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Computability Theory and Degree Structures

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Abstract

This thesis mainly studies the thin-free degrees. In particular, we will show the technique to construct a thin-free degree, and consider the relation between thin-free degrees with minimal degrees hyperimmune-free degrees, genericity, and nonbranching degrees.

In Chapter 1, we briefly review basics of computability theory, including degree structures and fundamentals of Π^0_1 classes.

In Chapter 2, we provide the method to construct a thin-free degree, using $\mathbf{0}''$ as oracle. We will show how a hyperimmune-free minimal thin-free degree is constructed below $\mathbf{0}''$. This degree is not below $\mathbf{0}'$. Our focus in this chapter is to describe the feature to show a recursive tree whose body is not thin. This will be used in Chapters 3 and 4.

In Chapter 3, we first prove that any 1-generic set below $\mathbf{0}'$ is not thin-free. The Σ_1-correct approximation of a 1-generic set below $\mathbf{0}'$ is the key feature to construct the recursive tree below $\mathbf{0}'$ whose body is thin. In contrast, any nonrecursive set below a 2-generic set is thin-free. In particular, all 2-generic sets are thin-free.

In Chapter 4, we will provide the construction of a nonbranching thin-free r.e. degree by full approximation argument.

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Contents

Chapter 1

Introduction

1.1 Basics of computibility theory

A function f from $A \subseteq \omega$ to ω is a partial recursive (p.r. for short) function if there is a Turing machine M such that with input x, M stops by finitely many steps and outputs $f(x)$ for $x \in A$, and never stops with input $x \notin A$. f is total if $A = \omega$. We can assign a code number, called Gödel number, to each Turing machine. Based on this, we can enumerate Turing machines effectively, and denote this enumeration by $\{M_e : e \in \omega\}$. We use φ_e to denote the p.r. function computed by M_e. Then $\{\varphi_e : e \in \omega\}$ is an effective enumeration of all p.r. functions.

For $s \in \omega$ and $x < s$, we write $\varphi_{e,s}(x) \downarrow= y$ if there is some $y < s$ such that $M_e(x)$ halts in less than s steps with output y. If such y does not exist, we write $\varphi_{e,s}(x) \uparrow$. We write $\varphi_e(x) \downarrow= y$ if there exists some s such that $\varphi_{e,s}(x) \downarrow= y$, and say that φ_e is defined on x. We write $\varphi_e(x) \uparrow$ if for all s, $\varphi_{e,s}(x) \uparrow$.

A set $A \subseteq \omega$ is recursively enumerable (r.e. for short) if A is the domain of a p.r. function. Let W_e be the domain of φ_e. Then we have an effective enumeration of r.e. sets $\{W_e : e \in \omega\}$, and we use $W_{e,s}$ to denote the domain of $\varphi_{e,s}$. It is well-known that A is r.e. if and only if A is either $\emptyset$ or the range of some total recursive function, which gives the meaning of "recursively enumerable". A is recursive if both A and its complement $\overline{A}$ are recursively enumerable. We can effectively determine the membership of a recursive set, by enumerating A and $\bar{A}$ simultaneously.

Turing's halting set $K = \{e : \varphi_e(e) \downarrow\}$ is r.e., but not recursive. We say that a set P is productive if there is a p.r. function $\varphi(x)$ such that for any x, if $W_x \subseteq P$, then $\varphi(x) \downarrow\in P - W_x$. Indeed, $\overline{K}$ is productive, and in terms of computability theory, K is creative. All creative sets are 1-reducible to each other (we say A is 1-reducible to B is there is a one-to-one recursive function f such that $x \in A$ if and only if $f(x) \in B$) and they form a nontrivial orbit in the lattice of recursively enumerable sets, when automorphisms are considered. In order to find some incomplete and nonrecursive r.e. sets, Post considered simple sets, hypersimple sets, and even hyperhypersimple sets, whose complements do not contain infinite r.e. sets.

We now give a brief review of fundamental properties of p.r. functions. Padding Lemma says that for any p.r. function φ_e, we can effectively find infinitely many different indices e' such that $\varphi_{e'} = \varphi_e$, and Parameter Theorem, also called the s_n^m-theorem, says that for any p.r. function $\psi(x, y)$ with variables x, y, there is a one-to-one total recursive function s such that $\varphi_{s(x)}(y) = \psi(x, y)$. It is named as "Parameter Theorem" because x is treated as parameters. Kleene's Recursion Theorem, i.e., Fixed Point Theorem, says that for any total recursive function f, there is an index e such that $W_e = W_{f(e)}$. Kleene's Recursion Theorem allows us to build a p.r. function assuming there is a index for it. A set A is an index set if for any $\varphi_x = \varphi_y$, $x \in A \Leftrightarrow y \in A$. There are a lot of interesting stories about index sets. We can definitely consider Turing degrees of index sets, and Rice's Theorem states that the only recursive index sets are $\emptyset$ and ω, so any nontrivial property of p.r. functions is undecidable. Some index sets, like Fin $= \{x : W_x \text{ is finite}\}$, Tot $= \{x : \varphi_x \text{ is total}\}$, etc., play an important role in the study of sets in the arithmetic hierarchy. For example, Fin is Σ_2-complete, and Tot is Π_2-complete, where a set is Σ_n-complete (Π_n-complete) if it is Σ_n (Π_n) and all Σ_n (Π_n) sets are 1-reducible to it.

In computability theory, there are several reductions among sets of numbers, and these reductions provide various classifications of degrees of unsolvability. Among these reduction, Turing reduction is considered as one natural reduction among problems and has been well-studied. Say that A is Turing reducible to B, denoted as $A \leq_T B$, if there is an oracle Turing machine that decides the membership in A, with B as oracle. We use Φ^B to denote the p.r. function computed by an oracle Turing machine Φ with oracle B. Say A and B are Turing incomparable, written as $\deg(A)|_T \deg(B)$, if $A \not\leq_T B$ and $B \not\leq_T A$.

$\leq_T$ is a pre-order, and to obtain a partial order, we consider Turing equivalence classes, called Turing degrees, and the ordering induced by $\leq_T$ on Turing degrees. We denote the Turing degree of A by $\deg(A)$, i.e., $\deg(A) = \{B \subseteq \omega : B \equiv_T A\}$, and usually, we just denote it as $\boldsymbol{a}$. We write $\boldsymbol{a} \leq \boldsymbol{b}$ if there are sets $A \in \boldsymbol{a}$ and $B \in \boldsymbol{b}$ such that $A \leq_T B$, and $\boldsymbol{a} < \boldsymbol{b}$ if $\boldsymbol{a} \leq \boldsymbol{b}$ but $\boldsymbol{b} \not\leq \boldsymbol{a}$.

We use $\mathbf{0}$ to denote the Turing degree of recursive sets, and $\mathbf{0'}$ as the Turing degree of the halting set. $\mathbf{0}$ is the least Turing degree. A degree is r.e. if it contains an r.e. set. Both $\mathbf{0}$ and $\mathbf{0'}$ are r.e. degrees, and $\mathbf{0}'$ is the greatest r.e. degree. Shoenfield's limit lemma says that if A is Turing reducible to K, then A has a recursive approximation.

Let A join B, written as $A \oplus B$, be $\{2x : x \in A\} \cup \{2x+1 : x \in B\}$. $\deg(A \oplus B)$ is the least upper bound of $\deg(A)$ and $\deg(B)$, and we call it the join of $deg(A)$ and $deg(B)$, which is well-defined. We use $\boldsymbol{a} \vee \boldsymbol{b}$ to denote the joint $\deg(A \oplus B)$, if $A \in \boldsymbol{a}$ and $B \in \boldsymbol{b}$. Thus, the Turing degrees form an uppersemilattice.

Oracle Turing machines can be effectively enumerated, as $\{\Phi_e : e \in \omega\}$. We also call each Φ_e a partial recursive functional, in the sense that for a set A, Φ_e^A is a partial recursive function. We write $\Phi_{e,s}^A(x) \downarrow= y$ if y is the output when we run

Turing machine Φ on input x in s steps, with A_s as oracle, where $\{A_s : s \in \omega\}$ is a recursive approximation of A, and $\Phi^A_{e,s}(x) \uparrow$ if such a y does not exist. We write $\Phi^A_e(x) \downarrow = y$ if there is some s such that $\Phi^A_{e,t}(x) \downarrow = y$ for all $t > s$, and $\Phi^A_e(x) \uparrow$ otherwise. The oracle here can be a finite string, say σ, and we agree that $\Phi^\sigma_e(x) \uparrow$, if $x > |\sigma|$, or the machine queries a number $> |\sigma|$ in the oracle. The use of a converging computation, say $\Phi^A_e(x) \downarrow$, is defined as 1+ the largest number z, where the machine queries the membership of $z \in A$ in the computation, and 0 if no such query is made. Note that in computation, it can happen that infinitely many queries are made, resulting a divergent computation. We use $\varphi^A_e(x)$ to denote this use. The use principle says that if $\Phi^A_e(x) \downarrow$ and $B \upharpoonright \varphi^A_e(x) = A \upharpoonright \varphi^A_e(x)$, then $\Phi^B_e(x) \downarrow = \Phi^A_e(x)$.

For a set A, the halting set relativized to A is defined as $A' = \{e : \Phi^A_e(e) \downarrow\}$ (The halting set K is thus the Turing jump of $\emptyset$). A' is called the Turing jump of A, and the jump operator $'$ is a function mapping A to A'. Here are some properties of the jump operator: A' is r.e. in A, and $A' \not\leq_T A$. We write $A^{(n)}$ to denote the set obtained by applying the jump operator n times to A. Many results for p.r. functions can be relativized, like Relativised Parameter Theorem, which states that for any p.r. functional Φ, there is a one-to-one total recursive function s such that $\Phi^A_{s(x)}(y) = \Phi^A(x, y)$ for any oracle A and variables x, y.

We can define the Turing jump of a degree $\mathbf{a}$ by letting $\mathbf{a}'$ as the Turing degree of A'. This is well-defined. We say a set is Turing complete if the halting set K is reducible to it, or equivalently, it computes K (K is thus Turing complete). Friedberg Jump Invertation Theorem says that the jump operator is an onto mapping from the set of Turing degrees to Turing complete degrees, Shoenfield Jump Inversion Theorem shows that that every Σ^0_2 degree $\geq \mathbf{0'}$ is the jump of some $\boldsymbol{a} \leq \mathbf{0'}$, and Sacks Jump Inversion Theorem strengthens Shoenfield's theorem by showing that every Σ^0_2 degree $\geq \mathbf{0'}$ is the jump of some r.e. degree $\boldsymbol{a}$.

Jump operator can be used to provide a classification of Turing degrees below

$\mathbf{0}'$, i.e., the high/low classes. A below $\emptyset'$ is low$_n$ if $A^{(n)} \equiv_T \emptyset^{(n)}$, and high$_n$ if $A^{(n)} \equiv_T \emptyset^{(n+1)}$. The low sets are "nearly" computable and the high sets are "nearly" complete in some sense. For instance, any r.e. degree splits above a low smaller one, and any high r.e. degree bounds a minimal pair. The high/low classification can be generalized as follows: A is generalized low$_n$, GL$_n$ for short, if $A^{(n)} \equiv_T (A \oplus \emptyset')^{(n-1)}$, and generalized high$_n$, GH$_n$, if $A^{(n)} \equiv_T (A \oplus \emptyset')^{(n)}$.

It is well-known that any nonempty r.e. set is m-reducible to K. Here, say that A is m-reducible (many-one reducible) to B, written as $A \leq_m B$, if there is a total recursive function f such that for all x,

$$x \in A \text{ if and only if } f(x) \in B.$$

The m-degree of A is defined as the collection of sets which are m-equivalent to A. The 1-reduction (one-one reduction) we mentioned before is a special case for m-reduction. We have a complete description of the structure on r.e. m-degrees, including some local structural properties, such as $\mathbf{0}'_m$ does not split, and every m-degree below $\mathbf{0}'_m$ has a strong minimal cover, etc. The m-reduction has variations in the areas of numbering theory and recursively enumerable equivalence relations.

The *wtt*-reduction is a weak version of Turing reduction, with the feature of adaptivity, a crucial property of Turing reduction. Here, *wtt* is an abbreviation of weak-truth-table. Say that A is *wtt*-reducible reducible to B, written as $A \leq_{wtt} B$, if there is an oracle Turing machine Φ_e and a recursive function f such that $A = \Phi_e^B$, and that its use $\varphi_e(x) \leq f(x)$ for all x. That is, f is a recursive bound of the use function, which is the reason for many people to call the weak-truth-table reduction bounded-Turing reduction. Let $\{B : B \equiv_{wtt} A\}$ be the *wtt*-degree of A and a *wtt*-degree is r.e. if it contains an r.e. set. $\mathbf{0}_{wtt}$ and $\mathbf{0}'_{wtt}$ are the least and the greatest elements in the r.e. *wtt*-degrees. As the use is recursively bounded, many pathological properties of r.e. Turing degrees disappear when we consider r.e. *wtt*-degrees. Sasso and Ladner proved that density and splitting can be combined in the r.e. *wtt*-degrees, and that any nonzero r.e. *wtt*-degree has the anti-cupping property. Downey, et al., also proved that each incomplete r.e. *wtt*-degree branches.

As seen that $\mathbf{0}'$ is strictly above $\mathbf{0}$, Post raised a problem in the 1930s whether there are r.e. degrees different from $\mathbf{0}$ and $\mathbf{0}'$, which is known as Post's Problem nowadays. Note that if A and B are both r.e., then $A \oplus B$ is also r.e., and hence the r.e. degrees form a uppersemilattice. Kleene and Post [28] constructed a pair of incomparable degrees below $\mathbf{0}'$ by using the so-called finite extension argument, which requires $\mathbf{0}'$ as oracle to decide the outcome of the requirement at each stage. What Kleene and Post did was to decompose $A \not\leq_T B$ into infinitely many requirements, one for defeating a possible reduction, and the construction is run in infinitely many stages, and at each stage, one requirement is met by a further approximation extending the given approximation one, a computability-theoretic version of Cohen's forcing in set theory. The degrees constructed by Kleene and Post are not necessarily to be r.e., even though they are below $\mathbf{0}'$, as the construction is not recursive.

Based on the fact that the complement of any creative set contains an infinite r.e. set, Post tried to construct r.e. sets with complements fairly sparse to make sure that it does not contain any infinite r.e. set, and Post called these sets simple sets. But Post himself found that simple sets could be Turing complete and hence failed to provide a solution to his question. Post continued his program by sparsing the complement of r.e. sets even further to hyperimmume and hyperhyperimmune sets. People found that all these effects fail again, as there are such sets in the degree $\mathbf{0}'$.

Even though Post did not solve his problem by sparsing the complements of constructed sets further and further, these sets are not complete for strong reductions. That is, simple sets are not m-complete, and hypersimple sets are not tt-complete. In the 1970s, Ershov [15] introduced η-simple sets, η-hypersimple sets, η-hyperhypersimple sets, and this idea leads to a final solution of Post's problem, following Post's original idea of making their complements sparse.

Post's problem was eventually solved by Friedberg [18] and Muchnik [39] independently around 1956, where injury argument was invented and applied.

Theorem 1.1.1 (Friedberg [18], Muchnik [39])**.** *There exist r.e. sets A and B such that* $\deg(A)|\deg(B)$.

In the construction, Friedberg and Muchnik assigned priority to the list of requirements and guaranteed that for each requirement, there are only finitely many requirement with higher priority, which ensures that each requirement can be injured by those requirements with higher priority at most finitely often, and gets satisfied eventually. The name 'finite injury' comes from this key feature of the construction.

Sacks [42] provided a much stronger result by showing that any nonzero r.e. degree is the join of two smaller ones.

Theorem 1.1.2 (Sacks [42])**.** *Every nonrecursive r.e. sets A can split to two incomparable r.e. sets A_0 and A_1 such that $A = A_0 \cup A_1$ amd $A_0 \cap A_1 = \emptyset$.*

The construction is again a finite injury argument, but unlike the Friedberg-Muchnik's construction, the number of injuries of each requirement in Sacks splitting cannot be bounded by any recursive function. In this construction, Sacks invented a preservation technique, to show that a given nonrecursive r.e. set is not Turing reducible to those constructed by us, and it is this kind of strategy that makes the injuries not bounded by any recursive function. This technique was later developed and used by Sacks himself to show that the r.e. Turing degrees are dense in [43].

Theorem 1.1.3 (Sacks [43])**.** *Between any two r.e. degrees $\boldsymbol{a} < \boldsymbol{b}$, there exist a r.e. degree $\boldsymbol{c}$ such that $\boldsymbol{a} < \boldsymbol{c} < \boldsymbol{b}$.*

In the proof, Sacks introduced the so-called the Sacks coding strategy, a pioneering work for infinite injury argument, which enables us to work above a given incomplete r.e. degree. Even though the injuries from requirements with higher priority is infinite, and hence the restraints raised by these requirements could be infinite, it also provides a finite infimum, allowing infinitely many chances for the given requirement to be satisfied.

After seeing these two theorems of Sacks, people consider whether these two constructions can be combined together. Robinson [41] proved that it can be done if the smaller degree is low. However, Lachlan [34] proved his nonsplitting theorem that it is not true in general, where he constructed two r.e. degrees $\mathbf{a} > \mathbf{b}$ such that $\mathbf{a}$ cannot split above $\mathbf{b}$. In the construction of $\mathbf{a}$ and $\mathbf{b}$, Lachlan invented a $0'''$ injury argument, which has the feature that a strategy can be injured along the true path at most finitely many times. In the 1980s, Lachlan's argument is commonly called monstrous method.

1.2 R.e. degrees with/without infima

While the r.e. degrees form a uppersemilattice, they do not form a lattice, due to the existence of two r.e. degrees without infimum, which was proved by Yates in 1966 in his paper [49], where an exact pair is constructed. Jockusch provided a direct construction (finite injury) in [22], which leads to the discovery of strongly noncappable degrees of Ambos-Spies (in [1]) and Harrington (not published), an r.e. Turing degree with no infimum with any r.e. Tuiring incomparable with it.

Lachlan and Yates proved independently that $\mathbf{0}$ is the infimum of two nonzero r.e. Turing degrees, using quite different methods, and such a pair is called a minimal pair. Cooper proved in [6] that every high r.e. degree bounds a minimal pair, showing the analogy between high r.e. degrees and $\mathbf{0}'$. An r.e. degree is called cappable if it is either $\mathbf{0}$ or one part of a minimal pair. Obviously, cappable degrees are downward closed and every nonzero r.e. degree bounds a nonzero cappable degree. Ambos-Spies, et al. proved that all cappable degrees form an ideal of r.e. degrees and an r.e. degree is cappable if and only if it cannot be cupped to $\mathbf{0}'$ by low r.e. degrees if and only if it does not contain promptly simple sets. Soares book [46] and Coopers book [7] provide a comprehensive study of r.e. sets and degrees.

