I \\ K_i^T \end{bmatrix}^\perp = \begin{bmatrix} K_i \\ I \end{bmatrix}^T$ .

Hence (7.12) is equivalent to

$$\begin{bmatrix} -I \\ K_i^T \end{bmatrix}^\perp \mathcal{P}_{ij} \begin{bmatrix} -I \\ K_i^T \end{bmatrix}^{\perp T} < 0$$

By using the Finsler's lemma, we have (7.10) or

$$\mathcal{P}_{ij} < \sigma_{ij} \begin{bmatrix} -I \\ K_i^T \end{bmatrix} \begin{bmatrix} -I & K_i \end{bmatrix} \quad (7.13)$$

for some  $\sigma_{ij} \in \mathcal{R}_+ \setminus \{0\}$ . Let  $\sigma = \max_{i,j \in \Omega_d} \sigma_{ij}$ . Now we redefine  $P_i$  as  $\frac{1}{\sigma}P_i$ ,  $U_{ij}$  as  $\frac{1}{\sigma}U_{ij}$ ,  $W_i$  as  $\frac{1}{\sigma}W_i$ , (7.11) follows. ■

**Remark 7.2.1** *The difference between (7.8) and (7.11) (or between (7.9) and (7.10)) is that in (7.8) and (7.9), the Lyapunov matrices are totally separated from the system matrices, which is not the case for (7.10) and (7.11).*

The above theorem converts the state feedback control problem into a BMI problem, which is known to be *NP-hard*. As stated in Appendix A, there exist some local and global algorithms for BMI problems. Here we may simply apply the V-K algorithm for (7.9) or (7.10). For example, using (7.9), we have the following algorithm:

**Algorithm 7.2.1** V-K ITERATION ALGORITHM FOR STATIC STATE FEEDBACK CONTROLLER DESIGN:

**Step 0** *Initialize the controller gain  $K_i$ , and other parameters:  $\kappa > 0$ ,  $k = 0$ , where  $\kappa$  is used to limit the maximum number of iterations and  $k$  is a counter.*

**Step 1** *Let  $K_i$  be fixed and  $\bar{A}_i = A_i + B_i K_i$ . Note that  $\Psi_j$  and  $\Upsilon_{ij}$  are variables in this step. Solve the LMI optimization problem:*

$$\begin{aligned} & \min_{P_i, U_{ij}, W_i, \Psi_i, \Upsilon_{ij}} \lambda_{ij}; \\ \text{s.t. } & (7.7) \text{ and } \Phi_{ij}^{(2)}(\bar{A}_i, P_i, U_{ij}, \Upsilon_{ij}, \Psi_i) < \lambda_{ij} I, \forall (i, j) \in \Omega_d \end{aligned} \quad (7.14)$$

*If the optimization leads to  $\lambda_{ij} \leq 0$ , then the system is stabilizable via static state feedback.  $K_i$  is a suitable stabilizing controller gain. Computation stops.*

*Otherwise, we set  $k := k + 1$  and go to Step 2.*

**Step 2** If  $k > \kappa$ , computation stops. Otherwise, set  $\Upsilon_{ij}$  and  $\Psi_j$  as obtained in Step 1, and solve the LMI optimization problem:

$$\begin{aligned} & \min_{P_i, U_{ij}, W_i, K_i} \lambda_{ij}; \\ \text{s.t. } & (7.7) \text{ and } \Phi_{ij}^{(2)}(\bar{A}_i, P_i, U_{ij}, \Upsilon_{ij}, \Psi_i) < \lambda_{ij} I, \forall (i, j) \in \Omega_d \end{aligned} \quad (7.15)$$

If  $\lambda_{ij} \leq 0$ ,  $K_i$  is a suitable controller gain. Computations stops. Otherwise, go to Step 1.

For (7.11), if an initial controller  $K_i^{(0)}$  is given, we have

$$(K_i^{(k)} - \alpha K_i^{(k-1)})^T (K_i^{(k)} - \alpha K_i^{(k-1)}) \geq 0, \alpha \in \mathcal{R}_+ \setminus \{0\} \quad (7.16)$$

i.e.,  $\mathcal{K}_i^{(k,k-1)} \geq -K_i^{(k)T} K_i^{(k)}$ ,  $\mathcal{K}_i^{(k,k-1)} \triangleq \alpha^2 K_i^{(k-1)T} K_i^{(k-1)} - \alpha K_i^{(k-1)T} K_i^{(k)} - \alpha K_i^{(k)T} K_i^{(k-1)}$

Hence

$$\begin{bmatrix} 0 & K_i^{(k)} \\ K_i^{(k)T} & -K_i^{(k)T} K_i^{(k)} \end{bmatrix} \leq \begin{bmatrix} 0 & K_i^{(k)} \\ K_i^{(k)T} & \mathcal{K}_i^{(k,k-1)} \end{bmatrix} \quad (7.17)$$

So we can conclude that

$$\mathcal{P}_{ij} + \begin{bmatrix} 0 & K_i^{(k)} \\ K_i^{(k)T} & \mathcal{K}_i^{(k,k-1)} \end{bmatrix} < 0 \implies \mathcal{P}_{ij} + \begin{bmatrix} 0 & K_i^{(k)} \\ K_i^{(k)T} & -K_i^{(k)T} K_i^{(k)} \end{bmatrix} < 0 \quad (7.18)$$

Normally, we simply choose  $\alpha = 1$ . Thus an iterative procedure similar to Algorithm 7.2.1 can be implemented too. We can also apply a similar procedure to (7.8). We omit the detail here.

**Example 7.2.1** Consider the system (7.1) with the following parameters:

$$\begin{aligned} A_1 = A_3 &= \begin{bmatrix} 1 & 0.1 \\ 0 & 1.6 \end{bmatrix}, & A_2 = A_4 &= \begin{bmatrix} 1 & 0.5 \\ 0 & 1.6 \end{bmatrix} \\ B_1 = B_3 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & B_2 = B_4 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ F_1 = -F_3 &= \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, & F_2 = -F_4 &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Note that if we use the design procedure given by Lemma 7.2.1 [43, 57, 126], we can not obtain a solution for the state feedback control. Now we use V-K Algorithm and obtain the following results with initial  $K_1 = K_2 = [0 \ 0]$  and  $K_3 = K_4 = [1 \ 1]$ :

$$\begin{aligned} P_1 &= \begin{bmatrix} 63.21 & -0.29 \\ -0.29 & 61.84 \end{bmatrix}, P_2 = \begin{bmatrix} 47.98 & 9.22 \\ 9.21 & 56.228 \end{bmatrix} \\ P_3 &= \begin{bmatrix} 75.23 & -1.60 \\ -1.60 & 65.92 \end{bmatrix}, P_4 = \begin{bmatrix} 37.19 & 28.85 \\ 28.85 & 59.43 \end{bmatrix} \\ K_1 &= \begin{bmatrix} -0.040 & -0.16 \end{bmatrix}, K_2 = \begin{bmatrix} -0.003 & -1.65 \end{bmatrix} \\ K_3 &= \begin{bmatrix} -1.208 & 0.071 \end{bmatrix}, K_4 = \begin{bmatrix} 0.007 & -2.000 \end{bmatrix} \end{aligned}$$

**Remark 7.2.2** *A significant disadvantage of some iteration algorithms, such as V-K Algorithm, is that the algorithm heavily depends on the initial value. That is, if the initial values of the feedback gains are not properly chosen, the algorithm may not produce required stabilizing feedback gains. Thus we need other efficient algorithm to solve the BMI problem. Eliminating some variables may improve the efficiency. Hence we provide another algorithm based on projection lemma [22] (see Appendix A).*

In order to apply the projection lemma, we may make a tradeoff between the computational efficiency and conservatism. Here we set  $W_i = 0$  which is equivalent to the requirement that  $P_i$  is positive definite. Quite often, we found that the obtained  $P_i$  be positive definite even though we did not set  $W_i = 0$  in simulations.

Note that with  $W_i = 0$ , (7.8) is simply equivalent to

$$\begin{bmatrix} -P_i + F_i^T U_{ij} F_i & \bar{A}_i^T P_j \\ P_j \bar{A}_i & -P_j \end{bmatrix} < 0, \quad \forall (i, j) \in \Omega_d \quad (7.19)$$

which can be rewritten as

$$\Sigma_{ij} + \mathcal{X}_{ij} K_j \mathcal{Y}^T + \mathcal{Y} K_j^T \mathcal{X}_{ij}^T < 0 \quad (7.20)$$

$$\text{where } \Sigma_{ij} = \begin{bmatrix} -P_i + F_i^T U_{ij} F_i & A_i^T P_j \\ P_j^T A_i & -P_j \end{bmatrix}, \mathcal{Y} = \begin{bmatrix} I_{n \times n} \\ 0_{n \times n} \end{bmatrix}, \mathcal{X}_{ij} = \begin{bmatrix} 0_{n \times m} \\ P_j B_i \end{bmatrix}.$$

We assume  $\text{rank}(B_i) = l_i$ , then  $\mathcal{X}_{ij}^\perp = \begin{bmatrix} I_{n \times n} & 0_{n \times (n-l_i)} \\ 0_{n \times n} & P_j^{-1} B_i^\perp \end{bmatrix} \doteq \begin{bmatrix} \vec{I}_{1i} \\ \vec{B}_{ij} \end{bmatrix}$ ,  $\mathcal{Y}^\perp = \begin{bmatrix} 0_{n \times n} \\ I_{n \times n} \end{bmatrix}$ . Based on the projection lemma (Appendix A), (7.20) is equivalent to the following two LMIs

$$\begin{aligned} \mathcal{X}_{ij}^{\perp T} \Sigma_{ij} \mathcal{X}_{ij}^\perp &= \vec{I}_{1i}^T (-P_i + F_i^T U_{ij} F_i) \vec{I}_{1i} + \\ \vec{B}_{ij}^T P_j A_i \vec{I}_{1i} + \vec{I}_{1i}^T A_i^T P_j \vec{B}_{ij} - \vec{B}_{ij}^T P_j \vec{B}_{ij} &< 0 \end{aligned} \quad (7.21)$$

where

$$\vec{B}_{ij}^T P_j A_i \vec{I}_{1i} = \begin{bmatrix} 0_{n \times n} \\ B_i^{\perp T} A_i \end{bmatrix} \vec{I}_{1i}, \vec{B}_{ij}^T P_j \vec{B}_{ij} = \begin{bmatrix} 0_{n \times n} & 0_{n \times (n-l_i)} \\ 0_{(n-l_i) \times n} & B_i^{\perp T} P_j^{-1} B_i^\perp \end{bmatrix}$$

and

$$\mathcal{Y}^{\perp T} \Sigma_{ij} \mathcal{Y}^\perp = -P_j < 0 \quad (7.22)$$

In (7.21), if we set  $Q_j = P_j^{-1}$ , it leads to a problem with LMI and equality constraints for some  $(P_j > 0, Q_j > 0, K_j)$  as follows:

$$\begin{bmatrix} -P_i + F_i^T U_{ij} F_i & A_i^T B_i^\perp \\ B_i^{\perp T} A_i & -B_i^{\perp T} Q_j B_i^\perp \end{bmatrix} < 0, \quad \forall (i, j) \in \Omega_d \quad (7.23)$$

$$P_j Q_j = I, \quad \forall j \in \mathcal{I} \quad (7.24)$$

Thus we have the following result.

**Theorem 7.2.3** *Consider the system (7.1). The following two statements are equivalent for  $\forall i \in \mathcal{I}, \forall (i, j) \in \Omega_d$ :*

- (1) *There exists a solution  $(P_i > 0, U_{ij}, K_i)$  satisfying (7.19);*

(2) There exists a solution  $(P_i > 0, U_{ij}, Q_i > 0)$  satisfying (7.23) and (7.24).

Now the problem is that how to solve the equality constraint (7.24). Note that (7.24) can be weakened to the following well-known semi-definite programming relaxation:

$$\begin{bmatrix} -P_j & I \\ I & -Q_j \end{bmatrix} \leq 0 \quad (7.25)$$

Observe that the condition  $P_j Q_j = I$  is equivalent to  $\text{trace}(P_j Q_j) = n$ , thus the equality constraint can be addressed by solving the following optimization problem

$$\min \sum_{j \in \mathcal{I}} \text{trace}(P_j Q_j), \text{ subject to (7.25)} \quad (7.26)$$

The above problem is not convex since the cost function in (7.26) is bilinear. This bilinear problem has been investigated by many researchers in static output feedback control for continuous-time systems. In fact, some efficient computational algorithms, such as the cone complementarity linearization method [71] and the sequential linear programming matrix method (SLPMM)[121], have been proposed. Here, we also apply the SLPMM because it always generates a strictly decreasing sequence of the objective function value which is bounded below by some integer, and thus it is convergent. Now by extending the SLPMM to solve the state feedback control problem, we have the following result.

### Algorithm 7.2.2 SLPMM BASED ON PROJECTION LEMMA

**Step 1** Obtain an initial set  $(P_i^0, Q_i^0)$  by solving (7.23) for  $\forall (i, j) \in \Omega_d$ .  $k=0$ .

**Step 2** Given  $P_i^k$  and  $Q_i^k$ , solve the following optimization problem for some  $P_i > 0, Q_i > 0$ :

$$\begin{aligned} \min \sum_{j \in \mathcal{I}} \text{trace}(P_j Q_j^k + P_j^k Q_j) \\ \text{s.t. (7.25) and (7.23) for } \forall (i, j) \in \Omega_d \end{aligned} \quad (7.27)$$

**Step 3** If  $\sum_{j \in \mathcal{I}} \text{trace}(P_j Q_j^k + P_j^k Q_j - 2P_j^k Q_j^k) \leq \epsilon$ , then stop, where  $\epsilon$  is a pre-defined sufficient small positive scalar.

**Step 4** Compute  $\alpha \in [0, 1]$  by solving

$$\min \sum_{j \in \mathcal{I}} \text{trace}((1 - \alpha)P_j + \alpha P_j^k)((1 - \alpha)Q_j + \alpha Q_j^k)$$

Set  $P_j^{k+1} = (1 - \alpha)P_j + \alpha P_j^k$  and  $Q_j^{k+1} = (1 - \alpha)Q_j + \alpha Q_j^k$ . Go to Step 2.

**Remark 7.2.3** The global convergence of the above algorithm can be easily deduced from Theorems 3.8 and 3.9 in [121] by letting  $P = \text{diag}\{P_1, P_2, \dots, P_{\|\mathcal{I}\|}\}$  and  $Q = \text{diag}\{Q_1, Q_2, \dots, Q_{\|\mathcal{I}\|}\}$ .

**Remark 7.2.4** Once we obtain a feasible solution from Algorithm 7.2.2, we can solve (7.19) to get  $K_i$ . Note that although (7.19) can be rewritten as (7.24) and

$$\begin{bmatrix} -P_i + F_i^T U_{ij} F_i & \bar{A}_i^T \\ \bar{A}_i & -Q_j \end{bmatrix} < 0, \quad \forall (i, j) \in \Omega_d \quad (7.28)$$

and SLPMM can be also applied here, it is less efficient than Algorithm 7.2.2, because of less variables involved.

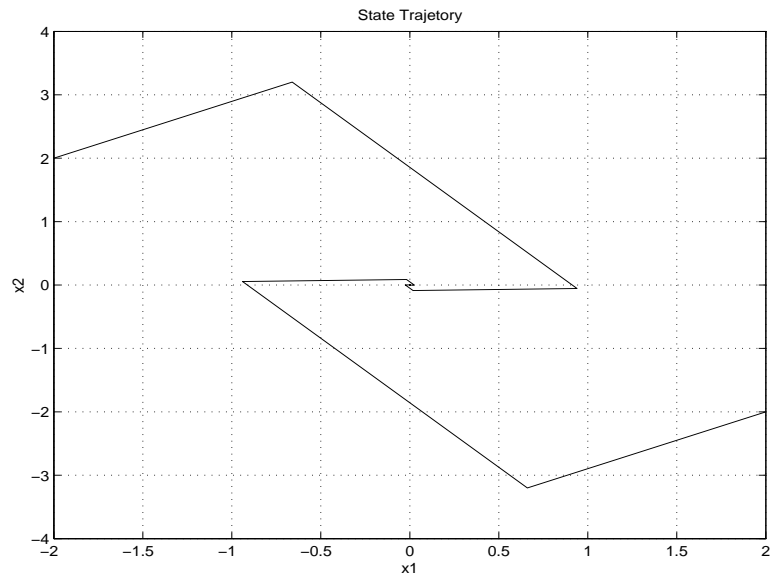
**Example 7.2.2** We still consider Example 7.2.1.

By Algorithm 7.2.2, we can easily obtain the following results only by 4 iterations.

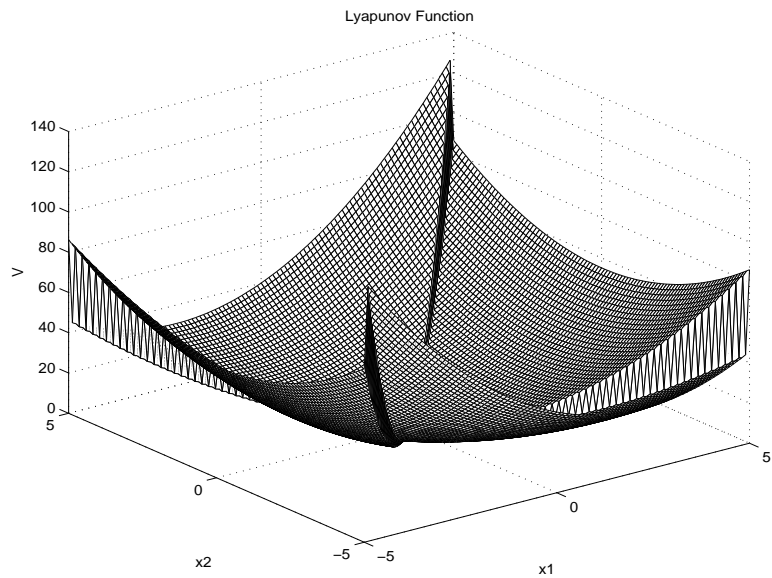
$$P_1 = P_3 = \begin{bmatrix} 2.20 & 2.94e - 017 \\ 2.94e - 017 & 1.25 \end{bmatrix}, \quad P_2 = P_4 = \begin{bmatrix} 1.89 & 0.84 \\ 0.84 & 1.60 \end{bmatrix}$$

$$K_1 = K_3 = \begin{bmatrix} -1.00 & -0.43 \end{bmatrix}, \quad K_2 = K_4 = \begin{bmatrix} -0.091 & -1.64 \end{bmatrix}$$

Figures 7.1 shows two state trajectories of the closed-loop system and the Lyapunov function, respectively.



State trajectory from  $[2 \ -2]$  and  $[-2 \ 2]$



Lyapunov function

Figure 7.1: State trajectory and Lyapunov function

**Remark 7.2.5** Besides the conservatism, Lemma 7.2.1 also cannot deal with the affine term, i.e., there exist some  $i \in \mathcal{I}_1$ . For Algorithm 7.2.2, although it is less conservative than Lemma 7.2.1 and more efficient than Theorem 7.2.2, it also faces the problem of the affine term. In fact, a constraint on  $B_i^\perp$  shall be made.<sup>1</sup> However, Theorem 7.2.2 can manage this problem.

**Example 7.2.3** We consider an inverted pendulum on a cart [59] stated in Example 2.2.5. Now we discretize the system using zero-order-hold with sampling time  $T_s = 0.01s$ . We assume that the system is controlled by the dominative mode, and then have following discrete-time PWA systems:

$$A_1 = A_4 = \begin{bmatrix} 1 & 0.01 \\ 0.002395 & 1 \end{bmatrix}, A_2 = A_3 = \begin{bmatrix} 1 & 0.01 \\ 0.1729 & 1 \end{bmatrix}$$

$$a_1 = -a_4 = \begin{bmatrix} 1.8395e - 005 \\ 0.003679 \end{bmatrix}, a_2 = a_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$B_1 = B_4 = \begin{bmatrix} -2.6e - 8 \\ -5.2e - 5 \end{bmatrix}, B_2 = B_3 = \begin{bmatrix} -8.826e - 7 \\ -1.766e - 3 \end{bmatrix}$$

In this example, due to the affine term, we cannot obtain a proper controller gain using SLPMM. However, with the result from SLPMM as initial condition for Theorem 7.2.2, we can easily obtain the following results:

$$P_1 = \begin{bmatrix} -67.8183 & -0.129806 & 163.416 \\ -0.129806 & 0.000995921 & -0.584656 \\ 163.416 & -0.584656 & 3263.33 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} -65.5415 & -0.0864739 & 112.472 \\ -0.0864739 & 0.000793427 & 0.619362 \\ 112.472 & 0.619362 & 2992.46 \end{bmatrix},$$

---

<sup>1</sup>In the case that  $(i, j)\Omega_d$ ,  $i \in \mathcal{I}_1$ ,  $j \in \mathcal{I}_0$ , if we let  $B_i^{\perp T} = \begin{bmatrix} B_i^{(11)} & B_i^{(12)} \\ B_i^{(21)} & B_i^{(22)} \end{bmatrix}$  and  $\bar{Q}_j = \begin{bmatrix} Q_j & 0 \\ 0 & 0 \end{bmatrix}$ , then a necessary condition for (7.23) is that  $B_i^{(11)} \neq 0$ ,  $B_i^{(21)} \neq 0$ .

$$P_2 = \begin{bmatrix} 0.2143 & 0.00105152 \\ 0.00105152 & 2.54319e - 005 \end{bmatrix}, P_3 = \begin{bmatrix} 0.2143 & 0.00105201 \\ 0.00105201 & 2.57877e - 005 \end{bmatrix}$$

$$K_1 = K_4 = \begin{bmatrix} 8585.81 & 18357.5 \end{bmatrix}, K_2 = K_3 = \begin{bmatrix} 330.968 & 568.452 \end{bmatrix}$$

We shall note that here  $P_1$  and  $P_4$  have negative eigenvalue. Once again, we see the power of local Lyapunov matrices, since there are not necessarily positive definite.

### 7.2.2 $H_\infty$ Analysis and Synthesis

The problem under consideration is stated as follows:

**$H_\infty$  CONTROL PROBLEM:** Consider the PWA system defined in (7.1). Given a scalar  $\gamma > 0$  and an integer  $N > 0$ , design a state feedback controller such that

$$\|y(t)\|_{\ell_2[0,N]} < \gamma \|w(t)\|_{\ell_2[0,N]} \quad (7.29)$$

for any non-zero  $w(t) \in \ell_2[0, N]$  and zero initial state.

**Lemma 7.2.4** *The PWA system defined in (7.1) satisfies the  $H_\infty$  performance requirement (7.29), if there exists a solution  $(K_i, \Upsilon_{ij}, P_i = P_i^T, \Psi_j, W_i \succeq 0, U_{ij} \succeq 0)$  such that*

$$P_i - F_i^T W_i F_i > 0, \quad \forall i \in \mathcal{I} \quad (7.30)$$

$$\begin{bmatrix} \bar{A}_i^T P_j \bar{A}_i - P_i + F_i^T U_{ij} F_i + \bar{C}_i^T \bar{C}_i & * \\ D_{i1}^T \bar{C}_i + B_{i1}^T P_j \bar{A}_i & \mathcal{B}_{ij} \end{bmatrix} < 0, \quad \forall i, j \in \Omega_d \quad (7.31)$$

where  $\mathcal{B}_{ij} = B_{i1}^T P_j B_{i1} + D_{i1}^T D_{i1} - \gamma^2 I$ .

**Proof:** Define the Lyapunov function as:

$$V(x(t)) = x(t)^T P_i x(t), \quad \forall x(t) \in S_i \quad (7.32)$$

According to (7.30) and the (1,1)-block of the inequality (7.31), we know that the unforced system of (7.1) (setting  $w(t) \equiv 0$ ) is piecewise quadratically stable<sup>2</sup>. Thus we have:

$$\begin{aligned}\Delta V(t) &= V(x(t+1)) - V(x(t)) \\ &= (\bar{A}_i x(t) + B_{i1} w(t))^T P_j (\bar{A}_i x(t) + B_{i1} w(t)) - x(t)^T P_i x(t)\end{aligned}\quad (7.33)$$

Defining  $\xi(t) = [x(t)^T \ w(t)^T]^T$ , and multiplying (7.31) by  $\xi(t)^T$  from the left and  $\xi(t)$  from the right, we have:

$$\begin{aligned}x(t)^T (\bar{A}_i P_j \bar{A}_i - P_i + F_i^T U_{ij} F_i + \bar{C}_i^T \bar{C}_i) x(t) + w(t)^T (D_{i1}^T \bar{C}_i + B_{i1}^T P_j \bar{A}_i) x(t) + \\ x(t)^T (D_{i1}^T \bar{C}_i + B_{i1}^T P_j \bar{A}_i)^T w(t) + w(t)^T (B_{i1}^T P_j B_{ij} + D_{i1}^T D_{i1} - \gamma^2 I) w(t) < 0\end{aligned}$$

Note that  $x(t)^T F_i^T U_{ij} F_i x(t) \geq 0$ , we get:

$$\begin{aligned}\Delta V(t) + x(t)^T \bar{C}_i^T \bar{C}_i x(t) + w(t)^T D_{i1} \bar{C}_i x(t) \\ + x(t)^T \bar{C}_i^T D_{i1} w(t) + w(t)^T (D_{i1}^T D_{i1} - \gamma^2 I) w(t) < 0\end{aligned}$$

That is

$$\Delta V(t) < \gamma^2 \|w(t)\|^2 - \|y(t)\|^2 \quad (7.34)$$

Summing (7.34) from 1 to  $N$ , we have:

$$0 \leq V(t+1) < \sum_{t=0}^N (\gamma^2 \|w(t)\|^2 - \|y(t)\|^2)$$

Thus we get the result. ■

Since the inequality (7.31) includes the items  $K_i^T B_{2i}^T P_j^T B_{2i} K_i$ , it is neither an LMI nor a BMI. Similar to Theorem 7.2.2, we should first obtain an analysis result which will enable an  $H_\infty$  state feedback design via BMI techniques. For this purpose, we make a transformation first. We define  $\tilde{A}_i = \begin{bmatrix} \bar{A}_i & B_{1i} \\ \bar{C}_i & D_{1i} \end{bmatrix}$ ,  $\tilde{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & I \end{bmatrix}$ ,

---

<sup>2</sup>If the unforced system (7.1) is piecewise quadratically stable, then there must exist a large enough  $\gamma$  satisfying (7.31)[55, 64].

$\tilde{E}_i = \begin{bmatrix} E_i^T W_i E_i & 0 \\ 0 & 0 \end{bmatrix}$  and  $\tilde{E}_{ij} = \begin{bmatrix} E_i^T U_{ij} E_i & 0 \\ 0 & (1 - \gamma^2)I \end{bmatrix}$ . Thus the inequalities (7.30) and (7.31) can be rewritten as:

$$\tilde{P}_i - \tilde{E}_i > 0, \quad i \in \mathcal{I} \quad (7.35)$$

$$\tilde{A}_i^T \tilde{P}_j \tilde{A}_i - \tilde{P}_i + \tilde{E}_{ij} < 0, \quad (i, j) \in \Omega_d \quad (7.36)$$

Hence, we can apply the similar techniques as stated in Theorem 7.2.2 to transform this problem into a BMI problem. For example, if the technique of (7.9) is used, then (7.31) is equivalent to

$$\begin{bmatrix} -I & * & * & * \\ \bar{C}_i^T & -P_i + F_i^T U_{ij} F_i + \bar{A}_i^T \Upsilon_{ij} + \Upsilon_{ij}^T \bar{A}_i & * & * \\ 0 & -\Upsilon_{ij} + \Psi_j^T \bar{A}_i & P_j - (\Psi_j + \Psi_j^T) & * \\ 0 & D_{i1}^T \bar{C}_i + B_{i1}^T P_j \bar{A}_i & 0 & B_{ij} \end{bmatrix} < 0 \quad (7.37)$$

for some  $(K_i, \Upsilon_{ij}, P_i = P_i^T, \Psi_j, W_i \succeq 0, U_{ij} \succeq 0)$ , for  $\forall (i, j) \in \Omega_d$ .

Of course, the SLPMM can be applied here, if  $P_i > 0$  and a proper transformation is introduced. Applying the Schur complement to (7.31), we obtain

$$\begin{bmatrix} -P_i + F_i^T U_{ij} F_i & \bar{A}_i^T P_j & \bar{C}_i^T & 0 \\ P_j \bar{A}_i & -P_j & 0 & P_j E_i \\ \bar{C}_i & 0 & -I & G_i \\ 0 & E_i^T P_j & G_i^T & -\gamma^2 I \end{bmatrix} < 0 \quad (7.38)$$

(7.38) can be rewritten as in the form of (7.20), where

$$\Sigma_{ij} = \begin{bmatrix} -P_i + F_i^T U_{ij} F_i & A_i^T P_j & C_i^T & 0 \\ P_j A_i & -P_j & 0 & P_j E_i \\ C_i & 0 & -I & G_i \\ 0 & E_i^T P_j & G_i^T & -\gamma^2 I \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} I_{n \times n} \\ 0_{n \times n} \\ 0_{r \times n} \\ 0_{l \times n} \end{bmatrix}, \quad \mathcal{X}_{ij} = \begin{bmatrix} 0_{n \times m} \\ P_j B_i \\ D_i \\ 0_{l \times m} \end{bmatrix}$$

We assume that  $\text{rank}([B'_i \ D'_i]) = m_i$ . Thus we have

$$\mathcal{Y}^\perp = \begin{bmatrix} 0_{n \times n} & 0_{n \times r} & 0_{n \times l} \\ I_{n \times n} & 0_{n \times r} & 0_{n \times l} \\ 0_{r \times n} & I_{r \times r} & 0_{r \times l} \\ 0_{l \times n} & 0_{l \times r} & I_{l \times l} \end{bmatrix},$$

$$\mathcal{X}_{ij}^\perp = \begin{bmatrix} I_{n \times n} & 0_{n \times (n+r-m_i)} & 0_{n \times l} \\ 0_{(n+r) \times n} & \begin{bmatrix} P_j^{-1} & 0 \\ 0 & I_{r \times r} \end{bmatrix} [B'_i \ D'_i]^\perp & 0_{(n+r) \times l} \\ 0_{l \times n} & 0_{l \times (n+r-m_i)} & I_{l \times l} \end{bmatrix} \quad (7.39)$$

Then (7.31) is transferred to the following problem:

$$\mathcal{Y}^{\perp T} \Sigma_{ij} \mathcal{Y}^\perp = \begin{bmatrix} -P_j & 0 & P_j E_i \\ 0 & -I & G_i \\ E_i^T P_j & G_i^T & -\gamma^2 I \end{bmatrix} < 0 \quad (7.40)$$

$$\mathcal{X}_{ij}^{\perp T} \Sigma_{ij} \mathcal{X}_{ij}^\perp = \begin{bmatrix} -P_i + F_i^T U_{ij} F_i & * & * \\ [B'_i \ D'_i]^{\perp T} \begin{bmatrix} A_i \\ C_i \end{bmatrix} & -[B'_i \ D'_i]^{\perp T} \begin{bmatrix} Q_j & 0 \\ 0 & I \end{bmatrix} [B'_i \ D'_i]^\perp & * \\ 0 & [E_i^T \ G_i^T] [B'_i \ D'_i]^\perp & -\gamma^2 I \end{bmatrix} < 0 \quad (7.41)$$

$$P_j Q_j = I \quad (7.42)$$

for which the SLPMM can be applied here.

**Example 7.2.4** Consider the system (7.1) with the following parameters:

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & 1 \end{bmatrix}, \quad B_{21} = B_{23} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_{22} = B_{24} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{11} = B_{13} = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, \quad B_{12} = B_{14} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix},$$

$$D_{1i} = 0, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_{2i} = 0.1, \quad i = 1, 2, 3, 4$$

$$F_1 = -F_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad F_2 = -F_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

This example is borrowed from [58] where an optimal  $\gamma = 0.2$  was reported. Using the SLPMM, we obtain a much better optimal  $\gamma$  of 0.0309 than that in [58]. Further, if we take the obtained result as initial value and apply it to (7.37) and (7.30), with the aid of the path-following algorithm [83], we can get a better result  $\gamma = 0.0300$ . One set of the stabilizing controller gains is

$$K_1 = K_3 = \begin{bmatrix} -0.514161 & -1.10141 \end{bmatrix}, \quad K_2 = K_4 = \begin{bmatrix} -1.01279 & -0.372147 \end{bmatrix}$$

Figure 7.2 shows the trajectories of  $y$  from the initial states  $[0 \ 0]^T$  and  $[0.1 \ 0.1]^T$ , respectively.

