

# On some locally 3-transposition graphs

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## Abstract

Let  $\Sigma_n^\epsilon$  be the graph defined on the (+)-points of an  $n$ -dimensional  $\text{GF}(3)$ -space carrying a nondegenerate symmetric bilinear form with discriminant  $\epsilon$ , points are adjacent iff they are perpendicular. We prove that if  $\epsilon = 1$ ,  $n \geq 6$  (resp.  $\epsilon = -1$ ,  $n \geq 7$ ) then  $\Sigma_{n+1}^\epsilon$  is the unique connected locally  $\Sigma_n^\epsilon$  graph. One may view this result as a characterization of a class of  $c^k \cdot C_2$ -geometries (or 3-transposition groups). We briefly discuss an application of the result to a characterization of Fischer's sporadic groups.

## 1. Introduction and results

The study of geometries of classical groups as point-line systems with fixed local structure is very extensive. For instance, see Tits [12]. We refer the reader to a paper of Cohen and Shult [4] for a brief survey on more recent results.

In this paper we use the aforementioned approach to characterize some "nonclassical" geometries, namely 3-transposition graphs that arise from orthogonal  $\text{GF}(3)$ -groups.

For the concept of a 3-transposition graph and a 3-transposition group, see Fischer [8]. In case of characteristic 2 groups such graphs come from classical geometries. It is worthwhile to mention the work of Hall and Shult [9] characterizing some class of 3-transposition graphs as *locally cotriangular* ones. Note that the graphs  $\Sigma_n^\epsilon$  considered in the present paper are not locally cotriangular (apart from finitely many exceptions for small  $n$ ).

There is a one-to-one correspondence between 3-transposition graphs and *Fischer spaces* (see, e.g. Buekenhout [2], Cuypers [6], Weiss [13], [14]), namely 3-transposition graphs are complements to the collinearity graphs of Fischer spaces. It turns out that it is possible to exploit this duality and classification of Fischer spaces in the final part of the proof of our theorem 1.1. Note, however, that locally  $\Sigma_n^\epsilon$  graphs not always arise from Fischer spaces.

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E.g., there are at least two nonisomorphic locally  $\Sigma_6^-$ -graphs, one of them has diameter 4, (i.e. it does not correspond to any Fischer space) see e.g. [1]. The author has shown (in a paper in preparation) that they are the only examples of such graphs.

Throughout the paper we consider undirected graphs without loops and multiple edges. Given a graph  $\Gamma$ , let us denote the set of vertices by  $V = V\Gamma$ , the set of edges by  $E = E\Gamma$ . Given two graphs  $\Gamma, \Delta$ , the graph  $\Gamma \cup \Delta$  (resp. the graph  $\Gamma \cap \Delta$ ) is the graph with the vertex set  $V\Gamma \cup V\Delta$  (resp.  $V\Gamma \cap V\Delta$ ) and the edge set  $E\Gamma \cup E\Delta$  (resp.  $E\Gamma \cap E\Delta$ ). Given  $v \in V\Gamma$ , we denote by  $\Gamma_i(v)$  the subgraph induced by vertices at distance  $i$  from  $v$ , and  $\Gamma_1(v) = \Gamma(v)$ . Furthermore,  $\Gamma(X) = \bigcap_{x \in X} \Gamma(x)$ . To simplify the notation we use  $\Gamma(v_1, \dots, v_k)$  instead of  $\Gamma(\{v_1, \dots, v_k\})$  and  $u \in \Gamma_i(\dots)$  instead of  $u \in V\Gamma_i(\dots)$ . As usual,  $v = v(\Gamma) = |V\Gamma|$ ,  $k = k(\Gamma) = v(\Gamma(x))$ , where  $x \in V\Gamma$ . Let  $y \in \Gamma_2(x)$ . We denote  $\mu = \mu(x, y) = \mu(\Gamma) = v(\Gamma(x, y))$ . Of course, we use  $k, \mu$  if it makes sense, i.e. if those numbers are independent on the particular choice of the corresponding vertices. If  $\Delta$  is a (proper) subgraph of  $\Gamma$  we denote this fact as  $\Delta \subseteq \Gamma$  (resp.  $\Delta \subset \Gamma$ ).