Definition 1.2.1. *An r.e. degree* $\boldsymbol{a}$ *is branching if* $\boldsymbol{a}$ *is the infimun of two incomparable r.e. degrees* $\boldsymbol{b}$ *and* $\boldsymbol{c}$*, and* $\boldsymbol{a}$ *is nonbranching otherwise.*

Obviously, $\mathbf{0}'$ is nonbranching. Lachlan [33] constructed a nonzero branching degree and an incomplete nonbranching degree. Then Lachlan and Soare [35] proved that in r.e degrees, the maximal element of a lattice M_5 is nonbranching, showing that the nondistributive lattice S_8 cannot be embeded into the r.e. degrees. Nonbranching degrees are dense in the r.e. degrees, by Fejer [17], where the nonbranching degree construction was combined with the density arguments. For branching degrees, Fejer [16] showed that there is a nonzero branching degree below any nonzero r.e. degree, and above any low r.e. degree. Slaman proved in [45] that branching degrees are dense in the r.e. degrees. In another word, nonbranching degrees are dense and co-dense in the r.e. degrees.

1.3 Generic degrees

Besides r.e. sets and r.e. degrees, we also consider generic sets and generic degrees, where the notion of genericity comes from set theory, originally developed by Cohen in his forcing construction of outer models when he proved that the Continuum Hypothesis is independent of ZFC.

Definition 1.3.1. *(1) A set $A \in 2^\omega$ forces $V_e \subseteq 2^{<\omega}$ if it satisfies the forcing requirement:*

$$(\exists \sigma \subset A)((\sigma \in V_e) \vee ((\forall \tau \supseteq \sigma)(\tau \notin V_e)).$$

(2) For $n \geq 1$, a set A is generic with respect to $\{V_e\}$ if A forces V_e for each e.

(3) For $n \geq 1$, a set A is n-generic ($n \geq 1$) if it is generic with respect to the collection of Σ_n^0 sets, and a degree $\mathbf{a}$ is n-generic if it contains an n-generic set.

In particular, when $n = 1$, A is 1-generic if it is generic with respect to the collection of r.e. sets $\{W_e : e \in \omega\}$. Note that the sets in Kleene-Post's construction are 1-generic sets.

In terms of Jockusch and Posner in [21] [40], A is n-generic if it meets or avoids every Σ_n^0 subset of $2^{<\omega}$.

Definition 1.3.2 (Jockusch [21], Posner [40])**.** *(1) A set A meets $V \subseteq 2^{<\omega}$ if there is a $\sigma \subset A$ such that $\sigma \in V$.*

(2) A set A avoids V if there is a $\sigma \subset A$ such that $\forall \tau \supseteq \sigma(\tau \notin V)$.

(3) A set A is n-generic if it meets or avoids every Σ_n^0 subset of $2^{<\omega}$.

In general, by applying finite extension method, we can construct n-generic sets below $0^{(n)}$, and construct arithmetically generic sets below $0^{(\omega)}$. n-generic sets are in GL_n, and each 2-generic degree bounds a degree in GL_2 but not in GL_1.

As Jockusch and Posner observed in [21] [40], a set is 1-generic if and only if it forces its jump, providing an equivalent definition of 1-genericity.

Definition 1.3.3 (Jockusch [21], Posner [40])**.** *Given e, σ and x, $\Phi_e^\sigma(x)$ is strongly undefined if $(\forall \tau \supseteq \sigma)(\Phi_e^\sigma(x) \uparrow)$. That is, the computation diverges on every possible extension of σ.*

Jockusch and Posner proved that A is 1-generic if for every e, there is $\sigma \subset A$ such that $\Phi_e^\sigma(e)$ is either defined or strongly undefined. A 1-generic degree below $\mathbf{0}'$ can be constructed by using oracle construction and full approximation argument. Moreover, every nonzero r.e. degree bounds a 1-generic degree. Jockusch proved in [21] that every 1-generic set A is r.e. in some $B <_T A$, though no 1-generic sets can be r.e.. $\mathbf{0}'$ is the join of two 1-generic degrees. In [48], Wu provided a stronger result that any nonzero r.e. degree is the join of two 1-generic degrees.

It is well-known that for $n \geq 2$, no n-generic degree bounds a minimal degree. Chong, Downey and Juckush investigated whether 1-generic degree can bound a minimal degree. Chong and Juckush first proved in [5] that no 1-generic degree below $\mathbf{0}'$ bounds a minimal degree. Haught strengthened this in [19] that 1-generic sets below $\mathbf{0}'$ is downwards closed, and a crucial part of the proof uses the so-called Shore's Lemma, asserting the existence of good approximations of 1-generic sets below $\mathbf{0}'$. After this, Chong and Downey proved in [4] that there is a 1-generic degree bounding a minimal degree, which was also obtained by Kumabe in his paper [29] independently. Their 1-generic degree bounding a minimal degree is not below $\mathbf{0}'$, but $\mathbf{0}''$.

A $\{0, 1\}$-valued total function f is diagonally nonrecursive (DNR for short), if $f(e) \neq \varphi_e(e)$ for all e. A well-known fact about DNR functions is Arslanov's completeness criterion: An r.e. set is Turing complete if and only if it computes a DNR function. The degrees of DNR functions below $\mathbf{0}'$ play an important role in the discovery of priority-free solution to Post's problem, as each DNR function recursive in $\mathbf{0}'$ computes a nonrecursive r.e. set.

For 1-generic sets, Demuth and Kučera proved in [9] that no 1-generic set can compute DNR functions. This implies that no 1-generic degree can compute PA degrees, since a degree is PA if and only if it computes a DNR function, and we can say that 1-generic sets are fairly weak in computing. The existence of hyperimmune-free degrees, which cannot be computed by any 1-generic degree, can be obtained as follows: by the hyperimmune-free basis theorem, there is a DNR function of hyperimmune-free degree, and this hyperimmune-free degree cannot be computed by any 1-generic degree. In contrast to this, by an arithmetical Sacks forcing argument, Downey and Yu proved in [14] the existence of 1-generic degrees bounding hyperimmune-free degrees, an analogy to Chong-Downey-Kumabe's result above for minimal degrees. We refer to [21] and [30] for more results.

Every $n+1$-generic set is also n-generic, but the converse may not be true. The proper hierarchy allows us to define properly n-genericity, that is, a set is properly n-generic if and only if it is n-generic and not $n+1$-generic. Csima, et al. [8] showed that every 1-generic set computes a properly 1-generic set, while for $m > n \geq 2$, an m-generic set cannot compute a properly n-generic real.

A related notion with properly n-genericity is weak n-genericity, which was first raised by Kurtz in [31], via dense subsets of $2^{<\omega}$. Here, $V \subseteq 2^{<\omega}$ is dense if for any $\sigma \in 2^{<\omega}$, there is some $\tau \supseteq \sigma$ such that $\tau \in V$. Then A is weakly n-generic it meets every dense Σ^0_n subset of $2^{<\omega}$. Kurtz pointed out in [31] that weak n-genericity is a notion weaker than n-genericity: every n-generic set is weakly n-generic, and every weakly $n+1$-generic set is n-generic, and that there are continuum many n-generic sets not weakly $(n+1)$-generic, and weakly $(n+1)$-generic sets but not $(n+1)$-generic. Kurtz also proved that a weakly $(n+1)$-generic degree is hyperimmune relative to $\mathbf{0^{(n)}}$, and a degree above $\mathbf{0}^{(n)}$ hyperimmune relative to $\mathbf{0^{(n)}}$ is the n-th jump of a weakly $(n+1)$-generic degree.

Several other notions of genericity are also considered. In [11, 12], Downey, Juckush and Stob introduced the *pb*-genericity in order to study multiple permitting arguments, which include simple permitting, high permitting and prompt permitting. Downey, Juckush and Stob proved that *pb*-genericity is strictly between 1-genericity and 2-genericity. It is also proved [11, 12] that *pb*-generic sets have array noncomputable degrees and that array noncomputable degrees bound *pb*-generic degrees.

Jockusch studied in [23] the e-genericity of r.e. sets, where finite injury priority arguments are applied in constructions. Jockusch proved that e-generic degrees are low, promptly simple, that there is an e-generic degree above any given low r.e. degree, and that every promptly simple degree bounds an e-generic degree. It is also known that every e-generic degree is nonbranching, and strongly noncappable. Following this, Ingrassia raised in his thesis [20] the notion of p-genericity as a variant of e-genericity.

1.4 Hyperimmune-free degrees

The notions of immunity, hyperimmunity, and hyperhyperimmunity were proposed by Post in the 1930s, when he tried to find a solution to find an r.e. degree strictly between $\mathbf{0}$ and $\mathbf{0}'$. He tried to construct 'simple set', which was in contrast to 'creative set', and then he noticed simple sets could be Turing complete. He then continued to construct hypersimple sets and hyperhypersimple sets. Even though those sets do not provide solutions to the original problem, they do provide solutions to the analogues of Post's problem for r.e. m-degrees and r.e. truth-table degrees.

Definition 1.4.1. *A set A is immune if it is infinite and does not contain any infinite r.e. set.*

A set is simple if it is r.e. and its complement is immune.

Note that a set A is immune if and only if it is infinite and $\bar{A} \cap rng(f) \neq \emptyset$ for any total recursive function f. Thus, for an immune set A, for any recursive function f, $A \cap \{f(x)\} = \emptyset$ for some x. This motivates Post to define hyperimmune sets by replacing singletons with strong array of finite sets.

Definition 1.4.2. *Let D_y denote a finite set $A = \{x_1, x_2, ..., x_k\}$ where $x_1 < x_2 < ... < x_k$, with the canonical index $y = 2^{x_1} + 2^{x_2} + ... + 2^{x_k}$. Let $D_0 = \emptyset$. A strong* *array of finite sets is a sequence* $\{D_{f(x)}\}_{x\in\omega}$ *for some recursive function* f*. An array is disjoint if its members are pairwise disjoint.*

Definition 1.4.3. *A set* A *is hyperimmune if it is infinite and for any disjoint strong array, there is some* x *s.t.* $A \cap D_{f(x)} = \emptyset$*. A degree is hyperimmune if it contains a hyperimmune set. A degree is hyperimmune-free iff it is not hyperimmune.*

Given two functions f and g, say that f majorizes g if $f(x) \geq g(x)$ for all x. Recall the principal function of a set $A = \{a_0 < a_1 < ...\}$ is denoted as p_A, where for each n, $p_A(n) = a_n$. Say that f majorizes A if f majorizes p_A. The following theorem gives an equivalent characterization of hyperimmunity in terms of majorization.

Theorem 1.4.4 (Kuznecov [32], Medvedev [38], Uspensky [47])**.** *An infinite set* A *is hyperimmune iff no recursive function* f *majorizes* A*.*

Definition 1.4.5. *A degree* $\mathbf{a}$ *is hyperimmune-free if for any* $A \in \mathbf{a}$ *and* $f \leq_T A$*,* f *is majorized by a recursive function.*

By this definition, $\mathbf{0}$ is hyperimmune-free and hyperimmune-free degrees are 'almost recursive'. A standard construction of a set with hyperimmune-free degree is to use the majorization property above and use $\mathbf{0}''$ as oracle in the construction. Martin and Miller [36] observed that if $\boldsymbol{a} < \boldsymbol{b} \leq \boldsymbol{a'}$ for some $\boldsymbol{a}$, then $\boldsymbol{b}$ is hyperimmune. Thus, $\mathbf{0} < \boldsymbol{b} \leq \mathbf{0'}$ is hyperimmune. By Friedberg jump inversion theorem, any $\boldsymbol{b} \geq \mathbf{0'}$ is hyperimmune. Hence, any nonzero degree comparable with $\mathbf{0}'$ is hyperimmune. In other words, if we construct a nonzero hyperimmune-free degree by using $\mathbf{0}''$ as oracle, then this degree is not below $\mathbf{0}'$, which provides an approach of constructing a degree below $\mathbf{0}''$, but not below $\mathbf{0}'$. Kurtz proved in [26] that a degree is weakly 1-generic if and only if it is hyperimmune, and that there is a 1-random set of hyperimmune-free degree. Jockusch and Soare proved in [25] the well-known hyperimmune-free basis theorem for degrees of members of Π^0_1 classes.

1.5 Π^0_1 classes

Π^0_1 classes is another central topic in the development of computability theory, a topic was first studied by Jockusch and Soare in [24, 25] in 1970s. Π^0_1 classes arise naturally in many areas of logic, like theories of Peano Arithmetic, and has many applications in algorithmic randomness and reverse mathematics. For example, the separating class for two disjoint r.e. sets is a Π^0_1 class, where the separating class is the collection of sets containing one set but disjoint with the other. Another example is the class of all complete extensions of Peano Arithmetic.

Definition 1.5.1. *A tree $T \subseteq 2^{<\omega}$ is a subset closed under initial segments. A path through T is an infinite sequence $A \in 2^\omega$ such that all initial segment of A is in T. $[T]$ is the set of all paths through T. We say $[T]$ is the body of T.*

A subset $P \subseteq 2^\omega$ is a $\boldsymbol{\Pi^0_1}$ class if P is the collection of paths $[T]$ through a (primitive) recursive tree $T \subseteq 2^{<\omega}$.

The notion of "Π^0_1" classes comes from the fact that whether an element $A \in P$, i.e., whether $(\forall n)(A \upharpoonright n \in T)$, is a Π^0_1 relation. In terms of effective topology on the Cantor space 2^ω, Π^0_1 classes are effectively closed sets. A function is primitive recursive if it is obtained from the basic functions by closure under composition and primitive recursion, and a tree is primitive recursive if its characteristic function is primitive recursive. As we can enumerate primitive recursive trees effectively, by Definition 1.5.1, we have an effective enumeration of all Π^0_1 classes. Π^0_1 classes in this thesis always refer to Π^0_1 classes in 2^ω, though Π^0_1 class can be defined in Baire Space.

One topic in Π^0_1 classes is to consider Turing degrees of members in Π^0_1 classes. It is well-known that in a nonempty Π^0_1 class, the leftmost member is of r.e. degree, and those isolated elements are computable (so all members in a finite Π^0_1 class are computable). As a consequence, a Π^0_1 class with no computable members has cardinality $2^{\aleph_0}$. We call a Π^0_1 class special if it does not have any computable member.

To find elements in a given Π^0_1 class with some property, we usually apply forcing on recursive trees. That is, we start with some recursive tree and prune it to a recursive subtree to force a requirement to be satisfied. Many basis theorems are all proved in this manner.

Say that a collection of degrees is a basis for Π^0_1 classes if every nonempty Π^0_1 class has an element in the collection. In [25], Jockusch and Soare proved the low (superlow) basis theorem: every nonempty Π^0_1 class contains a member of low (superlow) degree, and the hyperimmune-free basis theorem: every nonempty Π^0_1 class has an element of hyperimmune-free degree. They also proved that the upper cone avoidance basis theorem: for any nonrecursive set C, every nonempty Π^0_1 class contains a member which cannot compute C, and the lower cone avoidance basis theorem: for any C, every special Π^0_1 class contains a member cannot be computed from C. Together with the proof of low basis theorem, we can prove that for given noncomputable sets C_j for $j \in \omega$, every special Π^0_1 class contains a member whose degree is incomparable with each C_j, the so-called incomparability basis theorem. This leads to the minimal pair basis theorem: every special Π^0_1 class has a minimal pair. Scott proved in [44] a crucial result that the sets computable from a PA degree form a basis for Π^0_1 classes. Cenzer [2] proved the proper low_{n+1} basis theorem: every special Π^0_1 class contains an element which is low_{n+1} but not low_n.

In contrast to these basis theorems, there are also nonbasis theorems and antibasis theorems. A collection of sets is a nonbasis if there is some nonempty Π^0_1 class has an element not in the collection. For example, the collection of recursive sets is a nonbasis, due to the existence of recursively inseparable pair. Jockusch and Soare [24] showed that the collection of incomplete r.e. sets is a nonbasis, by constructing a Π_1^0 class such that if members with r.e. degrees can only have degree $\mathbf{0}'$.

An antibasis is a collection of sets such that if a nonempty Π_1^0 class has an element in an antibasis, then it has an element of any given degree. Kent and Lewis proved in [27] the low antibasis theorem: a Π_1^0 class containing members of all nonzero low degrees has members of all degrees.

In this thesis, we will study thin Π_1^0 classes: a Π_1^0 class P is thin if every Π_1^0 subclass Q of P is the intersection of P with some clopen set. Historically, thin classes were first constructed by Martin and Pour-El in their paper [37], when they constructed an axiomatizable essentially undecidable theory such that any axiomatizable extension of it is a finite extension, and the concept of thin Π_1^0 class was first raised explicitly by Downey in his PhD thesis [10]. In all Π_1^0 class, isolated elements are recursive, but not vice versa. While in thin Π_1^0 classes, recursive elements are all isolated, and hence, perfect thin Π_1^0 classes contain no recursive element.

The Cantor-Bendixson (CB for short) derivative of a closed set $C \subset 2^\omega$ is the collection of limit points in C, denoted as $D(C)$. $D(C)$ is a closed subset of C. The iterated CB derivative $D^\alpha(C)$ is defined for all ordinals α by transfinite induction. The CB rank of a closed set C is the least ordinal α such that $D^{\alpha+1}(C) = D^\alpha(C)$. The perfect set theorem states that any closed subset is the union of a perfect closed set (the perfect kernel) and a countable set. The perfect kernel of C is got by iterated CB derivative until reaching the CB rank of C. As C has at most $2^{\aleph_0}$ elements, the CB-rank of C is less than $2^{\aleph_0}$. Specially, the CB rank of a countable closed set is the least ordinal α such that the $(\alpha+1)$-st derivative is empty. Any countable closed set has countable rank and has an isolated element. The CB rank of an element in C is the least ordinal α such that the element is in the α-th derivative of the closed set but not in the $(\alpha+1)$-st derivative if α exists.

Cenzer, et al. [3] studies the degree structure of members of countable thin Π^0_1 classes. They constructed countable thin Π^0_1 classes with members of any given CB rank. They also constructed a thin Π^0_1 class whose derivative is a given set with degree $0'$, and that for any two comparable r.e. degrees, there exists a thin CB rank one degree between them. On the other hand, they were able to construct an r.e. degree whose element cannot be a member of any thin Π^0_1 class, and we call such degrees thin-free degrees. Downey, Wu and Yang proved in [13] that the r.e. thin-free degrees are both dense and co-dense in r.e. degrees, providing another example of natural classes both dense and co-dense in the r.e. degrees.