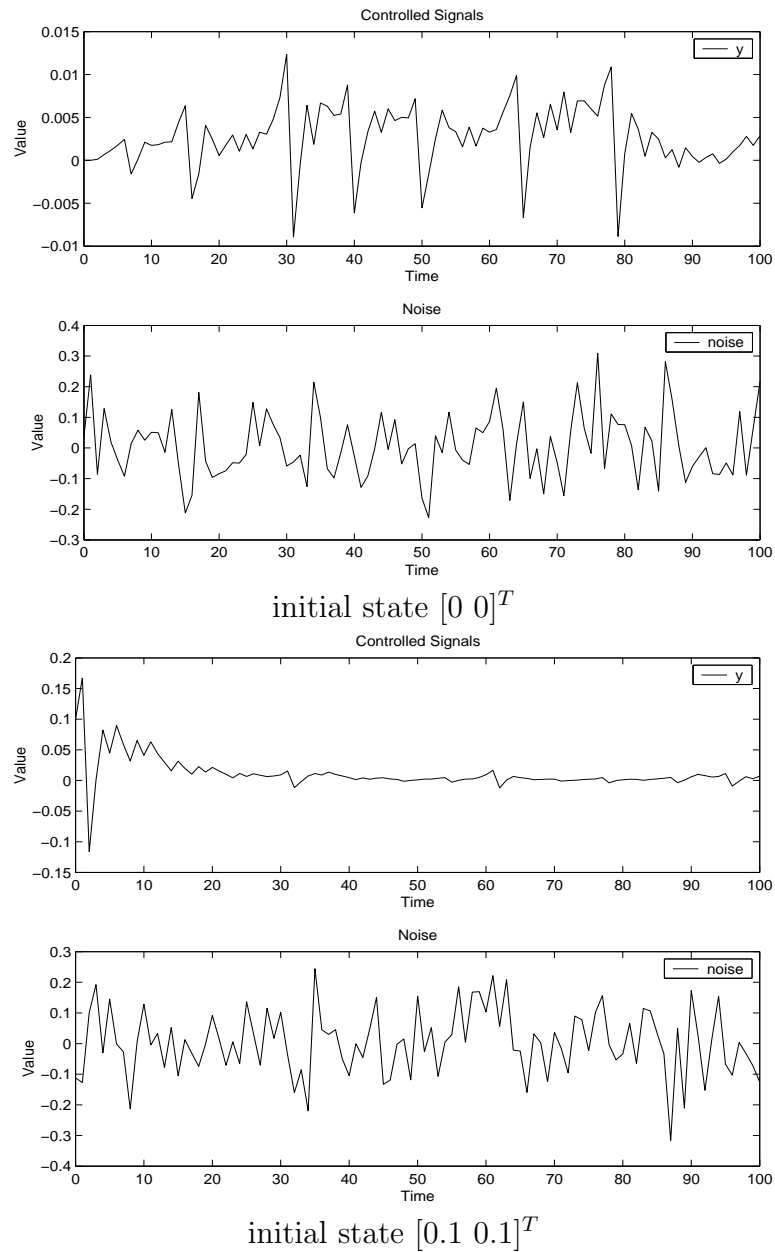
**Example 7.2.5** Consider the system (7.1) with the following parameters

$$A_1 = A_3 = \begin{bmatrix} 0.28 & 0.315 \\ 0.63 & -0.84 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 0.525 & 0.77 \\ -0.7 & -0.07 \end{bmatrix},$$

$$B_{21} = B_{23} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{22} = B_{24} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{11} = B_{13} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{12} = B_{14} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$D_{11} = D_{12} = D_{13} = D_{14} = 0, \quad C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$D_{21} = D_{22} = D_{23} = D_{24} = 0, \quad F_1 = -F_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_2 = -F_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Figure 7.2: Output  $y$  via the  $H_\infty$  controller (Example 7.2.4)

This example is borrowed from [64, 63]. If the SLPMM algorithm is applied to this example, the performance is significantly improved to  $\gamma = 1.37$  compared with the optimal  $\gamma = 1.73$  given in [63, 64]. The corresponding controller gains are

$$K_1 = K_3 = \begin{bmatrix} -0.241011 & -0.366977 \end{bmatrix}$$

$$K_2 = K_4 = \begin{bmatrix} 0.736137 & 0.123002 \end{bmatrix}$$

Figure 7.3 shows the trajectories of  $y$  from the initial states  $[0 \ 0]^T$  and  $[1 \ 1]^T$  respectively.

### 7.2.3 Generalized $H_2$ Analysis and Synthesis

The problem under consideration is stated as follows:

**GENERALIZED  $H_2$  CONTROL PROBLEM:** Consider the PWA system defined in (7.1). Given a scalar  $\tau > 0$  and an integer  $N > 0$ , design a state feedback controller such that

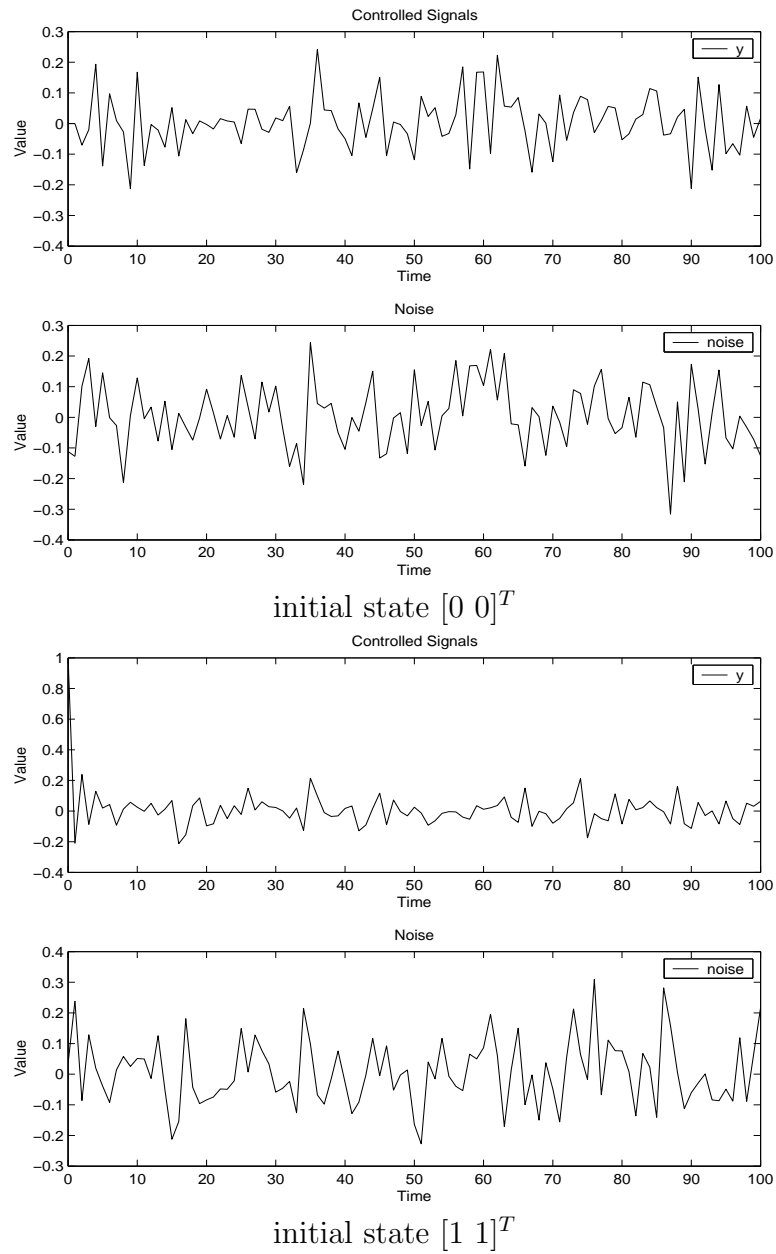
$$\|y(t)\|_{\ell_\infty[0,N]} < \tau \|w(t)\|_{\ell_2[0,N-1]} \quad (7.43)$$

for any non-zero  $w(t) \in \ell_2[0, N-1]$  and zero initial condition.

**Lemma 7.2.5** *The PWA system defined in (7.1) with  $D_{i1} = 0$  satisfies the generalized  $H_2$  performance requirement (7.43), if there exists a solution  $(\Upsilon_{ij}, P_i = P_i^T, K_i, \Psi_j, W_i \succeq 0, U_{ij} \succeq 0)$ , such that*

$$\begin{bmatrix} P_i - F_i^T W_i F_i & * \\ \bar{C}_i & \tau I \end{bmatrix} \succeq 0 \quad (7.44)$$

$$\begin{bmatrix} \bar{A}_i^T P_j \bar{A}_i - P_i + F_i^T U_{ij} F_i & * \\ B_{i1}^T P_j \bar{A}_i & B_{i1}^T P_j B_{i1} - I \end{bmatrix} \prec 0, \text{ for } i, j \in \Omega_d \quad (7.45)$$

Figure 7.3: Output  $y$  via the  $H_\infty$  controller (Example 7.2.5)

**Proof:** From the (1,1)-block of inequalities (7.44) and (7.45), we know the system (7.1) is piecewise quadratically stable. Multiplying (7.45) by  $\xi(t)^T$  from the left and  $\xi(t)$  from the right, we have:

$$\begin{aligned} & x(t)^T (\bar{A}_i^T P_j \bar{A}_i - P_i + F_i^T U_{ij} F_i) x(t) + w(t)^T (B_{i1}^T P_j \bar{A}_i) x(t) + \\ & x(t)^T (B_{i1}^T P_j \bar{A}_i)^T w(t) + w(t)^T (B_{i1}^T P_j B_{i1} - I) w(t) < 0 \end{aligned}$$

Note that  $x(t)^T F_i^T U_{ij} F_i x(t) \geq 0$ , we get:

$$\Delta V(t) \leq \Delta V(t) + x(t)^T F_i^T U_{ij} F_i x(t) < w(t)^T w(t) \quad (7.46)$$

Summing (7.46) from 0 to  $N$ , we have:

$$V(x(N+1)) < \sum_{t=0}^N \|w(t)\|^2 \quad (7.47)$$

From (7.44), we have:

$$V(x(t)) = x(t)^T P_i x(t) \geq x(t)^T (P_i - F_i^T W_i F_i) x(t) \geq \frac{1}{\tau} \|y(t)\|^2 \quad (7.48)$$

Combining (7.47) and (7.48), we get the result.  $\blacksquare$

Similar to the  $H_\infty$  control problem, we can transfer this problem as a BMI problem.

For example, (7.45) is equivalent to the following BMI:

$$\begin{bmatrix} -P_i + F_i^T U_{ij} F_i + \bar{A}_i^T \Upsilon_{ij} + \Upsilon_{ij}^T \bar{A}_i & * & * \\ -\Upsilon_{ij} + \Psi_j^T \bar{A}_i & P_j - (\Psi_j + \Psi_j^T) & * \\ B_{i1}^T P_j \bar{A}_i & 0 & B_{i1}^T P_j B_{i1} - I \end{bmatrix} < 0 \quad (7.49)$$

for some  $(K_i, \Upsilon_{ij}, P_i = P_i^T, \Psi_j, W_i \succeq 0, U_{ij} \succeq 0)$ , for  $\forall (i, j) \in \Omega_d$ .

**Theorem 7.2.6** *The PWA system defined by (7.1) with  $D_{i1} = 0$  satisfies the generalized  $H_2$  performance requirement (7.43), if there exists a solution  $(\Upsilon_{ij}, K_i, P_i = P_i^T, \Psi_j, W_i \succeq 0, U_{ij} \succeq 0)$ , such that (7.44) and (7.49) are satisfied.*

Of course, the SLPMM when  $P_i > 0$  and the BMI algorithms stated in Appendix when  $W_i \succeq 0$  can be applied to solve Theorem 7.2.6. We omit the detail here.

## 7.3 Conclusion

This chapter has proposed less conservative analysis and synthesis methods for  $H_\infty$  and generalized  $H_2$  state feedback controller designs for discrete-time PWA systems. When applying the analysis result to feedback controller design, a BMI problem has been formulated. Two algorithms, namely a direct iterative procedure and an iterative procedure based on the projection lemma (the SLPMM algorithm), are proposed. The main feature of our approaches, as compared with existing design methods for PWA systems, is that we take into account the partition information of the system and hence our method generally can produce a much less conservative design, which has been verified by simulation examples.

# Chapter 8

## $H_\infty$ and Generalized $H_2$ Estimator Design for PWA Systems

### 8.1 Introduction

Although the observer or estimator design problem for linear systems has a long history, there have been few existing works for PWA systems. In [1], a Luenberger observer for both discrete-time and continuous-time systems is proposed, and a design based on a projection method to minimize the estimation error is given in [2]. In [49], state estimation for a finite-state Markov chain has been studied. However, in these works, the modes of the systems are known “*a priori*”.

Feng proposes an observer design approach for a class of DPS in [57] and presents two filter design methods for the same class systems in [61], where he assumes that the partitions are defined in terms of the system output so that the plant and state estimator always switch to the same partition at the same time. However, many PWA systems are more likely partitioned based on the state space. Furthermore, there inevitably exists measurement noise in the output. In such situations, there is no guarantee that the system state and the estimated state always stay in the same partition at the same time. In other words, it is likely that the system state

might operate in a different region as the estimated state from time to time. This type of state estimation is referred to as non-synchronized state estimation.

For non-synchronized state estimation, Liu and Zhang [125] give a new filtering scheme using the most probable trajectory estimated and a shift transformation. Their work can be used to estimate the PWA systems, while the group of candidate distribution actually is not easy to be determined. Recently, Juloski *et al.* [108, 107] introduce a design procedure for the Luenberger type of observer, which does not require information on currently active dynamics of a bi-modal PWL system. Also partition information is included there to alleviate the design conservatism similar to the works of [102, 145].

In addition, Trecate *et al.* in [65] propose a state-smoothing algorithm for hybrid systems based on moving-horizon estimation (MHE) by exploiting the equivalence between hybrid systems modelled in the MLD form and PWA systems. Some sufficient conditions have been developed to guarantee asymptotic convergence of the moving horizon estimation algorithm, and some practical implementation issues have also been addressed. However, this approach actually requires a huge computational power.

In this chapter, we will study how to design the non-synchronized estimators with required performance for PWA systems in an efficient way.

We consider both continuous-time PWA system (CPS) and discrete-time PWA system (DPS):

$$\begin{aligned} \delta x(t) &= A_i x(t) + B_i w \\ y(t) &= C_i x(t) + D_i w \quad \text{for } x(t) \in S_i, i \in \mathcal{I} \\ z(t) &= E_i x(t) \end{aligned} \tag{8.1}$$

where  $x$ ,  $w$ ,  $y$  are defined in (2.4).  $z \in \mathcal{R}^{n_z}$  is the signal to be estimated.

## 8.2 Estimator Structure and Problem Statement

We introduce the following Luenberger-type of estimator

$$\begin{aligned} \delta \hat{x}(t) &= A_j \hat{x}(t) + L_j (y(t) - C_j \hat{x}(t)) & \hat{x}(t) \in S_j, j \in \mathcal{I} \\ \hat{z}(t) &= E_j \hat{x}(t) \end{aligned} \quad (8.2)$$

where  $L_j$  is the estimator gain to be designed.

In view of the system dynamics  $x(t) \in S_i$  and the estimator dynamics  $\hat{x}(t) \in S_j$ , the state estimation error dynamics can be obtained by combining the system (8.1) and the state estimator (8.2):

$$(\Sigma) : \begin{aligned} \delta \xi(t) &= \tilde{A}_{ij} \xi(t) + \tilde{B}_{ij} w(t) \\ \epsilon(t) &= z(t) - \hat{z}(t) = \tilde{E}_{ij} \xi(t) \end{aligned} \quad (8.3)$$

where

$$\begin{aligned} e(t) \triangleq x(t) - \hat{x}(t), \xi(t) \triangleq \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \tilde{E}_{ij} = [E_i - E_j \quad E_j], \\ \tilde{A}_{ij} = \begin{bmatrix} A_i & 0 \\ A_i - A_j - L_j C_i + L_j C_j & A_j - L_j C_j \end{bmatrix}, \tilde{B}_{ij} = \begin{bmatrix} B_i \\ B_i - L_j D_i \end{bmatrix} \end{aligned} \quad (8.4)$$

Note that the transition of the estimator state  $\hat{x}(t)$  is based on its estimated value. If  $x(t)$  and  $\hat{x}(t)$  are close enough such that  $e(t)$  can be ignored, i.e.,  $x(t)$  and  $\hat{x}(t)$  are synchronized in transition from one partition to another, then the method in [57, 146] may be applied here. However, in practice,  $x(t)$  and  $\hat{x}(t)$  may not always be located in the same partition, especially in the initial period. Thus they are non-synchronized.

**Remark 8.2.1** For the case  $x(t) \in S_i$  and  $\hat{x}(t) \in S_j$ , we have

$$\tilde{F}_{ij} \xi(t) \triangleq \begin{bmatrix} F_i & 0 \\ F_j & -F_j \end{bmatrix} \xi(t) \geq 0 \quad (8.5)$$

We define new partition  $\tilde{S}_{ij} = \{\xi(t) | \tilde{F}_{ij}\xi(t) \geq 0\}$ , and denote the index pair set for the CPS and DPS as  $\Phi_c$  and  $\Phi_d$ , respectively. It is easy to check that  $\tilde{S}_{ij}$  is still a convex polyhedron. We denote  $\vec{F}_{ij}$  as the continuity matrix of  $\tilde{F}_{ij}$ . We also denote the index set of the system jump from  $\tilde{S}_{ij}$  to  $\tilde{S}_{kl}$  for the CPS and DPS as  $\Psi_c$  and  $\Psi_d$ , respectively. Similar to (2.7), if  $(i, j, k, l) \in \Psi_c$  and  $\tilde{F}_{ij} \cap \tilde{F}_{kl} \neq 0$ , we define the boundary as  $\tilde{S}_{ijkl} = \{\xi(t) | \tilde{F}_{ijkl}\xi(t) + \tilde{f}_{ijkl} = 0\}$ .

The system state and estimated state in DPS is much more complex than that of CPS. As mentioned before, the two consecutive system states  $x(t)$  and  $x(t+1)$  may belong to any partitions in DPS. Thus we define a set  $\check{\Omega}$  that represents all transitions from one partition to another which happen in the system state and estimated state, that is,  $\check{\Omega} \triangleq \{i, j | x(t) \in S_i, \hat{x}_t \in S_j, i \neq j, i, j \in \mathcal{I}\}$ . As a consequence, there are four cases about the dynamics transitions of the combined system (8.3) from  $\xi(t)$  to  $\xi(t+1)$ , where  $x(t) \in S_i$ ,  $x(t+1) \in S_k$ ,  $\hat{x}(t) \in S_j$  and  $\hat{x}(t+1) \in S_l$ :

$$\Psi_d \triangleq \left\{ (i, j, k, l) \mid \begin{cases} \text{Case 1 : } i = j, k = l \in \mathcal{I} \\ \text{Case 2 : } i = j \in \mathcal{I}, k, l \in \check{\Omega} \\ \text{Case 3 : } i, j \in \check{\Omega}, k = l \in \mathcal{I} \\ \text{Case 4 : } i, j \in \check{\Omega}, k, l \in \check{\Omega} \end{cases} \right\} \quad (8.6)$$

Note that if  $x(t)$  and  $\hat{x}(t)$  always synchronize in transition from one partition to another, there only exists Case 1.

The problems under consideration are stated as follows:

**$H_\infty$  estimation problem:** Consider the PWA system (8.1). Given a scalar  $\gamma > 0$ , design an estimator of the form (8.2) such that the error system (8.3) is stable and

$$(CPS) \quad \|z - \hat{z}\|_{l_2[0, \infty)}^2 < \gamma^2 \|w\|_{l_2[0, \infty)}^2 + v(x_0, \hat{x}_0), \quad 0 \neq w \in l_2[0, \infty) \quad (8.7)$$

$$(DPS) \quad \|z - \hat{z}\|_{l_2[0, \infty)} < \gamma^2 \|w\|_{l_2[0, \infty)} + v(x_0, \hat{x}_0), \quad 0 \neq w \in l_2[0, \infty) \quad (8.8)$$

for some function  $v(\cdot, \cdot) \geq 0$ , where  $x_0$  and  $\hat{x}_0$  are initial states for  $x$  and  $\hat{x}$ , respectively.

**Generalized  $H_2$  estimation problem:** Assume that  $x_0 = 0$  and  $\hat{x}_0 = 0$ . Given a positive scalar  $\tau$ , design an estimator of the form (8.2) such that the error system (8.3) is stable and

$$(CPS) \quad \|z - \hat{z}\|_{l_\infty[0, \infty)} < \tau \|w\|_{l_2[0, \infty)}, \quad 0 \neq w \in l_2[0, \infty) \quad (8.9)$$

$$(DPS) \quad \|z - \hat{z}\|_{l_\infty[0, \infty)} < \tau \|w\|_{l_2[0, \infty)}, \quad 0 \neq w \in l_2[0, \infty) \quad (8.10)$$

### 8.3 $H_\infty$ Estimation

The following analysis results can be deduced based on the stability results and definition of the piecewise Lyapunov function components as  $v_{ij} = \xi(t)^T P_{ij} \xi(t)$ . We discuss the CPS first in Subsection 8.3.1, and then state the results for DPS in Subsection 8.3.2. In each subsection, we address the  $H_\infty$  estimation problem in two situations depending on whether the Lyapunov matrix  $P_{ij}$  is positive definite.

#### 8.3.1 Continuous-time Case

**Theorem 8.3.1** Consider the continuous-time PWA system defined by (8.1). A given estimator satisfies the  $H_\infty$  performance  $\gamma$ , if there exists a solution ( $T = T^T$ ,  $V_{ij} \succeq 0$ ,  $U_{ij} \succeq 0$ ) to the following inequalities:

$$P_{ij} = \overrightarrow{\tilde{F}}_{ij}^T T \overrightarrow{\tilde{F}}_{ij}, \quad \forall (i, j) \in \Phi_c \quad (8.11)$$

$$P_{ij} - \tilde{F}_{ij}^T V_{ij} \tilde{F}_{ij} > 0, \quad \forall (i, j) \in \Phi_c \quad (8.12)$$

$$\begin{bmatrix} \varpi_{ij}^{(1)} & P_{ij} \tilde{B}_{ij} \\ \tilde{B}_{ij}^T P_{ij} & -\gamma^2 I \end{bmatrix} < 0, \quad \forall (i, j) \in \Phi_c \quad (8.13)$$

where  $\varpi_{ij}^{(1)} = \tilde{A}_{ij}^T P_{ij} + P_{ij} \tilde{A}_{ij} + \tilde{F}_{ij}^T U_{ij} \tilde{F}_{ij} + \tilde{E}_{ij}^T \tilde{E}_{ij}$ .

**Proof:** We define a Lyapunov function candidate as follows:

$$v_{ij}(\xi) = \xi^T P_{ij} \xi, \quad \xi \in \tilde{S}_{ij}$$

Then, along the state trajectory of the system (8.3),

$$\dot{v}_{ij} = \zeta^T \begin{bmatrix} P_{ij} \tilde{A}_{ij} + \tilde{A}_{ij}^T P_{ij} & P_{ij} \tilde{B}_{ij} \\ \tilde{B}_{ij}^T P_{ij} & 0 \end{bmatrix} \zeta, \quad (i, j) \in \Phi_c \quad (8.14)$$

where  $\zeta = [\xi^T \quad w^T]^T$ .

The existence of  $(P_{ij} > 0, U_{ij} \succeq 0, i, j \in \mathcal{I})$  to (8.13) implies that

$$P_{ij} \tilde{A}_{ij} + \tilde{A}_{ij}^T P_{ij} + \tilde{E}_{ij}^T \tilde{E}_{ij} + \tilde{F}_{ij}^T U_{ij} \tilde{F}_{ij} < 0$$

which implies the stability of the system (8.3) [102].

Next, (8.13) can be rewritten as:

$$\begin{bmatrix} P_{ij} \tilde{A}_{ij} + \tilde{A}_{ij}^T P_{ij} & P_{ij} \tilde{B}_{ij} \\ \tilde{B}_{ij}^T P_{ij} & 0 \end{bmatrix} + \begin{bmatrix} \tilde{E}_{ij}^T \tilde{E}_{ij} & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} \tilde{F}_{ij}^T U_{ij} \tilde{F}_{ij} & 0 \\ 0 & 0 \end{bmatrix} < 0 \quad (8.15)$$

Multiplying (8.15) by  $\zeta^T$  and  $\zeta$  from the left and the right, respectively, we have:

$$\dot{v}_{ij} + (\|z - \hat{z}\|^2 - \gamma^2 \|w\|^2) \leq -\xi^T \tilde{F}_{ij}^T U_{ij} \tilde{F}_{ij} \xi < 0 \quad (8.16)$$

By piecewise integrating (8.16) from 0 to  $\infty$ , the global Lyapunov function  $v(\xi(t))$  satisfies

$$v(\xi(\infty)) - v(\xi(0)) + \int_0^\infty (\|z - \hat{z}\|^2 - \gamma^2 \|w\|^2) dt < 0 \quad (8.17)$$

which implies that

$$\|z - \hat{z}\|^2 < \gamma^2 \|w\|^2 + v(x_0, \hat{x}_0) \quad (8.18)$$

Thus we complete the proof. ■

**Remark 8.3.1** Note that if  $\Psi_c$  is predetermined, then the continuity condition (8.11) in Theorem 8.3.1 can be replaced by

$$P_{kl} - P_{ij} + \tilde{F}_{ijkl}^T p_{ijkl} + p_{ijkl}^T \tilde{F}_{ijkl} \leq 0, \quad \forall (i, j, k, l) \in \Psi_c \quad (8.19)$$

for some  $p_{ijkl}$ .

**Remark 8.3.2** Sliding mode may occur in the PWA system (8.1) and/or the observer (8.2), when the corresponding vector fields generated on the neighboring of the switching planes point towards to the switching planes. Following the discussions in [108, 67], we can conclude that the  $H_\infty$  performance is guaranteed even when the sliding mode occurs.

There are three cases of sliding mode. The sliding mode occurs: a) in the system (8.1) itself; b) in the estimator (8.2); and c) in both the system and the observer. We only analyze the third case here.

Suppose there is a sliding motion on the boundary of two cells  $\tilde{S}_{ij}$  and  $\tilde{S}_{kl}$ , and the corresponding dynamics of  $\xi$  is characterized by  $\tilde{A}_{ij}\xi + \tilde{B}_{ij}w$  and  $\tilde{A}_{kl}\xi + \tilde{B}_{kl}w$  respectively. The dynamics in  $\tilde{S}_{ijkl}$  is given by the following differential inclusion:

$$\dot{\xi} = \mathbf{Co}\{\tilde{A}_{ij}\xi + \tilde{B}_{ij}w, \tilde{A}_{kl}\xi + \tilde{B}_{kl}w\} \quad (8.20)$$

for  $\xi \in \tilde{S}_{ijkl}$  and  $(i, j, k, l) \in \Psi_c$ . Hence it is easy to see that Theorem 8.3.1 guarantees the  $H_\infty$  performance. A detailed discussion on bimodal cases can be found in [179].

Note that in Theorem 8.3.1,  $P_{ij}$  is not necessarily positive definite, which makes the design more difficult, though it can result in a less conservative design. In the following, based on whether  $P_{ij}$  is positive definite, we divide our discussion into two cases:  $V_{ij} = 0$  and  $V_{ij} \succeq 0$ .

(I)  $V_{ij} = 0$

In this case,  $P_{ij} > 0$ . A direct approach for the CPS is to let  $P_{ij} = \begin{bmatrix} P_{ij}^{(1)} & 0 \\ 0 & P_j^{(3)} \end{bmatrix}$ ,  $(i, j) \in \Phi_c$ , in (8.13). Though such approach can lead to a design based on convex optimization, it is very conservative in general, because we impose a strong structural constraint on  $P_{ij}$ . In the following, we consider the problem of how to alleviate the design conservatism.

**Lemma 8.3.2** *There exists a solution  $P_{ij} > 0$  to (8.13) if and only if for some positive scalar  $\varepsilon$ , there exists a solution  $(G_{ij}, P_{ij} > 0, (i, j) \in \Phi_c)$  to the following matrix inequality*

$$\begin{bmatrix} \varpi_{ij}^{(3)} & * & * \\ \tilde{B}_{ij}^T G_{ij} & -\gamma^2 I & * \\ G_{ij} - P_{ij} - \varepsilon G_{ij}^T \tilde{A}_{ij} & -\varepsilon G_{ij}^T \tilde{B}_{ij} & -\varepsilon (G_{ij} + G_{ij}^T) \end{bmatrix} < 0 \quad (8.21)$$

where  $\varpi_{ij}^{(3)} = G_{ij}^T \tilde{A}_{ij} + \tilde{A}_{ij}^T G_{ij} + \tilde{E}_{ij}^T \tilde{E}_{ij} + \tilde{F}_{ij}^T U_{ij} \tilde{F}_{ij}$ .

**Proof:** *Necessity:* By setting  $G_{ij} = G_{ij}^T = P_{ij}$  and letting  $\varepsilon \rightarrow 0$ , (8.21) follows from (8.13).

*Sufficiency:* Note that  $G_{ij}$  is invertible since  $G_{ij} + G_{ij}^T > 0$ . Define  $\Gamma \triangleq \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\tilde{A}_{ij} & -\tilde{B}_{ij} & I \end{bmatrix}$ .

Multiplying (8.21) by  $\Gamma^T$  from the left and  $\Gamma$  from the right, respectively, we have

$$\begin{bmatrix} \varpi_{ij}^{(3)} & * & * \\ \tilde{B}_{ij}^T P_{ij} & -\gamma^2 I & * \\ G_{ij} - P_{ij} + \varepsilon G_{ij} \tilde{A}_{ij} & \varepsilon G_{ij} \tilde{B}_{ij} & -\varepsilon (G_{ij} + G_{ij}^T) \end{bmatrix} < 0 \quad (8.22)$$

which implies that (8.13) holds. ■

**Remark 8.3.3** *The above result shows the equivalence between (8.13) and (8.21). This indicates that (8.21) has no advantage over (8.13) in terms of  $H_\infty$  performance analysis for a given estimator. It should, however, be noted that (8.21)*

will provide a less conservative design of an estimator due to: 1) no structural constraint required on the Lyapunov matrix  $P_{ij}$  as in the direct approach when applying (8.21) to the estimator design; 2) the introduction of the partition-dependent slack variables  $G_{ij}$ ,  $(i, j) \in \Phi_c$ .

In order to apply Lemma 8.3.2 to the design of estimator using an LMI approach, in (8.21) we let

$$P_{ij} = \begin{bmatrix} P_{ij}^{(1)} & P_{ij}^{(2)} \\ P_{ij}^{(2)T} & P_{ij}^{(3)} \end{bmatrix}, \quad G_{ij} = \begin{bmatrix} G_{ij}^{(1)} & G_{ij}^{(2)} \\ 0 & G_j^{(3)} \end{bmatrix} \quad (8.23)$$

Since  $G_{ij}$  is invertible, so is  $G_j^{(3)}$ . By substituting (8.23) into (8.21) and invoking (8.4), we have the following result.

**Theorem 8.3.3** Consider the continuous-time PWA system defined by (8.1). Given a scalar  $\gamma > 0$ , there exists an estimator (8.2) that solves the  $H_\infty$  estimation problem if for some  $\varepsilon > 0$ , there exists a solution  $(P_{ij} > 0, G_{ij}, U_{ij} \succeq 0, W_j, (i, j) \in \Phi_c)$  to (8.11) and the following LMI:

$$\begin{bmatrix} A_i^T G_{ij}^{(1)} + G_{ij}^{(1)T} A_i + \tilde{U}_{ij}^{(1)} + \tilde{E}_{ij}^{(1)} & * & * \\ \Theta_{ij}^{(1)} + \tilde{U}_{ij}^{(2)T} + \tilde{E}_{ij}^{(2)T} & \Theta_{ij}^{(2)} + \tilde{U}_{ij}^{(3)} + \tilde{E}_{ij}^{(3)} & * \\ B_i^T G_{ij}^{(1)} & B_i^T G_{ij}^{(2)} + B_i^T G_j^{(3)} - D_i^T W_j & * \\ G_{ij}^{(1)} - P_{ij}^{(1)} - \varepsilon G_{ij}^{(1)T} A_i & G_{ij}^{(2)T} - P_{ij}^{(2)} & * \\ -P_{ij}^{(2)T} - \varepsilon \Theta_{ij}^{(1)} & G_j^{(3)} - P_{ij}^{(3)} - \varepsilon (G_j^{(3)T} A_j - W_j^T C_j) & * \\ * & * & * \\ * & * & * \\ -\gamma^2 I & * & * \\ -\varepsilon G_{ij}^{(1)T} B_i & -\varepsilon (G_{ij}^{(1)} + G_{ij}^{(1)T}) & * \\ \Theta_{ij}^{(3)} & -\varepsilon G_{ij}^{(2)T} & -\varepsilon (G_j^{(3)} + G_j^{(3)T}) \end{bmatrix} < 0, \quad \forall (i, j) \in \Phi_c \quad (8.24)$$

where  $\Theta_{ij}^{(1)} = G_{ij}^{(2)T} A_i + G_{ij}^{(3)T} (A_i - A_j) - W_j (C_i - C_j)$ ,  $\Theta_{ij}^{(2)} = A_j^T G_j^{(3)} + G_j^{(3)T} A_j -$

$$C_j^T W_j - W_j^T C_j, \Theta_{ij}^{(3)} = -\varepsilon G_{ij}^{(2)T} B_i - \varepsilon (G_j^{(3)T} B_i - W_j^T D_i) \text{ and } \begin{bmatrix} \tilde{U}_{ij}^{(1)} & \tilde{U}_{ij}^{(2)} \\ \tilde{U}_{ij}^{(2)T} & \tilde{U}_{ij}^{(3)} \end{bmatrix} =$$

$$\tilde{F}_{ij}^T U_{ij} \tilde{F}_{ij}, \begin{bmatrix} \tilde{E}_{ij}^{(1)} & \tilde{E}_{ij}^{(2)} \\ \tilde{E}_{ij}^{(2)T} & \tilde{E}_{ij}^{(3)} \end{bmatrix} = \tilde{E}_{ij}^T \tilde{E}_{ij}.$$

In this situation, the estimator gains can be given by

$$L_j = G_j^{(3)-T} W_j^T, \quad j \in \mathcal{I} \quad (8.25)$$

**Remark 8.3.4** To get the minimum  $\gamma$ , Theorem 8.3.3 involves a one-dimensional search over the scaling parameter  $\varepsilon$ . It can be easily done by applying a numerical optimization algorithm, such as the program **fminsearch** in the Optimization Toolbox of Matlab.