We denote the complete multipartite graph with the  $m$  parts of equal size  $n$  by  $K_{n \times m}$ .  $\text{Aut}(\Gamma)$  denotes the automorphism group of  $\Gamma$ . Our group-theoretic notation is as in [5]. Let  $\Gamma, \Delta$  be two graphs. We say that  $\Gamma$  is *locally*  $\Delta$  if  $\Gamma(v) \cong \Delta$  for any  $v \in V(\Gamma)$ . Let  $\Gamma, \bar{\Gamma}$  be two graphs. We say that  $\Gamma$  is a *cover* of  $\bar{\Gamma}$  if there exists a mapping  $\varphi$  from  $V\Gamma$  to  $V\bar{\Gamma}$  which maps edges to edges. Suppose we have a chain of graphs  $\Sigma_1, \dots, \Sigma_n$ , such that  $\Sigma_i$  is locally  $\Sigma_{i-1}$ ,  $i = 2, \dots, n$ . Then for any complete  $k$ -vertex subgraph  $\Upsilon$  of  $\Sigma_m$ ,  $1 < k < m \leq n$  we have  $\Sigma_m(V\Upsilon) \cong \Sigma_{m-k}$ . We say that a graph  $\Gamma$  is a *triple* graph if for each nonedge  $(u, v)$  there exist a unique  $w \in V\Gamma$  such that  $\Gamma(u, v) = \Gamma(u, w) = \Gamma(v, w)$ .

We slightly adopt notation and several basic facts from [10]. Let  $T = T_n$  be an  $n$ -dimensional  $\text{GF}(3)$ -vector space carrying a nondegenerate symmetric bilinear form  $(,)$  with discriminant  $\varepsilon$ . We say that the point  $\langle v \rangle \subset T$  (or a nonzero vector  $v \in T$ ) is of type  $(+)$ ,  $(-)$  or *isotropic* according to  $(v, v) = 1, -1, 0$  respectively. Since the form is constant on a point, the notation like  $(p, q)$  for points  $p, q$  will be used freely. The orthogonal complement of  $X \subseteq T$  in  $T$  is denoted by  $X^\perp$ .

Define the graph  $\Sigma_n^\varepsilon = \Gamma(V, E)$  as follows. Let  $V$  be the set of  $(+)$ -points of  $T$ . Define  $E = \{(u, v) \subset V \times V \mid (u, v) = 0\}$ , i.e. the edges are pairs of perpendicular  $(+)$ -points. Given  $\Gamma = \Sigma_n^\varepsilon$ , we denote by  $T(\Gamma)$  the underlying  $\text{GF}(3)$ -space. Note that  $\Sigma_n^\varepsilon$ ,  $n \geq 5$  if  $\varepsilon = 1$ ,  $n \geq 4$  if  $\varepsilon = -1$ , is a rank 3 graph with automorphism group  $GO_n^\mu(3)$ , where Witt defect  $\mu$  is empty if  $n$  is odd, otherwise  $\mu = (-\varepsilon)^{n/2}$ . Since  $T_n$  can be represented as the orthogonal direct sum of a  $(+)$ -point and  $T_{n-1}$ , the graph  $\Sigma_n^\varepsilon$  is locally  $\Sigma_{n-1}^\varepsilon$ . We denote  $\Sigma_n^1$  by

$\Sigma_n^+$ , and  $\Sigma_n^{-1}$  by  $\Sigma_n^-$ . Note that  $\Sigma_n^\epsilon$  is a triple graph. We refer the interested reader to [10] for more detailed information about  $\Sigma_n^\epsilon$ . We will prove the following theorem.

**Theorem 1.1** *Let  $\Theta = \Theta_{n+1}$  be a connected locally  $\Sigma_n^\epsilon$ -graph, where  $\epsilon = 1, n \geq 6$ , or  $\epsilon = -1, n \geq 7$ . Then  $\Theta$  is isomorphic to  $\Sigma_{n+1}^\epsilon$ .*

**Remark 1**

One may view the graphs  $\Theta_{n+1}$  as the collinearity graphs of certain  $c^k \cdot C_2(s, t)$ -geometries  $\mathcal{G}(\Theta_{n+1})$  (here  $k = n - (\epsilon + 9)/2$ ), i.e. rank  $k + 2$  geometries with diagram



Conversely, the elements of the geometry may be viewed as  $i$ -cliques of the graph with natural incidence,  $i = 1, 2, \dots, k + 1, s + k + 1$ . Meixner has proved the following result [10].