1.6 Outline

In Chapter 1, we provide a brief overview of computability theory, including definitions and history needed in this thesis. In the Chapter 2, we first construct a thin-free degree below $\mathbf{0}''$, pointing out the key feature to obtain thin-freeness. Then we construct a hyperimmune-free minimal thin-free degree below $\mathbf{0}''$. Our thin-free degree is not below $\mathbf{0}'$, different from the one constructed by Cenzer, et al.'s in [3]. In Chapter 3, we consider the relationship between genericity and thin-free degrees. We show that 1-generic degrees below $\mathbf{0}'$ cannot be thin-free. Besides $\mathbf{0}$, we provide other degrees which are not thin-free. Afterwards, we prove that all 2-generic degrees are thin-free. Those 2-generic thin-free degrees are not below $\mathbf{0}'$. In chapter 4, we describe the methods of Cenzer, et al. [3] for the construction of thin-free degrees below $\mathbf{0}'$, especially by full approximation, and then construct an r.e. thin-free nonbranching degree.

Chapter 2

A Hyperimmune-free Minimal Thin-free Degree

The purpose of this chapter is to introduce the method of constructing thin-free degrees. We first describe how to construct thin-free degrees by forcing argument on recursive trees with $\mathbf{0''}$ as oracle, which enables the construction of hyperimmune-free minimal thin-free degrees below $\mathbf{0''}$.

Recall that a set A has thin-free degree if any set Turing equivalent to A is not in any thin Π^0_1 class. Note that $\mathbf{0}$ is not thin-free, as any recursive set A is in a thin Π^0_1 class $\{A\}$. Furthermore, Cenzer, et al. [3] showed that if A is a menber of a thin Π^0_1 class, then $A' \leq A \oplus \emptyset''$. Hence, degrees above $\mathbf{0''}$ are not thin-free.

In [3], Cenzer, et al. worked below $\mathbf{0'}$, constructing thin-free degrees below $\mathbf{0'}$, which can be minimal or r.e.. In this chapter, we work below $\mathbf{0''}$. Note that the thin-free degree constructed in this chapter cannot be below $\mathbf{0'}$ as it is hyperimmune-free, so it is different from the thin-free degrees constructed by Cenzer, et al..

2.1 A Thin-free degree below $\mathbf{0}''$

To show that there is a thin-free degree, we shall construct a set A such that for any $B \equiv_T A$, B is not in any thin Π^0_1 class. A is nonrecursive, as shown above. Let (Φ_e, P_e) be an effective list of tuples where for each e, Φ_e is a $\{0,1\}$-valued partial recursive functional, and P_e is a primitive recursive tree. To make A thin-free, we shall ensure for any e such that if Φ^A_e is total and Turing equivalent to A, then one of the following is guaranteed:

(1) $\Phi^A_e \notin [P_e]$, or

(2) $[P_e]$ is not thin.

Suppose that A is constructed on a total recursive perfect tree T. The idea to meet (1) is to find some string τ on T such that Φ^τ_e is not extendible on P_e, and if such a τ exists, we force A to extend τ, which will guarantee that $\Phi^A_e \supseteq \Phi^\tau_e$, if Φ^A_e is total, and hence $\Phi^A_e \notin [P_e]$, as no infinite path on P_e extends Φ^τ_e. If there is no such a τ, we will then need to find a Π^0_1 subclass of $[P_e]$ which is not the intersection of $[P_e]$ with any clopen set. With this in mind, we construct a recursive subtree S_e of P_e, such that Φ^A_e lies on S_e, and for any length l, there exists some $B \in [S_e]$ and $C \in [P_e] \backslash [S_e]$ such that $B \upharpoonright l = C \upharpoonright l = \Phi^A_e \upharpoonright l$ (refer to Figure 2.1). This implies that $\Phi^A_e \in [S_e]$. We claim that $[S_e]$ is not the intersection of $[P_e]$ with any clopen set. Otherwise, suppose that $[S_e] = [P_e] \cap U$ for some clopen U, where $U = \bigcup_{0 \le i \le n} N_{\beta_i}$, $N_{\beta_i} = \{X \supset \beta_i : X \in 2^\omega\}$, the cone above β_i. Then

$$[S_e] = [P_e] \cap U = [P_e] \cap (\bigcup_{0 \le i \le n} N_{\beta_i}) = \bigcup_{0 \le i \le n} ([P_e] \cap N_{\beta_i}),$$

and hence, $[P_e] \cap N_{\beta_i} \subseteq [S_e]$. As $\Phi^A_e \in [S_e]$, Φ^A_e is in $[P_e] \cap N_{\beta_i}$ for some i, which implies that Φ^A_e extends β_j. By the construction of S_e, we have some C in $[P_e]$ extending β_j and $C \notin [S_e]$, a contradiction. In this case, what we will do is to force A on a total recursive subtree T_e of T, such that for any α, $\Phi^{T_e(\alpha 0)(\alpha 1)}_e$ are incompatible in P_e (as we are assuming that our attempt to satisfy (1) fails, both $\Phi_e^{T_e(\alpha 0)(\alpha 1)}$ are extendible on P_e and thus there is at least one infinite path through P_e extending it), and there is a path on P_e extending $\Phi_e^{T_e(\alpha)}$ but not $\Phi_e^{T_e(\alpha 0)(\alpha 1)}$. The even subtree of the e-splitting subtree of T, if exists, is such a candidate for T_e. Suppose that the e-splitting subtree of T, $SP(T, e)$, exists, and Φ_e^A is total, then the even subtree of $SP(T, e)$, $E(SP(T, e))$, is a total recursive subtree of T, and $\Phi_e^{E(SP(T,e))}$ is a total recursive subtree of P_e witnessing that $[P_e]$ is not thin. If $SP(T, e)$ does not exist, then there is a string $T(\alpha)$ such that above $T(\alpha)$, no string e-splits, and hence, if A is on the full subtree of T above α, $Full(T, \alpha)$, then Φ_e^A is recursive, and hence making A and Φ_e^A not Turing equivalent.

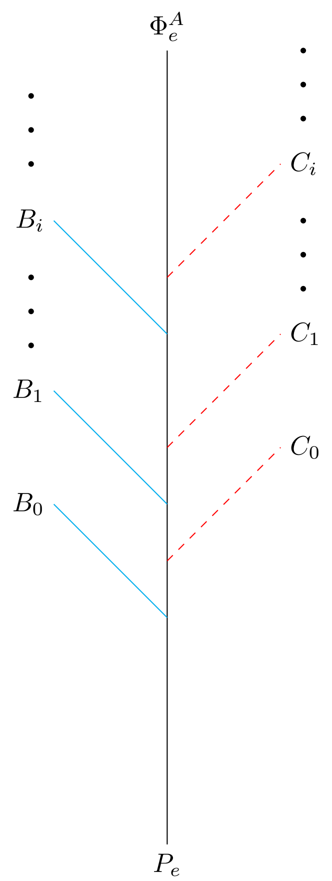

Figure 2.1: If such a S_e exists, it ensures infinitely many paths on P_e along Φ_e^A, say B_i, C_i, and $L(B_i, \Phi_e^A) < L(C_i, \Phi_e^A) < L(B_{i+1}, \Phi_e^A)$ for all i, where $L()$ is the length of common initial segment of two infinite sequence.

For convenience, we define the following notations.

Definition 2.1.1. *For σ on T,*

(1) σ is e-extendible, if for any $l > |\Phi_e^\sigma|$, there is a string τ on P_e of length l with $\tau \supseteq \Phi_e^\sigma$.

(2) σ is e-splittable on T, if there are two strings τ and ρ on T extending σ, and some x such that $\Phi_e^\tau(x)\downarrow$, $\Phi_e^\rho(x)\downarrow$ and $\Phi_e^\tau(x) \neq \Phi_e^\rho(x)$. The pair (τ, ρ) is called an e-splitting extension of σ on T, and τ, ρ e-splits σ.

A partial recursive tree T is an e-splitting tree if whenever $T(\alpha^\frown 0)(\alpha^\frown 1)\downarrow$, then $T(\alpha^\frown 0)(\alpha^\frown 1)$ is an e-splitting extension of $T(\alpha)$.

For a recursive tree T, we define:

- $Full(T, \alpha)$, the full subtree of T above $T(\alpha)$. That is, $Full(T, \alpha)(\beta) = T(\alpha^\frown\beta)$ for any β.
- $E(T)$, the even subtree of T. That is, $E(T)(\emptyset) = T(\emptyset)$, and for any $E(T)(\alpha) = T(\beta)$, $E(T)(\alpha 0)(\alpha 1) = T(\beta 00)(\beta 01)$.
- $SP(T, e)$, the e-splitting tree of T above $T(\alpha)$. That is, $SP(T, e)(\emptyset) = T(\alpha)$, and for any $SP(T, e)(\beta)\downarrow$, $SP(T, e)(\beta 0)(\beta 1)$ is defined to be the first e-splitting extension above $SP(T, e)(\beta)$ on T, if exists, or undefined otherwise.

Suppose a total recursive tree T is given, T_e is defined as follows:

1. Test whether all strings in T are e-extendible. That is, ask whether for all α on T, and for all $l > |\Phi_e^{T(\alpha)}|$, there exists $\tau \in P_e \cap 2^l$ such that $\tau \supseteq \Phi_e^{T(\alpha)}$.

   This is a Π_1^0 question, and is recursive in $\mathbf{0'}$ (hence, recursive in $\mathbf{0''}$).

   (a) If the answer is NO, i.e., there exists some string α on T not e-extendible, we let $T' = Full(T, \alpha)$ for the least string $T(\alpha)$ not e-extendible.

In this case, any path A on T' extends $T(\alpha)$ and thus $\Phi_e^A \notin [P_e]$, (1) is already met.

(b) If all strings on T are e-extendible, let $T' = T$ for further pruning.

2. Test whether there exists some β, such that for all $\gamma_0, \gamma_1 \supseteq \beta$ and x, $\Phi_e^{T'(\gamma_0)}(x)\uparrow$, $\Phi_e^{T'(\gamma_1)}(x)\uparrow$ or $\Phi_e^{T'(\gamma_0)}(x)\downarrow = \Phi_e^{T'(\gamma_1)}(x)\downarrow$.

   This is Σ_2^0 question and is recursive in $\mathbf{0''}$.

   - If there is no such a β, which means that above any $T'(\beta)$, there exists some e-splitting extension on T'. In this case, $SP(T', e)$ is a total recursive tree.
     - If T' is from case (b) above, then $\Phi_e^{SP(T',e)}$ is a recursive extendible subtree of P_e, and there are infinitely many branches along any path on $\Phi_e^{SP(T',e)}$.

       In this case, we let $T_e = E(SP(T', e))$, and $\Phi_e^{T_e}$ is the S_e we need to show that $[P_e]$ is not thin, (2) is met.

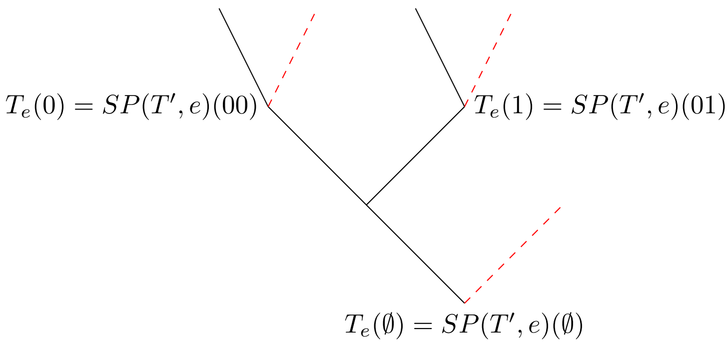

Figure 2.2: $T_e = E(SP(T', e))$.

     - If T' is from case (a) above, T_e is a subtree of T', and for any $A \in [T_e]$, A lies on T', (1) is met.
   - If there is some such β, let $T_e = Full(T', \beta)$ for the least such β. In this case, A is forced to be above a string not e-splittable, which implies

that Φ_e^A is recursive and thus not Turing equivalent to A. Recall that the thin-freeness of A implies that A is not recursive.

In summary, we first try to find some non-e-extendible string α on T. If such a string α exists, then we force A to extend the least such α, making Φ_e^A not in P_e, and the requirement is satisfied, no matter how the tree is pruned at step 2. If there is no such α, we try to find some string β not e-splittable on T, and force A to the least such β, if β exists. If there is no such β, we then know enough information about the structure of P_e by enumerating strings τ on T and compute Φ_e^τ, and the $\Phi_e^{E(SP(T,e))}$ is the recursive subtree we need.

Now we give the basic construction of a thin-free degree.

Theorem 2.1.2. *There is a nonzero thin-free degree* $\boldsymbol{a} \leq \mathbf{0}''$.

Proof. To construct a nonrecursive set A of thin-free degree, we shall meet for each e the following requirement:

$$Q_e : \text{If } \Phi_e^A \text{ total and Turing equivalent to } A, \text{ then either } \Phi_e^A \in [P_e] \text{ or } [P_e] \text{ is not thin.}$$

Construction: Let $T_{-1} = 2^{<\omega}$.

Stage s: Given a total recursive tree T_{s-1}.

1: Ask whether for all α and for all $l > |\Phi_s^{T(\alpha)}|$, there exists some $\tau \in P_s$ and $|\tau| = l$, such that $\tau \supseteq \Phi_s^{T(\alpha)}$.

If NO, let $T'_s = Full(T, \alpha)$ for the least α such that $T(\alpha)$ is non-e-extendible.

If YES, let $T'_s = T_s$.

2: Ask whether for all β on T'_s, there exists $\gamma_0, \gamma_1 \supseteq \beta$ and x, such that $\Phi_e^{T'_s(\gamma_0)}(x) \downarrow \neq \Phi_e^{T'_s(\gamma_1)}(x) \downarrow$.

If YES, let $T''_s = E(SP(T'_s, s))$,

If NO, let $T''_s = Full(T'_s, \beta)$ for the least such β.

3: Let $T_s = Full(T''_s, 0)$.

At the end of the construction, let $A = \bigcup_s T_s(\emptyset)$.

Verification: We first check that T_e is a total recursive tree for any e. Note that $T_{-1} = 2^{<\omega}$ is total recursive. Suppose T_{e-1} is given as a total recursive tree. We obtain T_e from T_{e-1} via three steps: At step 1, T'_e is defined as either T_{e-1} or the cone above a certain $T_{e-1}(\alpha)$. In either case, T'_e is a total recursive tree.

At step 2, T''_e is defined as a full subtree of T'_e above some $T'_e(\beta)$, or as the even subtree of the e-splitting subtree of T'_e, $E(SP(T'_e, e))$. As the e-splitting subtree of a total recursive tree is also total recursive, and the even subtree of a total recursive tree is total recursive, we know that T''_e is total recursive.

At step 3, we let T_e as the full subtree above $T''_e(0)$, which is again a total recursive tree.

This shows that all trees T_e, $e \in \omega$, are total recursive.

For all e, $T_e(\emptyset) \subset T_{e+1}(\emptyset)$ by step 3, and $T_e \supseteq T_{e+1}$. Since $A = \bigcup_e T_e(\emptyset)$, A lies on T_e for all e.

$A \leq_T \mathbf{0''}$ since the construction is recursive in $\mathbf{0''}$.

Finally, we prove that for any e, the requirement Q_e is met.

- If at step 1, T'_e is defined by the first case, then T_e is a subtree of T'_e, $T_e(\emptyset)$ is not e-extendible. If Φ_e^A is total, then Φ_e^A is not on P_e, by $A \supset T_e(\emptyset)$.
- If at step 2, T''_e is defined by the second case, and T_e is defined as a full subtree of T'_e, then $T_e(\emptyset)$ is non-e-splittable. Thus, T_e is a tree without e-splitting, and Φ_e^A is recursive if it is total, and hence not Turing equivalent to A as A is nonrecursive, a property guaranteed by the thin-freeness of A.

- If T''_e is defined by the second case in step 2, and T_e is defined as the even subtree of the e-splitting tree of T'_e, then let $S_e = \Phi_e^{T_e}$. That is, let $S_e(\alpha) = \Phi_e^{T_e(\alpha)}$ for all α. Since T_e is a total recursive tree, so is S_e. In this case, S_e is a subtree of P_e, and all strings on S_e are extendible on P_e. For any path A on T_e, Φ_e^A lies on S_e. For any l, suppose $T_e(\alpha) \subseteq A$ and $|S_e(\alpha)| > l$. For that α, suppose $T_e(\alpha) = SP(T'_e, e)(\beta)$, and $T_e(\alpha 0)(\alpha 1)$ extend $SP(T'_e, e)(\beta i)$ for some $i = 0, 1$. In this case, $\Phi_e^{SP(T'_e,e)(\beta(1-i))}$ is also extendible on P_e but not on S_e, which implies that there is an infinite path through P_e extending $S_e(\alpha)$ but not $S_e(\alpha 0)(\alpha 1)$. Suppose $T_e(\alpha j) \subseteq A$ for some $j = 0, 1$. Then there are two infinite paths $B \supseteq \Phi_e^{SP(T'_e,e)(\beta(1-i))}$ and $C \supseteq S_e(\alpha(1-j))$, where B lies on P_e but not on S_e, C lies on S_e, and $B \upharpoonright l = C \upharpoonright l = \Phi_e^A \upharpoonright l$. Then we know that $[S_e]$ is a Π_1^0 subclass of $[P_e]$ and is not the intersection of $[P_e]$ with any clopen set. So if Φ_e^A is total and Turing equivalent to A, then $[P_e]$ is not thin.

Hence, for any e, if Φ_e^A is total and $\Phi_e^A \equiv_T A$, then either $\Phi_e^A \notin [P_e]$ or $[P_e]$ is not thin. This shows that A has thin-free degree. □

2.2 A Thin-free Hyperimmune-free degree

Note that the strategy of obtaining a subclass witnessing that $[P_e]$ is not thin uses $\mathbf{0}''$ to tell whether a total recursive tree is e-extendible and e-splittable. In particular, if a tree T is e-splittable, $E(SP(T, e))$, the even subtree of $SP(T, e)$, is also an e-splitting tree. This shows that the thin-free degree we construct is minimal.

Corollary 2.2.1. *There is a minimal thin-free degree $\boldsymbol{a}$ below $\mathbf{0}''$.*

Proof. Construct A as in Theorem 2.1.2. To show that A has minimal degree, it suffices to prove that for any e, A lies on a recursive tree which is either e-splitting or without e-splitting.