Lemma 8.3.2 applies partition-dependent Lyapunov functions without constraints on  $P_{ij}$ , hence will be less conservative than the direct method. However, there is still structural constraint on  $G_{ij}$ . In order to remove the constraints completely, we must consider another way.

We make a transformation and propose another useful Lemma. Let  $\mathcal{C}_{ij} = \begin{bmatrix} \tilde{A}_{ij} & \tilde{B}_{ij} \\ 0 & 0 \end{bmatrix}$ ,

$\mathcal{P}_{ij} = \begin{bmatrix} P_{ij} & 0 \\ 0 & I \end{bmatrix}$  and  $\mathcal{U}_{ij} = \begin{bmatrix} \tilde{E}_{ij}^T \tilde{E}_{ij} + \tilde{F}_{ij}^T U_{ij} \tilde{F}_{ij} & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$ . Then (8.13) can be rewritten as

$$\mathcal{C}_{ij}^T \mathcal{P}_{ij} + \mathcal{P}_{ij} \mathcal{C}_{ij} + \mathcal{U}_{ij} < 0 \quad (8.26)$$

Using a tricky transformation as Lemma A.0.6, we have the following lemma.

**Lemma 8.3.4** *There exists a solution  $P_{ij} > 0$  to (8.13) if and only if for some positive scalar  $\epsilon$  the following LMI holds:*

$$\begin{bmatrix} -\epsilon \mathcal{P}_{ij}^{-1} & I + \epsilon \mathcal{C}_{ij} \\ I + \epsilon \mathcal{C}_{ij}^T & \mathcal{U}_{ij} - \epsilon^{-1} \mathcal{P}_{ij} \end{bmatrix} < 0 \quad (8.27)$$

If we define a new variable  $Q_{ij}$  to replace  $P_{ij}^{-1}$  in (8.27), the problem again becomes to solve an additional equation constraint  $Q_{ij}P_{ij} = I$ . Note that we use the discontinuous Lyapunov function as (8.19) here. Hence the problem is transferred into a BMI problem similar to Theorem 7.2.3, and we can use the SLPMM to solve it.

(II)  $V_{ij} \succeq 0$

Note that Theorem 8.3.1 is already a BMI problem. We can use some standard BMI algorithms to solve this problem. However, the LMI approaches are still possible.

An LMI approach providing more variables compared with Lemma 8.3.2, can be stated as follows. (8.3) can be rewritten as

$$\mathcal{D}_{ij}\varsigma = 0, \quad \mathcal{D}_{ij} \triangleq \begin{bmatrix} \tilde{A}_{ij} & \tilde{B}_{ij} & -I \end{bmatrix}, \quad \varsigma \triangleq \begin{bmatrix} \xi^T & \omega^T & \delta\xi^T \end{bmatrix}^T \quad (8.28)$$

Hence we have

$$\varsigma^T \begin{bmatrix} \mathcal{U}_{ij} & \begin{bmatrix} P_{ij} \\ 0 \end{bmatrix} \\ \begin{bmatrix} P_{ij} & 0 \end{bmatrix} & 0 \end{bmatrix} \varsigma < 0, \quad s.t. \quad (8.28) \quad (8.29)$$

Using Finsler's lemma, we have the following result.

**Lemma 8.3.5** *Inequality (8.29) is equivalent to*

$$\begin{bmatrix} \mathcal{U}_{ij} & \begin{bmatrix} P_{ij} \\ 0 \end{bmatrix} \\ \begin{bmatrix} P_{ij} & 0 \end{bmatrix} & 0 \end{bmatrix} + \mathcal{D}_{ij}^T \mathcal{X}_{ij} + \mathcal{X}_{ij}^T \mathcal{D}_{ij} < 0 \quad (8.30)$$

for some additional variables  $\mathcal{X}_{ij} = [X_{ij}, Y_{ij}, Z_{ij}]$ .

**Remark 8.3.5** *It is clear that if we let  $\mathcal{X}_{ij} = [G_{ij}, G_{ij}, -\epsilon G_{ij}]$ , then we have Lemma 8.3.2. It means that we provide an alternative way to prove Lemma 8.3.2.*

Meanwhile, we can easily prove that in the setting of  $V_{ij} \succeq 0$ , (8.21) is still equivalent to (8.13). Hence we have the following theorem.

**Theorem 8.3.6** Consider the continuous-time PWA system defined by (8.1). Given a scalar  $\gamma > 0$ , there exists an estimator (8.2) that solves the  $H_\infty$  estimation problem if for some  $\varepsilon > 0$ , there exists a solution  $(P_{ij} = P_{ij}^T, G_{ij}, U_{ij} \succeq 0, W_j, (i, j) \in \Phi_c)$  satisfying (8.12), (8.19) and (8.24). Furthermore, the estimator gain is given by (8.25).

### 8.3.2 Discrete-time Case

**Theorem 8.3.7** Consider the discrete-time PWA system defined by (8.1). A given estimator satisfies the  $H_\infty$  performance  $\gamma$ , if there exists a solution  $(P_{ij} = P_{ij}^T, V_{ij} \succeq 0, U_{ijkl} \succeq 0)$  to the following inequalities:

$$P_{ij} - \tilde{F}_{ij}^T V_{ij} \tilde{F}_{ij} > 0, \quad \forall (i, j) \in \Phi_d \quad (8.31)$$

$$\begin{bmatrix} \varpi_{ijkl}^{(2)} & \tilde{A}_{ij}^T P_{kl} \tilde{B}_{ij} \\ \tilde{B}_{ij}^T P_{kl} \tilde{A}_{ij} & \tilde{B}_{ij}^T P_{kl} \tilde{B}_{ij} - \gamma^2 I \end{bmatrix} < 0, \quad \forall (i, j, k, l) \in \Psi_d \quad (8.32)$$

where  $\varpi_{ijkl}^{(2)} = \tilde{A}_{ij}^T P_{kl} \tilde{A}_{ij} - P_{ij} + \tilde{F}_{ij}^T U_{ijkl} \tilde{F}_{ij} + \tilde{E}_{ij}^T \tilde{E}_{ij}$ .

**Proof:** We can follow the proof procedure of Theorem 8.3.1 and Lemma 7.2.4. Hence we omit the detail. ■

Following the way of CPS, we also discuss the  $H_\infty$  estimator design for DPS by two cases:  $V_{ij} = 0$  and  $V_{ij} \succeq 0$ .

$$(I) \quad V_{ij} = 0$$

By Schur complement, (8.32) can be represented as

$$\begin{bmatrix} -P_{kl} & P_{kl}\tilde{A}_{ij} & P_{kl}\tilde{B}_{ij} \\ * & \vartheta_{ijkl} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \quad (8.33)$$

where  $\vartheta_{ijkl} = -P_{ij} + \tilde{F}_{ij}^T U_{ijkl} \tilde{F}_{ij} + \tilde{E}_{ij}^T \tilde{E}_{ij}$ . Thus we can let  $P_{ij} = \begin{bmatrix} P_{ij}^{(1)} & 0 \\ 0 & P^{(3)} \end{bmatrix}$ ,  $(i, j, k, l) \in \Psi_d$  in (8.33) to get a direct design approach based on Theorem 8.3.7 using LMI. However, similar to CPS, such approach is too conservative due to the structural constraint on  $P_{ij}$ . The technique like Lemma 8.3.2 is expected. Nevertheless, different from CPS, there is an index coupling problem in (8.33) for DPS. Hence, we propose the following lemma.

**Lemma 8.3.8** *The inequality (8.32) of Theorem 8.3.7, can be implied by the following inequality for  $(P_{ij} > 0, U_{ijkl} \succeq 0, G_{ij})$ , for  $\forall (i, j, k, l) \in \Psi_d$ :*

$$\begin{bmatrix} P_{kl} - 2P_{ij} & G_{ij}^T \tilde{A}_{ij} & G_{ij}^T \tilde{B}_{ij} & G_{ij}^T - P_{ij} \\ * & \vartheta_{ijkl} & 0 & -\varepsilon \tilde{A}_{ij}^T G_{ij} \\ * & * & -\gamma^2 I & -\varepsilon \tilde{B}_{ij}^T G_{ij} \\ * & * & * & -\varepsilon (G_{ij} + G_{ij}^T) \end{bmatrix} < 0 \quad (8.34)$$

where  $\varepsilon$  is a positive scalar.

**Proof:** The proof can be completed by noting that  $P_{ij} P_{kl}^{-1} P_{ij} \geq 2P_{ij} - P_{kl}$  and following the way of Lemma 8.3.2.  $\blacksquare$

**Remark 8.3.6** *Similar to Lemma 8.3.2, the key idea of Lemma 8.3.8 is to eliminate the coupling between the Lyapunov matrices  $P_{kl}$  and the system matrices. Now we choose  $P_{ij}$  and  $G_{ij}$  as (8.23). Based on the Lemma 8.3.8, we can obtain a less conservative approach analogous to Theorem 8.3.3. We omit the detail here.*

We can also remove the constraints on  $G_{ij}$ . In fact, we can easily see that the following conditions are equivalent to the inequality (8.32) of the theorem 8.3.7 [173].

$$\begin{bmatrix} -Q_{kl} & \tilde{A}_{ij} & \tilde{B}_{ij} \\ * & \vartheta_{ijkl} & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0, \quad \forall (i, j, k, l) \in \Psi_d \quad (8.35)$$

$$P_{kl}Q_{kl} = I, \quad (k, l) \in \Phi_d \quad (8.36)$$

(8.36) is a BMI problem. SLPMM can be applied again to solve this problem.

Note that for DPS, a similar form to (8.26) can be written as follows.

$$\mathcal{C}_{ij}^T \mathcal{P}_{kl} \mathcal{C}_{ij} - \mathcal{P}_{ij} + \bar{\mathcal{U}}_{ijkl} < 0, \quad (i, j, k, l) \in \Psi_d \quad (8.37)$$

$$\text{where } \bar{\mathcal{U}}_{ijkl} = \begin{bmatrix} \tilde{E}_{ij}^T \tilde{E}_{ij} + \tilde{F}_{ij}^T U_{ijkl} \tilde{F}_{ij} & 0 \\ 0 & I - \gamma^2 I \end{bmatrix}.$$

(II)  $V_{ij} \succeq 0$

We observe that the terms  $\tilde{A}_{ij}^T P_{kl} \tilde{A}_{ij}$ ,  $\tilde{A}_{ij}^T P_{kl} \tilde{B}_{ij}$  and  $\tilde{B}_{ij}^T P_{kl} \tilde{B}_{ij}$  in (8.32) are not even bilinear as  $\tilde{A}_{ij}$  and  $\tilde{B}_{ij}$  involve the estimator gains to be determined. To overcome this obstacle, we propose the following lemma.

**Lemma 8.3.9** *Inequality (8.32) of Theorem 8.3.7, is equivalent to the following inequality for some  $(P_{ij} = P_{ij}^T, \Upsilon_{ijkl}, \Psi_{ij}, U_{ijkl} \succeq 0)$  for  $\forall (i, j, k, l) \in \Psi_d$ :*

$$\begin{bmatrix} \varrho_{ijkl} + \hat{A}_{ij}^T \Upsilon_{ijkl} + \Upsilon_{ijkl}^T \hat{A}_{ij} & -\Upsilon_{ijkl}^T + \hat{A}_{ij}^T \Xi_{kl} \\ * & P_{kl} - \Xi_{kl} - \Xi_{kl}^T \end{bmatrix} < 0 \quad (8.38)$$

$$\text{where } \varrho_{ijkl} = \begin{bmatrix} -P_{ij} + \tilde{F}_{ij}^T U_{ijkl} \tilde{F}_{ij} + \tilde{E}_{ij}^T \tilde{E}_{ij} & 0 \\ * & -\gamma^2 I \end{bmatrix}, \quad \hat{A}_{ij} = \begin{bmatrix} \tilde{A}_{ij} & \tilde{B}_{ij} \end{bmatrix}.$$

**Proof:** Here we give an outline of the proof. It is easy to see that (8.32) is equivalent to

$$\zeta^T \begin{bmatrix} \hat{\mathcal{U}}_{ijkl} - \begin{bmatrix} P_{ij} & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & P_{kl} \end{bmatrix} \zeta < 0, \text{ s.t. (8.28)} \quad (8.39)$$

where  $\hat{\mathcal{U}}_{ijkl} = \bar{\mathcal{U}}_{ijkl} - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ . Hence, using Finsler's Lemma again, we can conclude that (8.32) is equivalent to

$$\begin{bmatrix} \hat{\mathcal{U}}_{ijkl} - \begin{bmatrix} P_{ij} & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & P_{kl} \end{bmatrix} + \mathcal{D}_{ij}^T \mathcal{X}_{ijkl} + \mathcal{X}_{ijkl}^T \mathcal{D}_{ij} < 0 \quad (8.40)$$

By noting that the bottom-right block of (8.40) is only related to  $(k, l)$ , we can choose  $\mathcal{X}_{ijkl} = [\Upsilon_{ijkl} \ \Xi_{kl}]$ . ■

**Remark 8.3.7** Actually, the choice of  $\mathcal{X}_{ijkl} = [\Upsilon_{ij} \ \Xi_{kl}]$  is also necessary and sufficient to (8.32) of Theorem 8.3.7. Here we also show it by the means of proving Lemma 8.3.9. The sufficiency can be easily observed via multiplying (8.38) with  $\Upsilon_{ijkl} = \Upsilon_{ij}$  by  $\Lambda_{ij}^T = \begin{bmatrix} I & \hat{A}_{ij}^T \\ 0 & I \end{bmatrix}$  at the left side and  $\Lambda_{ij}$  at the right side:

$$\begin{bmatrix} \varrho_{ijkl} + \hat{A}_{ij}^T P_{kl} \hat{A}_{ij} & * \\ -\Upsilon_{ij} + P_{kl} \hat{A}_{ij} - \Xi_{kl}^T \hat{A}_{ij} & P_{kl} - \Xi_{kl} - \Xi_{kl}^T \end{bmatrix} < 0 \quad (8.41)$$

Now we can choose  $\Xi_{kl} = P_{kl} + \sigma I$  and  $\Upsilon_{ij} = -\sigma \hat{A}_{ij}$ , where  $\sigma$  is a sufficiently large scalar, to transfer (8.41) to

$$\begin{bmatrix} \varrho_{ijkl} + \hat{A}_{ij}^T P_{kl} \hat{A}_{ij} & * \\ 0 & -P_{kl} - 2\sigma I \end{bmatrix} < 0 \quad (8.42)$$

Since  $\Lambda_{ij}$  is invertible, the necessity is proved. Note that for the iterative LMI approaches to BMI problem, the less variables may achieve better results based

on our experience, if these conditions are equivalent. Thus we have the following result.

**Theorem 8.3.10** Consider the discrete-time PWA system defined by (8.1). Given a scalar  $\gamma > 0$ , there exists an estimator (8.2) that solves the  $H_\infty$  estimation problem if for  $\forall(i, j, k, l) \in \Psi_d$ , there exists a solution  $(L_j, P_{ij} = P_{ij}^T, \Upsilon_{ijkl} = \Upsilon_{ij}, \Psi_{ij}, V_{ij} \succeq 0, U_{ijkl} \succeq 0)$  to the LMIs (8.31) and BMIs (8.38).

**Remark 8.3.8** It is no doubt that if the region  $S_i$  is identified by output state space, i.e.,  $S_i = \{y | F_i y + f_i \geq 0\}$ , then the estimator design can be simplified, because the mode switching of the system dynamics and estimator dynamics are synchronized.

### 8.3.3 Examples

**Example 8.3.1** Consider the CPS (8.1) with the following parameters

$$A_1 = \begin{bmatrix} -1 & -0.2 \\ 0.2 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0.2 \\ -0.2 & -0.3 \end{bmatrix}, B_1 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D_1 = D_2 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, F_1 = -F_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T, E_1 = E_2 = I$$

Using the direct approach, we obtain the optimal  $H_\infty$  performance  $\gamma = 0.0251$ . Using Theorem 8.3.3, by employing the **fminsearch**, we obtain the optimal  $H_\infty$  performance  $\gamma = 0.0232$  when  $\varepsilon = 0.1050$ . Using Theorem 8.3.6, we can improve it to  $\gamma = 0.0215$  due to two facts: 1) different Lyapunov matrices are used for each partition; 2) the Lyapunov matrices are not necessarily positive definite. The optimal estimator gains are

$$L_1 = \begin{bmatrix} 112.04 & -112.03 \\ -133.82 & 133.82 \end{bmatrix}, L_2 = \begin{bmatrix} 226.15 & -226.21 \\ -206.83 & 206.84 \end{bmatrix}$$

To simulate the tracking performance, we let the input be a white noise with unit variance. The simulation results with initial state  $[-5 \ -1]^T$  for the actual system state and its estimated state are shown in Figure 8.1 where the estimated and the actual states are close to each other. We can see that the estimation error is large at some time instances due to switching.

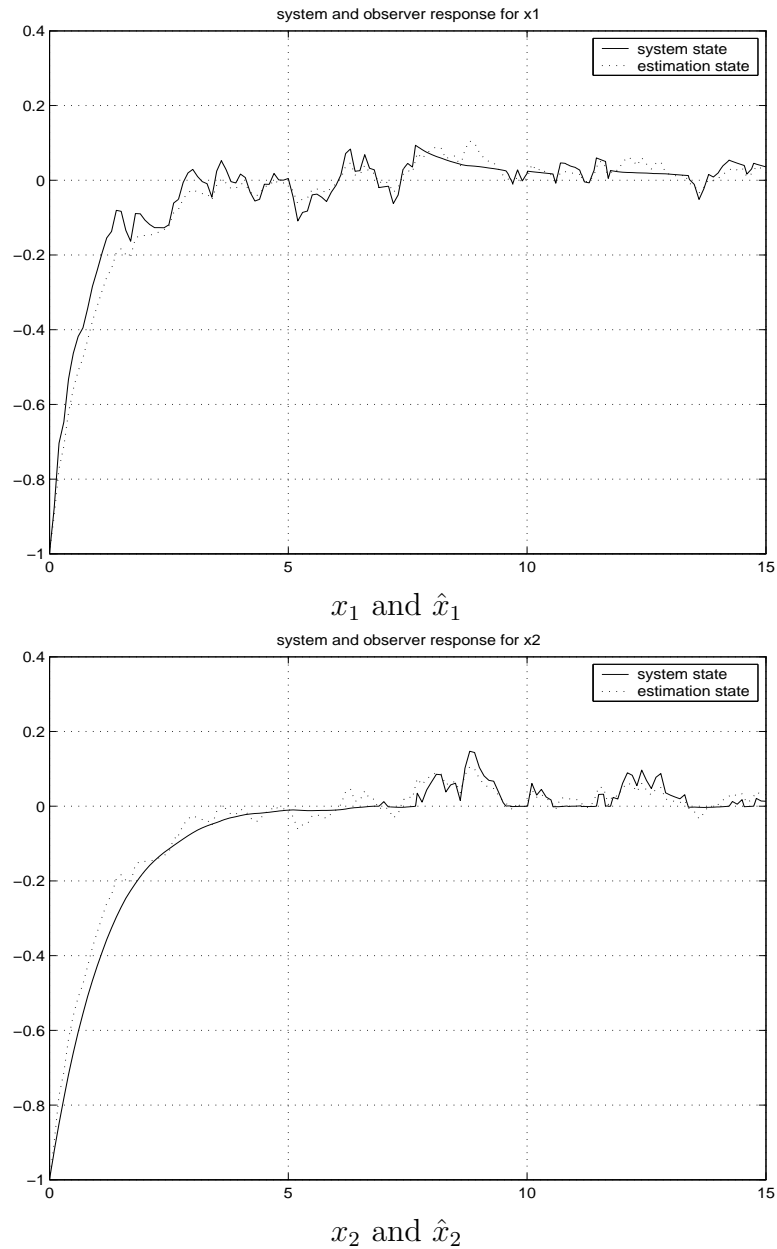


Figure 8.1: Actual state and estimated state via the  $H_\infty$  estimator.

**Example 8.3.2** Consider the following system slightly changed from the one in [61, 44]. It is shown in [61] the system model can be represented by DPS (8.1) with the following parameters

$$A_1 = \begin{bmatrix} 0 & 0.89 & 0.5 \\ -0.12 & 0.89 & 0 \\ -0.1 & 0 & 0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0.89 & 0.5 \\ 0.12 & 0.89 & 0 \\ -0.1 & 0 & 0.9 \end{bmatrix}, C_1 = C_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T,$$

$$B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, D_1 = D_2 = [1], F_1 = -F_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T, E_1 = E_2 = I$$

Theorem 8.3.3 generates the optimal  $\gamma$  of 0.17694, which is better than the result  $\gamma = 0.18750$  from the direct approach. Using SLPMM algorithm, we can get  $\gamma = 0.17112$ . However, Theorem 8.3.10 gives  $\gamma = 0.16945$ . The corresponding estimator gains are  $L_1 = \begin{bmatrix} 1.01846 \\ 0.99763 \\ 1.00151 \end{bmatrix}$ ,  $L_2 = \begin{bmatrix} 0.98911 \\ 1.00550 \\ 1.00323 \end{bmatrix}$ . Note that  $\Psi_d$  shall be  $\{(i, j, k, l) | i, j, k, l = 1, 2\}$ , i.e.,  $2^4 = 16$  pairs, in this example. One sample of estimation error  $e(t)$  when the input noise is a white noise with unit power and initial conditions  $x_0 = [10 \ 10 \ -10]^T$ ,  $\hat{x}_0 = x_0$ , is shown in Figure 8.2. When the initial condition  $x_0 \neq \hat{x}_0$ , we can still obtain a fast convergent result. For example, we choose  $x_0 = [1 \ 1 \ -1]^T$ ,  $\hat{x}_0 = -x_0$ , the estimation error  $e(t)$  is shown in Figure 8.3.

## 8.4 Generalized $H_2$ Estimation

In this section, we only discuss the generalized  $H_2$  estimation for CPS. Theorem 8.4.1 gives a BMI-based solution, while Theorem 8.4.3 gives a LMI-based solution. Using the solution from Theorem 8.4.3 as an initial condition for the iterative algorithms, we may expect a better result from Theorem 8.4.1. We omit the

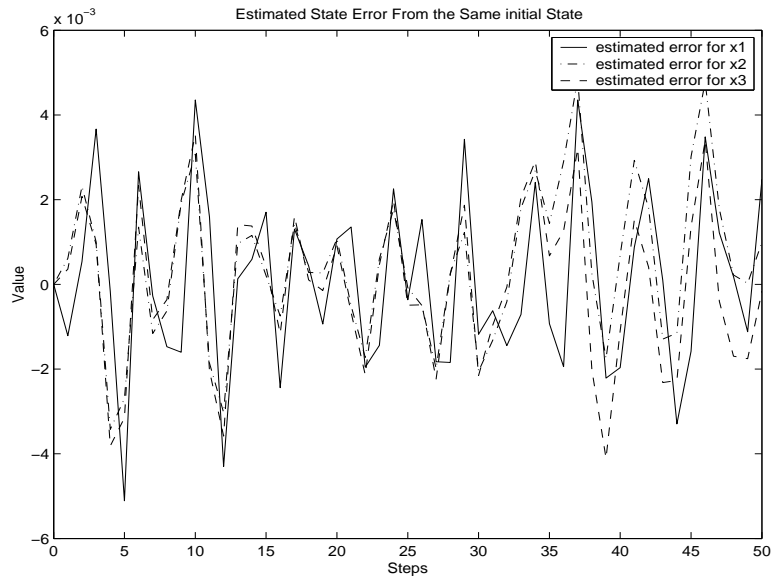


Figure 8.2: Trajectories of the state estimation error

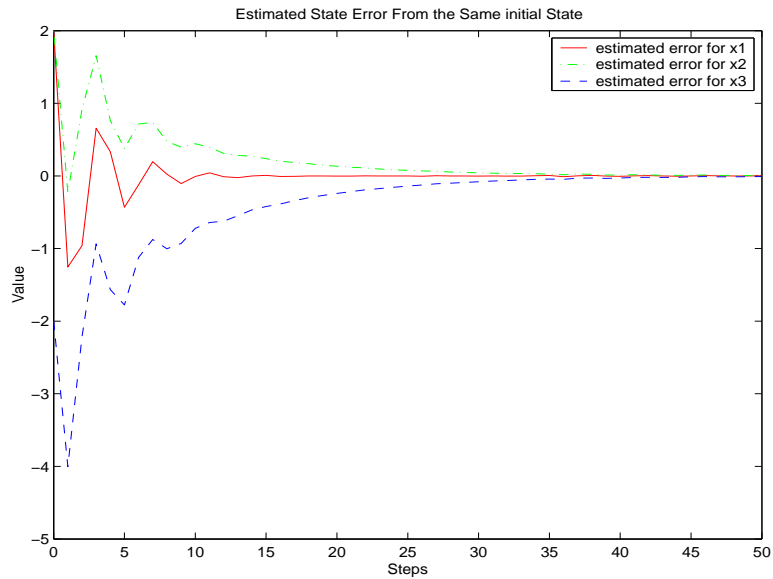


Figure 8.3: Trajectories of the state estimation error

proofs here, since they are similar to those of the last section. We can follow a similar way of the last section to discuss the counterpart for DPS.

**Theorem 8.4.1** *Consider the continuous-time PWA system defined by (8.1). Given a scalar  $\tau > 0$ , the estimator (8.2) solves the generalized  $H_2$  estimation problem if there exists a solution  $(T = T^T, U_{ij} \succeq 0, V_i \succeq 0, L_i, (i, j) \in \Phi_c)$  to (8.11), (8.12) and the following matrix inequalities*

$$P_{ij} - \frac{1}{\tau} \tilde{E}_{ij}^T \tilde{E}_{ij} \geq 0, \quad \forall (i, j) \in \Phi_c \quad (8.43)$$

$$\begin{bmatrix} P_{ij} \tilde{A}_{ij} + \tilde{A}_{ij}^T P_{ij} + \tilde{F}_{ij}^T U_{ij} \tilde{F}_{ij} & P_{ij} \tilde{B}_{ij} \\ \tilde{B}_{ij}^T P_{ij} & -I \end{bmatrix} < 0, \quad \forall (i, j) \in \Phi_c \quad (8.44)$$

Similar to Lemma 8.3.2, we have:

**Lemma 8.4.2** *There exists a matrix  $P_{ij} > 0$  to (8.44) if and only if for some positive scalar  $\varepsilon$ , there exists a solution  $(G_{ij}, P_{ij} = P_{ij}^T > 0, U_{ij} \succeq 0, L_i, i, j \in \mathcal{I})$  for  $\forall (i, j) \in \Phi_c$  satisfies:*

$$\begin{bmatrix} G_{ij}^T \tilde{A}_{ij} + \tilde{A}_{ij}^T G_{ij} + \tilde{F}_{ij}^T U_{ij} \tilde{F}_{ij} & * & * \\ \tilde{B}_{ij}^T G_{ij} & -I & * \\ G_{ij} - P_{ij} - \varepsilon G_{ij}^T \tilde{A}_{ij} & -\varepsilon G_{ij}^T \tilde{B}_{ij} & -\varepsilon (G_{ij} + G_{ij}^T) \end{bmatrix} < 0 \quad (8.45)$$

We choose  $G_{ij}$  with the same structure as in (8.23). Then, the following result follows.

**Theorem 8.4.3** *Consider the continuous-time system defined by (8.1). The optimal generalized  $H_2$  estimator can be obtained by the following optimization*

$$\min_{(\varepsilon, T, G_{ij}, W_j, (i, j) \in \Omega_c)} \tau \quad (8.46)$$



## Chapter 9

# A PWA System Approach to Control of Takagi-Sugeno Fuzzy Logic Systems

### 9.1 Introduction

Fuzzy logic control (FLC) techniques usually decompose a complex system into several subsystems according to the human experts' knowledge about the system. Meanwhile, a set of simple and straightforward control laws are used to emulate the human control strategy in each local operating region [56, 59, 60]. A global control law is then constructed by combining all local control actions through fuzzy membership functions, i.e., the weight functions [102, 56, 59, 60]. Visually, "IF-THEN" conditional statements in terms of human expert's qualitative linguistic variables together with simple models are employed in the analysis and synthesis instead of using one complex mathematical model.

Recently, there have appeared a number of systematic stability analysis and controller synthesis methods in fuzzy control theory, where Takagi-Sugeno (T-S) fuzzy model [160] is widely used. In [161], the stability of the system is determined by

checking whether there exists a common positive definite matrix  $P$  which satisfies a Lyapunov equation or a linear matrix inequality (LMI) for all local models. However, such approach is generally conservative, because it is difficult to find such a common Lyapunov matrix for all models, especially when the system is a highly nonlinear complex one.

Hence, piecewise Lyapunov functions have been employed to reduce the conservatism of stability analysis. Johansson [102] presents a stability condition for continuous-time T-S fuzzy systems using a differential inclusion that involves all consequent dynamics via operating regime based models. In each operating regime, there is a corresponding Lyapunov matrix. The discrete-time version is considered in [167]. In [59], Feng presents a stability analysis method for continuous-time T-S fuzzy dynamic systems via considering the difference between the dominant system mode and other system modes in the same partition as an uncertain term. The corresponding discrete-time version is discussed in [60]. These ideas, in fact, are inspired by Johansson and Rantzer [145], where piecewise affine (PWA) systems are analyzed. It has been shown that a wide class of fuzzy dynamic systems can be considered as PWA systems [102, 59, 60]. For controller synthesis, Wang and Feng [167] give a controller design method for discrete-time fuzzy systems. In [54], Feng presents a method to construct an  $H_\infty$  controller for fuzzy dynamic systems using a piecewise continuous Lyapunov function.

This chapter is motivated by recent stability results for PWA systems [102, 104, 55, 60, 64, 172, 167], polytopic uncertain systems [176, 46, 47], and controller design methods [167, 171]. We consider the T-S fuzzy system as a particular example of operating regime models [102, 167], and then formulate it as a PWA system. By introducing a new partition rule based on (individual) membership functions, the fuzzy system is converted to a PWA system. Using the so-called piecewise fuzzy Lyapunov functions, we present a less conservative stability condition for discrete-time fuzzy systems compared with the existing results [102, 59, 60, 167]. The corresponding stability conditions are further extended to give a less con-

servative controller design method. In the controller design, we first introduce a design method based on BMI whose solution depends on initial conditions. Some techniques for choosing a proper initial condition are given.

## 9.2 Takagi-Sugeno Fuzzy Model

In this chapter, we consider the T-S fuzzy dynamic model similar to the one proposed in [167, 104] which can be used to represent a wide class of complex systems with both fuzzy inference rules and local analytic linear models. This model is obtained by considering fuzzy systems as a particular example of operating regime based models [167, 104]. Operating regime based modelling is a series of techniques that construct a globally valid model of the system dynamics by combining simple local models, each valid within a proper operating regime [102].

$$\begin{array}{l}
 \text{Rule}^i : \\
 \text{IF} \quad \underbrace{x_1 \text{ is } L_1^i \text{ AND } \dots \text{ AND } x_n \text{ is } L_n^i}_{\text{operating regime specification}} \\
 \text{THEN} \quad \underbrace{\delta x(t) = A_i x(t) + B_i u(t) + a_i}_{\text{local dynamics}} \\
 \quad \quad \quad i \in \bar{\mathcal{I}} = \{1, 2, \dots, \bar{m}\}
 \end{array} \tag{9.1}$$

where  $rule^i$  denotes the  $i$ -th fuzzy inference rule.  $L_j^i$ , ( $j = 1, 2, \dots, n$ ) is the fuzzy set.  $\bar{\mathcal{I}}$  is the index set of inference rules.  $x = [x_1, \dots, x_n]^T \in \mathcal{R}^n$  is the system state.  $u \in \mathcal{R}^{n_u}$  is the control input vector.  $A_i, B_i$  are the system parameters of the  $i$ -th local model.