**Result 1.2** *Let  $\mathcal{G}$  be a residually connected flag-transitive  $c^k \cdot C_2$ -geometry. Then if the  $c^1 \cdot C_2$ -residues (resp.  $c^3 \cdot C_2$ -residues) of  $\mathcal{G}$  are isomorphic to  $\mathcal{G}(\Sigma_6^+)$  (resp. to  $\mathcal{G}(\Sigma_7^-)$ ) then  $\mathcal{G} = \mathcal{G}(\Sigma_{k+s}^+)$  (resp.  $\mathcal{G} = \mathcal{G}(\Sigma_{k+4}^-)$ ).*

Our theorem implies that the flag-transitivity assumption can be replaced to a geometric condition (X) from [10]. (X) states that in the collinearity graph of  $\mathcal{G}$  each  $i$ -clique is the shadow of some element,  $i = 1, 2, \dots, k + 1, s + k + 1$ .

**Remark 2**

The significance of  $\Sigma_{i+3}^+$  as subgraphs of 3-transposition graphs  $\Delta_{2i}$  of Fischer's sporadic simple groups  $F_{i_{2i}}$  ( $i = 2, 3, 4$ ) is well-known. Namely, for  $\Gamma = \Delta_{2i}$  the subgraph  $\Gamma(x, y)$ , where  $x$  and  $y$  are two vertices at distance 2, is isomorphic to  $\Sigma_{i+3}^+$ . The author used theorem 1.1 to prove the following result [11].

**Result 1.3** *Any connected locally  $\Delta_{22}$  (resp.  $\Delta_{23}$ ) graph is isomorphic to  $\Delta_{23}$  (resp. to  $\Delta_{24}$  or to its 3-fold antipodal cover).*

**2. Proof of the theorem**

*Preliminaries.* A proof of the following technical statement is omitted.

**Lemma 2.1** Let  $\Gamma = \Sigma_n^\epsilon$ ,  $n \geq 5$ , and let  $a, b, c$  be isotropic points of  $T = T(\Gamma)$ .

If  $(a, b) = 0$ ,  $a \neq b$ ,  $(a, c) \neq 0$  then

- (i)  $a^\perp \cap c^\perp \cap \Gamma \cong \Sigma_{n-2}^{-\epsilon}$ ,
- (ii)  $a^\perp \cap b^\perp \cap \Gamma$  is not isomorphic to  $\Sigma_{n-2}^{-\epsilon}$ .
- (iii) Denote  $\Upsilon_p = \Gamma \cap p^\perp$ . If  $\Upsilon_a \cap \Upsilon_b$  contains  $\{v\} \cup \Upsilon_a(v)$  for some  $v \in V\Upsilon_a$  then  $a = b$ .
- (iv) For any  $u \in V\Gamma \setminus V\Upsilon_a$ , the subgraph  $\Gamma(u) \cap \Upsilon_a$  is isomorphic to  $\Sigma_{n-2}^{-\epsilon}$ .

Neighbourhood of two vertices at distance two. We start with a simple general fact. Let  $\Delta$  be a connected graph satisfying the following property

- (\*) For any  $u \in V\Delta$  and  $v \in V\Delta \setminus (V\Delta(u) \cup \{u\})$  the subgraph  $\Delta(u, v)$  is isomorphic to some  $M_\Delta$ , whose isomorphism type is independent on the particular choice of  $u$  and  $v$ .

**Lemma 2.2** Let  $\Gamma$  be locally  $\Delta$  graph, where  $\Delta$  satisfies (\*). Then for any  $u \in V\Gamma$ ,  $v \in \Gamma_2(u)$  the graph  $\Gamma(u, v)$  is locally  $M_\Delta$ .

Note that  $\Gamma = \Sigma_n^\epsilon$  satisfies (\*), when  $\epsilon = 1, n \geq 5$  (resp.  $\epsilon = -1, n \geq 4$ ). Indeed, the stabilizer of  $u \in V\Gamma$  in  $\text{Aut}(\Gamma)$  acts transitively on  $\Gamma_2(u)$ . This implies (\*). Thus lemma 2.2 holds for locally  $\Gamma$  graphs. The next statement characterizes locally  $M_\Gamma$ -subgraphs of  $\Gamma$ .

**Proposition 2.3** Let  $\Gamma = \Sigma_n^\epsilon$ , and either  $\epsilon = 1, n \geq 6$ , or  $\epsilon = -1, n \geq 7$ . Let  $\Omega \subset \Gamma$  be a locally  $M_\Gamma$  graph. Then there exists a unique isotropic point  $p \subset T(\Gamma)$  such that  $\Omega = \Gamma \cap p^\perp$ .