For any e, there is some j such that $\Phi_j = \Phi_e$ and $P_j = 2^{<\omega}$. Then Φ_j^τ are j-extendible for all τ if converges. So at stage j, T_j is defined as either a full subtree above some non-j-splittable string, or the even subtree of the j-splitting tree, which is still j-splitting. That is, A lies on a total recursive trees which is either e-splitting or without e-splitting for each e. □

We now construct a hyperimmune-free thin-free degree, by combining the basic hyperimmune-free degree construction with the construction above using $\mathbf{0}''$ as oracle.

First we review how Martin and Miller [36] constructed a nonzero hyperimmune-free set A with $\mathbf{0''}$ as oracle by forcing. They characterized the hyperimmune-free degrees by using majorizing functions. That is, a degree $\boldsymbol{a}$ is hyperimmune-free if and only if for any $A \in \mathbf{a}$ and $f \leq_T A$, f is majorized by a recursive function. To make A hyperimmune-free, A is constructed via a sequence of nested total recursive trees $\{T_e : e \in \omega\}$, such that for each T_e, either (1) Φ_e^B is total for all $B \in [T_e]$, or (2) Φ_e^B is NOT total for all $B \in [T_e]$. Furthermore, if T_e satisfying (1), in order to find a recursive function majorizing Φ_e^A, T_e is required such that for any n, $\Phi_e^{T_e(\alpha)}(n)\downarrow$ for any $|\alpha| = n$. In this way, the value of $\Phi_e^A(n)$ must be one of these 2^n values.

Now we describe how to construct such trees. Given a total recursive tree T, we ask whether there exist some α and n such that for all $\beta \supseteq \alpha$, $\Phi_e^{T(\beta)}(n)\uparrow$. This is a Σ_2^0 question and is recursive in $\mathbf{0''}$.

If the answer is YES, we force A to extend the least such string $T(\alpha)$ by letting $T_e = Full(T, \alpha)$, which ensures that $A \supset T(\alpha)$ for any A on T_e. In this case, $\Phi_e^A(n)\uparrow$ and Φ_e^A is not total.

If the answer is NO, then for any $T(\alpha)$ and any n, there is some $\beta \supset \alpha$ such that $\Phi_s^{T(\beta)}(n)\downarrow$. In this case, we define T_e as follows. For convenience, the tree constructed in this procedure is denoted as $M(T,e)$. Let $M(T,e)(\emptyset) = T(\alpha)$ for the least α such that $\Phi_e^{T(\alpha)}(0)\downarrow$. For any $M(T,e)(\gamma)\downarrow$, suppose $M(T,e)(\gamma) = T(\beta)$, $M(T,e)(\gamma 0)(\gamma 1) = T(\tau)(\rho)$ for the least incompatible pair $\tau, \rho \supset \beta$ such that $\Phi_s^{T(\tau)}(|\gamma|+1)\downarrow$ and $\Phi_s^{T(\rho)}(|\gamma|+1)\downarrow$. Such a pair exists and can be found recursively by enumerating strings on T. $M(T,e)$ is a recursive subtree of T such that Φ_e^A is total for all paths A through it, and for any n, $\Phi_e^{M(T,e)(\gamma)}(n)\downarrow$ for any γ with length n. Furthermore, for any A lies on $M(T,e)$,

$$g(n) = \max\left\{\Phi_e^{M(T,e)(\gamma)}(n) : |\gamma| = n\right\}$$

is a recursive function, with $\Phi_e^A(n) \leq g(n)$ for all n.

Thus, in the construction of a set with hyperimmune-free degree, at stage s, given T_{s-1} (we let $T_{-1} = 2^{<\omega}$), we ask whether there exists some α and n such that for all $\gamma \supseteq \alpha$, $\Phi_e^{T_{s-1}(\gamma)}(n)\uparrow$. If YES, let $T'_s = Full(T_{s-1}, \alpha)$ for the least such α. Otherwise, $T'_s = M(T_{s-1}, s)$. In both cases, we let $T_s(\sigma) = T'_s(0\sigma)$ to get T_s. We let $A = \bigcup_s T_s(\emptyset)$.

By induction on e, we can show that each T_e is a total recursive tree. For any e, T_{e+1} is a subtree of T_e with $T_{e+1}(\emptyset) \supset T_e(\emptyset)$, and A lies on all T_e.

$A \leq_T \mathbf{0''}$ because the construction is recursive in $\mathbf{0''}$. A has hyperimmune-free degree, because for any e, if T_e is defined as a full subtree, then there is some n such that for all α, $\Phi_e^{T_e(\alpha)}(n)\uparrow$, which implies that Φ_s^A is not total for all A on T_e. Otherwise, we define a recursive function $g(n) = \max\left\{\Phi_e^{T_e(\alpha)}(n) : |\alpha| = n\right\}$ by the total recursive tree T_e, and $\Phi_e^A(n) \leq g(n)$ for all $A \in [T_e]$ and n.

Below we present a construction of a hyperimmune-free thin-free degree. We shall construct a set A which is thin-free and hyperimmune-free by satisfying for each e the requirements:

Q_e : if Φ_e^A total, Turing equivalent to A, and $\Phi_e^A \in [P_e]$, then $[P_e]$ is not thin.

R_e : if Φ_e^A total, then there is a recursive function g majorizing Φ_e^A, i.e.,

$g(n) \geq \Phi_e^A(n)$ for all n.

The basic idea to meet each single requirement is the same as shown above, and now we put them together. That is, given a total recursive tree T. At step 1, we ask whether there exists a non-e-extendible string on T. If YES, we define $T' = Full(T, \alpha)$ for the least such α and force A on T'. Otherwise, let $T' = T$. At step 2, we ask whether there is some string β on T' above which there is not e-splitting at all. If YES, we force A to the least such string, say $T'(\beta)$, and define $T'' = Full(T', \beta)$. Otherwise, we force A on $T'' = E(SP(T'', e))$. At step 3, we ask whether there is some σ and n such that $\Phi_e^{T''(\tau)}(n) \uparrow$ for all $\tau \supseteq \sigma$. If YES, we define $T_e = Full(T'', \sigma)$ for the least such σ, otherwise, let $T_e = M(T'', e)$, by which we can define a recursive majorizing function.

Note that the question at each step is recursive in $\mathbf{0''}$, and that all trees constructed above are total recursive. It is easy to see that each requirement R_e is satisfied.

For Q_e, if T' is defined by the first case at step 1, then for all paths A on T_e, if Φ_e^A is total, then it is not in $[P_e]$, and Q_e is met.

If T'' is defined by the first case at step 2, then T'' is a tree with no e-splitting, and so is T_e. Then Φ_e^A is recursive for all paths A on T_e, if Φ_e^A is total, and Q_e is met as A is not recursive.

If $T'' = E(SP(T', e))$, then as before, $S_e = \Phi_e^{T_e}$ is a subtree of P_e, and $[S_e]$, as a subclass of $[P_e]$, shows that $[P_e]$ is not thin.

Also as explained above, we can combine the construction above with the minimal degree construction, and we thus have the following theorem:

Theorem 2.2.2. *There is a hyperimmune-free, minimal, thin-free degree* $\boldsymbol{a}$ *below* $\mathbf{0}''$.

This thin-free degree is not below $\mathbf{0}'$ as it is hyperimmune-free.

Chapter 3

Genericity and Thin-free Degrees

We have seen that the construction of thin-free degrees below $\mathbf{0}''$ uses the e-splitting trees, which enables us to include and exclude infinitely paths of a given Π^0_1 class $[P_e]$, if it admits an e-splitting subtree. So from this nature, we can see that the construction of minimal degrees can be modified to obtain thin-free degrees. In contrast to this feature, we consider the relation between genericity and thin-free degrees. We will show that all 1-generic sets below $\mathbf{0}'$ are not thin-free, and that any nonrecursive set below a 2-generic degree is thin-free. In particular, all 2-generic sets are of thin-free degree.

As every 1-generic set A is GL_1, i.e., $A' \leq_T A \oplus \emptyset'$, 1-generic sets below $\mathbf{0}'$ are actually low. Moreover, 1-generic sets are nonrecursive, so our result that all 1-generic sets below $\mathbf{0}'$ are not thin-free provides some nonzero degrees below $\mathbf{0}'$ which are not thin-free. They are quite "close to" $\mathbf{0}$.

We know that every 2-generic degree bounds a nonzero degree which is not 1-generic from Jockusch's survey paper [21]. Together with the fact that 2-generic sets are also 1-generic and Haught's result in [19] that 1-generic sets below $\mathbf{0}'$ are downward closed, we see that no 2-generic degree is below $\mathbf{0}'$. So our result that all 2-generic degrees are thin-free makes further efforts in finding different thin-free degrees form Cenzer, et al.'s work in [3].

Recall that a set A is n-generic if it meets or avoids every Σ^0_n subset of $2^{<\omega}$.

3.1 1-generic sets below $\mathbf{0}'$ are not thin-free

In this section, we will show that any 1-generic degree below $\mathbf{0'}$ cannot be thin-free. Let A be a 1-generic set reducible to $\mathbf{0'}$. To show a set A is not thin-free, it suffices to establish a recursive thin tree T containing a path C which is Turing equivalent to A.

Shoenfield's limit lemma says that any set reducible to $\mathbf{0'}$ admits a Δ^0_2 approximation. For 1-generic sets reducible to $\mathbf{0}'$, they admit Δ^0_2 approximations with an extra property, the so-called Σ_1-correctness, which was due to Shore, and was used by Haught [19] to show that 1-generic degrees below $\mathbf{0}'$ are downwards closed. For completeness of this thesis, we present a proof of Shore's Lemma.

Lemma 3.1.1 (Shore). *Any recursive approximation of a 1-generic set $A <_T \mathbf{0'}$ has a Σ_1-correct approximation, where a recursive approximation $\{\sigma_s : s \in \omega\}$ of A is Σ_1-correct if for any infinite r.e. set S of natural numbers, there exists some $s \in S$ such that $\sigma_s \subset A$.*

Proof. As $A <_T \mathbf{0'}$, by Shoenfield's limit lemma, we can have a recursive approximation of A, $\{\sigma_s\}$ say. For any infinite r.e. set $S \subseteq \omega$, we define a set of strings $V = \{\sigma_s : s \in S\}$, which is r.e.. By 1-genericity, A either meets or avoids V. A cannot avoid V because for any initial segment σ of A, as $\{\sigma_s\}$ approximates A, there exists some s such that for all $t > s$, $\sigma_t \supseteq \sigma$. Thus, A meets V, i.e., there is a $\sigma \subset A$ such that $\sigma \in V$, which implies the existence of $s \in S$ with $\sigma_s \subset A$. □

In addition, we can actually have a Σ_1-correct approximation $\{\sigma_s\}$ of A satisfying that $|\sigma_{s+1}| > |\sigma_s|$ for all $s \in \omega$. To see this, for any Σ_1-correct approximation

$\{\alpha_s\}$ of A, we define a function $f : \omega \to \omega$ inductively by taking $f(0) = 0$ and $f(s+1) = \mu t > f(s)(|\alpha_t| > |\alpha_{f(s)}|)$. Such a t exists because $\{\alpha_s\}$ is a recursive approximation. Note that f is recursive and increasing. Let $\sigma_s = \alpha_{f(s)}$, then $\{\sigma_s\}$ is also a recursive approximation of A. For any infinite r.e. set S, $V = \{f(s) : s \in S\}$ is r.e. and infinite. Since $\{\alpha_s\}$ is Σ_1-correct, there is some $f(s) \in V$ such that $\alpha_{f(s)} \subset A$, which implies that $\sigma_s = \alpha_{f(s)} \subset A$. Thus $\{\sigma_s\}$ is also Σ_1-correct.

Theorem 3.1.2. *A 1-generic degree* $\boldsymbol{a} < \mathbf{0}'$ *is not thin-free.*

Let $\{\sigma_s : s \in \omega\}$ be a Σ_1-correct approximation of A such that $|\sigma_{s+1}| > |\sigma_s|$ for each s. We will construct a recursive tree T, such that $[T]$ is thin and there is a path C on $[T]$ with $C \equiv_T A$.

Before we provide the construction of T, we first consider the set S of all initial segments of σ_s for $s \in \omega$. S is a tree since it is closed under initial segment. Obviously, A is an infinite path through S. Actually, A is the only path on S. To see this, for $D \neq A$, let x be the least number such that $D(x) \neq A(x)$. For this x, there is a stage s large enough such that for any stage $t > s$, $\sigma_t \supseteq \sigma_s \restriction x+1$, which implies that $\sigma_s \restriction x+1 = A \restriction x+1$. It just says that after stage s, only strings extending $A \restriction x+1$ can be enumerated into S, implying that $D \notin [S]$.

On the other hand, S is r.e. as a string τ is put on S if and only if there is some stage s such that $\sigma_s \supseteq \tau$. As $\{\sigma_s : s \in \omega\}$ is a Δ^0_2 approximation of A, some strings on S may not be extendible. We want the tree T we construct to be recursive and extendible, and the construction of T "follows" the enumeration of S. With this in mind, we need an additional symbol B, standing for "blank", such that all strings up to some length, $l(s)$ say, are defined on T at each stage s. Then T is a subtree of $\{0,1,B\}^{<\omega}$, which can be coded into a binary tree recursively. Second, for a finite string $\tau \in \{0,1,B\}^{<\omega}$ (or an infinite sequence $C \in \{0,1,B\}^{\omega}$, respectively), τ^d (or C^d) is defined to be a string (or a finite string or an infinite subsequence, respectively) obtained by deleting all B from τ (or

C) while keeping the appearance of 0's and 1's the same in the given order. For example, $(0010BB10B01)^d = 00101001$.

We construct T as follows: At stage 0, let $\emptyset$ be the root of T, and $l(0) = 0$. At stage $s+1$, let ρ be the string on T of length $l(s)$ with $\rho^d \subseteq \sigma_s$, $m = |\sigma_s| - |\rho^d|$, and $l(s+1) = l(s) + m + 1$. Now there are $m+1$ substages for $i = 0, 1, ..., m$, and at substage i, for strings τ on T of length $l(s) + i$, if $\tau^d \subseteq \sigma_s$, put $\tau 0$ and $\tau 1$ into T, else put τB into T.

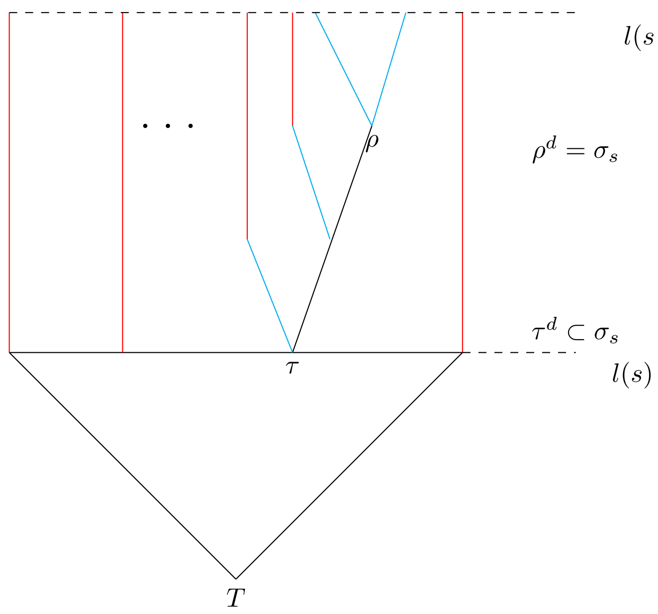

Figure 3.1: An example of how T is constructed at stage $s+1$. τ is the leaf on T such that $\tau^d \subset \sigma_s$, and in this example, $\sigma_s = \tau^d 11$. The meaning of different colors of strings: red means B, black and blue means 0 or 1. The black strings are added such that at the end of this stage, there is a string ρ on T such that $\rho^d = \sigma_s$, and the blue strings are added such that all possibile prefixes of $\sigma_{s'}$, $s' > s$ are included in $\{\pi^d : \pi \in T, |\pi| = l(s+1)\}$.

At the end of stage s, $l(s)$ and all strings τ on T of length $l(s)$ are already defined. Note that for all τ's above, $\tau^d \upharpoonright |\tau^d| - 1$ are already enumerated into S by stage s, while none of τ^d is on S yet. At stage $s+1$, with σ_s being put into S, for the string ρ of length $l(s)$ on T and $\rho^d \subseteq \sigma_s$, let π be the string such that $\sigma_s = \rho^d \pi$ and $m = |\pi|$. Then first, we put, if $m \geq 1$, $\rho^\frown(\pi \upharpoonright i)^\frown 0$ and $\rho^\frown(\pi \upharpoonright i)^\frown 1$ for $0 \leq i \leq m-1$, and $\rho\pi 0$ and $\rho\pi 1$ into T in order. In this manner, all strings ending with 0 or 1 on T longer than $l(s)$ are defined. Second, we extend all strings on T by B up to length $l(s)+m+1$, and then all strings on T of length $l(s)+m+1$ are defined.

What does T look like? Consider all the strings on T of length $s+1$, i.e., those strings on the $(s+1)$-st level of T. There are exactly $s+2$ strings on the $(s+1)$-st level, and among them, two strings end with 0 and 1 and share the common initial segment of length s, and the other strings end with B. Let τ_i for $0 \leq i \leq s+1$ be the strings on the $(s+1)$-st level of T, then τ_i^d are incompatible with each other, and only one of them, say τ_j^d, is extendible on S (equivalently, $\tau_j^d \subset A$). So above the $(s+1)$-st level of T, there are only finitely many paths above τ_i on T for each $i \neq j$, and infinitely many paths on T extending τ_j. Since A is the unique path on S, we know that there is only one path C on T which contains infinitely many 0's and 1's, with $(C)^d = A$. For the other paths D, there is some n such that for all $x > n$, $D(x) = B$, and $(D)^d$ is a string which is never enumerated into S.

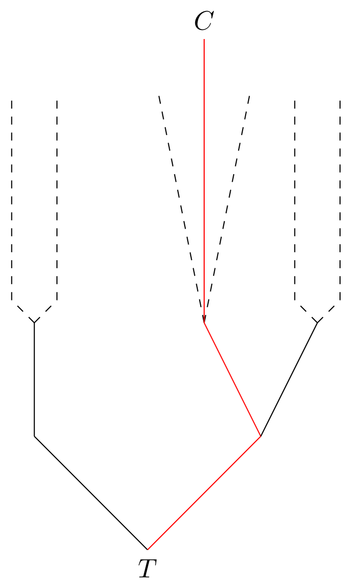

Figure 3.2: An example of T and C. C is the unique path on T contains infinitely many 0's and 1's. For any τ on T, if τ is not a prefix of C, then there are only finitely many different infinite paths on T extending τ, else there are infinitely many.