Let  $\mu_i(x(t))$  be the normalized membership function satisfying  $0 \leq \mu_i(x(t)) \leq 1$  and  $\sum_{i=1}^{\bar{m}} \mu_i = 1$  for the inferred fuzzy set  $L^i$  where  $L^i = \bigcap_{j=1}^n L_j^i$ . We use  $w_j^i$  to denote the corresponding membership weight for each  $L_j^i$  (i.e., individual membership function) and assume  $w_j^i$  is also normalized. Then we have  $\mu_i(x(t)) = \prod_{j=1}^n w_j^i$ . By using a center-average defuzzifier, product inference and singleton

fuzzifier, the fuzzy system (9.1) can be expressed as the following global model:

$$\delta x(t) = \sum_{i=1}^{\bar{m}} \mu_i(x(t)) \{A_i x(t) + B_i u(t) + a_i\}, \quad x(t) \in \mathcal{R}^n \quad (9.2)$$

Following the idea from [102, 104, 167], the rules of the fuzzy system (9.1) induce a set of partitions of the state space. Usually, we assume that these partitions are polyhedra/polytopes. However, in order to present more general fuzzy models, we introduce a different partition method from [102, 104, 167]<sup>1</sup>.

$$\bar{S}_\ell = \begin{cases} (i) : \{x | 0 < \mu_{i_k}(x) < 1, \sum_k \mu_{i_k} = 1, \forall i_k \in \bar{\mathcal{I}}\} \text{ or} \\ (ii) : \{x | \mu_{i_k}(x) = 1, i_k \in \bar{\mathcal{I}}\} \end{cases} \quad (9.3)$$

$\ell = 1, 2, \dots, m$

Denote the index set of  $\ell$  as  $\mathcal{I}$ . The size  $m$  of the set  $\mathcal{I}$  may not be equal to  $\bar{m}$ . Then the partition  $S_\ell$  can be obtained by considering the closure of  $\bar{S}_\ell$ . Note that we shall ignore the partition  $\bar{S}_\ell$  resulting from the formulation (ii) in (9.3), which can be absorbed by the closure of neighboring partitions. Note also that the zero value of  $\mu_{i_k}$  has no effect on the behavior of (9.2). Hence it is not considered in formulation (9.3).

**Remark 9.2.1** In [59, 60], a different representation of the fuzzy system (9.2) is presented by considering the differences between the dominant system mode and other system modes in a partition as uncertain terms and upper bounds are introduced for these uncertain terms. However, for the case that one partition consists of many system models, the upper bounds of the uncertain terms are not easy to obtain and generally conservative. Recall that the partition [59, 60] is described as

$$\tilde{S}_i = \{x | \mu_i(x) > \mu_j(x), i, j \in \bar{\mathcal{I}}, i \neq j\} \quad (9.4)$$

We denote the closure of  $\tilde{S}_i$  as  $\check{S}_i$ . Observe that with the partition as defined in (9.4), the partition number is always equal to the mode number.

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<sup>1</sup>In [102, 104, 167], the operating regime  $i$  is given by the intersection of the cells induced by the fuzzy sets that describe each propositional variable:  $\hat{S}_i = \{x | \mu_i(x) = 1\} = \bigcap_k \{x | \mu_{i_k}(x) = 1, i_k \in \bar{\mathcal{I}}\}$ .

We explain these partition methods in the following examples.

**Example 9.2.1** Consider a fuzzy dynamic system with affine terms from [60]:

$$\begin{array}{ll} \text{Rule}^i & \text{IF } x_1 \text{ is } L_1^i \\ & \text{THEN } \delta x = A_i x(t) + B_i u(t) + a_i \\ & i \in \bar{\mathcal{I}} = \{1, 2, \dots, \bar{m}\} \end{array} \quad (9.5)$$

where  $\bar{m} = 5$  and  $L_1^i$  represents a set of fuzzy labels {negative large, negative small, around zero, positive small, positive large} and the corresponding membership functions are shown in Figure 9.1. Using the proposed partition method of (9.3), we can divide the state space into 7 partitions:  $S_1 = \{x_1 \leq -2\}$ ,  $S_2 = \{-2 \leq x_1 \leq -1\}$ ,  $S_3 = \{-1 \leq x_1 \leq -0.5\}$ ,  $S_4 = \{-0.5 \leq x_1 \leq 0.5\}$ ,  $S_5 = \{0.5 \leq x_1 \leq 1\}$ ,  $S_6 = \{1 \leq x_1 \leq 2\}$ ,  $S_7 = \{2 \leq x_1\}$ . There are two dynamic models in  $S_2$  and  $S_6$ .

Note that although both the partition formulation in [102] and that in (9.4) [59, 60] give 5 partitions, the partition regions are not the same as ours. In fact, the former gives that  $\hat{S}_1 = S_1$ ,  $\hat{S}_2 = S_2$ ,  $\hat{S}_3 = S_3 \cup S_4 \cup S_5$ ,  $\hat{S}_4 = S_6$ ,  $\hat{S}_5 = S_7$ . And the latter gives that  $\check{S}_1 = \{x_1 \leq -1.5\}$ ,  $\check{S}_2 = \{-1.5 \leq x_1 \leq -0.5\}$ ,  $\check{S}_3 = \{-0.5 \leq x_1 \leq 0.5\}$ ,  $\check{S}_4 = \{0.5 \leq x_1 \leq 1.5\}$ ,  $\check{S}_5 = \{1.5 \leq x_1\}$ .

**Example 9.2.2** Consider a fuzzy dynamic system with affine terms (9.5) from [60], where  $\bar{m} = 7$  and  $L_1^i$  represents a set of fuzzy labels {negative very large, negative large, negative small, around zero, positive small, positive large, positive very large} and the corresponding membership functions are shown in Figure 9.2. Using the formulation of (9.3), we can divide the state space into 6 partitions:  $S_1 = \{x_1 \leq -2\}$ ,  $S_2 = \{-2 \leq x_1 \leq -1\}$ ,  $S_3 = \{-1 \leq x_1 \leq 0\}$ ,  $S_4 = \{0 \leq x_1 \leq 1\}$ ,  $S_5 = \{1 \leq x_1 \leq 2\}$ ,  $S_6 = \{2 \leq x_1\}$ . Note that if we use the partition formulation in [102], then there are 3 partitions. In the partition  $\{-2 \leq x_1 \leq 2\}$ , actually all modes are included. Hence it is generally conservative. The formulation (9.4) [59, 60] gives 7 partitions.

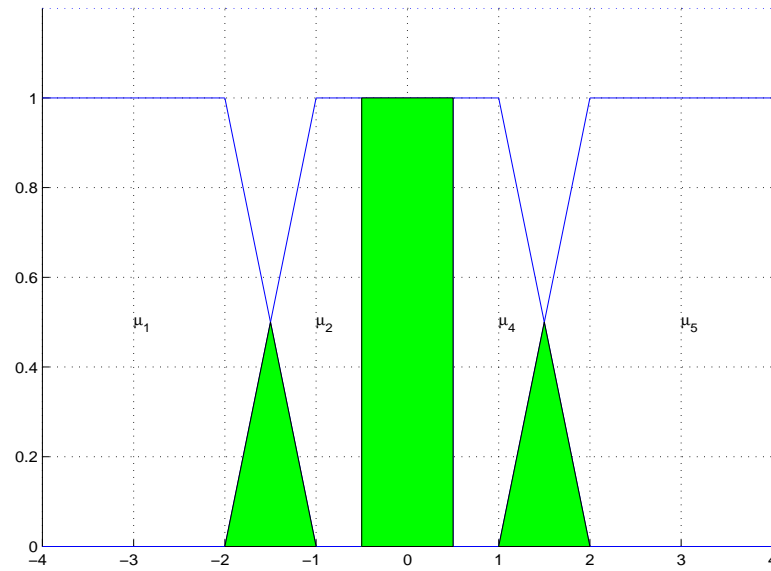


Figure 9.1: Membership functions of Example 9.2.1

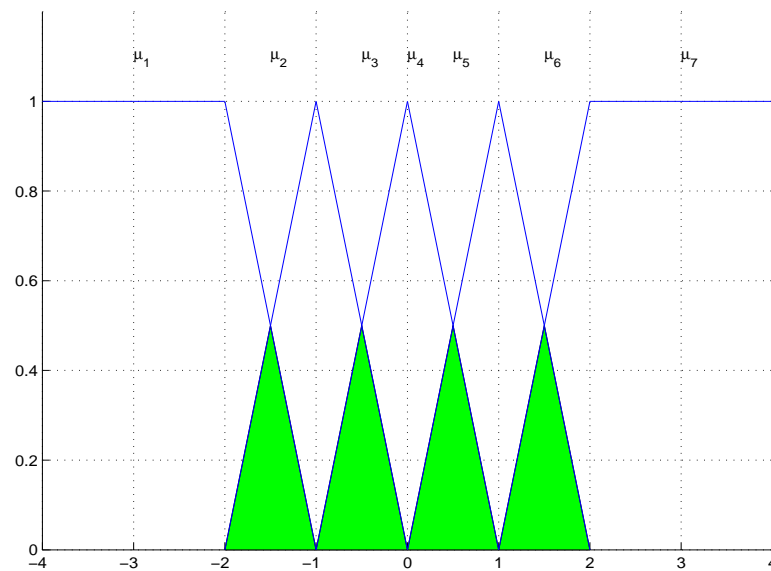


Figure 9.2: Membership functions of Example 9.2.2

i	$L_1^i$	$L_2^i$	$\mu_i$
1	0	0	$w_1^1 w_2^1$
2	0	$\pm 1.5$	$w_1^1 w_2^2$
3	1.5	0	$w_1^2 w_2^1$
4	1.5	$\pm 1.5$	$w_1^2 w_2^2$
5	-1.5	0	$w_1^3 w_2^1$
6	-1.5	$\pm 1.5$	$w_1^3 w_2^2$

Table 9.1: Fuzzy relationship

**Example 9.2.3** Now we consider the more general case (9.1). We assume that each  $L_j^i$  has  $l_j^i$  choices. For example,  $l_j^i$  in Example 9.2.1 and Example 9.2.2 are 5 and 7, respectively. The corresponding membership function is  $w_j^i$ . We assume that  $w_j^i$  is also normalized, i.e.,  $\sum_{j=1}^n w_j^i = 1$ . Then we can partition the space using the weight  $w_j^i$  based on (9.3). To illustrate, we consider the following fuzzy systems:

$$\begin{array}{l}
 \text{Rule}^i \quad \text{IF} \quad x_1 \text{ is about } L_1^i \text{ and } x_2 \text{ is about } L_2^i \\
 \text{THEN} \quad \delta x(t) = A_i x(t) + a_i \\
 \quad \quad \quad i \in \bar{\mathcal{I}} = \{1, 2, \dots, \bar{m}\}
 \end{array} \tag{9.6}$$

where  $L_1^i \in \{-1.5, 0, 1.5\}$  and  $L_2^i \in \{0, \pm 1.5\}$ . Hence  $\bar{m} = 2 * 3 = 6$ . The corresponding relationship is shown in Table 9.1.

The membership function corresponding to  $L_j^i$  is  $w_j^i$ ,  $j = 1, 2$ , defined as

$$w_1^1(x_1) = \begin{cases} 1 - \frac{8}{27}x_1^3 & x_1 \geq 0 \\ 1 + \frac{8}{27}x_1^3 & x_1 \leq 0 \end{cases}, \quad w_1^2(x_1) = \begin{cases} \frac{8}{27}x_1^3 & x_1 \geq 0 \\ 0 & x_1 \leq 0 \end{cases}$$

$$w_1^3(x_1) = \begin{cases} 0 & x_1 \geq 0 \\ -\frac{8}{27}x_1^3 & x_1 \leq 0 \end{cases}, \quad w_2^1 = 1 - \frac{4}{9}x_2^2, \quad w_2^2 = \frac{4}{9}x_2^2$$

The global membership function is defined in Table 9.1 and illustrated in Figure 9.3. The relationship for  $w_j^i$ ,  $j = 1, 2$  is shown in Figure 9.4, from which we can easily partition the state space into 4 regions, i.e.,  $\mathcal{I} = \{1, 2, 3, 4\}$ . In each region, there are four associated dynamic models. For example, for  $x \in \text{int}(S_1)$ ,

$w_1^1(x)$ ,  $w_1^2(x)$ ,  $w_2^1(x)$ ,  $w_2^2(x) \in (0, 1)$  and  $w_1^1(x) + w_1^2(x) = 1$ ,  $w_2^1(x) + w_2^2(x) = 1$ . Since  $((w_1^1(x) + w_1^2(x))(w_2^1(x) + w_2^2(x))) = 1$ , we have  $\dot{x} = \sum_{i=1}^4 \mu_i(A_i x + a_i)$  and  $\sum_{i=1}^4 \mu_i = 1$ . Let  $\mathcal{K}(i)$  be the index set for the system matrices that determine the system dynamics of that subspace. Then  $\mathcal{K}(1) = \{1, 2, 3, 4\}$ . Similarly,  $\mathcal{K}(2) = \{1, 2, 5, 6\}$ ,  $\mathcal{K}(3) = \{1, 2, 5, 6\}$ ,  $\mathcal{K}(4) = \{1, 2, 3, 4\}$ .

Note that if we use the partition formulation in [102], then there is only 1 region. In this example, if we use the formulation (9.4), the partitions are not easy to calculate. In fact, we have to calculate and compare each  $\mu_i$ , and then decide the partitions. Hence, for a general case, our partition method has more advantages.

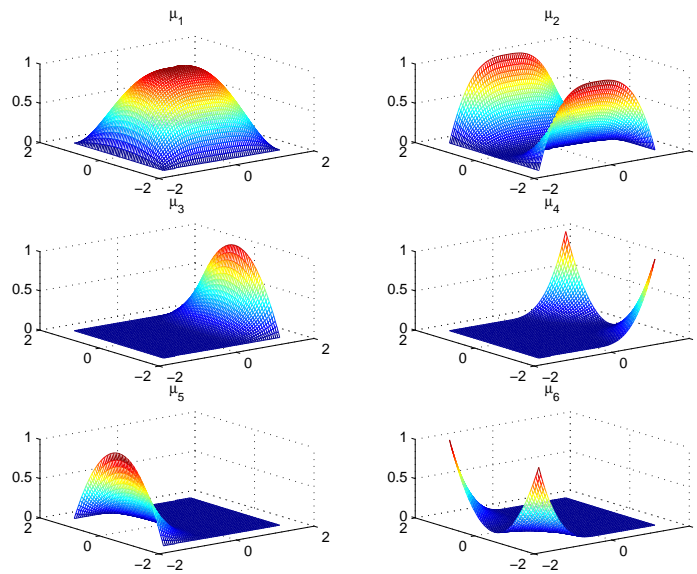


Figure 9.3: Global membership functions of Example 9.2.3

**Remark 9.2.2** Based on the aforementioned analysis, for a general case with rule from  $L_1^i$  to  $L_n^i$ , instead of using global membership formulation based on (9.3) to determine the partition, we can use the local membership functions. In fact, there are up to  $\prod_{l=1}^n \kappa_l$  partitions, where  $\kappa_l$  is the maximum partition number according to (9.3) using the local membership function:

$$\dot{S}_{l_j} = \begin{cases} (i) : \{x | 0 < w_{i_k}^i(x) < 1, \sum_k w_{i_k}^i = 1, \forall i_k \in \bar{\mathcal{I}}\} \text{ or} \\ (ii) : \{x | w_{i_k}^i(x) = 1, i_k \in \bar{\mathcal{I}}\} \end{cases}, \quad (9.7)$$

$$l_j = 1, 2, \dots, \kappa_l$$

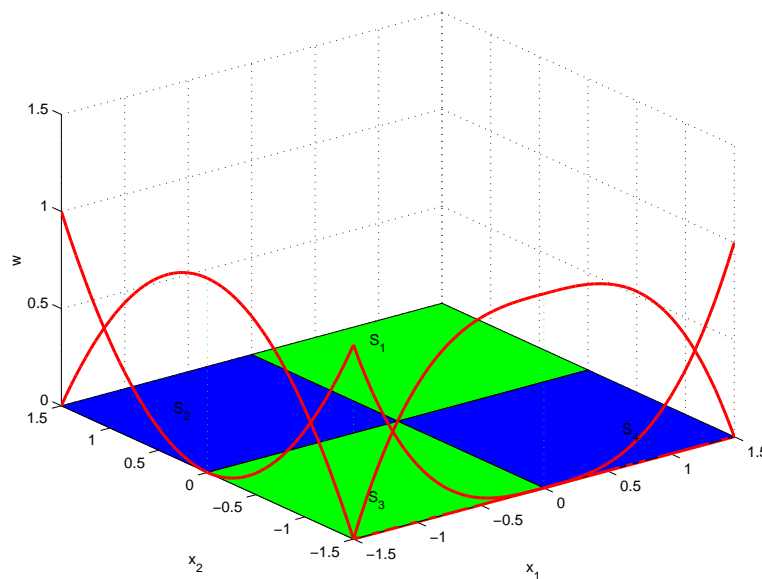


Figure 9.4: Individual membership functions of Example 9.2.3

For instance,  $\kappa_1 = 2$  and  $\kappa_2 = 2$  in Example 9.2.3. Again, we also ignore these  $w_{i_k}^i(x) = 1$  with only one point. Of course, we may combine some partitions together or only consider some rules while ignore the others in order to lower the computational complexity. In this example, if we only consider  $L_1^i$ , then there are 2 partitions.

Thus the fuzzy system (9.2) can be considered as a number of subsystems in a set of individual subspaces, each of which can be defined as

$$S_i = \{x(t) | F_i x(t) + f_i \geq 0\}, \quad i \in \mathcal{I} \quad (9.8)$$

Similar to previous discussion, we denote those index sets with the corresponding partitions containing the origin (assume  $a_i = 0$ ) as  $\mathcal{I}_0$ . Further, we define  $\mathcal{I}_1 = \mathcal{I} \setminus \mathcal{I}_0$ . Here we assume that  $F_i$  and  $f_i$  satisfy zero interpolation property [102] for  $i \in \mathcal{I}_0$ . Otherwise, we may reconstruct them using the method in [102]. Now we can write the fuzzy system (9.2) in the fuzzy subspaces as a convex combination of linear systems

$$\begin{aligned} \delta x(t) &= \mathcal{A}_i x(t) + \mathcal{B}_i u(t) + \alpha_i = \sum_{k \in \mathcal{K}(i)} \mu_k(x(t)) \{A_k x(t) + B_k u(t) + a_k\} \\ x(t) &\in S_i, \quad i \in \mathcal{I} \end{aligned} \quad (9.9)$$

where  $0 \leq \mu_k(x(t)) \leq 1$  and  $\sum_{k \in \mathcal{K}(i)} \mu_k(x(t)) = 1$ . For each subspace  $S_i$ , the set  $\mathcal{K}(i)$  contains the indices for the system matrices that determine the dynamics of the systems in that region. We also assume that given any initial condition  $x_0$ , the global fuzzy system (9.9) has a unique solution for all  $t \geq 0$  [102, 167].

For convenience, we introduce the extended presentation of (9.9):

$$\begin{aligned} \delta\zeta(t) &= \bar{\mathcal{A}}_i\zeta(t) + \bar{\mathcal{B}}_i = \sum_{k \in \mathcal{K}(i)} \mu_k(\zeta(t)) \{ \bar{A}_k\zeta(t) + \bar{B}_k u(t) \} \\ x(t) &\in S_i, \quad i \in \mathcal{I} \end{aligned} \quad (9.10)$$

where  $\zeta$ ,  $\bar{A}_k$ ,  $\bar{B}_k$ ,  $\bar{F}_i$ ,  $\bar{F}_{ij}$  are defined in (2.9).

### 9.3 Stability Analysis

The fuzzy system (9.9) now is transferred into the framework of the PWA systems with linear differential/difference inclusions (LDIs). In this section, combining the recent results on stability conditions [102, 55, 64] on PWA systems and dilated LMI characterization [47, 46, 176, 171, 173], we present some new stability criteria for discrete-time fuzzy systems (9.9). For the sake of simplification of notation, we denote  $RH(i)$  as the right hand side matrix of the equation or inequality ( $i$ ) and  $LH(i)$  as the left hand side matrix of ( $i$ ).

We consider the following Lyapunov function candidate:

$$v(x) = \begin{cases} x' P_i x & x \in S_i, \quad i \in \mathcal{I}_0 \\ \zeta^T \bar{P}_i \zeta & x \in S_i, \quad i \in \mathcal{I}_1 \end{cases} \quad (9.11)$$

where  $P_i = \sum_{k \in \mathcal{K}(i)} \mu_k P_{ki}$ ,  $i \in \mathcal{I}_0$  and  $\bar{P}_i = \sum_{k \in \mathcal{K}(i)} \mu_k \bar{P}_{ki}$ ,  $i \in \mathcal{I}_1$ .

We have the following result with the aid of (9.11).

**Theorem 9.3.1** *The discrete-time fuzzy system (9.9) with  $u \equiv 0$  is exponentially stable, if there exist  $P_{ki} = P_{ki}^T$ ,  $H_{ij}$ ,  $G_j$ ,  $U_{k_1 k_2 ij} \succeq 0$  and  $W_{ki} \succeq 0$  satisfying the following LMIs:*

$$P_{ki} - F_i^T W_{ki} F_i > 0, \quad \forall i \in \mathcal{I}_0, \quad \forall k \in \mathcal{K}(i) \quad (9.12)$$

$$\bar{P}_{ki} - \bar{F}_i^T W_{ki} \bar{F}_i > 0, \quad \forall i \in \mathcal{I}_1, \quad \forall k \in \mathcal{K}(i) \quad (9.13)$$

$$\begin{bmatrix} \Psi_{k_1 k_2 ij} & A_{k_1}^T G_{ij} - H_{ij}^T \\ G_{ij}^T A_{k_1} - H_{ij} & P_{k_2 j} - G_{ij} - G_{ij}^T \end{bmatrix} < 0 \quad (9.14)$$

$$\forall (i, j) \in \Omega_d, i, j \in \mathcal{I}_0, \quad \forall k_1 \in \mathcal{K}(i), \quad \forall k_2 \in \mathcal{K}(j)$$

$$\begin{bmatrix} \bar{\Psi}_{k_1 k_2 ij} & \bar{A}_{k_1}^T G_{ij} - H_{ij}^T \\ G_{ij}^T \bar{A}_{k_1} - H_{ij} & \bar{P}_{k_2 j} - G_{ij} - G_{ij}^T \end{bmatrix} < 0 \quad (9.15)$$

$$\forall (i, j) \in \Omega_d, \quad i \text{ or/and } j \in \mathcal{I}_1, \quad \forall k_1 \in \mathcal{K}(i), \quad \forall k_2 \in \mathcal{K}(j)$$

where  $\bar{A}_{k_j}, j = 1, 2$  and  $\bar{F}_i$  are defined in (2.9).  $\Psi_{k_1 k_2 ij} = -P_{k_1 i} + A_{k_1}^T H_{ij} + H_{ij}^T A_{k_1} + F_i^T U_{k_1 k_2 ij} F_i$ ,  $\bar{\Psi}_{k_1 k_2 ij} = -\bar{P}_{k_1 i} + \bar{A}_{k_1}^T H_{ij} + H_{ij}^T \bar{A}_{k_1} + \bar{F}_i^T U_{k_1 k_2 ij} \bar{F}_i$ .  $\bar{P}_{k_1 i} = \begin{bmatrix} I \\ 0 \end{bmatrix} P_{k_1 i} \begin{bmatrix} I \\ 0 \end{bmatrix}^T$  if  $i \in \mathcal{I}_0$ , and/or  $\bar{P}_{k_2 j} = \begin{bmatrix} I \\ 0 \end{bmatrix} P_{k_2 j} \begin{bmatrix} I \\ 0 \end{bmatrix}^T$  if  $j \in \mathcal{I}_0$  in (9.15).

**Proof:** It is obvious that

$$\begin{aligned} v(x) &= x^T P_i x = x^T (\sum_{k \in \mathcal{K}(i)} \mu_k P_{ki}) x \\ &> \sum_{k \in \mathcal{K}(i)} \mu_k (x^T F_i^T W_{ki} F_i x) \geq 0, \quad x \neq 0 \end{aligned} \quad (9.16)$$

for  $x \in S_i, i \in \mathcal{I}_0$  based on (9.12). Similarly,  $v(x) > 0$  for  $i \in \mathcal{I}_1$ .

For the case that  $(i, j) \in \Omega_d, i, j \in \mathcal{I}_0, k \in \mathcal{K}(i)$ , (9.14) implies

$$\begin{aligned} &\sum_{k_1 \in \mathcal{K}(i)} \mu_{k_1} \sum_{k_2 \in \mathcal{K}(i)} \mu_{k_2} LH(9.14) = \\ &\begin{bmatrix} -P_i + \mathcal{A}_i^T H_{ij} + H_{ij}^T \mathcal{A}_i + \Phi_{ij} & \mathcal{A}_i^T G_{ij} - H_{ij}^T \\ G_{ij}^T \mathcal{A}_i - H_{ij} & P_j - G_{ij} - G_{ij}^T \end{bmatrix} < 0 \end{aligned} \quad (9.17)$$

where  $\Phi_{ij} = \sum_{k_1 \in \mathcal{K}(i)} \mu_{k_1} \sum_{k_2 \in \mathcal{K}(i)} \mu_{k_2} F_i^T U_{k_1 k_2 ij} F_i$ . Multiplying  $\Lambda_i^T = \begin{bmatrix} I & \mathcal{A}_i^T \\ 0 & I \end{bmatrix}$

from the left and  $\Lambda_i$  from the right, it follows from (9.17) that

$$\begin{bmatrix} -P_i + \mathcal{A}_i^T P_j \mathcal{A}_i + \Phi_{ij} & \mathcal{A}_i^T P_j - \mathcal{A}_i^T G_{ij}^T - H_{ij}^T \\ * & P_j - G_{ij} - G_{ij}^T \end{bmatrix} < 0 \quad (9.18)$$

From the entry (1,1) of (9.18) we have

$$-P_i + \mathcal{A}_i^T P_j \mathcal{A}_i + \Phi_{ij} < 0 \quad (9.19)$$

which means

$$\begin{aligned} \Delta v(x(t)) &= v(x(t+1)) - v(x(t)) \\ &= (x(t)^T \mathcal{A}_i^T) P_j (\mathcal{A}_i x(t)) - x(t)^T P_i x(t) < 0, \quad x \neq 0 \end{aligned}$$

Similarly, for the case that  $(i, j) \in \Omega_d, i$  or/and  $j \in \mathcal{I}_1, k \in \mathcal{K}(i)$ , we also have  $\Delta v(t) < 0$  for  $x \neq 0$ . Thus we complete the proof. ■

**Remark 9.3.1** Note that (9.17) is also necessary for (9.19) to hold for certain combination of  $\mu_k(x(t))$ . In fact, we may choose  $G_{ij} = P_j + \sigma_j I$  and  $-H_{ij} = \sigma_j \mathcal{A}_i$ , where  $\sigma_j$  is a sufficiently large positive scalar.

**Remark 9.3.2** If we let  $P_{ki} = P_i, W_{ki} = 0$  and  $U_{kij} = 0$  in Theorem 9.3.1, then we obtain the stability condition in [167].

**Remark 9.3.3** Theorem 9.3.1 can actually be derived based on Finsler's lemma [22]. The key point is to introduce additional free variables into the LMIs so as to alleviate the conservatism of analysis for the case that one region involves several modes.

**Example 9.3.1** Consider a discrete-time fuzzy dynamic system (9.5) with following parameters [60]:

$$A_1 = \begin{bmatrix} 0.9 & -0.1 \\ 0.1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.95 & -0.06 \\ 0.06 & 0.95 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & -0.02 \\ 0.02 & 0.9 \end{bmatrix},$$

$$A_4 = A_2, \quad A_5 = A_1, \quad a_1 = \begin{bmatrix} 0 \\ -0.02 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ -0.01 \end{bmatrix},$$

$$a_4 = -a_2, \quad a_5 = -a_2$$

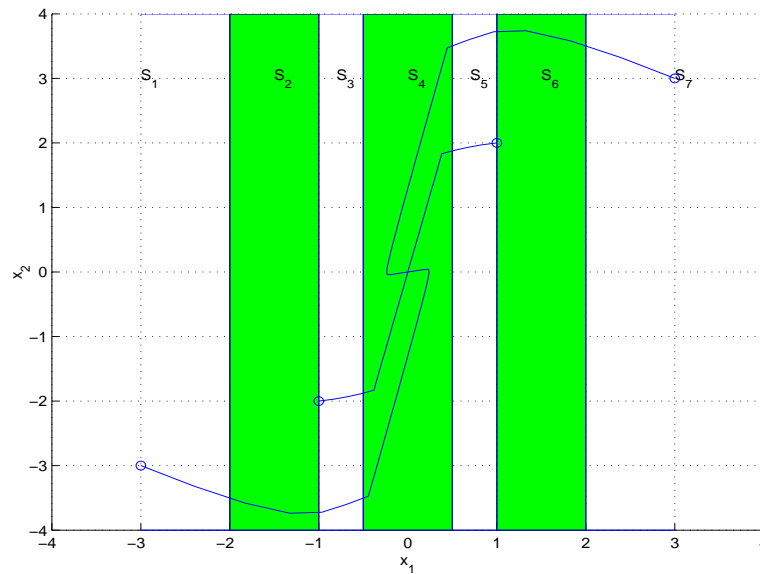


Figure 9.5: Trajectories from four initial conditions and region partitions

The system have already been proved to be stable in [60]. Hence, we consider the decay rate  $\alpha$  of Lyapunov function instead:  $\Delta v(x) < -\alpha v(x)$ . As illustrated in Figure 9.5, there are 7 partitions. Note that our partitions are different from [60]. Using Theorem 3.1 in [60], we obtain the maximum  $\alpha = 0.0083$ . Using the extended result with affine term for Theorem 1 in [167], we can obtain the maximum  $\alpha = 0.0082$ . However, our condition in Theorem 9.3.1 gives  $\alpha = 0.0093$ . In fact, it is also natural to reduce the partition number. For example, combine  $S_2$  and  $S_3$  as one partition  $\{x | -2 < x_1 \leq -0.5\}$ , and combine  $S_5$  and  $S_6$  as one partition  $\{x | 0.5 < x_1 \leq 2\}$ . If so, we can obtain the maximum  $\alpha = 0.0090$ .

**Remark 9.3.4** Note that  $\mu_k$  is a function of  $x$ , and hence  $P_i$  involves  $x$  according to (9.11). For the continuous-time fuzzy systems, we cannot directly apply the stability conditions of PWA in Chapter 3 due to the time derivative of  $\mu_k$ . However, if  $\mu_k$  is a even-order polynomial function with respect to  $x$  and differentiable within each partition  $S_i$ , we may use SOS theory discussed in Subsection 4.3.4 to derive some stability conditions similar to Theorem 4.3.5. We omit the detail here.

## 9.4 Controller Design

In this section, we consider the state feedback control for discrete-time fuzzy systems. For simplicity of presentation, we only consider the case that  $i \in \mathcal{I}_0$  and  $f_i = 0$ . The other case can be dealt with similarly using the transformation technique in the last section, see also [102, 55, 60].

Consider the following state feedback controller:

$$u(t) = \sum_{k \in \mathcal{K}(i)} \mu_k K_{ki} x(t), \quad x(t) \in S_i, \quad i \in \mathcal{I} \quad (9.20)$$

Thus system (9.1) can be rewritten as

$$x(t+1) = \sum_{k \in \mathcal{K}(i)} \mu_k(x(t)) \{ \tilde{A}_{ki} x(t) \}, \quad x(t) \in S_i, \quad i \in \mathcal{I} \quad (9.21)$$

where  $\tilde{A}_{ki} = A_k + B_k K_{ki}$ .

Based on Theorem 9.3.1, we can easily obtain the following result.

**Theorem 9.4.1** *The discrete-time fuzzy system (9.21) is exponentially stabilizable, if there exist  $P_{ki} = P_{ki}^T$ ,  $K_i$ ,  $H_{ij}$ ,  $W_{ki}$ ,  $G_{ij}$ ,  $U_{kij} \succeq 0$  and  $W_{ki} \succeq 0$  satisfying the following matrix inequalities:*

$$P_{ki} - F_i^T W_{ki} F_i > 0, \quad \forall i \in \mathcal{I}_0, \quad \forall k \in \mathcal{K}(i) \quad (9.22)$$

$$\begin{bmatrix} \Upsilon_{k_1 k_2 ij} & \tilde{A}_{k_1 i}^T G_{ij} - H_{ij}^T \\ G_{ij}^T \tilde{A}_{k_1 i} - H_{ij} & P_{k_2 j} - G_{ij} - G_{ij}^T \end{bmatrix} < 0 \quad (9.23)$$

$$\forall (i, j) \in \Omega_d, \quad i, j \in \mathcal{I}_0, \quad \forall k_1 \in \mathcal{K}(i), \quad \forall k_2 \in \mathcal{K}(j)$$

where  $\Upsilon_{k_1 k_2 ij} = -P_{k_1 i} + \tilde{A}_{k_1 i}^T H_{ij} + H_{ij}^T \tilde{A}_{k_1 i} + F_i^T U_{k_1 k_2 ij} F_i$ .

**Remark 9.4.1** *Theorem 9.4.1 are given in terms of BMIs. There are several existing iterative algorithms to the BMI problem as enumerated in Appendix A. However, it is well-known that the global algorithms for BMI are only suitable*

for a small-scale problem, and these local algorithms heavily depend on the initial conditions. Hence, we shall provide some methods for obtaining suitable initial conditions.