$A \leq_T C$ since $(C)^d = A$. Note that actually, A is coded into C on T while constructing T. That is, for nonempty string τ on the $(s+1)$-st level of T, if τ^d is enumerated into S at stage $s+1$, then τ is trying to code $A(x)$, which means that if $\tau \subset C$, then $A(x) = \tau^d(x)$, where $x = |\tau^d| - 1$. So for any nonempty string τ on T, if τ is extended by 0 or 1, then τ is trying to code $A(x)$ where $x = |\tau^d| - 1$, else τ is not trying to code $A(x)$ for any x.

$C \leq_T A$ because for any x, there is some s_x such that $l(s_x) \leq x \leq l(s_x + 1)$. Then find the least $s > s_x$ such that $\sigma_s \subset A$. Such an s exists since $\{\sigma_s\}$ is Σ_1-correct. For this s, find τ on T of length $l(s)$ with $\tau^d \subset A$. Then τ is an initial segment of C, and $\tau(x) = C(x)$.

The Σ_1-correctness of $\{\sigma_s\}$ guarantees that T is thin. Let U be a recursive subtree of T. There are two cases. If $C \notin [U]$, then there is some n such that $C \upharpoonright n \notin U$. As $C^d = A$, $(C \upharpoonright n)^d \subset A$, and thus there are only finitely many paths on U. Suppose that up to length m, strings on U do not split, let N be the union of cones above the strings on U of length m, then N is clopen and $[U]$ is the intersection of $[T]$ with N. For the other case, $C \in [U]$, consider $V = \{s : \tau \in T \backslash U, \tau^d \in S \text{ for some stage } s\}$. Then by the Σ_1-correctness, V is finite. Otherwise, there is some $s \in V$ such that $\sigma_s \subset A$, which means that there is some τ in $T \backslash U$ such that $\tau^d = \sigma_s$ is enumerated into S at stage s. By the construction, $\tau^d \subset A$, so $\tau \subset C$. However, $C \in [U]$ implies that for all $n \geq 0$, $C \upharpoonright n \in U$, a contradiction. Since V is finite, there are finitely many paths on T but not on U. Then there is a clopen set N such that $[T] \backslash [U] = [T] \cap N$, and hence $[U] = [T] \backslash N = [T] \cap \overline{N}$. The complement of N is what we need.

Now we give the proof of Theorem 3.1.2.

Proof. Fix a Σ_1-correct approximation $\{\sigma_s : s \in \omega\}$ of A, where $|\sigma_{s+1}| > |\sigma_s|$ for each s. T is constructed as follows:

Stage 0: Let $T_0 = \{\emptyset\}$ and $l(0) = 0$.

Stage $s+1$: T_s and $l(s)$ are given,

(1) For the leaf τ on T_s such that $\tau^d \subseteq \sigma_s$, let $m = |\sigma_s| - |\tau^d|$ and $l(s+1) = l(s) + m + 1$.

(2) There are $m+1$ substages for $i = 0, 1, ..., m$. Let $T_{s+1,0} = T_s$.

Substage i: For leaves τ on $T_{s+1,i}$ of length $l(s)+i$, if $\tau^d \subseteq \sigma_s$, put $\tau 0$ and $\tau 1$ into $T_{s+1,i+1}$, else put τB on $T_{s+1,i+1}$.

At the end of this stage, let $T_{s+1} = T_{s+1,m+1}$.

Let $T = \bigcup_s T_s$. Obviously, T is a recursive tree.

As pointed out above, T has the following properties:

- For any n, the $(n+1)$-st level of T contains $n+2$ many strings, two of which are in the form of $\tau 0$ and $\tau 1$ for some τ from the n-th level of T, and the other s strings end with B.
- For any n and two different strings on the $(n+1)$-st level of T, say τ and ρ, $\tau^d \neq \rho^d$.

These two properties can be proved by induction: It is trivial that the first level of T consists of two strings 0 and 1. Assume that it is true for level $n \geq 1$, and let τ_i, $0 \leq i \leq n$, be the strings on the n-th level of T, then $|\tau_i| = n$ for $0 \leq i \leq n$, and $\tau_i^d \neq \tau_j^d$ for any $0 \leq i < j \leq n$. Suppose all τ_i's are extended at stage $s+1$, and without loss of generity, we suppose τ_0^d is an initial segment of σ_{s+1}. Then none of τ_i^d for $0 < i \leq n$ can be an initial segment of σ_{s+1} because $\tau_i^d \neq \tau_0^d$. So $\tau_0 0$, $\tau_0 1$ and $\tau_i B$ for $0 < i \leq n$ are defined on T, which are the $n+2$ many strings of length $n+1$ on the $(n+1)$-st level of T. Except for $\tau_0 0$ and $\tau_0 1$, all strings in the $(s+1)$-st level of T end with B. Any two of $(\tau_0 0)^d$, $(\tau_0 1)^d$ and $(\tau_i B)^d$ for $0 < i \leq n$ are unequal, because $(\tau 0)^d = \tau^d 0$, $(\tau 1)^d = \tau^d 1$ and $(\tau B)^d = \tau^d$ for any τ.

We now show that T has the following good "splitting" property.

Lemma 3.1.3. *For any n, there is some $m > n$, such that for strings on the n-th level of T, all but one string do not split above level m on T.*

Proof. For any n, let τ_i for $0 \leq i \leq n$ be the strings on the n-th level of T, and $k = \max_i\{|\tau_i^d|\}$. For this k, by Σ_1-correctness, there is a stage $t \geq k$ such that $\sigma_t \subset A$. Then there is some stage s such that for all stages $s' > s$, $\sigma_{s'} \supset \sigma_t$. Let $m = l(s+1) - 1$ and ρ be the string on the m-th level of T such that $\rho^d = \sigma_s$. Then $\rho^d \supset \sigma_t$, and after stage s, all strings on the m-th level of T except ρ can only be extened by B, and thus if $\tau_i \nsubseteq \rho$, τ_i does not split above level m on T. □

Thus, all strings on T are extendible, and $[T]$ contains infinitely many paths. We prove below that among these infinitely paths, exact one path is Turing equivalent to A.

Lemma 3.1.4. *There exists an unique infinite path C through T containing infinitely many* 0 *and* 1*. Moreover, $C \equiv_T A$.*

Proof. Start from $n_0 = 0$. At stage $s+1$, n_s is given. Apply Σ_1-correctness to $\{n : n > n_s\}$, there is some n_{s+1} such that $\sigma_{n_{s+1}} \subset A$. By the construction of T, there is a string ρ_s on T such that $\rho_s^d = \sigma_{n_{s+1}}$ and the last digit of ρ_s is 0 or 1. As $|\sigma_{n+1}| > |\sigma_n|$ for all n and $\sigma_{n_{s+1}} \subset A$ for all s, $\rho_{s+1} \supset \rho_s$. Let $C = \cup_s \rho_s$, then C is a path on T containing infinitely many 0's and 1's.

Assume that there exists some $D \neq C$, $D \in [T]$ and D contains infinitely many 0 and 1. Let x be the least number such that $C(x) \neq D(x)$, then $C(x)$ is either 0 or 1. As D contains infinitely many 0 and 1, by the construction of T, there are infinitely many s such that $\sigma_s \supset D \upharpoonright x+1$. So $S = \{s : \sigma_s \supset (D \upharpoonright x+1)^d\}$ is an infinite r.e. set. By Σ_1-correctness, there is an $s \in S$ such that $\sigma_s \subset A$, which implies that $A(x) = \sigma_s(x) = D(x) \neq C(x)$. Similarly for C, we will have that $A(x) = C(x) \neq D(x)$, a contradiction. Thus, C is the only path in $[T]$ containing infinitely many 0's and 1's.

$A \leq_T C$ since $(C)^d = A$. We now show that $C \leq_T A$. For any x, let s_x be the number such that $l(s_x) \leq x \leq l(s_x + 1)$. s_x can be found recursively since l is a recursive function. Let s be the least number such that $s > s_x$ and $\sigma_s \subset A$. Such an s exists since $\{\sigma_s\}$ is Σ_1-correct, and s is recursive in A. Let τ be the string on T of length $l(s)$ with $\tau^d \subset A$, then τ is an initial segment of C, and $\tau(x) = C(x)$ as $l(s) \geq l(s_x + 1) \geq x$. Thus $C \equiv_T A$. □

Lemma 3.1.5. *T is thin.*

Proof. Let U be any recursive subtree of T. There are two cases.

- $C \notin [U]$.

In this case, there exists some n such that $C \restriction n \notin U$. For that n, there exists some m such that all strings on U of length n do not split above level m by Lemma 3.1.3. So there are only finitely many infinite paths through U. Enumerate all the strings in U of length m and let N be the union of the cones above those strings. N is a clopen set with $[U] = [T] \cap N$.

- $C \in [U]$.

In this case, for all $\tau \in T \backslash U$, $\tau^d \not\subset A$. Let $V = \{s : \sigma_s \supseteq \tau^d, \tau \in T \backslash U\}$. V is an r.e. set, since both T and U are recursive trees and $\{\sigma_s\}$ is a recursive approximation.

We claim that V is finite. Otherwise, by Σ_1-correctness, there is some $s \in V$ such that $\sigma_s \subset A$. That is, there is some $\tau \in T \backslash U$ such that $\tau^d \subseteq \sigma_s \subset A$. Then $\tau \subset C$, which means that $C \notin [U]$. A contradiction.

Thus, there are only finitely many infinite sequences in $[T] \backslash [U]$. Let $W = \{\tau : \tau \in T \backslash U, \tau^d \subseteq \sigma_s, s \in V\}$, then W is a finite set. Let N be the union of cones above $\tau \in W$, then as W is a finite set, N is clopen. As $[T] \backslash [U]$ is the intersection of $[T]$ with N, $[U] = [T] \cap \overline{N}$, where $\overline{N}$ is clopen as the complement of N. □

Therefore, we obtain a path C in a thin class $[T]$ with $C \equiv_T A$, and hence, A is not of thin-free degree. □

3.2 2-generic sets are thin-free.

The case for 2-generic sets is very different from 1-generic sets. Juckusch observed [21] that for any 2-generic set A, for any partial recursive functional Φ with Φ^A total and nonrecursive, A has an initial segment σ such that the set $T = \{\rho : \rho \subseteq \Phi^\tau, \tau \supseteq \sigma\}$ is a recursive extendible tree without isolated infinite paths.

Lemma 3.2.1 (Juckusch). *If A is 2-generic and Φ^A is total and nonrecursive, then there is $\sigma \subset A$ such that for all $\tau \supseteq \sigma$,*

- *for any x, there exists $\rho \supseteq \tau$ such that $\Phi^\rho(x)\downarrow$,*
- *there is a Φ-splitting extension above τ.*

Recall the construction in section 2.1, we can strengthen this property to show that a given Π^0_1 class is not thin.

Below we show that 2-generic sets always have thin-free degrees.

Theorem 3.2.2. *Any nonrecursive set below a 2-generic set is thin-free. Thus, 2-generic sets are of thin-free degree.*

Proof. Let A be a 2-generic set, and assume that Φ^A is total and nonrecursive, and that Φ^A lies on primitive recursive tree P, where Φ is a $\{0,1\}$-valued Turing functional. We will show that $[P]$ is not thin.

Let V be the set of strings τ such that:

- $\exists x$, $\forall \rho \supseteq \tau$, $\Phi^{\rho}(x)\uparrow$, or
- $\forall \rho, \pi \supseteq \tau$ and $\forall x$, if $\Phi^{\rho}(x)\downarrow$ and $\Phi^{\pi}(x)\downarrow$, then $\Phi^{\rho}(x) = \Phi^{\pi}(x)$, or
- $\Phi^{\tau} \notin P$.

V is Σ^0_2, and hence, A either meets or avoids V, as A is 2-generic. Note that A cannot meet V, because Φ^A is assumed to be total, nonrecursive, and lies on P. So A avoids V, which means that there is some string $\sigma \subset A$ such that for any $\tau \supseteq \sigma$,

(1) $\forall x$, $\exists \rho \supseteq \tau$ such that $\Phi^{\rho}(x)\downarrow$, and

(2) $\exists \rho, \pi \supseteq \tau$ and $\exists x$, such that $\Phi^{\rho}(x)\downarrow$, $\Phi^{\pi}(x)\downarrow$, and $\Phi^{\rho}(x) \neq \Phi^{\pi}(x)$, and

(3) $\Phi^{\tau} \in P$.

For σ above, consider $T = \{\rho : \tau \supseteq \sigma, \rho \subseteq \Phi^{\tau}\}$. T is an extendible tree. For any $\tau \supset \sigma$, Φ^{τ} is a finite string on T. For any $x \geq |\Phi^{\tau}|$ (meaning that $\Phi^{\tau}(x)\uparrow$), by (1), there is some $\pi \supseteq \tau$ such that $\Phi^{\pi}(x)\downarrow$. Then Φ^{π} extends Φ^{τ} and is on T. So Φ^{τ} is extendible on T.

Moreover, T is a recursive tree. For any l, if $l \leq |\Phi^{\sigma}|$, only one string of length l, i.e., $\Phi^{\sigma} \upharpoonright l$, is on T; If $l > |\Phi^{\sigma}|$, let τ be the least string extending σ such that $|\Phi^{\tau}| \geq l$. Such a τ exists by (1) and can be found recursively by enumerating strings extending σ. Let $u(l) = \max\{\varphi(x) : x < l\}$, where $\varphi(x)$ is the use of $\Phi^{\tau}(x)$. $u(l)$ is recursive. By enumerating all strings of length $u(l)$ extending σ, we get all strings on T of length smaller than l.

Now consider the leftmost path C though T. Since T is a recursive, extendible tree, C is recursive. C lies on T, so $C \supset \Phi^{\sigma}$. Let $\rho_0 = \sigma$. For given ρ_i, let ρ_{i+1}, τ_{i+1} and x_{i+1} be the first triple (ρ, τ, x) such that $\rho, \tau \supseteq \rho_i$, $\Phi^{\rho}(x)\downarrow$, $\Phi^{\tau}(x)\downarrow$, $\Phi^{\rho}(x) \neq \Phi^{\tau}(x)$, and $\Phi^{\rho} \upharpoonright x \subset C \upharpoonright x$. Such a triple exists because of (2) and $C \in [T]$. Furthermore, as C is recursive, the list of τ_i for $i \geq 1$ is recursive. By (3), T is a subtree of P, so for any $i \geq 1$, there is an infinite path through P extending Φ^{τ_i}. Let S be the collection of all initial segments of C and Φ^{τ_i} for all $i \geq 1$, and the strings on P extending Φ^{τ_i} for i even (refer to Figure 3.3). Then S is a recursive subtree of P, which is not the intersection of $[P]$ with any clopen set.

So for any 2-generic set A, for all partial recursive functionals Φ, if Φ^A is total, not recursive and $\Phi^A \in [P]$, then $[P]$ is not thin, completing the proof.

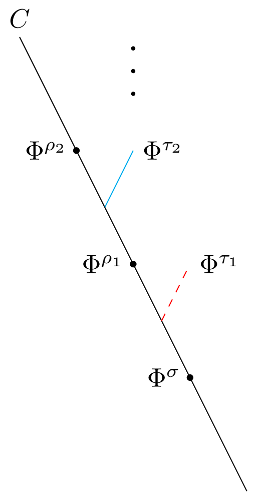

Figure 3.3: The construction of S.

□

Chapter 4

A Nonbranching Thin-free Degree

In Chapter 2, we have seen how to construct a thin-free degree, by using the even subtrees of e-splitting subtrees to show that given Π^0_1 classes are not thin. The construction is done with oracle $\mathbf{0}''$ to decide whether we can have a string not e-extendible in P_e, or whether we can have an e-splitting subtree of T. In some sense, the construction of thin-free degrees takes the advantage of features of minimal degrees, which enables us to construct a hyperimmune-free minimal thin-free degree, below $\mathbf{0}''$, of course. Furthermore, Cenzer, et al. [3] actually constructed a minimal thin-free degree below $\mathbf{0}'$ by Sacks forcing argument, and an r.e. thin-free degree by a full approximation argument In section 3.1, we have shown that thin-free degrees below $\mathbf{0}'$ cannot be 1-generic. In this chapter, we first analyze constructions of Cenzer, et al. in paper [3], and then construct a nonbranching thin-free r.e. degree.

4.1 Construction of thin-free degree below $\mathbf{0}'$ by Cenzer, et al.

We will briefly review the construction of Cenzer, et al. of a minimal thin-free degree below $\mathbf{0}'$. The construction is modified from Sacks construction of minimal degree below $\mathbf{0}'$, where some additional features of showing a given Π^0_1 class is not thin is provided. As we can only use $\mathbf{0}'$ as oracle, we cannot obtain new recursive subtrees as total from previous ones, when we run forcing arguments on a sequence of trees. Instead, we will process the construction through a sequence of partial recursive trees. The notation (Φ_e, P_e) and requirements Q_e are the same as in section 2.1, and for each e, the construction is to build a partial recursive tree which is either an e-splitting tree whose strings are all e-extendible, or a tree whose root is non-e-extendible or without e-splitting.

The basic idea to meet a single Q_e is that, with a given partial recursive tree T, we guess at the very beginning that there exist infinitely many consecutive e-extending and e-splitting extensions, and define $T_e = E(SP(T, e))$, as we did in section 2.1 ($\mathbf{0}''$ is powerful enough to tell whether our guess is true or not). In the following stages, we keep testing whether our guess is correct by using oracle $\mathbf{0}'$, and when we do so, we construct the wanted tree stage by stage. If at some stage our guess turns out to be wrong, i.e., we find a string on T_e is either non-e-extendible or non-e-splittable, we then switch to use the full subtree of the previous tree above that string. Otherwise, eventually we will have a partial recursive e-extending and e-splitting tree. We will guarantee that σ_s, the approximation of A at stage s, is defined at stage s, with σ_{s+1} always extending σ_s.

Say that T_e is good at stage $s + 1$ if either $T_e = Full(T, \alpha)$ for some α, or $T_e = E(SP(T, e))$ satisfying:

(1) For all α such that $SP(T, e)(\alpha)\downarrow$ until stage $s + 1$ and $|SP(T, e)(\alpha)| < s + 1$, there exists some $\tau \in P_e$ of length $s + 1$ and $\tau \supseteq \Phi_e^{SP(T,e)(\alpha)}$.

(*We try to approximate an e-extendible subtree, and at the moment, we are consider those strings up to level $s + 1$ on P_e.*)

As there are only finitely many such α, we can check it recursively.

(2) There exists an e-splitting extension on T_e above σ_s, i.e., for $\sigma_s = T(\alpha)$, there are β_0, β_1 extending α, and a number x_0, β_2, β_3 extending β_0 and a number

$x_1 > x_0$, such that $T(\beta_0) <_{lex} T(\beta_1)$, and

$$\Phi_e^{T(\beta_0)}(x_0) \downarrow \neq \Phi_e^{T(\beta_1)}(x_0) \downarrow, \text{ and } \Phi_e^{T(\beta_2)}(x_1) \downarrow \neq \Phi_e^{T(\beta_3)}(x_1) \downarrow.$$

(*We are targeting to looking for an e-splitting subtree and then use the even subtree.*)

Let $T_{e,s}$ denote the guess of T_e at stage s.

- If $T_{e,s}$ is defined as $Full(T, \alpha)$, then we do have a string $T(\alpha)$ which is either non-e-extendible or non-e-splitting. So $T_{e,s}$ is built not by a guess and thus must be good. Let $T_{e,s+1} = T_{e,s}$.
- If $T_{e,s} = E(SP(T, e))$ is good at stage $s + 1$, then we keep our guess and let $T_{e,s+1} = T_{e,s}$.
- If $T_{e,s} = E(SP(T, e))$ becomes not good at stage $s + 1$.
  - If there is some α with $SP(T, e)(\alpha) \downarrow$ and $|SP(T, e)(\alpha)| < s + 1$, but no $\tau \in P_e$ with $|\tau| = s+1$ extends $\Phi_e^{SP(T,e)(\alpha)}$, then we get a non-e-extendible string $SP(T, e)(\alpha)$ on T and we change our guess about T_e by letting $T_{e,s+1} = Full(T, \beta)$, where $SP(T, e)(\alpha) = T(\beta)$. Thus, if A is on $T_{e,s+1}$, $\Phi_e^A \notin [P_e]$.
  - If there is no extension above σ_s on T_e, which implies that $T(\alpha)$ or $T(\beta_0)$ is not e-splittable. Let $T_{e,s+1}$ be $Full(T, \alpha)$ if $T(\alpha)$ is not e-splittable, or $Full(T, \beta_0)$ else. In this case, Φ_e^A is recursive for any $A \in [T_{e,s+1}]$.

Finally, let $T_e = \lim_s T_{e,s}$. T_e is defined as $E(SP(T, e))$ at first, and can change at most once to a full subtree of T if some non-e-extendible or non-e-splittable string on T is found. Thus, T_e settles down eventually.

The interaction between actions for different requirements involves priority argument. That is, once T_e is changed, T_{e+1} must be changed accordingly, and the guess of T_{e+1} is injured. A standard argument shows that the guess of T_e can be changed at most 2^e many times, and eventually, we will have a correct guess of T_e, satisfying Q_e.

4.2 An r.e. Thin-free degree

We have seen that in the construction of thin-free degrees, to prove that a Π_1^0 class is not thin, we ask for the existence of e-splitting trees, and then use the even part of it. This method shows a close relation between thin-free degrees and minimal degrees. A natural question is to ask whether there are other degrees which are also thin-free. We have seen in section 3.1 that 1-generic degrees below $\mathbf{0}'$ are not thin-free, and that all 2-generic degrees are thin-free. In [3], Cenzer, et al. proved that thin-free degrees can be r.e.. In their construction, to show that a Π_1^0 class is not thin, infinitely many paths are constructed in an explicit way, which leads to the construction of a subclass witnessing that the given Π_1^0 class is not thin. This method was improved by Downey, Wu and Yang in [13], showing that r.e. thin-free degrees are dense in the r.e. degrees, like branching degrees. We first briefly describe the idea of Cenzer, et al.'s construction, and will provide a construction of an r.e. nonbranching thin-free degree in the next section.

Let (Φ_e, Ψ_e, P_e) be an effective list of triples where Φ_e and Ψ_e are $\{0,1\}$-valued partial recursive functionals, and P_e is a primitive recursive tree. Let φ_e and ψ_e be the uses of Φ_e and Ψ_e respectively. A set A will be constructed to meet the requirements:

$$Q_e : \text{If } \Phi_e^A, \Psi_e^{\Phi_e^A} \text{ are total, } \Psi_e^{\Phi_e^A} = A, \text{ and } \Phi_e^A \in [P_e], \text{ then } [P_e] \text{ is not thin.}$$

If all requirements are met, then A is of thin-free degree.

During the construction, if one of the following conditions is true, then Q_e is satisfied:

(1) Φ_e^A or $\Psi_e^{\Phi_e^A}$ is not total, or $\Psi_e^{\Phi_e^A} \neq A$,

(2) Φ_e^A is not in $[P_e]$,

(3) $[P_e]$ is not thin.

Let A_s be the recursive approximation of A at stage s, and we use $\Phi_{e,s}^A(n)$ to denote $\Phi_{e,s}^{A_s \upharpoonright s}(n)$. Suppose l is the least number such that $\Phi_{e,s}^{A_s}(n)\downarrow$ for all $n < l$ and $\Phi_{e,s}^{A_s}(l)\uparrow$, we use $\Phi_{e,s}^A$ to denote the finite string $\Phi_{e,s}^{A_s \upharpoonright s} \upharpoonright l$, for convenience. We compute $\Phi_{e,s}^A(n)$ and $\Psi_{e,s}^{\Phi_{e,s}^A}(n)$ for $n \leq s$ at stage s. Also we enumerate strings on P_e up to length s and see whether $\Phi_{e,s}^A$ lies on P_e or not. Unless there are some s and n such that $\Psi_{e,s}^{\Phi_{e,s}^A}(n)\downarrow\neq A_s(n)$ and A does not change below the use $\varphi_{e,s}(\psi_{e,s}(n))$, we do not know whether (1) is met by any finite stage. If $\Phi_{e,s}^A \upharpoonright n \notin P_e$ and A does not change below the use $\varphi_{e,s}(n)$, then (2) is met. Besides this, if $\Phi_{e,s}^A(n)\uparrow$, or $\Psi_{e,s}^{\Phi_{e,s}^A}(n)\uparrow$ at some n, then finally (1) is met. So during the construction, if $\Phi_{e,s}^A \upharpoonright n \downarrow\in P_e$ and $\Psi_{e,s}^{\Phi_{e,s}^A} \upharpoonright n \downarrow= A_s \upharpoonright n$ up to larger and larger n, we have three options:

(a) make a diagonalization $\Psi_{e,s}^{\Phi_{e,s}^A}(n)\downarrow\neq A_s(n)$ for some n, and preserve $A_s \upharpoonright \varphi_e(\psi_e(n))$,

(b) force $\Phi_{e,s}^A$ to extend some string on P_e, which is found to be non-extendible at a stage,

(c) keep constructing a recursive tree S_e with the properties described in 2.1 in order to show that $[P_e]$ is not thin.

To achieve (c) (we are not using any oracle), we construct a subtree S_e recursively by P_e via infinitely many stages in the construction. The target is that if Φ_e^A is total and lies on P_e, then along Φ_e^A, we can find a sequence of infinite paths on P_e, say B_i, such that:

- $B_i \neq \Phi_e^A$ for all i.

- Let l_i be least number such that $B_i \upharpoonright l_i = \Phi_e^A \upharpoonright l_i$ but $B_i(l_i + 1) \neq \Phi_e^A(l_i + 1)$, then $l_i < l_{i+1}$ for all i. This means that the common initial segment of B_i and Φ_e^A extends with i increases, and $B_i \neq B_j$ for $i \neq j$.

We will construct S_e so that $B_i \in [S_e]$ for any even i, and $B_i \notin [S_e]$ for any odd i. This ensures that $\Phi_e^A \in [S_e]$. Note that $[S_e]$ is not the intersection of $[P_e]$ with any clopen set, and hence $[P_e]$ is not thin.

Also to achieve (a), a problem is that when a number n is enumerated into A, $\Phi_{e,s}^A$ may change on a small number, and change $\Psi_{e,s}^{\Phi_{e,s}^A}(n)$, invalidate our attempt of making diagonalization at n.

To solve these problems, in the construction, we define a sequence of intervals $\{(x_{e,i}, z_{e,i}] : x_{e,i} < z_{e,i} < x_{e,i+1}, i \in \omega\}$ on P_e, one by one, with the target to secure an infinite path on P_e extending $\Phi_e^A \upharpoonright x_{e,i}$ and incompatible with $\Phi_e^A \upharpoonright z_{e,i}$. For convenience, we say such strings are in the interval $(x_{e,i}, z_{e,i}]$ of P_e. We choose a number $y_{e,i} \in (x_{e,i}, z_{e,i}]$ for each i in case that our effort of securing an infinite path in this interval fails, and the enumeration of $y_{e,i}$ into A will provide a wanted diagonalization.

The interval $(x_{e,i}, z_{e,i}]$ is defined as follows: choose $x_{e,i}$ first, wait for A and $\Psi_e^{\Phi_e^A}$ to agree up to $x_{e,i}$, and then choose $y_{e,i}$ bigger than the use $\varphi_e(\psi_e(x_{e,i}))$; after this, wait for A and $\Psi_e^{\Phi_e^A}$ to agree up to $y_{e,i}$, and then choose $z_{e,i}$ bigger than the use $\varphi_e(\psi_e(y_{e,i}))$. Once we find all strings on P_e in the interval $(x_{e,i}, z_{e,i}]$ are non-extendible, we put $y_{e,i}$ into A and we will argue that at least one of (a), (b), (c) is true.

We are assuming that the length of agreement between A and $\Psi_e^{\Phi_e^A}$ eventually longer than $z_{e,i}$, as otherwise, we do not need to anything as (1) is already met. As we are trying to secure an infinite path on P_e in the interval $(x_{e,i}, z_{e,i}]$, if we find that all strings in this interval become non-extendible on P_e, we enumerate $y_{e,i}$ into A, and one of the following two outcomes will happen:

- either Φ_e^A keeps the same value up to $z_{e,i}$, and we get $A(y_{e,i}) \neq \Psi_e^{\Phi_e^A}(y_{e,i})$; or
- Φ_e^A changes below $z_{e,i}$, and Φ_e^A extends some string non-extendible in P_e, and as a consequence, Φ_e^A is not in $[P_e]$.

In these two cases, we satisfy (1) or (2) accordingly, and both are global win for the requirement Q_e.

Suppose that in the construction, no $y_{e,i}$ has chance to be enumerated into A. Then each interval $(x_{e,i}, z_{e,i}]$ gives an infinitely path in $[P_e]$. In the construction of S_e, we enumerate all strings on P_e on S_e, except those strings in the interval $(x_{e,i}, z_{e,i}]$ for i odd. By doing so, $[S_e]$ does not contain those infinite paths in the intervals $(x_{e,i}, z_{e,i}]$, i odd, but contains all other infinite paths in $[P_e]$. This shows that $[P_e]$ is not thin, and (3) is met.

Suppose that in the construction, no $y_{e,i}$ has chance to be enumerated into A. Then each interval $(x_{e,i}, z_{e,i}]$ gives an infinitely path in $[P_e]$. In the construction of S_e, we enumerate all strings on P_e on S_e, except those strings in the interval $(x_{e,i}, z_{e,i}]$ for i odd. By doing so, $[S_e]$ does not contain those infinite paths in the intervals $(x_{e,i}, z_{e,i}]$, i odd, but contains all other infinite paths in $[P_e]$. This shows that $[P_e]$ is not thin, and (3) is met.

With this in mind, we divide Q_e into subrequirements, $N_{e,i}$, $i \geq 0$, and each $N_{e,i}$ is responsible for the construction of one interval.

$N_{e,i}$: If $\Phi_e^A, \Psi_e^{\Phi_e^A}$ total, $\Psi_e^{\Phi_e^A} = A$, and $\Phi_e^A \in [P_e]$, then $\exists (x_{e,i}, z_{e,i}]$, $[P_e]$ contains at least one element extending $\Phi_e^A \upharpoonright x_{e,i}$ but not $\Phi_e^A \upharpoonright z_{e,i}$.

$N_{e,i}$ is satisfied as follows:

Step 1:

Select some new large $x_{e,i}$ as the initial point of the interval. Wait for $\Phi^A_{e,s} \upharpoonright x_{e,i} \downarrow$. We assume that $\Phi^A_{e,s} \upharpoonright x_{e,i} \downarrow$ is on P_e, as otherwise no need to do anything. We will assume the same argument for this in the following, and if not necessary, we will not mention this again.

Step 2:

Select $y_{e,i} > \varphi_e(x_{e,i})$ (the candidate $y_{e,i}$ may be put into A, so we want to ensure this operation cannot injure $\Phi^A_e \upharpoonright x_{e,i}$). Wait for $\Phi^A_{e,s} \upharpoonright \psi_e(y_{e,i}) \downarrow$, and $A_s \upharpoonright y_{e,i} = \Psi^{\Phi^A_{e,s}}_{e,s} \upharpoonright y_{e,i}$.

Step 3:

Select $z_{e,i} > \psi_e(y_{e,i})$ as the end point of the interval. Wait for $\Phi^A_{e,s} \upharpoonright z_{e,i} \downarrow$.

Step 4:

Once $\Phi^A_{e,s} \upharpoonright z_{e,i}$ converges, if i is odd, we terminate all strings on S_e in the interval $(x_{e,i}, z_{e,i}]$. If i is even, we enumerate all other strings on P_e of length s into S_e as usual.

When i is odd, we call this interval a minus-strategy interval, and otherwise a plus-strategy interval.

Step 5:

Once we see all strings in the interval $(x_{e,i}, z_{e,i}]$ of P_e are non-extendible, we enumerate $y_{e,i}$ into A. With $A(y_{e,i})$ changing from 0 to 1, there are two possible cases (refer to Figures 4.1, 4.2):

(1) $\Phi^A_e \upharpoonright z_{e,i}$ cannot recover to the string before we put $y_{e,i}$ into A, then there are two subcases:

(1.1) $\Phi^A_e(v) \uparrow$ for some $v \leq z_{e,i}$, then Φ^A_e is not total.

(1.2) $\Phi^A_e \upharpoonright z_{e,i}$ cannot recover to the string before $y_{e,i}$ is enumerated into A, then as $\Phi^A_{e,s} \upharpoonright x_{e,i}$ does not change, Φ^A_e extends some non-extendible string in P_e and hence $\Phi^A_e \notin [P_e]$.

(2) $\Phi_e^A \upharpoonright z_{e,i}$ recovers to the string before $y_{e,i}$ is enumerated into A, then

$$A(y_{e,i}) = 1 \neq 0 = \Psi_e^{\Phi_e^A}(y_{e,i}).$$

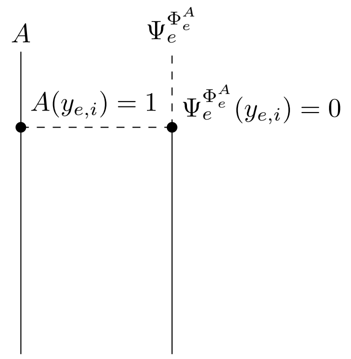

Figure 4.1: Possible case (2).

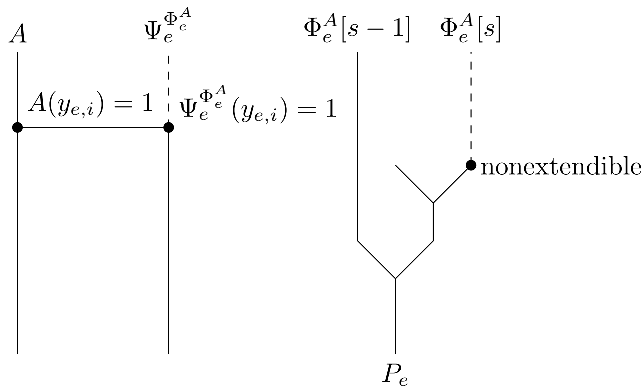

Figure 4.2: Possible case (1.2), $y_{e,i}$ is put into A at stage s.

During the construction, if we wait at step 1, 2 or 3 of some $N_{e,i}$ forever, or some $y_{e,i}$ is put into A at some stage, then one of (1), (2) is true and Q_e is satisfied. Otherwise, if all subrequirements $N_{e,i}$ succeed in finding a wanted interval, we obtain a recursive subtree proving that $[P_e]$ is not thin.

It is easy to see that subrequirements of one Q_e requirement work consistently in defining their intervals. For subrequirements for different Q-requirements, Q_{e_1} and Q_{e_2} say, with $e_1 < e_2$, for example, we need to consider the consistency between them. Suppose there are two stages s and t with $t > s$ such that at stage s, $N_{e_1,i}$ terminates the strings in the interval $(x_{e_1,i}, z_{e_1,i}]$ of S_{e_1}, and at stage t, $N_{e_2,j}$ puts some small number $y_{e_2,j}$ into A, which changes the computation of $\Phi^A_{e_1}$, leading $\Phi^A_{e_1}$ to enter the interval $(x_{e_1,i}, z_{e_1,i}]$ of S_{e_1}, which is a problem, as we are making S_e a recursive subtree.

To avoid this, we add a confirmation section for $N_{e_2,j}$ when it selects its $y_{e_2,j}$. That is, confirmation-strategy intervals are added to the sequence for Q_{e_1}, and any $y_{e_2,j}$ is selected with confirmation if exists. In detail, for $N_{e_1,i}$, if $i = 3k$ or $i = 3k + 1$, we define plus- or minus-strategy intervals as stated before. For $i = 3k + 2$, we define a confirmation-strategy interval by the first four steps in a plus-strategy interval. Then a confirmation-strategy does not put element into A and is set to provide element to Q_{e_2}. When $N_{e_2,j}$ wants to choose $y_{e_2,j}$, it first looks for a confirmation-strategy interval $(x_{e_1,i'}, z_{e_1,i'}]$, such that $y_{e_1,i'}$ is larger than both $\varphi_{e_1}(x_{e_1,i'})$ and $\varphi_{e_2}(x_{e_2,j})$, then let $y_{e_2,j}$ be equal to $y_{e_1,i'}$, and say that $y_{e_2,j}$ is confirmed when $\Phi^A_{e_1}(z_{e_1,i'})$ converges. If $\Phi^A_{e_1}(z_{e_1,i'})$ never converges, then $\Phi^A_{e_1}$ is not total, and Q_{e_1} is satisfied.