### 9.4.1 Initial Conditions

In this subsection, we discuss the initial conditions for the BMI problems in Theorem 9.4.1. These initial conditions can be obtained by LMI or efficient iterative LMI algorithms.

Firstly, we give the most conservative but most efficient algorithm. If we let  $W_{ki} = 0$ , i.e.,  $P_{ik} > 0$  and  $U_{ijk} = 0$ ,  $K_{ki} = K_i$  and  $P_{ik} = P_i$ , then we can transfer the design problem as an LMI problem [102, 167].

#### Algorithm 1

$$\min_{Q_i, R_i, T_{ij}} \kappa, \text{ s.t. (9.25)} \quad (9.24)$$

$$\left\{ \begin{array}{l} Q_i > 0, \quad i \in \mathcal{I}_0 \\ \left[ \begin{array}{cc} -Q_j & A_{ki}Q_i + B_{ki}R_i \\ Q_i^T A_{ki}^T + R_i^T B_{ki}^T & -Q_i \end{array} \right] < \kappa I \\ \forall (i, j) \in \Omega_d, \quad \forall k \in \mathcal{K}(i) \end{array} \right. \quad (9.25)$$

If  $\kappa \leq 0$ , then  $K_i = R_i Q_i^{-1}$  is a required stabilizing solution; otherwise, we take such  $K_i$  as initial controller gains for iterative algorithms, say, the V-K algorithm.

Secondly, motivated by the results in last two chapters [173, 178, 177], we propose the following algorithm.

#### Algorithm 2

$$\min_{P_i, Q_i, W_i, K_i, T_{ij}, U_{ki}(U_{kij})} \kappa, \text{ s.t. (9.27)} \quad (9.26)$$

$$\left\{ \begin{array}{l} P_i > 0, Q_i > 0, \forall i \in \mathcal{I}_0 \\ \left[ \begin{array}{cc} -Q_j & A_k + B_k K_i \\ A_{ki}^T + K_i^T B_k^T & -P_i + F_i^T U_{kij} F_i \end{array} \right] < \kappa I \\ \forall (i, j) \in \Omega_d, \forall k \in \mathcal{K}(i) \\ P_i Q_i = I, \forall i \in \mathcal{I}_0 \end{array} \right. \quad (9.27)$$

where  $\epsilon$  is a sufficiently small positive scalar. (9.27) are BMIs. Again, we can solve them by the sequential linear programming matrix method (SLPMM) [121, 173]. The correctness of this algorithm can be easily proof using the similar arguments in Chapter 8. Hence, we omit the details. Note that this algorithm is less conservative than Algorithm 1 because the partition information  $F_i$  is used.

**Example 9.4.1** *We still consider the inverted pendulum on a cart [59] in Example 2.2.5. However, we make a slight change on the membership function:*

$$\mu_i = f\left(\frac{h_i}{\sum_1^3 h_i}\right), \quad i = 1, 2, 3, \quad f(\nu) = \begin{cases} 1 & \nu > 1 - \epsilon \\ f & \epsilon \leq \nu \leq 1 - \epsilon \\ 0 & \nu < \epsilon \end{cases} \quad (9.28)$$

$\epsilon$  is a sufficiently small positive scalar, say  $10^{-3}$ . With  $\epsilon = 10^{-3}$ , the membership function is shown in Figure 2.7. The membership function is shown in Figure 9.6. We assume the working range of state is  $x_1 = [-\frac{\pi}{2}, \frac{\pi}{2}]$ . Hence, we have the following closure of these subspaces based on range of  $x_1$ :

$$\begin{aligned} &[-\frac{\pi}{2}, -1.0998], [-1.0998, -0.47080], [-0.47080, 0], [0, 0.47080], \\ &[0.47080, 1.1002], [1.1002, \frac{\pi}{2}] \end{aligned}$$

Similar to Example 7.2.3, we discretize the system using zero-order-hold with sampling time  $T_s = 0.01s$ . Denote these regions as  $S_i$ ,  $i = 1, \dots, 6$ , respectively. In  $S_1, S_3, S_4$  or  $S_5$ , there is only one system model involved, while there are two system models in  $S_2$  or  $S_5$ , i.e.,

$$A_1 = A_6 = A_{22} = A_{52} = \begin{bmatrix} 1 & 0.01 \\ 0.002395 & 1 \end{bmatrix}, \quad A_3 = A_4 = A_{21} = A_{51} = \begin{bmatrix} 1 & 0.01 \\ 0.1729 & 1 \end{bmatrix}$$

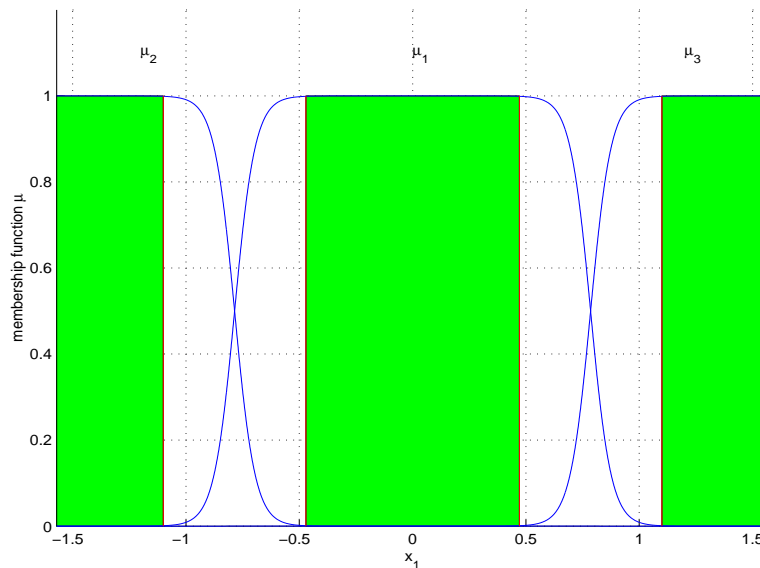


Figure 9.6: Membership functions of Example 9.4.1

$$a_1 = -a_6 = a_{22} = -a_{52} = \begin{bmatrix} 1.8395e - 005 \\ 0.003679 \end{bmatrix}, \quad a_2 = a_3 = a_{21} = a_{51} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$B_1 = B_6 = B_{22} = B_{52} = \begin{bmatrix} -2.6e - 8 \\ -5.2e - 5 \end{bmatrix}, \quad B_2 = B_3 = B_{21} = B_{51} = \begin{bmatrix} -8.826e - 7 \\ -1.766e - 3 \end{bmatrix}$$

Hence, we denote the local controllers for  $S_1$ ,  $S_3$ ,  $S_4$  and  $S_6$  as  $K_1$ ,  $K_3$ ,  $K_4$  and  $K_6$ , respectively, while the local controllers for  $S_2$  and  $S_5$  as  $K_{k2}$ ,  $k = 1, 2$ , and  $K_{k5}$ ,  $k = 4, 5$ . Using Theorem 9.4.1 with the initial controller gain from Algorithm 1, we have the following controller gain:

$$\begin{aligned} K_1 = K_6 &= \begin{bmatrix} 309304 & 19273.9 \end{bmatrix}, \quad K_{21} = K_{51} = \begin{bmatrix} 98764.7 & 6672.35 \end{bmatrix}, \\ K_3 = K_4 &= \begin{bmatrix} 2166.32 & 155.944 \end{bmatrix}, \quad K_{22} = K_{52} = \begin{bmatrix} 8024.95 & 572.975 \end{bmatrix}. \end{aligned} \quad (9.29)$$

Figure 9.7 gives the simulation result for the system (2.20) with the initial conditions  $[\pm 1.5 \ 0.8]^T$  and  $[\pm 0.5 \ 0.8]^T$ .

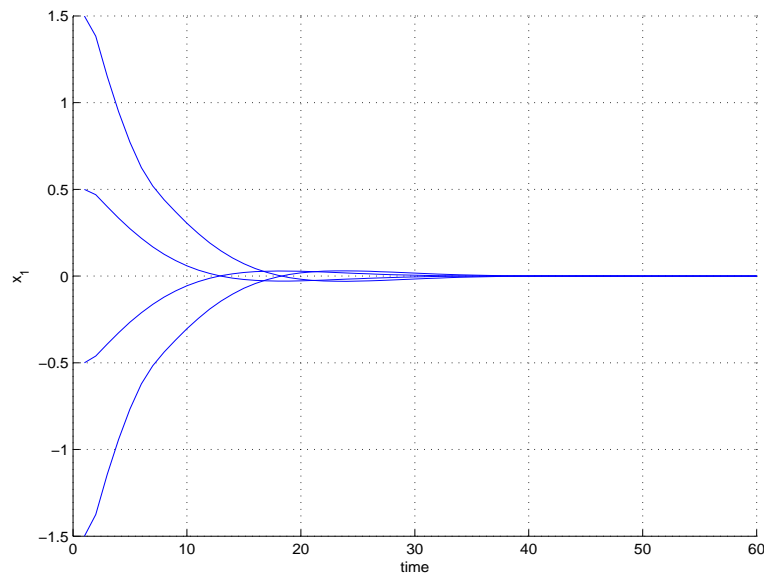


Figure 9.7: State trajectories of  $x_1$  with different initial conditions

### 9.4.2 $H_\infty$ Control Problem

In this subsection, we only consider the  $H_\infty$  state feedback control problem for the discrete-time fuzzy systems. The system is described as follows.

*Rule<sup>i</sup> :*

*IF*  $x_1$  is  $L_1^i$  AND ...  $x_n$  is  $L_n^i$

$$x(t+1) = A_i x(t) + B_{2i} u(t) + B_{1i} w \quad (9.30)$$

*THEN*  $y(t) = C_i x(t) + D_{2i} u(t) + D_{1i} w$

$$i \in \bar{\mathcal{I}} = \{1, 2, \dots, \bar{m}\}$$

where  $w$  is the noise, and the rest is the same as in (9.1).

The  $H_\infty$  performance is defined in Subsection 7.2.2. The state feedback controller is designed as (9.20).

**Theorem 9.4.2** *The fuzzy system defined in (9.30) satisfies the  $H_\infty$  performance, if there exist  $P_{ki} = P_{ki}^T$ ,  $K_i$ ,  $H_{ij}$ ,  $W_{ki}$ ,  $G_{ij}$ ,  $U_{kij} \succeq 0$  and  $W_{ki} \succeq 0$  satisfying (9.22)*

and the following inequality:

$$\begin{bmatrix} \tilde{A}_{k_i}^T P_j \tilde{A}_{k_1 i} - P_{k_1 i} + F_i^T U_{k_1 k_2 i j} F_i + \tilde{C}_{k_1 i}^T \tilde{C}_{k_1 i} & * \\ B_{1k_1}^T P_{k_2 j} \tilde{A}_{k_1 i} + D_{k_1 i}^T \tilde{C}_{k_1 i} & \varpi_{k_1 k_2 i j} \end{bmatrix} < 0 \quad (9.31)$$

$$\forall i, j \in \Omega_d, i, j \in \mathcal{I}_0, \forall k_1 \in \mathcal{K}(i), \forall k_2 \in \mathcal{K}(j)$$

where  $\varpi_{k_1 k_2 i j} = B_{1k_1}^T P_{k_2 j} B_{1k_1 i} + D_{k_1 i}^T D_{k_1 i} - \gamma^2 I$

**Proof:** Inequalities (9.22) and (9.31) can be rewritten as

$$\tilde{\mathcal{P}}_{k_i} - \mathcal{W}_{k_i} > 0 \quad (9.32)$$

$$\tilde{\mathcal{A}}_{k_1}^T \tilde{\mathcal{P}}_{k_2 j} \tilde{\mathcal{A}}_{k_1} - \tilde{\mathcal{P}}_{k_2 j} + \tilde{\mathcal{U}}_{k_1 k_2 i j} < 0 \quad (9.33)$$

where  $\tilde{\mathcal{A}}_{k_1} = \begin{bmatrix} \tilde{A}_{k_1 i} & B_{1k_1 i} \\ \tilde{C}_{k_1 i} & D_{1k_1 i} \end{bmatrix}$ ,  $\tilde{\mathcal{C}}_{k_1 i} = C_{k_1} + D_{2k_1 i} K_{k_1 i}$ ,  $\tilde{\mathcal{P}}_{k_l i} = \begin{bmatrix} P_{k_l i} & 0 \\ 0 & I \end{bmatrix}$ ,  $l = 1, 2$ ,  $\mathcal{W}_{k_i} = \begin{bmatrix} F_i^T W_{k_i} F_i & 0 \\ 0 & 0 \end{bmatrix}$  and  $\tilde{\mathcal{U}}_{k_1 k_2 i j} = \begin{bmatrix} F_i^T U_{k_1 k_2 i j} F_i & 0 \\ 0 & (1 - \gamma^2) I \end{bmatrix}$ . Now the problem is transferred into a similar form of stability analysis. Using the stability results, we immediately have

$$\Delta v(t) < \gamma^2 \|w\|^2 - \|y\|^2$$

Hence with the zero initial conditions, we have

$$0 < \sum_{t=0}^N \gamma^2 \|w\|^2 - \|y\|^2$$

Thus we complete the proof. ■

Based on (9.32) and (9.33), Theorem 9.4.2 can be easily transferred into a BMI problem. However, using projection lemma, we can also have a complementary condition similar to (7.26). We omit the details here.

**Example 9.4.2** Now we consider the example borrowed from [167]. The fuzzy system is with three rules, which are defined as

Rule<sup>1</sup> : IF  $x_1$  is about negative

Rule<sup>2</sup> : IF  $x_1$  is about zero

Rule<sup>3</sup> : IF  $x_1$  is about positive

The membership function is similar to that of Example 9.2.1. The system parameters are given as follows

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0.5 & -0.6 \\ 0.6 & 0.5 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0.3 \\ -0.1 & 1 \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_{23} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, B_{12} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, B_{13} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, C_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D_{11} = 1, D_{12} = -1, D_{13} = 1, D_{21} = D_{22} = D_{23} = 0$$

According to [167], the system has 5 modes, each attached to a partition. Here we only design one controller for each partition. Theorem 3 in [167] gives  $\gamma = 2.1$ . Using the controller gains obtained in [167] as initial values for V-K iterative algorithm for Theorem 9.4.2, we can slightly improve the result to  $\gamma = 1.99$ . However SLPMM provides a better result:  $\gamma = 1.7$ . The controller gains are

$$K_1 = \begin{bmatrix} 0.376 & -0.996 \end{bmatrix}, K_2 = \begin{bmatrix} 0.324 & -0.100 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -0.570 & -0.435 \end{bmatrix}, K_4 = \begin{bmatrix} -0.793 & -0.420 \end{bmatrix},$$

$$K_5 = \begin{bmatrix} -1.000 & -0.417 \end{bmatrix}$$

Figure 9.8 shows the simulation result with the noise as  $w_t = \sin(t)e^{-0.02t}$ , where we choose five different initial conditions:  $[0; 0]$  and  $[\pm 3; \pm 3]$ .

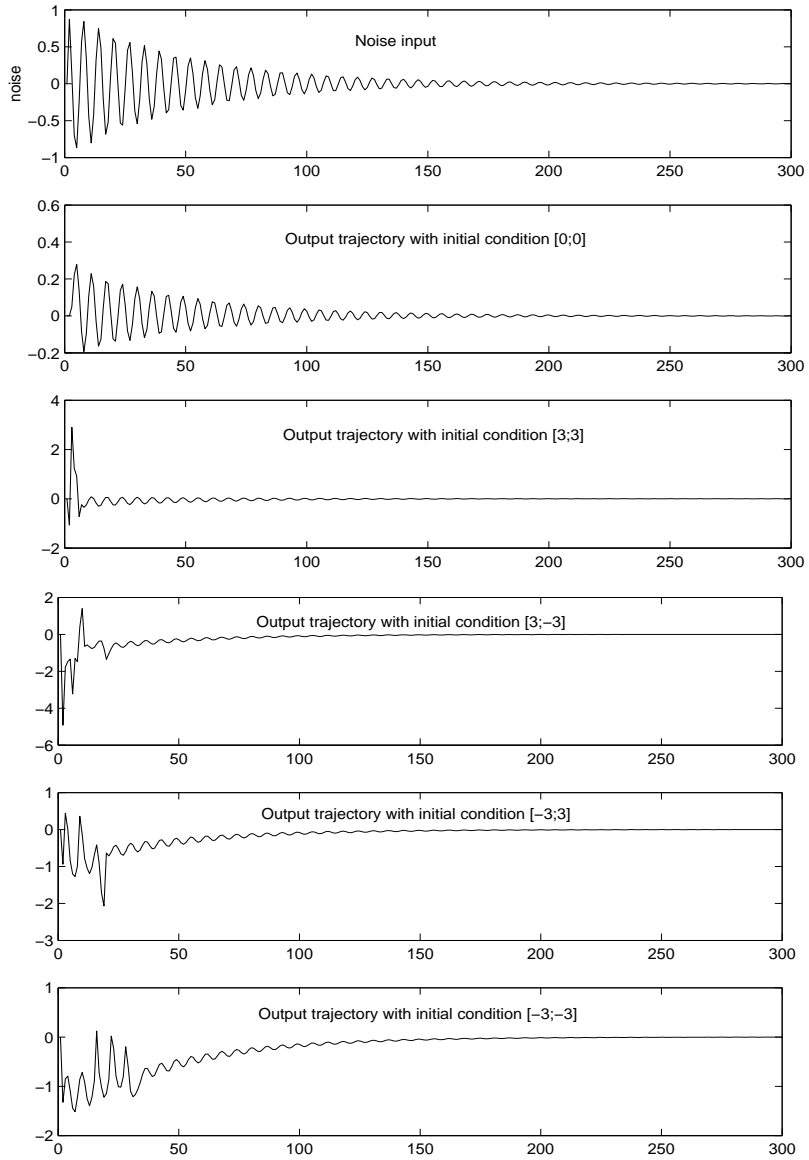


Figure 9.8: System state via the  $H_\infty$  controller.

## 9.5 Conclusion

This chapter has provided sufficient conditions for exponential stability of a class of discrete-time T-S fuzzy systems based on a new analytical model, which in a sense is a nested linear difference inclusion. By converting the fuzzy system into a PWA system, less conservative stability analysis results using fuzzy Lyapunov functions for the fuzzy system have been obtained. We have also considered the stabilization problem via state feedback controller. Some examples illustrate the advantages of our methods.

# Chapter 10

## Conclusion and Recommendations

### 10.1 Conclusion

Piecewise affine (PWA) systems are receiving increasing attention from the control society because a large class of nonlinear systems, switched systems and hybrid systems [25, 122] can be modelled as PWA systems. In this thesis, we have studied several problems on PWA systems. In particular, the properties of PWA systems, such as stability, controllability and reachability, and the control and estimation problems have been discussed. We summarize the contributions of this thesis in the following aspects.

We improved the existing results on stability analysis of both continuous-time and discrete-time PWA systems by using piecewise homogeneous polynomial Lyapunov functions (PHPLFs) with the SOS theory. This constructional approach is in terms of a set of LMIs by applying power transformations to both systems dynamic and partition description. Improved stability analysis results can be obtained by increasing the degree of the PHPLF.

We proposed two stability criteria of PWA systems using the partition information from vertex representation of polytope. These criteria actually take the advantage of parameterized Lyapunov functions. Especially for the vertex-dependant Lya-

punov function (VDLF), although it is not in a quadratic form with the vertex-dependant parameters, we proposed two LMI approaches to overcome this obstacle. In particular, the second approach provides an asymptotically necessary stability condition with respect to the degree of the VDLF.

We derived explicit necessary and sufficient conditions for the controllability and reachability of discrete-time planar bimodal piecewise linear systems. The main ideas lie in the general classification methods and set mapping for polyhedra to calculate the controllability set and the reachability set. We also briefly discussed the high-order and multi-model cases, as well as the associated complexity.

We proposed some less conservative approaches to feedback controller design for discrete-time PWA systems using the less conservative results of stability analysis based on PQLFs, which is summarized and reformulated in Chapter 3. Our results lead to some design approaches based on LMIs and BMIs. The latter can be solved by some efficient iterative LMI algorithms.  $H_\infty$  and generalized  $H_2$  control have been considered in a similar way.

We considered the  $H_\infty$  and generalized  $H_2$  state estimation problems for PWA systems. A Luenberger type estimator has been proposed to achieve the  $H_\infty$  and generalized  $H_2$  performances. The less conservatism is achieved by applying the *S-procedure* to include the partition information and introducing slack variables in optimization.

We provided sufficient conditions for exponential stability of a class of Takagi-Sugeno fuzzy systems based on a new analytical model, which in a sense is a nested linear differential/difference inclusion. By converting these fuzzy systems into PWA systems, less conservative stability analysis results for the fuzzy systems have been obtained. We have also considered the stabilization problem via state feedback controller, as well as the  $H_\infty$  state feedback problem.

## 10.2 Recommendations

Although this thesis has made some progress towards analysis and synthesis of PWA systems, many problems remain open. In this section, we recommend some topics for further research.

- Since uncertainty in PWA systems is generally inevitable, the robust stability of uncertain PWA systems is worth investigating. Sensitiveness of the stability criteria to the exact knowledge of the system parameters (including the partition) shall be studied. Although some results for bounded uncertainties in the systems parameters (not in the partition) using PQLF technique can be found in [60, 54], we may expect better analysis performance with the SOS technique.
- In Chapter 4 and Chapter 5, although we provide computational methods to solve the HPLF and VDLF, we do not give a comparison. Another interesting question is whether there exists some bound  $\hat{d}$  for  $d$  such that if (5.28) does not admit a solution for  $d \leq \hat{d}$ , the instability of the PWA systems can be concluded.
- In Chapter 6, we only give the explicit conditions of controllability and reachability for bimodal planar systems [175, 174]. It would be of great interest to extend the results to bimodal high-order systems or multi-modal planar systems, like the stability conditions given in [96, 97].
- Although the SOS technique is proved to be a very powerful tool in analysis, the design procedure using this technique is still noncommittal. In [99], several iterative algorithms are proposed to design a controller. In [141, 51], some special cases are discussed. However, in practice, the solution is very difficult to be obtained due to the complexity. Hence how to reduce the complexity becomes a challenging issue.

- It is also worth studying the dynamic output feedback control and sliding mode control for PWA systems.
- If the partition itself is also a design objective, then the synthesis problem will be much more practical but complicated than the case that partition is preset as considered in this thesis. For example, when we approximate a smooth nonlinear system using a PWA model, we have to choose proper operating points/regions, which leads to the partition problem. In [180], a second-order continuous-time switched system is discussed and a stabilization scheme is presented. However, the result [180] seems less general.
- Another important topic is the probabilistic issues, as only deterministic conditions are discussed in this thesis. Assigning probability to the partition is applicable during the system identification and modelling [78]. The results in [33, 7, 6, 125] deserve further investigations.

# Appendix A

## Semidefinite Programming

In this appendix, we will give some preliminaries on semidefinite programming. However, some basic concepts, such as convex set/hull/function, are omitted. One may refer to [22, 23]

A semidefinite program is a convex optimization problem with a linear objective and constraints specified in terms of linear matrix inequalities (LMIs). Linear inequalities, convex quadratic inequalities, matrix norm inequalities and various constraints from control theory, such as Lyapunov and Riccatic inequalities, can be represented as LMIs [22]. In general, an LMI is an inequality on the form

$$F(x) = F_0 + \sum_{i=1}^m x_i F_i \geq 0 \quad (\text{A.1})$$

where  $x = [x_1, \dots, x_m]^T$  is the variable,  $F_i = F_i^T \in R^{n \times n}$  are given symmetric matrices. A semidefinite programming (SDP) problem takes the form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & F(x) \geq 0 \end{aligned} \quad (\text{A.2})$$

i.e., optimization problem over the semidefinite cone.

The following results are useful when formulating and manipulating LMIs.

**Lemma A.0.1** (*Schur Complement*) [102, 22] Suppose  $S(x)$ ,  $Q(x) = Q(x)^T$  and

$R(x) = R^T(x)$  are matrices whose entries depend affinely on the vector  $x$ . Then the following condition holds.

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0 \iff R \geq 0, Q - SR^+S^T \geq 0, S(I - RR^+) = 0 \quad (\text{A.3})$$

Specially, if  $R$  is invertible, then

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0 \iff R > 0, Q - SR^{-1}S^T > 0 \quad (\text{A.4})$$

**Lemma A.0.2** (*S-procedure*)[102, 22] Let  $Q_0, \dots, Q_p$  be quadratic functions of  $x \in \mathcal{R}^n$ ,

$$Q_i(x) = x^T P_i x + 2b_i^T x + c_i, \quad i = 0, \dots, p \quad (\text{A.5})$$

where  $P_i = P_i^T$ . If there exist positive scalars  $\tau_i \geq 0$  such that

$$Q_0(x) \geq \sum_{i=1}^p \tau_i Q_i(x) \quad (\text{A.6})$$

then  $Q_0(x) \geq 0$  for all  $x$  such that  $Q_i(x) \geq 0, i = 1, \dots, p$ . Moreover, if  $p = 1$  and there exists an  $x_0$  such that  $Q_0(x_0) > 0$ , then condition (A.6) is also necessary.

**Lemma A.0.3** (*Finsler's Lemma*)[22] The following expressions are equivalent:

- (1)  $x^T A x > 0$  for  $\forall x \neq 0$ , subject to  $Bx = 0$ ;
- (2)  $B^{\perp T} A B^{\perp} > 0$ , where  $B^{\perp}$  is the kernel of  $B^T$ , i.e.,  $B^{\perp} B^T = 0$ ;
- (3)  $A + \sigma B^T B > 0$ , for some scale  $\sigma \in \mathcal{R}$ ;
- (4)  $A + X B + B^T X^T > 0$ , for some matrix  $X$ .

**Lemma A.0.4** (*Projection Lemma*)[22] Given a symmetric matrix  $\Sigma$  and two matrices  $\mathcal{X}$  and  $\mathcal{Y}$ . Let  $\mathcal{X}^{\perp}$  and  $\mathcal{Y}^{\perp}$  be the basis of nullspaces of  $\mathcal{X}^T$  and  $\mathcal{Y}^T$ , respectively. Then, there exists some matrix  $\Theta$  such that

$$\Sigma + \mathcal{X} \Theta \mathcal{Y}^T + \mathcal{Y} \Theta^T \mathcal{X}^T < 0 \quad (\text{A.7})$$

if and only if

$$\mathcal{X}^{\perp T} \Sigma \mathcal{X}^{\perp} < 0; \quad \mathcal{Y}^{\perp T} \Sigma \mathcal{Y}^{\perp} < 0 \quad (\text{A.8})$$

**Lemma A.0.5** [100] Let  $P > 0$  and  $\varrho(P)$  be a matrix expression that may relate to  $P$  or may have nothing to do with  $P$ . Then the following statement stands:

$$\begin{bmatrix} -\varrho(P) & A^T P \\ PA & -P \end{bmatrix} < 0 \iff \begin{bmatrix} -\varrho(P) & A^T Z \\ ZA & P - Z - Z^T \end{bmatrix} < 0 \quad (\text{A.9})$$

**Lemma A.0.6** Let  $P > 0$  and  $\varrho(P)$  be a matrix expression that may relate to  $P$  or may have nothing to do with  $P$ . Then the following statements are equivalent:

$$A^T P + PA + \varrho(P) < 0 \quad (\text{A.10})$$

$$A^T P + PA + \varrho(P) + \epsilon A^T P A < 0 \quad (\text{A.11})$$

$$\begin{bmatrix} -\epsilon P^{-1} & I + \epsilon A \\ I + \epsilon A^T & \varrho(P) - \epsilon^{-1} P \end{bmatrix} < 0 \quad (\text{A.12})$$

where  $\epsilon > 0$  is a sufficient small scale.

**Proof:** This result is an extension of the result in [153]. “(A.10) $\iff$ (A.11)” is obvious. “(A.11) $\iff$ (A.12)” can be obtained by Schur complement. ■

Based on the above lemmas, some useful results for control theory, such as bounded real lemma and positive real lemma [22], can be derived. However, most control synthesis problems cannot be written in LMI form, while in terms of a more general form known as a bilinear matrix inequality (BMI):

$$F(x, y) = F_0 + \sum_{i=1}^m x_i F_i + \sum_{j=1}^l y_j G_j + \sum_{i=1}^m \sum_{j=1}^l x_i y_j H_{ij} \geq 0 \quad (\text{A.13})$$

where  $G_j$  and  $H_{ij}$  are symmetric matrices of the same dimension as  $F_i$ . Computations over BMI constraints are known as NP-hard such that there does not exist

off-the-shelf algorithms for solving BMI problems. However, There are several existing iterative algorithms to BMI problems, such as V-K iterative algorithm [72]<sup>1</sup>, path-following algorithm [83]<sup>2</sup>, concave minimization [5] and method-of-centers-like algorithm [111] for local region, branch and bound/cut algorithm [18, 68], the generalized benders decomposition algorithm [73], the team algorithm (GA+LMI) [3] and trust region strategy [162] for global optimization. We can also apply the commercial software: PENBMI to solve this problem [158].

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<sup>1</sup>The basic idea of the V-K iteration is that a BMI can be converted into an LMI when some of the parameters in the BMI are fixed to be constant. This method has been called the alternating method or V-K iteration because the parameters that are fixed and solved in the alternating steps are typically the Lyapunov function  $V$  and controller gain  $K$ .

<sup>2</sup>This method performs a first order approximation by linearizing the BMI, and then iteratively computes a perturbation that "slightly" improves the controller performance by solving an LMI problem. This process is repeated until the desired performance is achieved, or the performance cannot be improved any further. Note that the updates to the parameters shall be constrained to be small enough so that the linearization approximation is valid.

# Appendix B

## Some Proofs

### B.1 Proof of Theorem 6.2.4

**Lemma B.1.1** *The set  $\{x | A_{12}^T x = 0, C^T x \geq 0\}$  is equivalent to  $\Omega_1 = \{x = [\delta_1, -\gamma_1]^T s \mid \forall s \in \mathcal{S}_+(\lambda_1)\}$ . The set  $\{x | A_{21}^T x = 0, C^T x < 0\}$  is equivalent to  $\Omega_2 = \{x = [\beta_2 - \alpha_2]^T s \mid \forall s \in \mathcal{S}_-(\lambda_2)\}$ .*

**Proof:** The result is obvious by defining  $x = [\delta_1, -\gamma_1]^T s$  and  $x = [\beta_2 - \alpha_2]^T s$ , respectively. ■

Now we consider the NC set  $\mathcal{X}_k$ . Of course, we begin with  $k = 1$ . For any  $x \in \mathcal{X}_1$ , there must exist  $u$  such that

$$\begin{cases} 0 = A_1 x + B_1 u \\ 0 \leq C^T x \end{cases} \quad \text{or} \quad \begin{cases} 0 = A_2 x + B_2 u \\ 0 > C^T x \end{cases} \quad (\text{B.1})$$

Thus if  $A_{12} \neq 0$  and  $A_{21} \neq 0$ , the input can be chosen as

$$u = -[\alpha_1 \ \beta_1]x, \quad x \in \Omega_1 \quad \text{or} \quad u = -[\gamma_2 \ \delta_2]x, \quad x \in \Omega_2 \quad (\text{B.2})$$

to satisfy (B.1). Thus we have

$$\mathcal{X}_1 = \Omega_1 \cup \Omega_2 \quad (\text{B.3})$$

According to  $\lambda_1$  and  $\lambda_2$ , there are three cases:

- (1)  $\lambda_1 = 0$
- (2)  $\lambda_1 \neq 0$  &  $\lambda_2 = 0$
- (3)  $\lambda_1 \neq 0$  &  $\lambda_2 \neq 0$

For the case that  $\lambda_1 = 0$ ,  $\Omega_1 = \{x | C^T x = 0\}$ . By Lemma 6.2.1,  $\mathcal{X}_2 = \mathcal{R}^2$  and the system (6.1) is NC. The remaining two cases are complex and we will discuss them in the following two subsections.

### B.1.1 $\lambda_1 \neq 0$ & $\lambda_2 = 0$

Due to  $\lambda_2 = 0$ , we have  $c_1 = \eta\alpha_2$  and  $c_2 = \eta\beta_2$  and  $\Omega_2 = \phi$ , thus  $\mathcal{X}_1 = \Omega_1$ . Now we consider  $\mathcal{X}_2$ . Assume that  $x \in \mathcal{X}_2$ .