Suppose that $N_{e_1,i}$ terminated all strings in the interval $(x_{e_1,i}, z_{e_1,i}]$ of S_{e_1} at a stage s, and at a later stage $t > s$, $N_{e_2,j}$ puts $y_{e_2,j}$ into A, where $y_{e_2,j}$ is confirmed as $y_{e_1,i'}$, then if this enumeration drives $\Phi^A_{e_1}$ to enter the interval $(x_{e_1,i}, z_{e_1,i}]$ on the tree P_{e_1}, then, by $\Phi^A_{e_1,t} \restriction x_{e_1,i} \supseteq \Phi^A_{e_1,t} \restriction z_{e_1,i'}$, we have

$$\Phi^A_{e_1,t} \restriction z_{e_1,i'} = \Phi^A_{e_1,s} \restriction z_{e_1,i'},$$

i.e., $\Phi^A_{e_1,t} \restriction z_{e_1,i'}$ recovers to the string before $y_{e_2,j}$, i.e., $y_{e_1,i'}$ was put into A, and hence

$$A(y_{e_2,j}) = 1 \neq 0 = \Psi_{e_1}^{\Phi^A_{e_1}}(y_{e_2,j}),$$

and S_{e_1} is no longer needed, as Q_{e_1} is satisfied

It may happen that $\Phi^A_{e_1,t} \restriction z_{e_1,i'}$ becomes different from $\Phi^A_{e_1,s} \restriction z_{e_1,i'}$, and if so, since the uses of corresponding computation do not change, we can still use the intervals we have defined for Q_{e_1}. So we go back to $N_{e_1,i'}$, recompute and take actions as in the basic steps. It is fine if we wait at step 1, 2 or 3. If we reach step 4, we continue to check the conditions for $N_{e_1,i'+1}$. As $N_{e_1,i'}$ is a confirmation-strategy, it never terminates strings in the definition of S_{e_1}, so S_{e_1} is still recursive.

In this manner, the requirement $N_{e_2,j}$ cannot injure $N_{e_1,i}$, who has global priority higher than $N_{e_2,j}$, and local priority lower than $N_{e_2,j}$.

In the construction, an $N_{e,i}$-requirement can have global priority lower than several Q-requirements. What we do is to nest the subrequirements to ensure that the candidate $y_{e,i}$ is confirmed by all Q-requirements with higher global priority. For example, when $N_{2,i}$ for some i wants to select $y_{2,i}$, it first requires confirmation from Q_1, to get a confirmed $y_{1,j} > \varphi_2(x_{2,i})$, but to get $y_{1,j}$, we also need to get confirmation from Q_0, which needs to make $y_{0,k}$ bigger than both $\varphi_1(x_{1,j})$ and $\varphi_2(x_{2,i})$, and then take

$$y_{2,i} = y_{1,j} = y_{0,k}.$$

So $y_{2,i}$ first asks for Q_1 confirmation. There are three possible cases:

Case 1:

$y_{2,i}$ is confirmed by both Q_0 and Q_1 as above.

$$x_{2,i} \longrightarrow x_{1,j} \longrightarrow x_{0,k} \longrightarrow y_{2,i} = y_{1,j} = y_{0,k} \longrightarrow z_{0,k} \xrightarrow{\text{Confirmed by } Q_0} z_{1,j} \xrightarrow{\text{Confirmed by } Q_1} z_{2,i}$$

Case 2:

$y_{2,i}$ gets progress at Q_1 and gets stuck for the confirmation of Q_0. We say that the confirmation passes Q_1 but not Q_0. In this case, Q_0 is satisfied.

Case 3:

$y_{2,i}$ gets stuck at Q_1 and hence has not chance to ask for confirmation of Q_0. We say that the confirmation stops at Q_1. In this case, Q_1 is satisfied.

For cases 1 and 2, Q_0 confirmation is completed after Q_1 confirmation is done. For case 3, $y_{2,i}$ needs to ask for Q_0 confirmation as we discussed above for two $Q-$requirements.

The process is a direct generalization when more Q-requirements are involved.

We are now ready to provide a construction of our result about the existence of nonbranching thin-free degrees.

4.3 A nonbranching thin-free degree

We now construct an r.e. set whose degree is nonbranching and thin-free. First recall that an r.e. degree $\boldsymbol{a}$ is nonbranching if $\boldsymbol{a}$ is not the infimum of any two incomparable r.e. degrees. To construct an r.e. set whose degree is nonbranching, we shall ensure that for each i and j, if $A <_T A \bigoplus W_i$ and $A <_T A \bigoplus W_j$, then there is an r.e. set $B_{i,j}$ such that $B_{i,j} \leq_T A \bigoplus W_i$, $B_{i,j} \leq_T A \bigoplus W_j$ and $B_{i,j} \not\leq_T A$. For convenience, we use notations $\widetilde{W_i} = A \bigoplus W_i$ and $\widetilde{W_j} = A \bigoplus W_j$, and let $\omega^{[n]} = \{\langle z, n\rangle : z \in \omega\}$.

To make $B_{i,j} \not\leq_T A$, we ensure that $\Phi_e^A \neq B_{i,j}$ for all e by diagonalization. That is, for fixed i, j and e, choose some fresh witness $x = x_{i,j,e} \in \omega^{[\langle i,j,e\rangle]}$ and wait for a stage s such that $\Phi_{e,s}^A(x) \downarrow= 0$. Once it happens, we set a restraint $q_{i,j,e}^s = \varphi_e(x)$, the use of $\Phi_{e,s}^A(x)$, on A to preserve the computation, and put x into $B_{i,j}$, to make

$$\Phi_{e,s}^A(x) = 0 \neq 1 = B_{i,j}(x).$$

To make $B_{i,j} \leq_T \widetilde{W_i}$ and $B_{i,j} \leq_T \widetilde{W_j}$, we want to define two partial recursive functionals Ψ_i and Ψ_j such that $B_{i,j} = \Psi_i^{\widetilde{W_i}} = \Psi_j^{\widetilde{W_j}}$. So whenever some x is put into $B_{i,j}$, we put a trace y_x into A, and thus into $\widetilde{W_i}$ and $\widetilde{W_i}$, to undefine the computation of $\Psi_i^{\widetilde{W_i}}(x)$ and $\Psi_j^{\widetilde{W_j}}(x)$.

In order to preserve the computation $\Phi_{e,s}^A(x) = 0$ from being destroyed by the enumeration of y_x into A, we need to select y_x greater than $q_{i,j,e}^s$. On the other hand, if $W_i \leq_T A$ (a similar problem for W_j), then $\widetilde{W_i} \equiv_T A$ and thus $B_{i,j} = \Psi_i^{\widetilde{W_i}}$ shows that $B_{i,j} \leq_T A$, which means that we cannot achieve $B_{i,j} \not\leq_T A$. With this in mind, if we cannot make $\Phi_e^A \neq B_{i,j}$, we need to argue that $W_i \leq_T A$.

For $x \in \omega^{[\langle i,j,e\rangle]}$, we set a movable marker Γ_x for y_x. Γ_x can move in $\omega^{[\langle i,j,e\rangle]}$ only if get permission from W_i, and can be put into A only if get permission from W_j. More specifically, let Γ_x^s be the candidate of Γ_x at the end of stage s.

Initially, let $\Gamma_x^0 = x$ for all $x \in \omega^{[\langle i,j,e\rangle]}$, and $A_0 = B_{i,j,0} = \emptyset$.

Stages $s+1$ has two steps:

Step 1:

Check whether there exists some $x \leq s$, $x \in \omega^{[\langle i,j,e\rangle]} - B_{i,j,s}$ such that $\Phi_{e,s}^A(x) \downarrow= 0$, and some element smaller than x appears in $W_{i,s+1} - W_{i,s}$, then for the least such x, if exists, move Γ_x^{s+1} to the least number in $\omega^{[\langle i,j,e\rangle]} - A_s$ greater than Γ_x^s and the restraints from higher priority. For $x' > x$ and $x' \in \omega^{[\langle i,j,e\rangle]} - A_s$, move $\Gamma_{x'}^{s+1}$ in order to fresh numbers in $\omega^{[\langle i,j,e\rangle]} - A_s$.

Step 1:

Consider $x \leq s$ and $x \in \omega^{[\langle i,j,e\rangle]} - B_{i,j,s}$. If $\Phi_{e,s}^A(x) \downarrow= 0$, and Γ_x^{s+1} is greater than its use $\varphi_e(x)$ and not restrained from higher priority, we say that x is eligible.

If there exists some x which is eligible, and some element smaller than x appears in $W_{j,s+1} - W_{j,s}$, we say that this requirement requires attention. For the least such x, put x into $B_{i,j}$ and Γ_x^{s+1} into A, and set a restraint $\varphi_e(x)$ on A.

Once some x is put into $B_{i,j}$, we say that this requirement receives attention.

- If a requirement already receives attention, then it can be injured by those requirements with higher priority, as the enumerations of these requirements can change the computation $\Phi_e^A(x)$. In particular, if x is enumerated into A already, then we need to choose another x' as a new witness.
- If a requirement already receives attention, then it will not require attention unless it is injured. By an induction proof, we can show that a requirement can require attention, and hence receive attention at most finitely often.

In this way, for each x, because $\Gamma_x^{s+1} \neq \Gamma_x^s$ only if $W_{i,s+1} \upharpoonright x \neq W_{i,s} \upharpoonright x$ for any s, which means Γ_x^s can change at most finitely many times, so $\Gamma_x = \lim_s \Gamma_x^s$ exists and is recursive in W_i. Furthermore, $B_{i,j} \leq_T \widetilde{W_i}$ as $x \in B_{i,j}$ if and only if $x \in B_{i,j,s}$ for stage $s = \mu t[A_t \upharpoonright (\Gamma_x + 1) = A \upharpoonright (\Gamma_x + 1)]$.

We can show that $B_{i,j} \leq_T W_j$ by the usual permission, that is, $x \in B_{i,j}$ if and only if $x \in B_{i,j,s}$ for $s = \mu t[W_{j,t} \upharpoonright x = W_j \upharpoonright x]$. Thus $B_{i,j} \leq_T \widetilde{W_j}$

Finally, by double use of the following lemma, we can see that if $A <_T W_i$ and $A <_T W_j$, then there must be some eligible x.

Lemma 4.3.1. *If A is r.e., $W_i \not\leq_T A$, U is an infinite set which is r.e. in A, and s_e is a function with input x which is recursive in A, then $V = \{x : x \in U, W_i \upharpoonright x \neq W_{i,s_e(x)} \upharpoonright x\}$ is infinite and r.e. in A.*

Proof. Using oracle A, we can enumerate elements x of U, and recusively compute $s_e(x)$, as U is r.e. in A and s_e is recursive in A. Then x is in V if $W_{i,s+1} \neq W_{i,s}$ at some stage $s > s_e(x)$, so we can enumerate V and thus V is r.e. in A.

Assume V is finite, let y be the largest element in V. Given oracle A, for any x, since U is infinite and r.e. in A, we can enumerate U and get some element z in U such that $z \geq x$ and $z > y$. By the definition of V, $W_i \upharpoonright z = W_{i,s_z} \upharpoonright z$, so $x \in W_i$ if and only if $x \in W_{i,s_e(z)}$. Thus $W_i \leq_T A$, contradicted to our assumption. Hence V is infinite. □

Now we handle requirements for all $\langle i, j, e\rangle$:

$R_{i,j,e}$: if $A <_T \widetilde{W_i}$ and $A <_T \widetilde{W_j}$, then there is an r.e. set $B_{i,j}$ such that $B_{i,j} \leq_T \widetilde{W_i}$, $B_{i,j} \leq_T \widetilde{W_i}$ and $B_{i,j} \neq \Phi_e^A$.

At stage 0: Put Γ_x^0 on x for all $x \in \omega$, let $A_0 = \emptyset$, $B_{i,j,0} = \emptyset$ for all i and j, and set $q_{i,j,e}^0 = 0$ for all i, j and e.

At stage $s+1$: Γ_x^s, A_s, $B_{i,j,s}$ and $q_{i,j,e}^s$ are given.

Step 1, for each triple (i, j, e) such that $\langle i, j, e\rangle \leq s$, find the least x such that

$$\begin{cases} x \leq s, \\ x \in \omega^{[\langle i,j,e\rangle]} - B_{i,j,s}, \\ \Phi_{e,s}^A(x) \downarrow = 0, \\ W_{i,s+1} \upharpoonright x \neq W_{i,s} \upharpoonright x. \end{cases}$$

If exists, for the least such x, let Γ_x^{s+1} be the least y such that

$$\begin{cases} y \in \omega^{[\langle i,j,e\rangle]} - A_s, \\ y > \varphi_e(x), \\ y \geq \Gamma_x^s, \\ y > \max\{q_{i',j',e'}^s\} \text{ for } \langle i', j', e'\rangle \leq \langle i, j, e\rangle. \end{cases}$$

And for $x' \geq x$, let $\Gamma_{x'+1}^{s+1}$ be the least number in $\omega^{[\langle i,j,e\rangle]} - A_s$ larger than $\Gamma_{x'}^{s+1}$.

Step 2, we say that x is eligible if

$$\begin{cases} x \leq s, \\ x \in \omega^{[\langle i,j,e\rangle]} - B_{i,j,s}, \\ \Phi_{e,s}^A(x) \downarrow = 0, \\ \Gamma_x^{s+1} > \varphi_e(x), \\ \Gamma_x^{s+1} > \max\{q_{i',j',e'}^s\} \text{ for } \langle i', j', e'\rangle \leq \langle i, j, e\rangle. \end{cases}$$

If there exists some $\langle i, j, e\rangle \leq s$ such that $R_{i,j,e}$ requires attention (that is, $R_{i,j,e}$ has some eligible x with $W_{j,s+1} \upharpoonright x \neq W_{j,s} \upharpoonright x$, and $q^s_{i,j,e} = 0$), choose the least such $\langle i, j, e\rangle$, and the least x making $R_{i,j,e}$ require attention, let $B_{i,j,s+1} = B_{i,j,s} \cup \{x\}$ and $A_{s+1} = A_s \cup \{\Gamma^{s+1}_x\}$, and set restraint $q^{s+1}_{i,j,e} = \varphi_e(x)$. We say that $R_{i,j,e}$ receives attention.

For $\langle i', j', e'\rangle < \langle i, j, e\rangle$, we set $q^{s+1}_{i',j',e'} = q^s_{i',j',e'}$, and for $\langle i', j', e'\rangle > \langle i, j, e\rangle$, set $q^{s+1}_{i',j',e'} = 0$.

We claim that each $R_{i,j,e}$ could receive attention for at most finitely many times. Fixed i, j and e, suppose the claim is true for $\langle i', j', e'\rangle < \langle i, j, e\rangle$, then there is a certain stage t after which $R_{i',j',e'}$ for $\langle i', j', e'\rangle < \langle i, j, e\rangle$ never put elements into A. If $R_{i,j,e}$ never receives attention after stage t, done. Otherwise, suppose $R_{i,j,e}$ receives attention at stage $s > t$, then by the choice of t, $R_{i,j,e}$ cannot be injured. Then since the restraint $q^s_{i,j,e} \neq 0$ for $s > t$, $R_{i,j,e}$ will never receive attention after stage s.

Finally, we prove that if $A <_T \widetilde{W_i}$ and $A <_T \widetilde{W_j}$, then $R_{i,j,e}$ must receive attention for all e. Suppose not, i.e., there is some e such that $B_{i,j} = \Phi^A_e$, then for any x, let s_e be a function such that $s_e(x)$ is the least stage t such that $\varphi_e(x)$ is defined. s_e is an A-recursive function. Then by applying lemma 4.3.1, $U = \{x : (x \in \omega^{[\langle i,j,e\rangle]}) \wedge (W_i \upharpoonright x \neq W_{i,s_e(x)} \upharpoonright x)\}$ is an infinite A-r.e. set. So step 1 is done for infinitely many $x \in \omega^{[\langle i,j,e\rangle]}$ at some stage $t_e(x) \geq s_e(x)$, and x is eligible at all stages after $t_e(x)$. Furthermore, t_e is an A-recursive function. Let $V = \{x : (x \in U) \wedge (W_j \upharpoonright x \neq W_{j,t_e(x)} \upharpoonright x)\}$. Again, by lemma 4.3.1, V is an infinite A-r.e. set. Since $\max\{q^s_{i',j',e'}\}$ for all $\langle i', j', e'\rangle \leq \langle i, j, e\rangle$ is finite, after $R_{i',j',e'}$ for all $\langle i', j', e'\rangle \leq \langle i, j, e\rangle$ settle down, $R_{i,j,e}$ must receive attention, which means that we put some x into $B_{i,j}$, which implies that $B_{i,j}(x) = 1 \neq 0 = \Phi^A_e(x)$, a contradiction.

We now combine two arguments to construct an r.e. set A which has thin-free nonbranching degree. We shall meet the requirements for all i, j, e:

$R_{i,j,e}$: if $A <_T \widetilde{W_i}$ and $A <_T \widetilde{W_j}$, then there is an r.e. set $B_{i,j}$ such that $B_{i,j} \leq_T \widetilde{W_i}$, $B_{i,j} \leq_T \widetilde{W_i}$ and $B_{i,j} \neq \Phi_e^A$.

Q_e : If $\Phi_e^A, \Psi_e^{\Phi_e^A}$ are total, $\Psi_e^{\Phi_e^A} = A$, and $\Phi_e^A \in [P_e]$, then $[P_e]$ is not thin.

For each Q_e, we consider the following subrequirements:

$N_{e,n}$: If $\Phi_e^A, \Psi_e^{\Phi_e^A}$ total, $\Psi_e^{\Phi_e^A} = A$, and $\Phi_e^A \in [P_e]$, then $\exists (x_{e,n}, z_{e,n}]$, $[P_e]$ contains at least one element extending $\Phi_e^A \upharpoonright x_{e,n}$ but not $\Phi_e^A \upharpoonright z_{e,n}$.

The method to meet a single $Q-$ and $R-$requirement has been shown in section 4.2 and above, and now we consider the interaction between them.

First, we give $\omega^{[0]}$ section to all $Q-$requirements, and $\omega^{[\langle i,j,e\rangle+1]}$ section to each $R_{i,j,e}$. Namely, Q_e for all e can only put elements from $\omega^{[0]}$ into A, and $R_{i,j,e}$ must choose x and y_x in $\omega^{[\langle i,j,e\rangle+1]}$ for each $\langle i,j,e\rangle$.

Second, note that $R_{i,j,e}$ could put some y_x into A and make $S_{e'}$ nonrecursive for some $e' \leq \langle i,j,e\rangle$ for the same reason shown in 4.2. To solve this, $Q-$requirements need to define more confirmation intervals, which can provide confirmation to some $R-$requirements. We first consider two requirements, say $R_{i,j,e}$ and $Q_{e'}$ for $e' \leq \langle i,j,e\rangle$. When $Q_{e'}$ defines confirmation intervals, those intervals select $y_{e',3k+2}$ in $\omega^{[0]}$ and $\omega^{[\langle i,j,e\rangle+1]}$ interweavingly. Then for $R_{i,j,e}$, at each stage s, if in step 1, Γ_x for some x should move, it moves to the position which is greater than the largest $z_{e',n'}$ defined by $Q_{e'}$ until stage s, and $\varphi_e(x)$. And in step 2, y_x is chosen as the least $y > \varphi_e(x)$ in $\omega^{[\langle i,j,e\rangle+1]}$ such that y is either confirmed or greater than the last $z_{e',n'}$ defined by stage s, and smaller than Γ_x^s. Γ_x^s is not the position of y_x, but the upper bound of y_x here.