If  $C^T x \geq 0$  then there exists  $u$  such that  $A_1 x + B_1 u \in \Omega_1$ , i.e.,

$$\begin{cases} [\alpha_1, \beta_1]x = \delta_1 s - u \\ [\gamma_1, \delta_1]x = -\gamma_1 s, s \in \mathcal{S}_+(\lambda_1) \end{cases} \quad (\text{B.4})$$

where  $\mathcal{S}_+(\lambda_1)$  is defined in (6.4). Since we can always find a suitable  $u$  to satisfy the first equation of (B.4), we only consider the second equation, which consists of the following cases:

$$[\gamma_1, \delta_1]x \begin{cases} \geq 0 & \text{if } \gamma_1 \lambda_1 < 0 \\ = 0 & \text{if } \gamma_1 \lambda_1 = 0 \\ \leq 0 & \text{if } \gamma_1 \lambda_1 > 0 \end{cases} \quad (\text{B.5})$$

Thus

$$\mathcal{X}_2 \cap \{x | C^T x \geq 0\} = \begin{cases} \{x | A_{12}^T x \geq 0, C^T x \geq 0\} & \text{if } \gamma_1 \lambda_1 < 0 \\ \Omega_1 & \text{if } \gamma_1 \lambda_1 = 0 \\ \{x | A_{12}^T x \leq 0, C^T x \geq 0\} & \text{if } \gamma_1 \lambda_1 > 0 \end{cases} \quad (\text{B.6})$$

If  $C^T x < 0$  then there exists  $u$  such that  $A_2 x + B_2 u \in \Omega_1$ , i.e.,

$$\begin{cases} [\alpha_2, \beta_2]x = \delta_1 s, & s \in \mathcal{S}_+(\lambda) \\ [\gamma_2, \delta_2]x = -\gamma_1 s - u \end{cases} \quad (\text{B.7})$$

Similar to (B.4), (B.7) consists of the following cases

$$[\alpha_2, \beta_2]x \begin{cases} \geq 0 & \text{if } \delta_1 \lambda_1 > 0 \\ = 0 & \text{if } \delta_1 \lambda_1 = 0 \\ \leq 0 & \text{if } \delta_1 \lambda_1 < 0 \end{cases} \quad (\text{B.8})$$

Since  $C^T x = [c_1, c_2]x = [\eta\alpha_2, \eta\beta_2]x = [\alpha_2, \beta_2]x\eta < 0$ , according to  $\eta$ , we have

$$[\alpha_2, \beta_2]x \begin{cases} < 0 & \text{if } \eta > 0 \\ \phi & \text{if } \eta = 0 \\ > 0 & \text{if } \eta < 0 \end{cases} \quad (\text{B.9})$$

Thus

$$\begin{aligned} & \mathcal{X}_2 \cap \{x | C^T x < 0\} = \\ & \begin{cases} \{x | C^T x < 0\} & \text{if } \delta_1 \lambda_1 \eta < 0 \\ \phi & \text{if } \delta_1 \lambda_1 \eta \geq 0 \end{cases} \end{aligned} \quad (\text{B.10})$$

If  $\delta_1 \lambda_1 \eta < 0$ , according to Lemma 6.2.1 (1), the system (6.1) is NC. If  $\delta_1 \lambda_1 \eta \geq 0$  and  $\gamma_1 = 0$ , according to (B.10) and (B.6), respectively, we have  $\mathcal{X}_2 = \mathcal{X}_1$ , and according to Lemma 6.2.1 (4), the system (6.1) is not NC. Then  $\mathcal{X}_2$  is just an angular region. Hence in the following, we only discuss the situation with  $\delta_1 \lambda_1 \eta \geq 0$  and  $\gamma_1 \neq 0$ .

**Remark B.1.1** *According to the relative position of the lines  $C^T x = 0$  and  $A_{12}^T x = 0$ , there are 6 types. And each  $\{x | A_{12}^T x \geq 0, C^T x \geq 0\}$  or  $\{x | A_{12}^T x \leq 0, C^T x \geq 0\}$  has 4 situations. Though we have the 32 different choices of  $c_1, c_2, \gamma_1$  and  $\delta_1$  according to their signs, there are only 24 types based on the location of  $C^T x = 0$  and  $A_{12}^T x = 0$ , because some are overlapped. Here  $A_{12}^T x = 0$  may coincide with the axes.*

Thus it is rather difficult to analyze because of too many cases. However, since  $\lambda_1 \neq 0$  and  $\eta \neq 0$ , without loss of generality, we may assume that  $\lambda_1 > 0$  and  $\eta > 0$ . In fact, if  $\eta < 0$ , we may choose  $\bar{x} = -x$  and  $\bar{u} = -u$  to replace the corresponding  $x$  and  $u$  in the system (6.1), which leads to  $\bar{C} = -C$ , while keep other parameters unchanged. If  $\lambda_1 < 0$ , choose  $P_1 = \text{diag}\{-1, 1\}$ , and let  $\bar{x} = P_1 x$  and some  $u$  as  $-u$ . Then we can always have  $\lambda_1 > 0$  and  $\eta > 0$ .

According to (B.6), there are two cases:  $\gamma_1 \lambda_1 < 0$  or  $\gamma_1 \lambda_1 > 0$ . Since  $\lambda_1 > 0$ ,  $\eta > 0$  and  $\delta_1 \lambda_1 \eta \geq 0$ , we have  $\delta_1 \geq 0$ . Thus, we may only consider 8 cases as shown in Table B.1.

Case	$\gamma_1 \lambda_1$	$\delta_1$	$\gamma_1$	$c_1$	$c_2$	$\mathcal{X}_3$
1.1.1	$< 0$	$\geq 0$	$< 0$	$> 0$	$< 0$	$\mathcal{X}_2$
1.1.2				$< 0$	$> 0$	$\mathcal{R}^2$ or $\{x   C^T x \geq 0\} \subseteq \mathcal{X}_3$
1.1.3				$> 0$	$> 0$	$\{x   C^T x < 0\} \subseteq \mathcal{X}_3$
1.1.4				$< 0$	$< 0$	impossible
1.2.1	$> 0$	$\geq 0$	$> 0$	$> 0$	$> 0$	$\mathcal{X}_2$
1.2.2				$< 0$	$< 0$	$\mathcal{R}^2$ or $\{x   C^T x \geq 0\} \subseteq \mathcal{X}_3$
1.2.3				$> 0$	$< 0$	$\{x   C^T x < 0\} \subseteq \mathcal{X}_3$
1.2.4				$< 0$	$> 0$	impossible

Table B.1: 8 cases when  $\lambda_1 > 0$  &  $\lambda_2 = 0$  &  $\eta > 0$  &  $\delta_1 \lambda_1 \eta > 0$

In the following we discuss the 8 cases. Since the cases 1.2.1-1.2.4 are similar to cases 1.1.1-1.1.4, we only address cases 1.1.1-1.1.4.

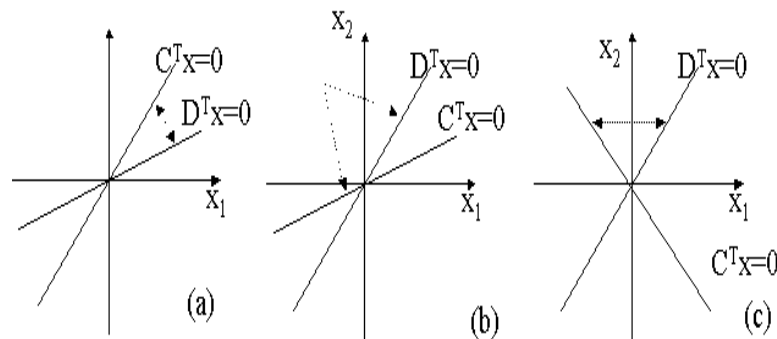


Figure B.1: Case 1.1.1-1.1.3

**Case 1.1.1** The boundary of  $\mathcal{X}_2$  is in the first quadrant as shown in Figure B.1

(a). We shall prove that  $\mathcal{X}_3 = \mathcal{X}_2$ . Otherwise, for any  $x \in \mathcal{R}^2 \setminus \mathcal{X}_2$ , either  $x \in \{C^T x \geq 0, A_{12}^T x < 0\}$  or  $x \in \{C^T x = A_{21}^T x < 0\}$ . For the case that  $x \in \{C^T x \geq 0, A_{12}^T x < 0\}$ , if there exists a  $u$  such that  $\bar{x} = A_1 x + B_1 u \in \mathcal{X}_2$ , then  $C^T \bar{x} \geq 0$  and  $[\gamma_1, \delta_1] \bar{x} \geq 0$  according to (B.6). Denote  $\bar{x} = [\bar{x}_1, \bar{x}_2]^T$ , we have  $\bar{x}_2 = A_{12}^T x < 0$ . Thus

$$\begin{cases} c_1 \bar{x}_1 + c_2 \bar{x}_2 \geq 0 \\ \gamma_1 \bar{x}_1 + \delta_1 \bar{x}_2 \geq 0 \end{cases} \implies \begin{cases} \bar{x}_1 \geq \frac{-c_2 \bar{x}_2}{c_1} \\ \bar{x}_1 \leq \frac{-\delta_1 \bar{x}_2}{\gamma_1} \end{cases} \implies \frac{c_1}{-c_2} < \frac{-\gamma_1}{\delta_1}$$

which contradicts with  $\lambda_1 > 0$ . So  $\{x | C^T x \geq 0, [\gamma_1, \delta_1] x < 0\}$  is not a subset of  $\mathcal{X}_3$ . Similarly, for the case that  $x \in \{C^T x < 0\}$ , we have

$$\begin{cases} c_1 \bar{x}_1 + c_2 \bar{x}_2 \geq 0 \\ \gamma_1 \bar{x}_1 + \delta_1 \bar{x}_2 \geq 0 \end{cases}, \bar{x} = A_2 x + B_2 u = \begin{bmatrix} A_{21}^T x \\ [\gamma_2, \delta_2] x + u \end{bmatrix}$$

If  $\delta_1 \neq 0$ , we can also deduce that  $\frac{c_1}{-c_2} < \frac{-\gamma_1}{\delta_1}$ , which contradicts with  $\lambda_1 > 0$ . Thus  $\{C^T x < 0\}$  is not a subset of  $\mathcal{X}_3$ . Otherwise if  $\delta_1 = 0$ , we have  $\lambda_1 = -c_2 \gamma_1 < 0$ , which contradicts with the assumption that  $\lambda_1 > 0$ .

**Case 1.1.2** The boundary of  $\mathcal{X}_2$  is in the first and third quadrants as shown in Figure B.1 (b). Considering the case that  $x \in \{x | C^T x \geq 0\}$  and  $x \in \{x | C^T x < 0\}$  respectively, we may find  $\mathcal{X}_3 = \mathcal{R}^2$ , if  $\delta_1 \neq 0$ . That is because we can always find suitable  $u_i$  such that  $A_i x + B_i u_i \in \mathcal{X}_2$ ,  $i = 1, 2$ . For any  $x \in \mathcal{X}_3$ , if  $C^T x \geq 0$ , for  $\bar{x} = [\bar{x}_1, \bar{x}_2]^T = A_1 x + B_1 u$ , we have

$$\begin{cases} c_1 \bar{x}_1 + c_2 \bar{x}_2 \geq 0 \\ \gamma_1 \bar{x}_1 + \delta_1 \bar{x}_2 \geq 0 \end{cases} \implies \begin{cases} \bar{x}_1 \leq \frac{-c_2 \bar{x}_2}{c_1} \\ \bar{x}_1 \leq \frac{-\delta_1 \bar{x}_2}{\gamma_1} \end{cases}$$

Since  $\bar{x}_1 = [\alpha_1 \ \beta_1] x + u$ , choose a proper  $u$ , such that  $\bar{x}_1 \leq \min\{\frac{-c_2 \bar{x}_2}{c_1}, \frac{-\delta_1 \bar{x}_2}{\gamma_1}\}$ , the above inequalities are always satisfied. For  $x \in \mathcal{X}_3$ , if  $C^T x < 0$ , for  $\bar{x} = [\bar{x}_1, \bar{x}_2]^T = A_2 x + B_2 u$ , we may choose a proper  $u$  such that  $\bar{x}_2 \geq \max\{\frac{-c_1 \bar{x}_1}{c_2}, \frac{-\gamma_1 \bar{x}_1}{\delta_1}\}$  to satisfy the corresponding inequalities.

If  $\delta_1 = 0$ , we have  $\mathcal{X}_3 = \{x | C^T x \geq 0\} \cup \{x | C^T x < 0, A_{21}^T x \leq 0\}$ . According to Lemma 6.2.1 (i), we have  $\mathcal{X}_4 = \mathcal{R}^2$ . Thus in this case, the system is NC.

**Case 1.1.3** The boundary of  $\mathcal{X}_2$  is in the first and second quadrants as shown in Figure B.1 (c). Considering the case that  $x \in \{x|C^T x \geq 0\}$  and  $x \in \{x|C^T x < 0\}$  respectively, we may find  $\mathcal{X}_3 \supseteq \{x|C^T x < 0\}$ . According to Lemma 6.2.1 (1),  $\mathcal{X}_4 = \mathcal{R}^2$ .

**Case 1.1.4** Since we assume that  $\lambda_1 > 0$ , this case is impossible.

### B.1.2 $\lambda_1 \neq 0$ & $\lambda_2 \neq 0$

In this case, based on  $\mathcal{X}_1$  from (B.3), we study  $\mathcal{X}_2$  first. For any  $x \in \mathcal{X}_2$ , if  $C^T x \geq 0$ , there exists  $u$  such that  $A_1 x + B_1 u \in \Omega_1 \cup \Omega_2$ , i.e.,

$$\begin{cases} [\alpha_1, \beta_1]x = \delta_1 s - u \\ [\gamma_1, \delta_1]x = -\gamma_1 s, s \in \mathcal{S}_+(\lambda_1) \end{cases} \quad \text{or} \quad \begin{cases} [\alpha_1, \beta_1]x = \beta_2 s - u \\ [\gamma_1, \delta_1]x = -\alpha_2 s, s \in \mathcal{S}_-(\lambda_2) \end{cases} \quad (\text{B.11})$$

Considering the sign of  $\gamma_1 \lambda_1$  and  $\alpha_2 \lambda_2$ , we have four cases as shown in Table B.2.

Case	$\gamma_1 \lambda_1$	$\alpha_2 \lambda_2$	$\mathcal{X}_2$
3.1	$> 0$	$> 0$	$\{x C^T x \geq 0\} \subseteq \mathcal{X}_2$
3.2	$< 0$	$< 0$	$\{x C^T x \geq 0\} \subseteq \mathcal{X}_2$
3.3	$\leq 0$	$> 0$	$\{x C^T x \geq 0 \cap A_{12}^T x \geq 0\} \subseteq \mathcal{X}_2$
3.4	$> 0$	$\leq 0$	$\{x C^T x \geq 0 \cap A_{12}^T x \leq 0\} \subseteq \mathcal{X}_2$

Table B.2: Four cases when  $\lambda_1 \neq 0$  &  $\lambda_2 \neq 0$  &  $C^T x \geq 0$

**Case 3.1** If  $s \in \mathcal{S}_+(\lambda_1)$ , we have  $\gamma_1 s \geq 0$ . If  $s \in \mathcal{S}_-(\lambda_2)$ , we have  $\alpha_2 s < 0$ . For any  $x \in \{x|C^T x \geq 0\}$ , if  $-[\gamma_1, \delta_1]x \geq 0$ , we may find  $s \in \mathcal{S}_+(\lambda_1)$ , such that  $[\gamma_1, \delta_1]x = -\gamma_1 s$ ; otherwise,  $-[\gamma_1, \delta_1]x < 0$ , then we can also find  $s \in \mathcal{S}_-(\lambda_2)$ , such that  $[\gamma_1, \delta_1]x = -\alpha_2 s$ . This implies that  $\{x|C^T x \geq 0\} \subseteq \mathcal{X}_2$ , then system (6.1) is NC.

**Case 3.2** Similarly, we can find proper  $s \in \mathcal{S}_+(\lambda_1)$  such that  $-[\gamma_1, \delta_1]x < 0$ , and  $s \in \mathcal{S}_-(\lambda_2)$  such that  $-[\gamma_1, \delta_1]x \geq 0$ , for any  $x \in \{x|C^T x \geq 0\}$ . Hence the system (6.1) is NC.

Case 3.3 & Case 3.4 (B.11) implies

$$[\gamma_1, \delta_1]x \begin{cases} \geq 0, & \text{if } \gamma_1\lambda_1 \leq 0, \alpha_2\lambda_2 > 0 \\ \leq 0, & \text{if } \gamma_1\lambda_1 > 0, \alpha_2\lambda_2 \leq 0 \end{cases} \quad (\text{B.12})$$

For any  $x \in \mathcal{X}_2$ , if  $C^T x < 0$ , there exists  $u$  such that  $A_2 x + B_2 u \in \Omega_1 \cup \Omega_2$ , i.e.,

$$\begin{cases} [\alpha_2, \beta_2]x = \delta_1 s, & s \in \mathcal{S}_+(\lambda_1) \\ [\gamma_2, \delta_2]x = -\gamma_1 s - u \end{cases} \quad \text{or} \quad \begin{cases} [\alpha_2, \beta_2]x = \beta_2 s, & s \in \mathcal{S}_-(\lambda_2) \\ [\gamma_2, \delta_2]x = -\alpha_2 s - u \end{cases} \quad (\text{B.13})$$

According to the sign of  $\delta_1\lambda_1$  and  $\beta_2\lambda_2$ , we also have four cases, which are listed in Table B.3.

Case	$\delta_1\lambda_1$	$\beta_2\lambda_2$	$\mathcal{X}_2$
4.1	$> 0$	$> 0$	$\{x   C^T x < 0\} \subseteq \mathcal{X}_2$
4.2	$< 0$	$< 0$	$\{x   C^T x < 0\} \subseteq \mathcal{X}_2$
4.3	$> 0$	$\leq 0$	$\{x   C^T x < 0 \cap A_{21}^T x \geq 0\} \subseteq \mathcal{X}_2$
4.4	$\leq 0$	$> 0$	$\{x   C^T x < 0 \cap A_{21}^T x \leq 0\} \subseteq \mathcal{X}_2$

Table B.3: Four cases when  $\lambda_1 \neq 0$  &  $\lambda_2 \neq 0$  &  $C^T x < 0$

From Tables B.2 and B.3, we can see Cases 3.1, 3.2, 4.1 and 4.2, i.e.,  $\gamma_1\lambda_1\alpha_2\lambda_2 > 0$  or  $\delta_1\lambda_1\beta_2\lambda_2 > 0$  guarantees that the system is NC. In the following, we only need to consider Case 3.3, 3.4, 4.3 and 4.4. Denote

$$\mathcal{V}_1 = \{x | C^T x \geq 0 \cap (x \text{ satisfies Case 3.3 or Case 3.4})\}$$

$$\mathcal{V}_2 = \{x | C^T x < 0 \cap (x \text{ satisfies Case 4.3 or Case 4.4})\}$$

If  $\gamma_1\lambda_1\alpha_2\lambda_2 \leq 0$  and  $\delta_1\lambda_1\beta_2\lambda_2 \leq 0$ , we have  $\mathcal{X}_2 = \mathcal{V}_1 \cup \mathcal{V}_2$ . So there are total four cases as shown in Table B.4. In the following, without loss of generality, we assume  $\lambda_1 > 0$  and  $\lambda_2 > 0$ .

**Remark B.1.2** *If one of  $\lambda_i$  is less than zero, then we choose  $P_2 = \text{diag}\{1, -1\}$  for  $\lambda_2$  or  $P_1 = \text{diag}\{-1, 1\}$  for  $\lambda_1$ , and let some  $u$  as  $-u$ . For example, if  $\lambda_1 < 0$*

*and  $\lambda_2 > 0$ , then  $\bar{A}_i = P_1 A_i P_1^{-1} = \begin{bmatrix} \alpha_i & -\beta_i \\ -\gamma_i & \delta_i \end{bmatrix}$ ,  $\bar{C} = P_1^{-T} C = [-c_1, c_2]$ . Now*

$\bar{\lambda}_1 = [-c_1, c_2] \begin{bmatrix} \delta_1 \\ \gamma_1 \end{bmatrix} = -\lambda_1 > 0$ , while  $\bar{\lambda}_2 = [-c_1, c_2] \begin{bmatrix} -\beta_2 \\ -\alpha_2 \end{bmatrix} = \lambda_2 > 0$ . If both of them are less than zero, we can let  $\bar{x} = -x$ ,  $\bar{u} = -u$ , thus  $\bar{C} = -C$ .

Case	$\gamma_1\lambda_1, \alpha_2\lambda_2$	$\delta_1\lambda_1, \beta_2\lambda_2$	$\gamma_1, \delta_1$	$\alpha_2, \beta_2$	$c_1, c_2$	$\mathcal{X}_3$
5.1.1	$\leq 0, > 0$	$> 0, \leq 0$	$\leq 0, > 0$	$> 0, \leq 0$	$> 0, < 0$	$\mathcal{X}_2$
5.1.2					$< 0, > 0$	$\mathcal{R}^2$
5.1.3					$> 0, > 0$	impossible
5.1.4					$< 0, < 0$	impossible
5.2.1	$\leq 0, > 0$	$\leq 0, > 0$	$\leq 0, \leq 0$	$> 0, > 0$	$> 0, > 0$	$\mathcal{R}^2$
5.2.2					$< 0, < 0$	$\mathcal{R}^2$
5.2.3					$> 0, < 0$	impossible
5.2.4					$< 0, > 0$	impossible
5.3.1	$> 0, \leq 0$	$> 0, \leq 0$	$> 0, > 0$	$\leq 0, \leq 0$	$< 0, < 0$	$\mathcal{R}^2$
5.3.2					$> 0, > 0$	$\mathcal{R}^2$
5.3.3					$< 0, > 0$	impossible
5.3.4					$> 0, < 0$	impossible
5.4.1	$> 0, \leq 0$	$\leq 0, > 0$	$> 0, \leq 0$	$\leq 0, > 0$	$> 0, < 0$	$\mathcal{X}_2$
5.4.2					$< 0, > 0$	$\mathcal{R}^2$
5.4.3					$< 0, < 0$	impossible
5.4.4					$> 0, > 0$	impossible

Table B.4: Four cases and sixteen subcases

Now each case has four subcases according to  $c_1$  and  $c_2$  shown in Table B.4. We only consider case 5.1, since the remaining are similar. In case 5.1,  $\mathcal{V}_1 = \{x | C^T x \geq 0, A_{12}^T x \geq 0\}$  and  $\mathcal{V}_2 = \{x | C^T x < 0, A_{21}^T x \geq 0\}$ .

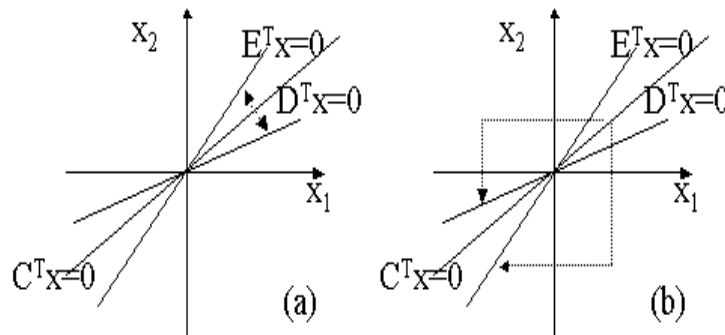


Figure B.2: Case 5.1.1-5.1.2

**Case 5.1.1** It is obvious that  $\begin{bmatrix} \delta_1 \\ -\gamma_1 \end{bmatrix}$ ,  $\begin{bmatrix} -\beta_2 \\ \alpha_2 \end{bmatrix}$ ,  $\begin{bmatrix} -c_2 \\ c_1 \end{bmatrix}$  are in the first quadrant, which is shown in Figure B.2 (a). We can easily see that  $\mathcal{X}_2 = \{A_{12}^T x \geq 0, A_{21}^T x \geq 0\}$ . Now we prove that  $\mathcal{X}_3 = \mathcal{X}_2$ . For any  $x \in \mathcal{R}^2 \setminus \mathcal{X}_2$ , either  $x \in \{x | C^T x \geq 0, A_{12}^T x < 0\}$  or  $x \in \{x | C^T x < 0, A_{21}^T x < 0\}$ . For  $x \in \{x | C^T x \geq 0, A_{12}^T x < 0\}$ , if there exists  $u$  such that  $\bar{x} = A_1 x + B_1 u \in \mathcal{X}_2$ , then,

$$\begin{cases} \alpha_2 \bar{x}_1 + \beta_2 \bar{x}_2 \geq 0 \\ \gamma_1 \bar{x}_1 + \delta_1 \bar{x}_2 \geq 0 \end{cases} \implies \begin{cases} \gamma_1 \alpha_2 \bar{x}_1 + \gamma_1 \beta_2 \bar{x}_2 \leq 0 \\ \alpha_2 \gamma_1 \bar{x}_1 + \alpha_2 \delta_1 \bar{x}_2 \geq 0 \end{cases} \\ \implies \gamma_1 \beta_2 \bar{x}_2 \leq \alpha_2 \delta_1 \bar{x}_2 \implies [\alpha_2, \beta_2][\delta_1, -\gamma_1]^T \leq 0$$

However, since  $[\delta_1, -\gamma_1]^T \in \{x | A_{21}^T x > 0\}$ , so we have  $[\alpha_2, \beta_2][\delta_1, -\gamma_1]^T > 0$ . Hence this is a contradiction. Thus  $\{x | C^T x \geq 0, A_{12}^T x < 0\}$  is not a subset of  $\mathcal{X}_3$ . In a similar way, we can deduce that  $\{x | C^T x < 0, A_{21}^T x < 0\}$  is not a subset of  $\mathcal{X}_3$ . Therefore,  $\mathcal{X}_3 = \mathcal{X}_2$  and the system (6.1) is not NC.

**Case 5.1.2** It is obvious that  $\begin{bmatrix} \delta_1 \\ -\gamma_1 \end{bmatrix}$ ,  $\begin{bmatrix} -\beta_2 \\ \alpha_2 \end{bmatrix}$  are in the first quadrant and  $\begin{bmatrix} -c_2 \\ c_1 \end{bmatrix}$  in third quadrant, which is shown in Figure B.2 (b). Note that in this case,  $\gamma_1 \neq 0$  and  $\beta_2 \neq 0$ . Otherwise, they will lead to  $\lambda_1 < 0$  and  $\lambda_2 < 0$ , respectively. Based on the analysis of Case 5.1.1, we can easily see that there exist  $u_1$  and  $u_2$  such that  $A_1 x + B_1 u_1 \in \mathcal{X}_2$  or/and  $A_2 x + B_2 u_2 \in \mathcal{X}_2$ , for any  $x \in \mathcal{R}^2$  using a similar argument of Case 1.2. So  $\mathcal{X}_3 = \mathcal{R}^2$ .

**Case 5.1.3 & Case 5.1.4** Since  $\lambda_2 > 0$ , case 5.1.3 is impossible, and since  $\lambda_1 > 0$ , case 5.1.4 is impossible.

Now we have already considered all situations, which lead to the system (6.1) is either NC or not NC.

**Remark B.1.3** *During the analysis, we make two assumptions "without loss of generality (WLOG)":*

$$(1) \lambda_1 > 0 \text{ and } \eta = \frac{c_1}{\alpha_2} = \frac{c_2}{\beta_2} > 0, \text{ if } \lambda_1 \neq 0 \text{ and } \lambda_2 = 0;$$

(2)  $\lambda_1 > 0$  and  $\lambda_2 > 0$ , if  $\lambda_1 \neq 0$  and  $\lambda_2 \neq 0$

which make the analysis much easier and clear. However, it is not difficult to remove the above assumptions in the analysis. In fact, we can analyze the NC set based on the geometry properties using the similar argument, and have the following results. In case (i), if (and only if) the region  $\{x|A_{12}^T x \geq 0(\text{or } A_{12}^T x \leq 0), C^T x \geq 0\}$  in (B.6) is bounded by  $A_{12}^T x = 0$  and  $C^T x = 0$  within one quadrant (8 situations), then we have  $\mathcal{X}_3 = \mathcal{X}_2 \neq \mathcal{R}^2$ . Analogously, in case (ii) that  $\lambda_1 \neq 0$  and  $\lambda_2 \neq 0$ , if (and only if) the region described by Cases 3.3, 3.4, 4.3 and 4.4 is bounded by  $A_{12}^T x = 0$  and  $A_{21}^T x = 0$  within one quadrant, then we have  $\mathcal{X}_3 = \mathcal{X}_2 \neq \mathcal{R}^2$ . Under corresponding situations, the system with such geometry properties is not NC. Thus we can deduce the conditions without any assumptions. To see this, we give a brief proof for case (ii). We shall first note that  $C^T = 0$ ,  $A_{12}^T x = 0$  and  $A_{21}^T x = 0$  are always in the same two quadrants, i.e.,  $\gamma_1\beta_1$ ,  $\alpha_2\beta_2$  and  $c_1c_2$  always have same sign, otherwise, we can get a conflict of the sign of one  $\lambda_i$ ,  $i = 1, 2$ . And then, we use the analogous argument of Case 5.1.1 to obtain the result.

## B.2 Proof of Theorem 6.2.5

Firstly, we consider the situation that  $\alpha_2\delta_1 = \beta_2\gamma_1 \neq 0$ . In this situation, we may assume that  $\alpha_2 = \tau\gamma_1$  and  $\beta_2 = \tau\delta_2$ , where  $\tau \in R$ . Then  $\lambda_2 = \tau\lambda_1$ . Hence, if  $\lambda_1 = 0$  and  $\lambda_2 = 0$ ,  $\mathcal{X}_2 = \mathcal{R}^2$ . Now we only need to consider the case  $\lambda_1 \neq 0$  and  $\lambda_2 \neq 0$ . Note that Case 3.3, Case 3.4, Case 4.3 and Case 4.4 of Section B.1.2 disappear under this condition. In fact,  $\gamma_1\lambda_1\alpha_2\lambda_2 = \tau^2\gamma^2\lambda_1^2 > 0$  and  $\delta_1\lambda_1\beta_2\lambda_2 = \tau^2\delta_1^2\lambda_1^2 > 0$ . As discussed in Section B.1.2,  $\mathcal{X}_3 = \mathcal{R}^2$ .

Secondly, if  $\alpha_2\delta_1 = \beta_2\gamma_1 = 0$ , there are several situations as shown in Table B.5. We only consider Case 2.1. As discussed in Section B.1.2, we can see that  $\mathcal{X}_1 = \{x|x_2 = 0\}$ . In fact,  $\Omega_1 = \{x|C^T x \geq 0, x_2 = 0\}$  and  $\Omega_2 = \{x|C^T x < 0, x_2 = 0\}$ , thus  $\mathcal{X}_1 = \Omega_1 \cup \Omega_2 = \{x|x_2 = 0\}$ . Then we have  $\mathcal{X}_2 = \{x|C^T x \geq 0, x_2 = 0\} \cup \{x|C^T x < 0\}$ . Thus  $\mathcal{X}_3 = \mathcal{R}^2$  according to Lemma 6.2.1. Case 2.2 is similar.

Case	$\gamma_1$	$\delta_1$	$\alpha_2$	$\beta_2$	$\mathcal{X}_2$
2.1	0	$\neq 0$	0	$\neq 0$	$\mathcal{X}_3 = \mathcal{R}^2$
2.2	$\neq 0$	0	$\neq 0$	0	$\mathcal{X}_3 = \mathcal{R}^2$
2.3	0	0	any	any	$\mathcal{X}_2 = \mathcal{R}^2$
2.4	any	any	0	0	$\mathcal{X}_2 = \mathcal{R}^2$

Table B.5:  $\alpha_2\delta_1 = \beta_2\gamma_1 = 0$  &  $c_i \neq 0$ ,  $i = 1, 2$ 

As for Case 2.3 (respectively, Case 2.4), they are under condition that  $A_{12} = 0$  (respectively,  $A_{21} = 0$ ), while based on Lemma 6.2.2, we can see the system is NC. Hence the system is NC.

### B.3 Proof of Theorem 6.2.6

Firstly, we consider  $c_1 \neq 0, c_2 = 0$ . Note that (B.2) and (B.1) still hold for  $\mathcal{X}_1$ . Now  $\lambda_1 = c_1\delta_1$  and  $\lambda_2 = c_1\beta_2$ . Considering whether  $\lambda_1$  and/or  $\lambda_2$  are/is equal to zero, there are four cases as shown in Table B.6.

Case	$\lambda_1$	$\lambda_2$	NC set
7.1	$= 0$	$\neq 0$	$\mathcal{X}_3 = \mathcal{R}^2$
7.2	$= 0$	$= 0$	$\mathcal{X}_4 = \mathcal{R}^2$ , if $\alpha_2 < 0$ $\mathcal{X}_4 = \mathcal{X}_3 \neq \mathcal{R}^2$ , if $\alpha_2 > 0$
7.3	$\neq 0$	$= 0$	$\mathcal{X}_3 = \mathcal{R}^2$ , if $\gamma_1 c_1 > 0$ $\mathcal{X}_3 = \mathcal{X}_2 \neq \mathcal{R}^2$ , if $\gamma_1 c_1 < 0$
7.4	$\neq 0$	$\neq 0$	$\mathcal{X}_3 = \mathcal{R}^2$

Table B.6:  $c_i = 0, i = 1$  or  $2$ 

**Case 7.1** If  $\lambda_1 = 0$  and  $\lambda_2 \neq 0$ , we have  $\delta_1 = 0, \gamma_1 \neq 0$  and  $\beta_2 \neq 0$ .  $\Omega_1 = \{x | C^T x = 0\} \subseteq \mathcal{X}_1$ . Now we consider  $\mathcal{X}_2$ . For any  $x \in \mathcal{X}_2$ , if  $C^T x \geq 0$ ,  $C^T(A_1 x + B_1 u) = 0$  can always be satisfied by a proper  $u$ . For any  $x \in \mathcal{X}_2$ , if  $C^T x < 0$ , we shall have  $C^T(A_2 x + B_2 u) = 0$ , i.e.  $A_{21}^T x = 0$ . Hence, we have  $\mathcal{X}_2 \supseteq \{x | C^T x \geq 0\} \cup \{x | C^T x < 0, A_{21}^T x = 0\}$ . According to Lemma 6.2.1 (2.b),  $\mathcal{X}_3 = \mathcal{R}^2$ .

**Case 7.2** Similarly, if  $\lambda_1 = 0$  and  $\lambda_2 = 0$ , then  $\delta_1 = 0, \beta_2 = 0$ ,  $\mathcal{X}_1 = \{x | C^T x = 0\}$  and  $\mathcal{X}_2 = \{x | C^T x \geq 0\} \cup \{x | C^T x < 0, A_{21}^T x = 0\}$ . Now we consider  $\mathcal{X}_3$ . For any  $x \in \mathcal{X}_3$ , we have (6.5) and (6.6). We only need to consider  $\{x | C^T x < 0\}$ , since

$\{x|C^T x \geq 0\}$  is NC. Hence,  $\mathcal{X}_3 = \{x|C^T x \geq 0\} \cup \{x|C^T x < 0, c_1 A_{21}^T x \geq 0\}$ . Now we consider  $\mathcal{X}_4$ . For any  $x \in \mathcal{X}_4$ , if  $C^T x < 0$ , we have (6.5) and (6.6). If  $\alpha_2 < 0$ , we have  $\mathcal{X}_4 = \mathcal{R}^2$ . Otherwise, if  $\alpha_2 > 0$ ,  $\mathcal{X}_4 = \mathcal{X}_3 \neq \mathcal{R}^2$ .

**Case 7.3** If  $\lambda_1 \neq 0$  ( $\delta_1 \neq 0$ ) and  $\lambda_2 = 0$  ( $\beta_2 = 0$ ,  $\alpha_2 \neq 0$ ), then  $\mathcal{X}_1 = \Omega_1$ . We assume that  $c_1 = \eta\alpha_2$ , where  $\eta \in R$  and  $\eta \neq 0$ . We have (B.4), (B.5), (B.7) and (B.10). Note that  $\delta_1 \lambda_1 \eta = \delta_1^2 c_1 \eta = \delta_1^2 \alpha_2 \eta^2$ . Thus if  $\alpha_2 < 0$ , the whole system is NC according to Lemma 6.2.1 (ii)(a). If  $\alpha_2 > 0$  and  $\gamma_1 = 0$ , then  $\mathcal{X}_3 = \mathcal{X}_2 = \Omega_1$ , and the whole system is not NC. Now we only need to consider  $\alpha_2 > 0$  and  $\gamma_1 \neq 0$ . There are two cases shown in Table B.7:  $\gamma_1 c_1 \delta_1 > 0$  and  $\gamma_1 c_1 \delta_1 < 0$ . With the similar argument from Section B.1.1, we can see that if  $\gamma_1 c_1 \neq 0$  and  $\delta_1 < 0$ , then  $\mathcal{X}_2 = \mathcal{R}^2$ ; otherwise if  $\gamma_1 c_1 \neq 0$  and  $\delta_1 > 0$ , then  $\mathcal{X}_3 = \mathcal{X}_2 \neq \mathcal{R}^2$ .

Case	$\gamma_1 c_1 \delta_1$	$\gamma_1$	$c_1$	$\delta_1$	$\mathcal{X}_3$
7.3.1	$> 0$	$> 0$	$> 0$	$> 0$	$\mathcal{X}_2$
7.3.2		$> 0$	$< 0$	$< 0$	$\mathcal{R}^2$
7.3.3		$< 0$	$< 0$	$> 0$	$\mathcal{X}_2$
7.3.4		$< 0$	$> 0$	$< 0$	$\mathcal{R}^2$
7.3.5	$< 0$	$> 0$	$> 0$	$< 0$	$\mathcal{R}^2$
7.3.6		$> 0$	$< 0$	$> 0$	$\mathcal{X}_2$
7.3.7		$< 0$	$< 0$	$< 0$	$\mathcal{R}^2$
7.3.8		$> 0$	$< 0$	$> 0$	$\mathcal{X}_2$

Table B.7: 8 subcases of Case 7.3

**Case 7.4** If  $\lambda_1 \neq 0$  and  $\lambda_2 \neq 0$ , then  $\mathcal{X}_1 = \Omega_1 \cup \Omega_2 = \{x|C^T x \geq 0, A_{12}^T x = 0\} \cup \{x|C^T x < 0, A_{21}^T x = 0\}$ . Considering Table B.2 and B.3 and noting that  $\delta_1 \lambda_1 \beta_2 \lambda_2 = c_1^2 \delta_1^2 \beta_2^2 > 0$ , Thus  $\{x|C^T x < 0\} \subseteq \mathcal{X}_2$ . According to Lemma 6.2.1 (ii)(a), we have  $\mathcal{X}_3 = \mathcal{R}^2$  and the system is NC.

We may also discuss the situation that  $c_1 = 0$  and  $c_2 \neq 0$  using the similar argument. Note that the results are not symmetric for these two situations, because the region  $\mathcal{S}_1$  and  $\mathcal{S}_2$  actually are not symmetric.

## B.4 Proof of Theorem 6.3.2

As we can see that  $\mathcal{Y}_{k-1} \subseteq \mathcal{Y}_k$ , so we only need to consider the region  $\mathcal{Y}_k \setminus \mathcal{Y}_{k-1}$  in each step. However, for convenience sometimes, we use the non-strict version instead of strict version in  $\mathcal{Y}_k$ , say,  $\{x|x_1 \geq 0\}$  instead of  $\{x|x_1 > 0\}$ , though  $\{x|x_1 = 0\} \subseteq \mathcal{Y}_{k-1}$ . Here we abuse the notation  $\bar{x}$ ,  $\bar{x}_1$  and  $\bar{x}_2$ , which denote the state in each recursive step, though they are different in each step. When we consider  $\bar{x} \in \mathcal{Y}_k$  for  $x \in \mathcal{Y}_{k-1} \cap \mathcal{S}_1$ , we need only discuss  $\bar{x}_2 = \gamma_1 x_1 + \delta_1 x_2$ , since  $\bar{x}_1$  can be freely chosen. Similarly, for  $x \in \mathcal{Y}_{k-1} \cap \mathcal{S}_2$ , we only discuss  $\bar{x}_1 = \alpha_2 x_1 + \beta_2 x_2$ .

### B.4.1 Case 1

Since  $\mathcal{Y}_1 = \{x|x_2 = 0\}$ , we only take a look at the region  $\{x|x_1 = 0\}$ . If  $x \in \mathcal{S}_1$ , i.e.,  $c_2 x_2 \geq 0$ ,  $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 x_1 + \beta_1 x_2 + u_2 \\ \delta_1 x_2 \end{bmatrix}$ , thus we have  $\bar{x}_2 \begin{cases} \geq 0 & \delta_1 c_2 > 0 \\ \leq 0 & \delta_1 c_2 < 0 \end{cases}$ . Note that if  $c_2 = 0$ , then  $\{\bar{x}\} = \mathcal{R}^2$ . So in the rest part of this subsection, we only consider  $c_2 \neq 0$ . Note that in this proof, we always omit the known conditions for either "reachability" or "unreachability" for further discussion without explanations.

Similarly, for  $x \in \mathcal{S}_2$ , we have  $\bar{x}_1 \begin{cases} > 0 & \beta_2 c_2 < 0 \\ < 0 & \beta_2 c_2 > 0 \end{cases}$ . So we obtain  $\mathcal{Y}_3$ , for example,  $\mathcal{Y}_3 = \{x|x_1 \leq 0\} \cup \{x|x_2 \geq 0\}$  if  $\beta_2 c_2 > 0$  and  $\delta_1 c_2 > 0$ .

We consider the region  $\{x|x_1 \leq 0\}$ . Note that  $\lambda_1 = c_1 \delta_1 \neq 0$  and  $\lambda_2 = c_1 \beta_2 \neq 0$ . If  $x \in \mathcal{S}_1$ , then  $\delta_1 \lambda_1 < 0$ , i.e.,  $c_1 < 0$ , guarantees  $\mathcal{Y}_4 = \mathcal{R}^2$ . If  $x \in \mathcal{S}_2$ , then  $\beta_2 \lambda_2 > 0$ , i.e.,  $c_1 > 0$ , guarantees  $\mathcal{Y}_4 = \mathcal{R}^2$ . Analogously, for the region  $\{x|x_1 \geq 0\}$ , no matter  $c_1 > 0$  or  $c_1 < 0$ , the system is reachable.

## B.4.2 Case 2

We first consider the case that  $c_1\alpha_2 > 0$  under  $\{x|x_1 < 0\}$ . Now we consider  $\mathcal{Y}_3$ .

Like case 1, when  $\delta_1\lambda_1 < 0$  or  $\beta_2\lambda_2 > 0$ ,  $\mathcal{Y}_3 = \mathcal{R}^2$ . We have

$$\bar{x}_2 \begin{cases} \text{any} & \delta_1\lambda_1 < 0 \cup (c_2 = 0, c_1 < 0, \delta_1 \neq 0) \\ \phi & c_2 = 0, c_1 > 0 \\ < 0 & \left\{ \begin{array}{l} \delta_1 < 0, c_2 > 0, \lambda_1 \leq 0 \\ \delta_1 > 0, c_2 < 0, \lambda_1 \geq 0 \end{array} \right\} := \Phi_{11} \\ > 0 & \left\{ \begin{array}{l} \delta_1 > 0, c_2 > 0, \lambda_1 \geq 0 \\ \delta_1 < 0, c_2 < 0, \lambda_1 \leq 0 \end{array} \right\} := \Phi_{12} \end{cases} \quad (\text{B.14})$$

$$\bar{x}_1 \begin{cases} \text{any} & \beta_2\lambda_2 > 0 \cup (c_2 = 0, c_1 > 0, \beta_2 \neq 0) \\ \phi & c_2 = 0, c_1 < 0 \\ > 0 & \left\{ \begin{array}{l} \beta_2 < 0, c_2 > 0, \lambda_2 \geq 0 \\ \beta_2 > 0, c_2 < 0, \lambda_2 \leq 0 \end{array} \right\} := \Phi_{13} \\ < 0 & \left\{ \begin{array}{l} \beta_2 > 0, c_2 > 0, \lambda_2 \leq 0 \\ \beta_2 < 0, c_2 < 0, \lambda_2 \geq 0 \\ \beta_2 = 0, c_2 = 0, c_1 > 0, \alpha_2 > 0 \end{array} \right\} := \Phi_{14} \end{cases} \quad (\text{B.15})$$

Note that in (B.14) and (B.15), the conditions in each row within  $\{\}$  have the relationship of "and/ $\cap$ ", while conditions between rows of  $\{\}$  have the relationship of "or/ $\cup$ ". For example,  $\Phi_{11} = (\delta_1 < 0, c_2 > 0, \lambda_1 \leq 0) \cup (\delta_1 > 0, c_2 < 0, \lambda_1 \geq 0)$ . In (B.14), the value set of  $\bar{x}_2$  have four situations. "any" means that under condition " $\delta_1\lambda_1 < 0 \cup (c_2 = 0, c_1 < 0, \delta_1 \neq 0)$ ",  $\bar{x}_2$  can be assigned to any value in  $R$ . Under other conditions, the value set of  $\bar{x}_2$  may be empty set,  $\{\bar{x}_2|\bar{x}_2 > 0\}$  or  $\{\bar{x}_2|\bar{x}_2 < 0\}$ . We can explain (B.15) in the similar way.

Condition  $(c_2 = 0, c_1 < 0)$  makes the system reachable. So for  $\mathcal{Y}_4$ , we only need to consider the situations under  $(\Phi_{11} \cup \Phi_{12}) \cap \Phi_{14}$ . We consider  $\{x|x_2 \geq 0\}$  ( $\{x|x_2 \leq 0\}$ ). For  $x \in \mathcal{S}_1$ , if  $\delta_1 < 0$ , then  $\mathcal{Y}_4 = \mathcal{R}^2$ . Now we only consider  $\delta_1 > 0$ . For

$x \in \mathcal{S}_2$ , if  $x_2 \leq 0$ , we have

$$\bar{x}_1 \begin{cases} \text{any} & \alpha_2 \lambda_1 < 0 \\ & \left\{ \begin{array}{l} \alpha_2 < 0, c_1 > 0, \lambda_2 \leq 0 \\ \alpha_2 > 0, c_1 < 0, \lambda_2 \geq 0 \end{array} \right\} := \Phi_{21} \\ & \left\{ \begin{array}{l} \alpha_2 > 0, c_1 > 0, \lambda_2 \geq 0 \\ \alpha_2 < 0, c_1 < 0, \lambda_2 \leq 0 \end{array} \right\} := \Phi_{22} \end{cases} \quad (\text{B.16})$$

If  $x_2 \geq 0$ , we have

$$\bar{x}_1 \begin{cases} \text{any} & \alpha_2 \lambda_1 > 0 \\ & \left\{ \begin{array}{l} \alpha_2 < 0, c_1 > 0, \lambda_2 \geq 0 \\ \alpha_2 > 0, c_1 < 0, \lambda_2 \leq 0 \end{array} \right\} := \Phi_{23} \\ & \left\{ \begin{array}{l} \alpha_2 > 0, c_1 > 0, \lambda_2 \leq 0 \\ \alpha_2 < 0, c_1 < 0, \lambda_2 \geq 0 \end{array} \right\} := \Phi_{24} \end{cases} \quad (\text{B.17})$$

Thus we can see that the system under Case 2 with  $c_1 \alpha_2 > 0$  is not reachable, if and only if  $\Phi_{81} := ((\Phi_{11} \cap \Phi_{14} \cap \Phi_{22}) \cup (\Phi_{12} \cap \Phi_{14} \cap \Phi_{24})) \cap (\delta_1 > 0)$ .  $\Phi_{81}$  seems to be complex, in fact, only contains two conditions:  $(\delta_1 > 0, \alpha_2 > 0, \beta_2 < 0, c_1 > 0, c_2 < 0, \lambda_2 \geq 0) \cup (\delta_1 > 0, \alpha_2 > 0, \beta_2 > 0, c_1 > 0, c_2 > 0, \lambda_2 \leq 0)$ , if we note that  $\lambda_1 = c_1 \delta_1$ . However, in the rest of this part, we do not make such simplification for these "short" expressions because it is trivial.

Similarly, we can consider the case  $c_1 \alpha_2 < 0$  under  $\{x | x_1 > 0\}$ . Noting that  $\bar{x}$  in  $\mathcal{Y}_3$

$$\bar{x}_2 \begin{cases} \text{any} & \delta_1 \lambda_1 > 0 \cup (c_2 = 0, c_1 > 0, \delta_1 \neq 0) \\ \phi & c_2 = 0, c_1 < 0 \\ \leq 0 & \left\{ \begin{array}{l} \delta_1 < 0, c_2 > 0, \lambda_1 \geq 0 \\ \delta_1 > 0, c_2 < 0, \lambda_1 \leq 0 \end{array} \right\} := \Phi_{31} \\ \geq 0 & \left\{ \begin{array}{l} \delta_1 > 0, c_2 > 0, \lambda_1 \leq 0 \\ \delta_1 < 0, c_2 < 0, \lambda_1 \geq 0 \end{array} \right\} := \Phi_{32} \end{cases} \quad (\text{B.18})$$

$$\bar{x}_1 \left\{ \begin{array}{l} \text{any } \beta_2 \lambda_2 < 0 \cup (c_2 = 0, c_1 < 0, \beta_2 \neq 0) \\ \phi \quad c_2 = 0, c_1 > 0 \\ > 0 \left\{ \begin{array}{l} \beta_2 < 0, c_2 > 0, \lambda_2 \leq 0 \\ \beta_2 > 0, c_2 < 0, \lambda_2 \geq 0 \\ \beta_2 = 0, c_1 < 0, c_2 = 0, \alpha_2 > 0 \end{array} \right\} := \Phi_{33} \\ < 0 \left\{ \begin{array}{l} \beta_2 > 0, c_2 > 0, \lambda_2 \geq 0 \\ \beta_2 < 0, c_2 < 0, \lambda_2 \leq 0 \end{array} \right\} := \Phi_{34} \end{array} \right. \quad (\text{B.19})$$

we have the following result: the system is not reachable if and only if  $\Phi_{82} := ((\Phi_{31} \cap \Phi_{33} \cap \Phi_{21}) \cup (\Phi_{32} \cap \Phi_{33} \cap \Phi_{23})) \cap (\delta_1 > 0)$ .

### B.4.3 Case 3

We first consider the case that  $c_1 \gamma_1 > 0$ . Now we think about  $\mathcal{Y}_3$ . For  $x \in \{x | x_2 \geq 0\}$ , we have

$$\bar{x}_2 \left\{ \begin{array}{l} \text{any } \gamma_1 \lambda_1 < 0 \\ \leq 0 \left\{ \begin{array}{l} \gamma_1 > 0, c_1 < 0, \lambda_1 \geq 0 \\ \gamma_1 < 0, c_1 > 0, \lambda_1 \leq 0 \end{array} \right\} := \Phi_{41} \\ \geq 0 \left\{ \begin{array}{l} \gamma_1 > 0, c_1 > 0, \lambda_1 \geq 0 \\ \gamma_1 < 0, c_1 < 0, \lambda_1 \leq 0 \end{array} \right\} := \Phi_{42} \end{array} \right. \quad (\text{B.20})$$

$$\bar{x}_1 \left\{ \begin{array}{l} \geq 0 \quad \beta_2 > 0 \\ \leq 0 \quad \beta_2 < 0 \end{array} \right. \quad (\text{B.21})$$

and for  $x \in \{x | x_1 = 0\}$ , we have

$$\bar{x}_2 \left\{ \begin{array}{l} \text{any } \delta_1 \neq 0, c_2 = 0 \\ = 0 \quad \delta_1 = 0, c_2 = 0 \\ \leq 0 \quad \delta_1 c_2 < 0 \\ \geq 0 \quad \delta_1 c_2 > 0 \end{array} \right. \quad (\text{B.22})$$

$$\bar{x}_2 \begin{cases} < 0 & \beta_2 c_2 > 0 \\ \phi & c_2 = 0 \\ > 0 & \beta_2 c_2 < 0 \end{cases} \quad (\text{B.23})$$

Noting that  $c_1 \gamma_1 > 0$ , we only need to consider the conditions  $\Phi_{42} \cap (\beta_2 > 0, c_2 \leq 0) \cap (c_2 \delta_1 > 0)$ , which makes  $\mathcal{Y}_3 = \{x | x_2 \geq 0\} \cup \{x | x_1 \geq 0\}$ , or  $\Phi_{42} \cap (\beta_2 < 0, c_2 \leq 0) \cap (c_2 \delta_1 > 0)$ , which makes  $\mathcal{Y}_3 = \{x | x_2 \geq 0\} \cup \{x | x_1 \leq 0\}$ , since the rest conditions guarantee that the system is reachable. Further, we consider  $\mathcal{Y}_4$ , the system is not reachable, if and only if  $\Phi_{51} := \Phi_{42} \cap ((\beta_2 > 0, c_2 < 0, \lambda_2 \geq 0) \cup (c_2 = 0, c_1 > 0)) \cap \Phi_{32}$  or  $\Phi_{52} := \Phi_{42} \cap ((\beta_2 < 0, c_2 < 0, \lambda_2 \geq 0) \cup (c_2 = 0, c_1 < 0)) \cap \Phi_{12}$ .

Similarly, for case that  $c_1 \gamma_1 < 0$ , we can see that  $\Phi_{43} \cap (\beta_2 > 0, c_2 \geq 0) \cap (c_2 \delta_1 < 0)$ , which makes  $\mathcal{Y}_3 = \{x | x_2 \leq 0\} \cup \{x | x_1 \leq 0\}$ , or  $\Phi_{43} \cap (\beta_2 < 0, c_2 > 0) \cap (c_2 \delta_1 < 0)$ , which makes  $\mathcal{Y}_3 = \{x | x_2 \leq 0\} \cup \{x | x_1 \geq 0\}$ , where  $\Phi_{43} := \begin{cases} \gamma_1 < 0, c_1 > 0, \lambda_1 \geq 0 \\ \gamma_2 > 0, c_1 < 0, \lambda_1 \leq 0 \end{cases}$ .

Hence the system is not reachable if and only if  $\Phi_{53} := \Phi_{43} \cap ((\beta_2 > 0, c_2 > 0, \lambda_2 \leq 0) \cup (c_2 = 0, c_1 < 0)) \cap \Phi_{31}$ , or  $\Phi_{54} := \Phi_{43} \cap ((\beta_2 < 0, c_2 > 0, \lambda_2 \leq 0) \cup (c_2 = 0, c_1 > 0)) \cap \Phi_{11}$ .

#### B.4.4 Case 4

Firstly, we consider the case that  $c_1 \alpha_1 > 0$  and  $c_1 \gamma_1 < 0$ , i.e.,  $\mathcal{Y}_2 = \{x | x_1 < 0\} \cup \{x | x_2 \leq 0\}$ . Now we discuss  $\mathcal{Y}_3$ . Defining  $\Phi_{61} = \Phi_{11} \cup (\delta_1 = 0, c_2 = 0, c_1 < 0, \gamma_1 > 0)$  and  $\Phi_{62} = \Phi_{12} \cup (\delta_1 = 0, c_2 = 0, c_1 < 0, \gamma_1 < 0)$ , and then replacing  $\Phi_{11}$ - $\Phi_{12}$  by  $\Phi_{61}$ - $\Phi_{62}$ , respectively, we have (B.24) and (B.15) for  $\{x | x_1 < 0\}$ .

$$\bar{x}_2 \begin{cases} any & \delta_1 \lambda_1 < 0 \cup (c_2 = 0, c_1 < 0, \delta_1 \neq 0) \\ \phi & c_2 = 0, c_1 > 0 \\ < 0 & \Phi_{61} \\ > 0 & \Phi_{62} \end{cases} \quad (\text{B.24})$$

For  $\{x|x_2 \leq 0\}$ , we have (B.16) and

$$\bar{x}_2 \begin{cases} \text{any} & \gamma_1 \lambda_1 > 0 \\ \leq 0 & \left\{ \begin{array}{l} \gamma_1 > 0, c_1 < 0, \lambda_1 \leq 0 \\ \gamma_1 < 0, c_1 > 0, \lambda_1 \geq 0 \end{array} \right\} := \Phi_{43} \\ \geq 0 & \left\{ \begin{array}{l} \gamma_1 > 0, c_1 > 0, \lambda_1 \leq 0 \\ \gamma_1 < 0, c_1 < 0, \lambda_1 \geq 0 \end{array} \right\} := \Phi_{44} \end{cases} \quad (\text{B.25})$$

Thus the system is not reachable if and only if  $\Phi_{71} := (\Phi_{61} \cup (c_2 = 0, c_1 > 0)) \cap \Phi_{43} \cap (\Phi_{14} \cup (c_2 = 0, c_1 < 0)) \cap \Phi_{22}$ .

Secondly, we consider the case  $c_1 \alpha_1 < 0$  and  $c_1 \gamma_1 > 0$ , i.e.,  $\mathcal{Y}_2 = \{x|x_1 > 0\} \cup \{x|x_2 \geq 0\}$ . Defining  $\Phi_{63} = \Phi_{31} \cup (\delta_1 = 0, c_2 = 0, c_1 > 0, \gamma_1 < 0)$ ,  $\Phi_{64} = \Phi_{32} \cup (\delta_1 = 0, c_2 = 0, c_1 > 0, \gamma_1 > 0)$ , and then replacing  $\Phi_{31}$ - $\Phi_{32}$  by  $\Phi_{63}$ - $\Phi_{64}$ , respectively, we have (B.26) and (B.19) for  $\{x|x_1 > 0\}$ :

$$\bar{x}_2 \begin{cases} \text{any} & \delta_1 \lambda_1 < 0 \cup (c_2 = 0, c_1 > 0, \delta_1 \neq 0) \\ \phi & c_2 = 0, c_1 < 0 \\ < 0 & \Phi_{63} \\ > 0 & \Phi_{64} \end{cases} \quad (\text{B.26})$$

For  $\{x|x_2 \geq 0\}$ , we have (B.17) and (B.20). Thus the system is not reachable if and only if  $\Phi_{72} := (\Phi_{64} \cup (c_2 = 0, c_1 < 0)) \cap \Phi_{42} \cap (\Phi_{33} \cup (c_2 = 0, c_1 > 0)) \cap \Phi_{23}$ .

Thirdly, we consider the case  $c_1 \alpha_1 > 0$  and  $c_1 \gamma_1 > 0$ , i.e.,  $\mathcal{Y}_2 = \{x|x_1 < 0\} \cup \{x|x_2 \geq 0\}$ . The system is not reachable if and only if  $\Phi_{73} := (\Phi_{62} \cup (c_2 = 0, c_1 > 0)) \cap \Phi_{42} \cap (\Phi_{14} \cup (c_2 = 0, c_1 < 0)) \cap \Phi_{24}$ .

Finally, we consider the case  $c_1 \alpha_1 < 0$  and  $c_1 \gamma_1 < 0$ , i.e.,  $\mathcal{Y}_2 = \{x|x_1 > 0\} \cup \{x|x_2 \leq 0\}$ . The system is not reachable if and only if  $\Phi_{74} := (\Phi_{63} \cup (c_2 = 0, c_1 < 0)) \cap \Phi_{43} \cap (\Phi_{33} \cup (c_2 = 0, c_1 > 0)) \cap \Phi_{21}$ .

## Author's Publications

- (1) J. Xu and L. Xie, *Null controllability of discrete-time planar bimodal piecewise linear systems*, International Journal of Control, vol. 78, no. 18, pp. 1486-1496, 2005.
- (2) J. Xu and L. Xie, *Non-synchronized  $H_\infty$  estimation of discrete-time piecewise linear systems*, Proc. the IFAC World Congress, Prague, Czech, July 2005.
- (3) J. Xu and L. Xie,  *$H_\infty$  state feedback control of discrete-time piecewise affine systems*, Proc. the IFAC World Congress, Prague, Czech, July 2005.
- (4) J. Xu and L. Xie, *Homogeneous polynomial Lyapunov functions for piecewise affine systems*, Proc. American Control Conference, Portland, USA, June 2005, pp. 581 – 586.
- (5) J. Xu, L. Xie, and G. Feng, *Feedback control design for discrete-time piecewise affine systems*, Proc. 5th International Conference on Control and Automation, Budepest, Hungary, June, 2005.
- (6) J. Xu and L. Xie,  *$H_\infty$  estimation of discrete-time piecewise linear systems*, Proc. 8th International Conference on Control, Automation, Robotics and Vision (Yunan, China), December 2004.
- (7) J. Xu, L. Xie, and Y. C. Soh,  *$H_\infty$  and generalized  $H_2$  estimation of continuous-time piecewise linear systems*, Proc. the 5th Asian Control Conference, IEEE, Melbourne, Australia, July 2004.

- (8) J. Xu and L. Xie, *Controllability and reachability of discrete-time planar bimodal piecewise linear systems*, American Control Conference, Minneapolis, Minnesota USA, June 2006.
- (9) J. Xu and L. Xie, *Dilated LMI characterization and a new stability criterion for polytopic uncertain systems*, IEEE World Congress on Intelligent Control and Automation, Dalian, China, June 2006.
- (10) J. Xu and L. Xie, *Non-synchronized  $H_\infty$  estimation of piecewise linear system*, IEEE Conference on Industrial Electronics and Applications, Singapore, September, 2006.
- (11) J. Xu, L. Xie, J.B. Zhang, T. J. B. Ng, and M. Luo. *A Novel Framework of Decentralized and Distributed Manufacturing Systems with Dedicated Decision Making Module*, the 4th International IEEE Conference on Industrial Informatics Singapore, August, 2006.
- (12) J. Xu and L. Xie. *Synthesis of discrete-time nonlinear systems: a SOS-based approach*, Accepted to American Control Conference 2007.
- (13) J. Xu and L. Xie, *Simultaneous stabilization and robust control for uncertain nonlinear systems using SOS*, Accepted to American Control Conference 2007.
- (14) J. Xu, L. Xie and C. E. de Souza, *A Scaling LMI Approach to Output Feedback Control of Discrete-time LTI Systems*, Accepted to International Conference on Control and Automation 2007.

# Bibliography

- [1] A. Alessandri and P. Colleta, “Design of Luenberger observers for a class of hybrid linear systems,” in *Proc. of Hybrid Systems: Computation and Control*, Rome, Italy, Jun 2001.
- [2] —, “Switching observers for continuous-time and discrete-time linear systems,” in *Proc. American Control Conference*, Arlington, VA, Jun 2001, pp. 2516–2521.
- [3] H. L. S. Almeida, A. Bhaya, D. M. Falcão, and E. Kaszkurewicz, “A team algorithm for robust stability analysis and control design of uncertain time-varying linear systems using piecewise quadratic Lyapunov function,” in *Proc. IEEE Conference on Decision and Control*, Tampa, Florida USA, Dec 1998, pp. 4410–4415.
- [4] A. A. Andronov and C. E. Chaikin, *Theory of Oscillations*. Princeton, NJ: Princeton University Press, 1949.
- [5] P. Apkarian and H. D. Tuan, “Robust control via concave minimization local and global algorithms,” *IEEE Trans. on Automatic Control*, vol. 45, no. 2, pp. 299–305, 2000.
- [6] S. Azuma and J. Imura, “A probabilistic approach to controllability/reachability analysis of hybrid systems,” in *Proc. IEEE Conference on Decision and Control*, Bahamas, US, Dec 2004, pp. 485–490.
- [7] —, “Probabilistic controllability analysis of sampled-data/discrete-time piecewise affine systems,” in *Proc. American Control Conference*, Boston, USA, 2004, pp. 2528–2533.

- [8] M. Babaali, “Switched linear systems: Observability and observers,” Ph.D. dissertation, Georgia Institute of Technology, Mar 2004.
- [9] A. I. Barkin and A. L. Zelentsovsky, “Method of power transformations for analysis of stability of nonlinear control systems,” *Systems and Control Letters*, vol. 3, pp. 303–310, 1983.
- [10] S. P. Bean, D. F. Coutinho, A. Trofino, and J. E. R. Cury, “Stability analysis and guaranteed domain of attraction for a class of hybrid systems: A LMI approach,” *International Journal of Robust and Nonlinear Control*, vol. 13, pp. 465–481, 2003.
- [11] A. Bemporad, G. Bianchini, F. Brogi, and G. Chesi, “Passivity analysis of discrete-time hybrid systems using piecewise polynomial storage functions,” in *Proc. IEEE Conference on Decision and Control*, Seville, Spain, Dec 2005, pp. 5421–5426.
- [12] A. Bemporad, P. Borodani, and M. Mannelli, “Hybrid control of an automotive robotized gearbox for reduction of consumptions and emissions,” in *Hybrid Systems: Computation and Control*, ser. *Lecture Notes in Computer Science*, vol. 2623. Springer Verlag, 2003, pp. 81–96.
- [13] A. Bemporad, F. Borrelli, and M. Morari, “Optimal controller for hybrid systems: Stability and piecewise linear explicit form,” in *Proc. IEEE Conference on Decision and Control*, Dec 2000.
- [14] A. Bemporad, G. Ferrari-Trecate, and M. Morari, “Observability and controllability of piecewise affine and hybrid systems,” *IEEE Trans. on Automatic Control*, vol. 45, no. 10, pp. 1864–1876, Oct 2000.
- [15] A. Bemporad, A. Garulli, S. Paoletti, and A. Vicino, “A bounded-error approach to piecewise affine system identification,” *IEEE Trans. on Automatic Control*, vol. 50, no. 10, pp. 1567–1580, 2005.
- [16] A. Bemporad, A. Giua, and C. Seatzu, “Synthesis of state-feedback optimal controllers for continuous-time switched linear systems,” in *Proc. IEEE Conference on Decision and Control*, Las Vegas, USA, 2002, pp. 3182–3187.

- [17] A. Bemporad, D. F. Torrisi, and M. Morari, "Optimization-based verification and stability characterization of piecewise affine and hybrid systems," in *Hybrid systems: Computation and control*, B. H. Korsh and N. Lynch, Eds., 2000, vol. 1790.
- [18] E. Beran, L. Vandenberghe, and S. Boyd, "A global BMI algorithm based on the generalized benders decomposition," in *European Control Conference*, Brussels, Belgium, 1997.
- [19] V. D. Blondel and J. N. Tsitsklis, "Complexity of stability and controllability of elementary hybrid systems," *Automatica*, vol. 35, no. 3, pp. 479–489, Mar 1999.
- [20] V. D. Blondel, J. N. Tsitsklis, C. P. P. Koiran, and J. Tsitsiklis, "Deciding stability and mortality of piecewise affine dynamical systems," *Theoretical Computer Science*, vol. 255, no. 1-2, pp. 687–696, 2001.
- [21] I. M. Bomze and E. de Klerk, "Solving standard quadratic optimization problems via linear, semidefinite and copositive programming," *Journal of Global Optimization*, vol. 24, pp. 163–185, 2002.
- [22] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishna, *Linear Matrix Inequalities in System and Control Theory*, ser. Studies in applied mathematics. SIAM, 1994.
- [23] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.
- [24] M. S. Branicky, "Analysis of continuous switching systems: Theory and examples," in *Proc. American Control Conference*, Baltimore, MD, Jun 1994, pp. 3110–3114.
- [25] —, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. on Automatic Control*, vol. 43, no. 4, pp. 475–483, Apr 1998.
- [26] R. W. Brockett, "Lie algebra and lie groups in control theory," in *Geometric Method in Systems Theory*, D. Q. Mayne and R. W. Brockett, Eds. Dordrecht, Reidel, 1973, pp. 43–82.