In this way, on one hand, if $R_{i,j,e}$ puts some element y_x into A at a certain stage s, it is either confiremed by $Q_{e'}$, or larger than the largest $z_{e',n}$ has been defined until stage s. In the first case, y_x would not injure $Q_{e'}$ as discussed in 4.2. In the second case, y_x would not change the computation of Φ_e^A up to the end point of the last interval defined until stage s, and thus would not injure the defined part of S_e. On the other hand, as Γ_x^s is the upper bound of y_x at stage s, it leaves enough room for the choice of y_x, such that waiting for confirmation from $Q_{e'}$ does not kill all eligible x.

In order to provide confirmation for all requirements with lower priority, we ensure that $Q_{e'}$ has enough $y_{e',3k+2}$ in $\omega^{[0]}$ and $\omega^{[\langle i,j,e\rangle+1]}$ for $\langle i,j,e\rangle > e'$ if $Q_{e'}$ has infinitely many intervals. We establish it by defining levels for all confirmation intervals: the first level consists of the first confirmation interval, the second level contains the second and third confirmation intervals, and so on. Then for m-th level, there are m confirmation intervals, and they select $y_{e',3k'+2}$ in $\omega^{[0]}$, $\omega^{[e'+1]} - A_s$ to $\omega^{[e'+m-1]} - A_s$ in order.

Theorem 4.3.2. *There is a nonbranching thin-free r.e. degree.*

Proof. Let $(\Phi_e, \Psi_e, P_e, W_i, W_j)$ be an effective list of quintuple, where Φ_e and Ψ_e are Turing functionals, P_e are primitive recursive trees, and W_i and W_j are r.e. sets. We construct an r.e. set A to meet all Q_e and $R_{i,j,e}$.

Construction: At stage 0: Let $A_0 = B_{i,j,0} = \emptyset$ for all i and j. Put Γ_x^0 on x for all $x \in \omega$. Set $r_e^0 = 0$ and $q_{i,j,e}^0 = 0$ for all i, j and e. Define a sequence of recursive functions f_e for all e, such that $f_e(k) = 0$ if $k = \Sigma_{p=0}^m p$ for some m, and $f_e(k) = e+l$ if $k = \Sigma_{p=0}^m p + l$ for some m and $0 < l < m$.

At stage $s+1$: A_s, $B_{i,j,s}$, Γ_x^s, $q_{i,j,e}^s$ and r_e^s are given. Consider $R_{i,j,e}$ for $\langle i,j,e\rangle \leq s$, and Q_e for $e \leq s$ with $r_e^s = 0$.

Let $U_0(e)$ be $\max\{q_{i',j',e'}\}$ for $\langle i',j',e'\rangle < e$, and $U_1(i,j,e)$ be $\max\{q_{i',j',e'}^s, r_{e''}^s\}$ for $\langle i',j',e'\rangle \leq \langle i,j,e\rangle$ and $e'' \leq \langle i,j,e\rangle$

For a subrequirement $N_{e,n}$ of Q_e:

Step 1:

If $x_{e,0}$ is undefined for $n = 0$, or $z_{e,n-1}$ is defined, $\Phi^A_{e,s} \upharpoonright z_{e,n-1} \downarrow \in P_e$ and $x_{e,n}$ is undefined for $i > 0$, then select some new large $x_{e,n}$.

Wait for $\Phi^A_{e,s} \upharpoonright x_{e,n} \downarrow$.

Step 2:

For $e > 0$, if $y_{e,n} \leq z_{e-1,n'}$ for the largest n' such that $z_{e-1,n'}$ is defined, and $y_{e,n} \neq y_{e-1,3k+2}$ for the least k such that $y_{e-1,3k+2} > \varphi_e(x_{e,n})$, cancel $y_{e,n}$ and all nodes defined above it.

If $\Phi^A_{e,s} \upharpoonright x_{e,n} \downarrow$ and on P_e, but $y_{e,n}$ is undefined, select $y_{e,n}$ to be

- if $e = 0$, select $y_{e,n}$ to be the least y such that $y > \varphi_e(x_{e,n})$, $y > U_0(e)$, and $y \in \omega^{[f_e(k)]} - A_s$;
- if $e > 0$ and $y_{e-1,3k'+2} \leq \varphi_e(x_{e,n})$ for all $y_{e-1,3k'+2}$ defined, select $y_{e,n}$ to be the least y such that $y > \varphi_e(x_{e,n})$, $y > U_0(e)$, $y > z_{e-1,n'}$ for the largest n' such that $z_{e-1,n'}$ is defined, and $y \in \omega^{[f_e(k)]} - A_s$;
- if $e > 0$ and there exists k' such that $y_{e-1,3k'+2} > \varphi_e(x_{e,n})$, $y_{e-1,3k'+2} > U_0(e)$, and $y_{e-1,3k'+2} \in \omega^{[f_e(k)]} - A_s$, select $y_{e,n}$ to be $y_{e-1,3k'+2}$ for the least such k'.

Wait for $\Phi^A_{e,s} \upharpoonright y_{e,n} \downarrow$, $A_s \upharpoonright y_{e,n} = \Psi^{\Phi^A_{e,s}}_{e,s} \upharpoonright y_{e,n}$ and $\Phi^A_{e,s} \upharpoonright \psi_e(y_{e,n}) \downarrow$ in P_e.

Step 3:

If $\Phi^A_{e,s} \upharpoonright y_{e,n} \downarrow$, $A_s \upharpoonright y_{e,n} = \Psi^{\Phi^A_{e,s}}_{e,s} \upharpoonright y_{e,n}$ and $\Phi^A_{e,s} \upharpoonright \psi_e(y_{e,n}) \downarrow \in P_e$, but $z_{e,n}$ is undefined, select a new large $z_{e,n} > \psi_e(y_{e,n})$.

Wait for $\Phi^A_{e,s} \upharpoonright z_{e,n} \downarrow \in P_e$.

Step 4:

If $\Phi^A_{e,s} \upharpoonright z_{e,n} \downarrow \in P_e$, For $i = 3k + 1$, we terminate all strings in the interval $(x_{e,n}, z_{e,n}]$ of P_e in S_e. For $i = 3k$ or $3k + 2$, we copy all alive string of length s from P_e into S_e.

Step 5:

For $i = 3k + 2$, skip this step. For $i = 3k$ or $3k + 1$, test whether all strings in the interval $(x_{e,n}, z_{e,n}]$ of P_e are non-extendible in P_e. If yes, put $y_{e,n}$ into A. Set $r_e^{s+1} = z_{e,n}$, $r_{e'}^{s+1} = r_{e'}^s$ for $e' < e$ and $= 0$ for $e' > e$. Let $q_{i',j',e'}^{s+1} = q_{i',j',e'}^s$ for $\langle i', j', e'\rangle < e$, and be 0 for $\langle i', j', e'\rangle \geq e$. Cancel all intervals defined by $Q_{e'}$ for $e' > e$, and say $R_{i',j',e'}$ does not receive attention for $\langle i', j', e'\rangle > e$.

For $R_{i,j,e}$, $\langle i, j, e\rangle \leq s$:

Step 1:

Find the least x such that

$$\begin{cases} x \leq s, \\ x \in \omega^{[\langle i,j,e\rangle+1]} - B_{i,j,s}, \\ \Phi_{e,s}^A(x) \downarrow = 0, \\ W_{i,s+1} \upharpoonright x \neq W_{i,s} \upharpoonright x. \end{cases}$$

If exists, for that x, let Γ_x^{s+1} be the least y such that

$$\begin{cases} y \in \omega^{[\langle i,j,e\rangle+1]} - A_s, \\ y > \varphi_e(x), \\ y \geq \Gamma_x^s, \\ y > U_1(i, j, e), \\ y > z_{e',n} \text{ for } e' \leq \langle i, j, e\rangle \text{ and } n \text{ such that } z_{e',n} \text{ is defined.} \end{cases}$$

For $x' \geq x$, let $\Gamma_{x'+1}^{s+1}$ be the least number in $\omega^{[\langle i,j,e\rangle+1]} - A_s$ larger than $\Gamma_{x'}^{s+1}$.

Step 2:

We say that x is eligible if $x \leq s$, $x \in \omega^{[\langle i,j,e\rangle+1]} - B_{i,j,s}$, $\Phi_{e,s}^A(x) \downarrow= 0$, and there is some y such that $y \in \omega^{[\langle i,j,e\rangle+1]} - A_s$, $\varphi_e(x) < y \leq \Gamma_x^{s+1}$,

$y > U_1(i,j,e)$, and for each $e' \leq \langle i,j,e \rangle$, for the largest n such that $z_{e',n}$ is defined, either $y > z_{e',n}$, or $y = y_{e',3k+2}$ for some k.

If there exists some $R_{i,j,e}$ has eligible x with $W_{j,s+1} \upharpoonright x \neq W_{j,s} \upharpoonright x$, and $q^s_{i,j,e} = 0$, we say $R_{i,j,e}$ requires attention.

For the least $\langle i,j,e \rangle \leq s$ such that $R_{i,j,e}$ requires attention, select the least eligible x, and y^{s+1}_x be the least y making x eligible. Let $B_{i,j,s+1} = B_{i,j,s} \cup \{x\}$ and $A_{s+1} = A_s \cup \{y^{s+1}_x\}$. Set $q^{s+1}_{i,j,e} = \varphi_e(x)$, $q^{s+1}_{i',j',e'} = q^s_{i',j',e'}$ for $\langle i',j',e' \rangle < \langle i,j,e \rangle$ and $= 0$ for $\langle i',j',e' \rangle > \langle i,j,e \rangle$, and $r^{s+1}_{e'} = r^s_{e'}$ for $e' \leq \langle i,j,e \rangle$ and $= 0$ for $e' > \langle i,j,e \rangle$. We say $R_{i,j,e}$ receives attention, and for $\langle i',j',e' \rangle > \langle i,j,e \rangle$, $R_{i',j',e'}$ does not receive attention.

Veryfication:

Lemma 4.3.3. *For all i and j, $B_{i,j} \leq_T \widetilde{W_i}$ and $B_{i,j} \leq_T \widetilde{W_i}$.*

Proof. Fixed i and j, for each x and s, $\Gamma^{s+1}_x \neq \Gamma^s_x$ only if $W_{i,s+1} \upharpoonright x \neq W_{i,s} \upharpoonright x$, so $\Gamma_x = \lim_s \Gamma^s_x$ exists and recursive in W_i. By the choice of y_x, $y^s_x \leq \Gamma^s_x \leq \Gamma_x$. Let s_x be the least stage s such that $W_{i,s} \upharpoonright x = W_i \upharpoonright x$ and $A_s \upharpoonright \Gamma_x = A \upharpoonright \Gamma_x$, then s_x is recursive in $\widetilde{W_i}$. Because $x \in B_{i,j}$ if and only if $x \in B_{i,j,s_x}$, we have $B_{i,j} \leq_T \widetilde{W_i}$.

$B_{i,j} \leq_T W_j$ as $x \in B_{i,j}$ if and only if $x \in B_{i,j,s}$ for the least stage s such that $W_{j,t} \upharpoonright x = W_j \upharpoonright x$. Thus $B_{i,j} \leq_T \widetilde{W_j}$. □

Lemma 4.3.4. *Any Q_e and $R_{i,j,e}$ can put a certain element into A for at most finitely many times.*

Proof. We prove it by induction. Fix e and suppose the result is correct for $Q_{e'}$ for all $e' < e$ and $R_{i',j',e'}$ for all $\langle i',j',e' \rangle < e$. Then there is a stage t after which these requirements never put elements into A. At some stage $s > t$, if Q_e put some element $y_{e,n}$ into A, then $r^s_e = z_{e,n}$. Then at any stage $> s$, by the choice of

t, requirements with higher priority never put elements into A, and requirements with lower priority choose no elements smaller than $z_{e,n}$, so Q_e cannot be injured. Hence the result is true for Q_e.

Fix i, j and e, suppose the result is correct for $Q_{e'}$ for all $e' \leq \langle i, j, e\rangle$ and $R_{i',j',e'}$ for all $\langle i', j', e'\rangle < \langle i, j, e\rangle$, and thus suppose t is the stage after which these requirements never put elements into A. If $R_{i,j,e}$ receives attention at some stage $s > t$, then by the choice of t, $R_{i,j,e}$ cannot be injured, so $R_{i,j,e}$ will never receive attention after stage s, and thus the result is true for $R_{i,j,e}$. □

Lemma 4.3.5. *For any e and n, $N_{e,n}$ can be injured for at most finitely many times.*

Proof. $N_{e,n}$ could be injured in two cases: The first case is that $Q_{e'}$ for some $e' < e$ puts some element into A, or $R_{i',j',e'}$ for some $\langle i', j', e'\rangle < e$ receives attention. The second case is that for $e > 0$, $y_{e,n}$ is defined greater that the last $z_{e-1,n'}$ at stage t (without confirmation), and at stage $s > t$, $z_{e-1,n''} > z_{e-1,n'}$ is defined and $y_{e,n} \leq z_{e-1,n''}$.

We prove that $N_{e,n}$ are finitely injuried by induction first on n then on e. It is obvious that $N_{0,i}$ cannot be injured. Fix $e > 0$, suppose $N_{e',n'}$ for $e' < e$ and all n' are finitely injured. Conside $N_{e,0}$. Suppose after stage t, $Q_{e'}$ for $e' < e$ and $R_{i',j',e'}$ for $\langle i', j', e'\rangle < e$ would not put elements into A. Then $x_{e,0}$ cannot be injured if defined after stage t. Then if $N_{e,0}$ reaches step 2, and there exists some $y_{e-1,3k+2} > \varphi_e(x_{e,0})$, suppose the least such $y_{e-1,3k+2}$ settles down at some stage $s > t$, then $y_{e,0}$ is defined with conformation $y_{e-1,3k+2}$, and $y_{e-1,3k+2}$ never change after stage s, which means that $y_{e,0}$ cannot be injured after stage s, and so is $z_{e,0}$; if $N_{e,0}$ reaches step 2 but all $y_{e-1,3k+2} \leq \varphi_e(x_{e,0})$, then $y_{e,0}$ would not be injured by the second case, and so is $z_{e,0}$. In the similar way, $N_{e,n}$ for $n > 0$ can be proved to be finitely injured. □

Lemma 4.3.6. *Q_e is satisfied for all e.*

Proof. Fix e, Q_e has three outcomes:

(1) Wait at some $N_{e,n}$ after some stage s. That is, after $N_{e,n}$ settles down, it waits at one of the step 1, 2 or 3 for all stage $> s$, which implies either Φ_e^A or $\Psi_e^{\Phi_e^A}$ is not total, or Φ_e^A is not Turing equivalent to A, so Q_e is met.

(2) After the stage $Q_{e'}$ for all $e' < e$ and $R_{i',j',e'}$ for all $\langle i', j', e'\rangle < e$ never put elements into A, some $N_{e,n}$ puts $y_{e,n}$ into A. Then Q_e is met by the same analysis as showed in Figure 4.1 4.2.

(3) $N_{e,n}$ reach step 5 and never put $y_{e,i}$ into A for all n. In this case, we get a recursive subtree S_e of P_e by the recursive sequence of intervals $\{(x_{e,n}, z_{e,n}]\}$. Φ_e^A lies on P_e, and along Φ_e^A, there are at least one infinte path in both $[S_e]$ and $[P_e]$, and one infinte paths in $[P_e]$ but not in $[S_e]$ for every three consective intervals of the sequence. By the construction of the sequence of intervals, we know that none of any two of these infinite paths are compatible with each other. So $[S_e]$ is a Π_1^0 subclass of $[P_e]$ but not the intersection of $[P_e]$ with any clopen set.

So Q_e is satisfied in all three cases. □

Lemma 4.3.7. *For any i, j and e, if $A <_T \widetilde{W_i}$ and $A <_T \widetilde{W_j}$, then finally $R_{i,j,e}$ must receive attention.*

Proof. Suppose not, i.e., $B_{i,j} = \Phi_e^A$, then for any x, define a function s_e by letting $s_e(x)$ be the least stage t such that $\varphi_e(x)$ is defined, then s_e is recursive in A. Let $U = \{x : (x \in \omega^{[\langle i,j,e\rangle+1]}) \wedge (W_i \upharpoonright x \neq W_{i,s_e(x)} \upharpoonright x)\}$. Then U is an infinite A-r.e. set by lemma 4.3.1. This means that step 1 is done for infinitely many $x \in \omega^{[\langle i,j,e\rangle+1]}$ at some stage $t_e(x) > s_e(x)$, and t_e is an A-recursive function.

By lemma 4.3.4, $U_1(i,j,e)$ is finite. Consider $Q_{e'}$ for $e' < \langle i,j,e\rangle$. If $Q_{e'}$ only define finitely many intervals, then for those $x \in \omega^{[\langle i,j,e\rangle+1]}$ larger than the largest $z_{e',n}$ defined, the corresponding y_x is not restrained by $Q_{e'}$. As the restraint from higher priority is finite, those x are always eligible after stage $t_e(x)$.

Otherwise, there are infinitely many elements in $\omega^{[\langle i,j,e\rangle+1]}$ being confirmed. Suppose the least such is y'. For each x, at stage $t_e(x)$, Γ_x moves to some position above the largest $z_{e',n}$ defined until stage $t_e(x)$. As there are infinitely many confirmed elements in this case, so for $x > y'$, we can select y_x from confirmed elements. Then for x larger than the restraint from higher priority, x is always eligible after stage $t_e(x)$. So in this case, there are infinitely many eligible x after stage $t_e(x)$.

Let $V = \{x : (x \in U) \wedge (W_j \upharpoonright x \neq W_{j,t_e(x)} \upharpoonright x)\}$. By applying lemma 4.3.1 again, V is an infinite A-r.e. set.

So after $R_{i',j',e'}$ for all $\langle i',j',e'\rangle \leq \langle i,j,e\rangle$ settle down, and $Q_{e'}$ for all $e' < \langle i,j,e\rangle$ never puts elements into A, $R_{i,j,e}$ must receive attention, which means that there is some x such that $B_{i,j}(x) = 1 \neq 0 = \Phi_e^A(x)$, contradicted to $B_{i,j} = \Phi_e^A$. □

Hence, A has thin-free nonbranching degree. □

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