- [27] B. Brogliato, “Some perspectives on the analysis and control of complementarity systems,” *IEEE Trans. on Automatic Control*, vol. 48, no. 6, pp. 918–935, Jun 2003.
- [28] —, “Some results on the controllability of planar variational inequalities,” *Systems and Control Letters*, vol. 54, no. 1, pp. 65–71, Jan 2005.
- [29] M. K. Çamlibel, W. P. M. H. Heemels, and J. M. Schumacher, “Stability and controllability of planar bimodal linear complementarity systems,” in *Proc. IEEE Conference on Decision and Control*, Maui, Hawaii USA, Dec 2003, pp. 1651–1656.
- [30] —, “On the controllability of bimodal piecewise linear systems,” in *Hybrid Systems: Computation and Control*, ser. LNCS, R. Alur and G. Pappas, Eds. Berlin: Springer, 2004, vol. 2993.
- [31] —, “Algebraic necessary and sufficient conditions for the controllability of conewise linear systems,” Eindhoven University of Technology, The Netherlands, Tech. Rep., 2005.
- [32] M. K. Çamlibel, J. Pang, and J. Shen, “Observability of conewise linear systems,” in *Proc. IEEE Conference on Decision and Control*, Sevilla, Spain, Dec 2005.
- [33] D. Chatterjee and D. Liberzon, “On stability of stochastic switched systems,” in *Proc. IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, 2004, pp. 4125–4127.
- [34] C. Chen, G. Feng, D. Sun, and X. Guan, “ $h_\infty$  output feedback control of discrete-time fuzzy systems with application to chaos control,” *IEEE Trans. on Fuzzy Systems*, vol. 13, no. 4, pp. 531–543, 2005.
- [35] G. Chesi, “Estimating the domain of attraction for uncertain polynomial systems,” *Automatica*, vol. 40, no. 6, pp. 1981–1986, Jun 2004.
- [36] —, “Characterizing the positive polynomials which are not sos,” in *Proc. IEEE Conference on Decision and Control*, Seville, Spain, Dec 2005, pp. 1642–1647.
- [37] —, “Establishing stability and instability of matrix hypercubes,” *Systems and Control Letters*, vol. 54, pp. 381–388, 2005.

- [38] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, "Homogeneous Lyapunov functions for system with structure uncertainties," *Automatica*, vol. 39, no. 6, pp. 1027–1035, Jun 2003.
- [39] —, "Robust stability of polyopic systems via polynomial parameter-dependent Lyapunov functions," in *Proc. IEEE Conference on Decision and Control*, Dec 2003, pp. 4670–4675.
- [40] G. Chesi, A. Tesi, A. Vicino, and A. Garulli, "An LMI approach to constrained optimization with homogeneous forms," *Systems and Control Letters*, vol. 42, pp. 11–19, 2001.
- [41] L. O. Chua and A. C. Deng, "Canonical piecewise-linear representation," *IEEE Trans. on Circuits and Systems*, vol. 35, no. 1, pp. 101–111, 1998.
- [42] L. O. Chua, C. A. Desoer, and E. S. Kuh, *Linear and Nonlinear Circuits*. McGraw-Hill, New York, 1987.
- [43] F. A. Cuzzola and M. Morari, "A generalized approach for analysis and control of discrete-time piecewise affine and hybrid systems," in *HSCC 2001, LNCS 2034*. Springer-Verlag Berlin Heidelberg, 2001, pp. 189–203.
- [44] J. Daafouz and G. Millerioux, "Parameter dependent lyapunov function for global chaos synchronization of discrete time hybrid systems," in *Control of Oscillations and Chaos, Proceedings. 2nd International Conference*, vol. 2, Russia, 2000, pp. 339–342.
- [45] M. de Oliveira, "Decomposition of a polynomial as a sum-of-squares of polynomials and the s-procedure," in *Proc. IEEE Conference on Decision and Control*, Seville, Spain, Dec 2005, pp. 1654–1659.
- [46] M. C. de Oliveira, J. Bernussou, and J. C. Geromel, "A new discrete-time robust stability condition," *Systems and Control Letters*, vol. 37, pp. 261–265, 1999.
- [47] M. C. de Oliveira, J. C. Geromel, and L. Hsu, "LMI characterization of structural and robust stability: the discrete-time case," *Linear Algebra and its Application*, vol. 296, pp. 27–38, 1999.

- [48] R. A. Decarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, “Perspectives and results on the stability and stabilizability of hybrid systems,” *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, Jun 2000.
- [49] A. Doucet, A. Logothetis, and V. Krishnamurthy, “Stochastic sampling algorithms for state estimation of jump markov linear systems,” *IEEE Trans. on Automatic Control*, vol. 45, pp. 188–202, Feb 2000.
- [50] S. Drulhe, G. Ferrari-Trecate, H. de Jong, and A. Viari, *Reconstruction of switching thresholds in piecewise-affine models of genetic regulatory networks*, ser. Hybrid Systems: Computation and Control, ser. Lecture Notes in Computer Science. Springer Verlag, 2006, vol. 3927, pp. 184–199.
- [51] C. Ebendauer, J. Renz, and F. Allgöwer, “Polynomial feedback and observer design using nonquadratic Lyapunov functions,” in *Proc. IEEE Conference on Decision and Control*, Seville, Spain, Dec 2005, pp. 7587–7591.
- [52] J. Ezzine and A. H. Haddada, “On the controllability and observability of hybrid systems,” *International Journal of Control*, vol. 49, pp. 2045–2055, 1989.
- [53] C. Fantuzzi, S. Simani, S. Beghelli, and R. Rovatti, “Identification of piecewise affine models in noisy environment,” *International Journal of Control*, vol. 75, no. 18, pp. 1472–1485, 2002.
- [54] G. Feng, “Controller design and analysis of uncertain piecewise-linear systems,” *IEEE Trans. on Circuit and Systems I: Fundamental theory and applications*, vol. 49, no. 2, pp. 224–232, Feb 2002.
- [55] —, “Stability analysis of piecewise discrete-time linear system,” *IEEE Trans. on Automatic Control*, vol. 47, no. 7, pp. 1108–1112, 2002.
- [56] —, “Controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions and bilinear matrix inequalities,” *IEEE Trans. on Fuzzy Systems*, vol. 11, no. 5, pp. 605 – 612, 2003.
- [57] —, “Observer-based output feedback controller design of piecewise discrete time linear systems,” *IEEE Trans. on Circuit and Systems I: Fundamental theory and applications*, vol. 50, no. 3, pp. 446–450, 2003.

- [58] —, “ $H_\infty$  controller analysis and synthesis of piecewise discrete time linear systems,” in *8th IEEE Int. Conf. on Control, Automation, Robotics and Vision*, Dec 2004.
- [59] —, “ $H_\infty$  controller design of fuzzy dynamic systems based on piecewise Lyapunov functions,” *IEEE Trans. on Systems Man and Cybernetics - Part B: Cybernetics*, vol. 34, no. 1, pp. 283–292, Feb 2004.
- [60] —, “Stability analysis of discrete-time fuzzy dynamic systems based on piecewise Lyapunov functions,” *IEEE Trans. on Fuzzy Systems*, vol. 12, no. 1, pp. 22–28, Feb 2004.
- [61] —, “Robust filtering design of piecewise discrete time linear systems,” *IEEE Trans. on Signal Processing*, vol. 53, no. 2, pp. 599–605, Feb 2005.
- [62] G. Feng and D. Sun, “Generalized  $H_2$  controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions,” *IEEE Trans. on Circuit and Systems I: Fundamental theory and applications*, vol. 49, no. 12, pp. 1843–1850, Dec 2002.
- [63] G. Ferrari-Trecate, F. A. Cuzzola, D. Migone, and M. Morari, “Analysis and control with performance of piecewise affine and hybrid system,” in *Proc. American Control Conference*, vol. 38, Jun 2001, pp. 2139–2146.
- [64] —, “Analysis of discrete-time piecewise affine and hybrid system,” *Automatica*, vol. 38, pp. 2139–2146, 2002.
- [65] G. Ferrari-Trecate, F. A. Cuzzola, and M. Morari, “Moving horizon estimation for hybrid systems,” *IEEE Trans. on Automatic Control*, vol. 47, no. 10, pp. 1663–1676, 2002.
- [66] G. Ferrari-Trecate, M. Muselli, D. Liberati, and M. Morari, “A clustering technique for the identification of piecewise affine systems,” *Automatica*, vol. 39, pp. 205–217, 2003.
- [67] A. F. Filippov, “Differential equations with discontinuous righthand sides,” *Mathematics and its Applications*, 1988, kluwer, Dordrecht, Netherlands.

- [68] M. Fukuda and M. Kojima, “Branch-and-cut algorithms for the bilinear matrix inequality eigenvalue problem,” *Computational Optimization and Applications*, vol. 19, no. 1, pp. 79 – 105, 2001.
- [69] M. Gati and G. Ferrari-Trecate, “Computation of observability regions for piecewise affine systems: A projection-based algorithm,” in *Proc. IEEE Conference on Decision and Control*, Seville, Spain, Dec 2005, pp. 6922–6927.
- [70] T. Geyer, “Low complexity model predictive control in power electronics and power systems,” Ph.D. dissertation, Swiss Federal Institute of Technology (ETH), ZURICH, 2005.
- [71] L. E. Ghaoui, F. Oustry, and M. Aitrami, “A cone complementarity linearization algorithm for static output feedback and related problems,” *IEEE Trans. on Automatic Control*, vol. 42, no. 8, pp. 870–878, 2001.
- [72] K. C. Goh, J. H. Ly, L. Turand, and M. G. Safonov, “Biaffine matrix inequality properties and computational methods,” in *Proc. American Control Conference*, Jun 1994, pp. 850–855.
- [73] K. C. Goh, M. G. Safonov, and G. P. Papavassilopoulos, “A global optimization approach for the BMI problem,” in *Proc. IEEE Conference on Decision and Control*, Dec 1994, pp. 2009–2014.
- [74] J. M. Gonçalves, A. Megretski, and A. Dahleh, “Global analysis of piecewise linear systems using impact maps and surface Lyapunov functions,” *IEEE Trans. on Automatic Control*, vol. 48, no. 12, pp. 2089–2106, 2003.
- [75] D. Goshen-Meskin and I. Y. Bar-Itzhack, “Observability analysis of piece-wise constant systems with application to inertial navigation,” in *Proc. IEEE Conference on Decision and Control*, Honolulu, Hawaii, Dec 1990, pp. 821–826.
- [76] P. Grieder, “Efficient computation of feedback controllers for constrained systems,” Ph.D. dissertation, Swiss Federal Institute of Technology (ETH), ZURICH, 2004.
- [77] L. Gruyitch, J. Richapd, P. Borne, and J. C. Gentina, *Stability domain*, ser. Non-linear systems in aviation aerospace aeronautics astronautics, S. Sivasundaram, Ed. Chapman & Hall/CRC, 2004, vol. 1.

- [78] T. K. H. Nakada., K. Takaba, “Identification of piecewise affine systems based on statistical clustering technique,” *Automatica*, vol. 41, pp. 905 – 913, 2005.
- [79] L. Habets and J. van Schuppen, “A control problem for affine dynamical systems on a full-dimensional polytope,” *Automatica*, vol. 40, pp. 21–35, 2004.
- [80] ———, “Reachability of affine systems on polytopes in the plane,” Centrum voor Wiskunde en Informatica, The Netherlands, Tech. Rep. MAS-E0423, Dec 2004.
- [81] G. Hardy, J. Littlewood, and G. Pólya, *Inequalities*, 2nd ed. Cambridge, UK: Cambridge University Press, 1952.
- [82] A. Hassibi, “Lyapunov methods in the analysis of complex dynamical systems,” Ph.D. dissertation, Stanford University, USA, Aug 2000.
- [83] A. Hassibi, J. How, and S. Boyd, “A path-following method for solving BMI problem in control,” in *Proc. American Control Conference*, Jun 1999, pp. 1385 – 1389.
- [84] W. P. M. H. Heemels, “Linear complementary systems: a study in hybrid dynamics,” Ph.D. dissertation, Eindhoven University of Technology, 1999.
- [85] W. P. M. H. Heemels, B. de Schutter, and A. Bemporad, “Equivalence of hybrid dynamical models,” *Automatica*, vol. 37, pp. 1085–1091, 2001.
- [86] D. Henrion and A. Garulli, Eds., *Positive polynomials in control*, ser. LNCS. Springer, 2005, vol. 312.
- [87] D. Henrion and J. B. Lasserre, “Gloptipoly: Global optimization over polynomials with Matlab and SeDuMi,” *ACM Transactions on Mathematical Software*, vol. 29, no. 2, pp. 165–194, 2003.
- [88] T. Hu and Z. Lin, “A complete stability analysis of planar discrete-time linear systems under saturation,” *IEEE Trans. on Circuit and Systems I: Fundamental theory and applications*, vol. 48, no. 6, pp. 710–725, Jun 2001.
- [89] ———, “A complete stability analysis of planar linear systems under saturation,” *IEEE Trans. on Circuit and Systems I: Fundamental theory and applications*, vol. 47, no. 4, pp. 498–513, Apr 2001.

- [90] —, *Control system with actuator saturation: analysis and design*. Boston Basel Berlin: Birkhäuser, 2001.
- [91] T. Hu, Z. Lin, and B. Chen, “An analysis and design method for linear systems subject to actuator saturation and disturbance,” *Automatica*, vol. 38, pp. 351–359, 2002.
- [92] J. Imura, “Classification and stabilizability analysis of bimodal piecewise affine systems,” *International Journal of Robust and Nonlinear Control*, vol. 12, pp. 897–926, 2002.
- [93] —, “Well-posedness analysis of switch-driven piecewise affine systems,” *IEEE Trans. on Automatic Control*, vol. 48, no. 11, pp. 1926–1935, Nov 2003.
- [94] —, “Optimal control of sampled-data piecewise affine systems,” *Automatica*, vol. 40, pp. 661–669, 2004.
- [95] J. Imura and A. V. der Schaft, “Characterization of well-posedness of piecewise-linear systems,” *IEEE Trans. on Automatic Control*, vol. 45, no. 9, pp. 1600–1619, Sep 2000.
- [96] Y. Iwatani and S. Hara, “Stability test based on eigenvalue loci for bimodal piecewise linear systems,” in *Proc. American Control Conference*, Jul 2004, pp. 1879 – 1884.
- [97] —, “An exact stability tests for planar and multi-modal piecewise linear systems,” in *IFAC World Congress*, Prague, Czech, Jul 2005.
- [98] Z. W. Jarvis-Wloszek, “Matrix representation of polynomials: Theory and applications,” Master’s thesis, University of California, Berkeley, May 2001.
- [99] —, “Lyapunov based analysis and controller synthesis for polynomial systems using sum-of-squares optimization,” Ph.D. dissertation, University of California, Berkeley, Fall 2003.
- [100] Y. Jia, “Alternative proofs for improved LMI representations for the analysis and the design of continuous-time systems with polytopic type uncertainty: A predic-

- tive approach,” *IEEE Trans. on Automatic Control*, vol. 48, no. 8, pp. 1413–1416, Aug 2003.
- [101] T. A. Johansen, “Computation of Lyapunov functions for smooth nonlinear systems using convex optimization,” *Automatica*, vol. 36, no. 11, pp. 1617–1626, 2000.
- [102] M. Johansson, *Piecewise Linear Control System*. Berlin Heidelberg NewYork: Springer-Verlag, 2003.
- [103] M. Johansson and A. Rantzer, “Computation of piecewise quadratic Lyapunov functions for hybrid systems,” *IEEE Trans. on Automatic Control*, vol. 43, no. 4, pp. 555–559, Apr 1998.
- [104] M. Johansson, A. Rantzer, and K.-E. Arzen, “Piecewise quadratic stability of fuzzy systems,” *IEEE Trans. on Fuzzy Systems*, vol. 7, no. 6, pp. 713 – 722, 1999.
- [105] M. Johansson, A. Rantzer, and K.-E. Årzén, “A piecewise quadratic approach to stability analysis of fuzzy systems,” in *Stability Issues in Fuzzy Control*, J. Aracil and F. Gordillo, Eds. Germany: Physica-Verlag, Heidelberg, 2000, pp. 299–322.
- [106] P. Julián., “A high level canonical piecewise linear representation: Theory and applications,” Ph.D. dissertation, Universidad Nacional del Sur, Bahia Blanca, Argentina, 1999.
- [107] A. L. Juloski, W. P. M. H. Heemels, Y. Boers, and F. Verschure, “Two approaches to state estimation for a class of piecewise affine systems,” in *Proc. IEEE Conference on Decision and Control*, Maui, Hawaii USA, Dec 2003, pp. 143–148.
- [108] A. L. Juloski, W. P. M. H. Heemels, and S. Weiland, “Observer design for a class of piece-wise affine system,” in *Proc. 41st IEEE Conference on Decision and control*, Las Vegas, USA, Dec 2002, pp. 2602–2611.
- [109] A. Juloski, H. W.P.M.H., and G. Ferrari-Trecate, “Data based hybrid modelling of the component placement process in pick-and-place machines,” *Control Engineering Practice*, vol. 12, no. 10, pp. 1241–1252, Oct 2004.

- [110] R. E. Kalman, "Phase-plane analysis of automatic control systems with nonlinear gain elements," *Trans. AIEE Part II: Applications and Industry*, vol. 73, pp. 383–390, 1954.
- [111] S. Kanev, C. Scherer, M. Verhaegen, and B. de Schutter, "Robust output-feedback controller design via local BMI optimization," *Automatica*, vol. 40, pp. 1115–1127, 2004.
- [112] M. Kantner, "Robust stability of piecewise linear discrete time systems," in *Proc. American Control Conference*, vol. 2, Albuquerque, New Mexico, Jun 1997, pp. 1241 – 1245.
- [113] M. Kantner and J. Doyle, "Robust simulation and nonlinear performance," in *Proc. IEEE Conference on Decision and Control*, vol. 3, Kobe, Japan, Dec 1996, pp. 2622 – 2623.
- [114] E. C. Kerrigan, "Robust constraint satisfaction: Invariant sets and predictive control," Ph.D. dissertation, Department of Engineering, University of Cambridge, 2000.
- [115] H. K. Khalil, *Nonlinear Systems*, 3rd ed. New Jersey, USA: Prentice Hall, 2002.
- [116] X. Koutsoukos and P. Antsaklis, "Design of hybrid system regulators," in *Proc. IEEE Conference on Decision and Control*, vol. 4, Phoenix, Arizona USA, Dec 1999, pp. 3990 – 3995.
- [117] J. B. Lasserre, "SOS approximation of polynomials nonnegative on an algebraic set," in *Proc. IEEE Conference on Decision and Control*, Seville, Spain, Dec 2005, pp. 5837–5841.
- [118] K. K. Lee and A. Arapostathis, "On the controllability of piece-wise linear hyper-surface systems," *Systems and Control Letters*, vol. 9, pp. 89–96, 1987.
- [119] D. M. W. Leenaerts, "Further extensions to chuas explicit piecewise linear function descriptions," *International Journal of Circuit Theory and Applications*, vol. 24, pp. 621–633, 1996.

- [120] D. M. W. Leenaerts and W. M. G. V. Bokhoven, *piecewise linear modeling and analysis*. Boston,Dorderecht,London: Kluwer academic publishers, 1999.
- [121] F. Leibfritz, “An LMI-based algorithm for design suboptimal static  $H_2/H_\infty$  output feedback controllers,” *SIAM Journal of Control and Optimization*, vol. 39, no. 6, pp. 1711–1735, 2001.
- [122] D. Liberzon and A. S. Morse, “Basic problems in stability and design of switched systems,” *IEEE Control Magazine*, vol. 19, no. 5, pp. 59–70, Oct 1999.
- [123] R. Lin, J.-N.; Unbehauen, “Canonical piecewise-linear approximations,” *IEEE Trans. on Circuit and Systems I: Fundamental theory and applicatons*, vol. 39, no. 8, pp. 697 – 699, Aug 1992.
- [124] —, “Canonical representation: from piecewise-linear function to piecewise-smooth functions,” *IEEE Trans. on Circuit and Systems I: Fundamental theory and applicatons*, vol. 40, no. 7, pp. 461 – 468, Jul 1993.
- [125] R. H. Liu and Q. Zhang, “Nonlinear filtering: A hybrid approximation scheme,” *IEEE Trans. on Aerospace and Electronic Systems*, vol. 37, no. 2, pp. 470–480, Apr 2001.
- [126] D. Mignone, G. Ferrari-Trecate, and M. Morari, “Stability and stabilization of piecewise affine and hybrid system: An LMI approach,” in *Proc. IEEE Conference on Decision and Control*, vol. 1, Sydney, Australia, Dec 2000, pp. 504–509.
- [127] B. E. A. Milani, “Piecewise-affine Lyapunov functions for discrete-time linear systems with saturating controls,” in *Proc. American Control Conference*, Arlington, VA, Jun 2001, pp. 4206–4210.
- [128] Y. Ohta, H. Imanishi, L. Gong, and H. Haneda, “Computer generated Lyapunov functions for a class of nonlinear systems,” *IEEE Trans. on Circuit and Systems I: Fundamental theory and applicatons*, vol. 40, no. 5, pp. 343–354, 1993.
- [129] R. C. L. F. Oliveira and P. L. D. Peres, “Stability of polytopes of matrices via affine parameter-dependent Lyapunov functions: asymptotically exact LMI conditions,” *Linear Algebra and its Applications*, vol. 405, pp. 209–228, 2005.

- [130] J. L. P. Biswas, P. Griedder and M. Morari, “A survey on stability analysis of discrete-time piecewise affine systems,” in *IFAC world congress*, Prague, Czech Republic, Jul 2005, pp. 319–330.
- [131] S. Parjna, P. A. Parrilo, and A. Rantzer, “Nonlinear control synthesis by convex optimization,” *IEEE Trans. on Automatic Control*, vol. 49, no. 2, pp. 310–314, Feb 2004.
- [132] P. A. Parrilo, “Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization,” Ph.D. dissertation, California Institute of Technology, Pasadena, California, USA, May 2000.
- [133] A. Pavlov, A. Pogromsky, N. van de Wouw, H. Nijmeijer, and K. Rooda, “Convergent piecewise affine systems: analysis and design part II: discontinuous case,” in *Proc. IEEE Conference on Decision and Control*, Seville, Spain, Dec 2005, pp. 5397–5402.
- [134] A. Pavlov, N. van de Wouw, and H. Nijmeijer, “Convergent piecewise affine systems: analysis and design part II: discontinuous case,” in *Proc. IEEE Conference on Decision and Control*, Seville, Spain, Dec 2005, pp. 5397–5402.
- [135] S. Pettersson, “Analysis and design of hybrid systems,” Ph.D. dissertation, Chalmers University of Technology, Sweden, 1999.
- [136] —, “Controller design of switched linear systems,” in *Proc. American Control Conference*, Boston, Massachusetts, Jun 2004, pp. 3869–3874.
- [137] N. B. O. L. Pettit, *Analysis of piecewise linear dynamical systems*, ser. UMIST control systems center series, M. B. Zarrop and P. E. Wellstead, Eds. Research studies press LTD and John Wiley & sons inc., 1995.
- [138] V. Powers and B. Reznick, “A new bound for pólyas theorem with applications to polynomials positive on polyhedra,” *J. Pure Appl. Algebra*, vol. 164, pp. 221–229, 2001.
- [139] S. Prajna and A. Papachristodoulou, “Analysis of switched and hybrid systems - beyond piecewise quadratic methods,” in *Proc. American Control Conference*, Denver, Colorado, Jun 2003, pp. 2779–2784.

- [140] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, “SOSTOOLS: Sum of squares optimization toolbox for matlab,” California Institute of Technology, Pasadena, California, User manual, 2002.
- [141] S. Prajna, A. Papachristodoulou, and F. Wu, “Nonlinear control synthesis by sum of squares optimization: A Lyapunov-based approach,” in *5th Asian Control Conference*, Melbourne, Australia, Jul 2004, pp. 157–165.
- [142] E. Prempain, “An application of the sum of squares decomposition to the  $l_2$  gain computation for a class of non linear systems,” in *Proc. IEEE Conference on Decision and Control*, Seville, Spain, Dec 2005, pp. 6865–6868.
- [143] A. Prestel and C. N. Delzell, *Positive polynomial*, ser. Springer monographs in mathematics. Springer, 2001.
- [144] A. Rantzer, “A dual to Lyapunov’s stability theorem,,” *Systems and Control Letters*, vol. 42, pp. 161–168, 2001.
- [145] A. Rantzer and M. Johansson, “Piecewise linear quadratic optimal control,” *IEEE Trans. on Automatic Control*, vol. 45, pp. 629–637, Apr 2000.
- [146] L. Rodrigues, “Dynamic output feedback controller synthesis for piecewise-affine systems,” Ph.D. dissertation, Stanford University, 2002.
- [147] —, “Sampled-data state feedback control for piecewise-affine systems,” in *Proc. IEEE Conference on Decision and Control*, Seville, Spain, Dec 2005, pp. 4487–4492.
- [148] L. Rodrigues and S. Boyd, “Piecewise-affine state feedback for piecewise-affine slab systems using convex optimization,” *Systems and Control Letters*, vol. 54, pp. 835–853, 2005.
- [149] L. Rodrigues and J. How, “Observer-based control of piecewise-affine systems,” *International Journal of Control*, vol. 76, no. 5, pp. 459–477, 2003.
- [150] J. Roll, A. Bemporad, and L. Ljung, “Identification of piecewise affine systems via mixed-integer programming,” *Automatica*, vol. 40, pp. 37–50, 2004.

- [151] W. L. Rugh, *Linear system theory*, 2nd ed., T. Kailath, Ed. New Jersey, USA: Prentice Hall, 1996.
- [152] J. W. Schwartz, "Piecewise linear servomechanisms," *AIEE Transactions*, vol. 72, pp. 401–405, 1953.
- [153] U. Shaked, "Improved LMI representations for the analysis and the design of continuous-time systems with polytopic type uncertainty," *IEEE Trans. on Automatic Control*, vol. 46, no. 1, pp. 652–656, Apr 2001.
- [154] H. Siegelmann and E. Sontag, "On the computational power of neural nets," *Journal of Computer and System Sciences*, pp. 132–150, 1995.
- [155] E. D. Sontag, "Interconnected automata and linear systems: A theoretical framework in discrete-time," in *Hybrid systems III: Verification and control*, R. Alur, T. A. Henzinger, and E. D. Sontag, Eds. Springer, 1996, pp. 436–448.
- [156] E. Sontag, "Nonlinear regulation: the piecewise linear approach," *IEEE Trans. on Automatic Control*, vol. 26, no. 2, pp. 346–357, 1981.
- [157] D. P. Spanos and J. M. Gonçalves, "Finding surface Lyapunov functions through sum-of-squares programming," in *Proc. American Control Conference*, Boston, Massachusetts, Jun 2004, pp. 1891–1892.
- [158] M. K. M. Stingl, *PENBMI User Manual*, 1st ed., www.penopt.com, Jan 2004.
- [159] B. Sturmfels, *Solving systems of polynomial equations*, ser. CNMS. American Mathematical society, 2002, no. 97.
- [160] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its application to modeling and control," *IEEE Trans. on Systems Man and Cybernetics*, vol. SMC-15, pp. 116–132, 1985.
- [161] K. Tanaka, T. Ikeda, and H. O. Wang, "Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: quadratic stabilizability," *IEEE Trans. on Fuzzy Systems*, vol. 8, pp. 1–13, Feb 1996.

- [162] J. B. Thevenet, D. Noll, and P. Apkarian, “Non linear spectral SDP method for BMI-constrained problems: Applications to control design,” in *International Conference on Informatics in Control, Automation and Robotics*, vol. 1, 2004, pp. 237–248.
- [163] A. Trofino and C. E. de Souza, “Bi-quadratic stability for nonlinear systems,” in *Proc. IEEE Conference on Decision and Control*, Phoenix, Arizona, Dec 1999, pp. 5016–5021.
- [164] V. I. Utkin, *Sliding mode in control and optimization*. Springer-Verlag, 1992.
- [165] R. Vidal, S. Schaffert, O. Shakernia, J. Lygeros, and S. Sastry, “Decidable and semi-decidable controller synthesis for classes of discrete time hybrid systems,” in *Proc. IEEE Conference on Decision and Control*, vol. 2, Orlando, Florida USA, Dec 2001, pp. 1243 – 1248.
- [166] R. Vidal, S. Soatto, and S. Sastry, “An algebraic geometric approach to the identification of linear hybrid systems,” in *Proc. IEEE Conference on Decision and Control*, 2003, pp. 167–172.
- [167] L. Wang and G. Feng, “Piecewise  $H_\infty$  controller design of discrete time fuzzy systems,” *IEEE Trans. on Systems Man and Cybernetics - Part B: Cybernetics*, vol. 34, no. 1, pp. 682–686, Feb 2004.
- [168] G. Xie and L. Wang, “Necessary and sufficient conditions for controllability of switched linear systems,” in *Proc. American Control Conference*, Anchorage, AK, May 2003, pp. 1897–1902.
- [169] G. Xie, L. Wang, B. Xun, and L. Zhao, “Null controllability of planar piecewise linear systems,” in *IEEE International conference on systems, man and cybernetics*, Washington D.C., USA, 2003, pp. 154–159.
- [170] L. Xie, L. Lu, H. Zhang, and D. Zhang, “Improved robust  $H_2$  and  $H_\infty$  filtering for uncertain discrete-time systems,” *Automatica*, vol. 40, pp. 873–880, 2004.
- [171] J. Xu and L. Xie, “ $H_\infty$  state feedback control of discrete-time piecewise affine systems,” in *IFAC World Congress*, Prague, Czech, Jul 2005.

- [172] —, “Homogeneous polynomial Lyapunov functions for piecewise affine systems,” in *Proc. American Control Conference*, Jun 2005, pp. 581 – 586.
- [173] —, “Non-synchronized  $H_\infty$  estimation of discrete-time piecewise linear systems,” in *IFAC World Congress*, Prague, Czech, Jul 2005.
- [174] —, “Null controllability of discrete-time planar bimodal piecewise linear systems,” *International Journal of Control*, vol. 78, no. 18, pp. 1486–1496, 2005.
- [175] —, “Controllability and reachability of discrete-time planar bimodal piecewise linear systems,” in *Proc. American Control Conference*, Minneapolis, Minnesota USA, Jun 2006, pp. 4387–4392.
- [176] —, “Dilated LMI characterization and a new stability criterion for polytopic uncertain systems,” in *IEEE World Congress on Intelligent Control and Automation*, Dalian, China, Jun 2006.
- [177] —, “Non-synchronized  $H_\infty$  estimation of piecewise linear systems,” in *the 1st IEEE Conference on Industrial Electronics and Applications*, Singapore, May 2006.
- [178] J. Xu, L. Xie, and G. Feng, “Feedback control design for discrete-time piecewise affine systems,” in *Proc. the 5th International Conference on Control and Automation*, 2005.
- [179] J. Xu, L. Xie, and Y. C. Soh, “ $H_\infty$  and generalized  $H_2$  estimation of continuous-time piecewise linear systems,” in *the 5th Asian Control Conference*. Melbourne, Australia: IEEE, Jul 2004.
- [180] X. Xu, “Analysis and design of switched systems,” Ph.D. dissertation, University of Notre Dame, USA, 2001.
- [181] X. Xu and P. J. Antsaklis, “Optimal control of switched systems via non-linear optimization based on direct differentiations of value functions,” *International Journal of Control*, vol. 75, no. 16/17, pp. 1406–1426, 2002.
- [182] H. Ye, A. N. Michel, and L. Hou, “Stability theory for hybrid dynamical systems,” *IEEE Trans. on Automatic Control*, vol. 43, no. 4, pp. 461–474, 1998.

- [183] C. Yfoulis, A. Muir, N. Pettit, and P. Wellstead, “Stabilization of orthogonal piecewise linear systems using piecewise linear Lyapunov-like functions,” in *Proc. IEEE Conference on Decision and Control*, Tampa, Florida USA, 1998, pp. 1476–1480.
- [184] C. Yfoulis, A. Muir, P. Wellstead, and N. Pettit, “Stabilization of orthogonal piecewise linear systems: Robustness analysis and design,” in *HSCC*, ser. LNCS. Berlin Heidelberg: Springer-Verlag, 1999, vol. 1569, pp. 256–271.
- [185] A. L. Zelentsovsky, “Nonquadratic Lyapunov functions for robust stability analysis of linear uncertain systems,” *IEEE Trans. on Automatic Control*, vol. 39, pp. 135–138, 1994.