**Proof.** We proceed by induction on  $n$ . It is straightforward to check its basis. We leave it to the reader, noticing that for  $\epsilon = 1, n = 6$  (resp.  $\epsilon = -1, n = 6$ ) it suffices to classify locally  $K_{3 \times 2} \cup K_{3 \times 2} \cup K_{3 \times 2}$  (resp.  $K_{3 \times 4} \cup K_{3 \times 4} \cup K_{3 \times 4}$ ) subgraphs of  $\Gamma$ .

Now let us check the inductive step. Let  $\Omega$  be a connected component of a locally  $M_\Gamma$  subgraph of  $\Gamma$ ,  $v_1 \in V\Omega$ ,  $v_2 \in \Omega(v_1)$ . By the inductive hypothesis,  $\Omega(v_i) = \Gamma(v_i) \cap p_i^\perp$ , where  $p_i$  is an isotropic point of  $T(\Gamma(v_i))$ , hence of  $T(\Gamma)$  ( $i = 1, 2$ ). Since  $p_2 \subset v_1^\perp$ , it defines a locally  $M_{\Gamma(v_1)}$  subgraph  $\Upsilon$  of  $\Gamma(v_1)$ , and  $\Upsilon(v_2) = \Omega(v_1, v_2)$ . Hence by lemma 2.1 (iii), applied to  $\Gamma(v_1)$ ,  $p_1 = p_2$ . Therefore  $\Omega = \Gamma \cap p_1^\perp$ . Finally, it is easy to check that  $\Omega$  is a unique connected component of the subgraph under consideration.  $\square$

*Final part of the proof.* Let  $\Theta$  be a connected locally  $\Gamma = \Sigma_n^\epsilon$  graph. Here we assume either  $\epsilon = 1, n \geq 6$  or  $\epsilon = -1, n \geq 7$ . Pick a vertex  $u \in V\Theta$ .

**Lemma 2.4**

- (i)  $\mu(\Theta) = \mu(\Sigma_{n+1}^\epsilon), v(\Theta) = v(\Sigma_{n+1}^\epsilon)$ .
- (ii)  $\Theta$  is a triple graph.

**Proof.** (i). The first claim follows from proposition 2.3. Indeed, counting in two ways the edges between  $\Theta(u)$  and  $\Theta_2(u)$ , we obtain the precise value of the number of vertices of  $\Theta$ . Let  $v \in \Theta_2(u), w \in \Theta(v) \setminus \Theta(u)$ . By lemma 2.1 (iv) we obtain  $\Theta(u, v, w) \cong \Sigma_{n-2}^{-\epsilon}$ , hence nonempty. Thus the diameter of  $\Theta$  equals two, and we are done.

(ii). Assume that there exist three distinct vertices  $v_i \in \Theta_2(u), i = 1, 2, 3$  such that  $\Upsilon = \Theta(u, v_1) = \Theta(u, v_2) = \Theta(u, v_3)$ . It contradicts the fact that  $\Theta(w)$ , where  $w \in V\Upsilon$ , is a triple graph. Observe that  $|\Theta_2(u)|$  is exactly twice the number of isotropic points in  $T(\Gamma)$ . Hence we have no choice determining the edges between  $\Theta(u)$  and  $\Theta_2(u)$ . Now since  $\Sigma_{n+1}^\epsilon$  is a triple graph, the same is true for  $\Theta$ . □

Let us denote  $\Gamma = \Theta(u), \Xi = \Theta_2(u)$ . Let  $\Upsilon$  be the graph defined on isotropic points of  $T(\Gamma)$ , two points are adjacent if they are not perpendicular.

**Proposition 2.5**  $\Xi$  is a two-fold cover of  $\Upsilon$ .

**Proof.** Let  $(v_1, v_2) \in E\Xi$ . By lemma 2.1 (iv) we have  $\Theta(u, v_1, v_2) \cong \Sigma_{n-2}^{-\epsilon}$ . Denote by  $p_i$  the isotropic point of  $T(\Gamma)$  such that  $p_i^\perp \cap \Gamma = \Theta(u, v_i)$  ( $i = 1, 2$ ). By proposition 2.3 such a point  $p_i$  exists and is unique ( $i = 1, 2$ ). By lemma 2.1 (i), (ii),  $(p_1, p_2) \neq 0$ .

Conversely, assume  $p_1, p_2$  are nonperpendicular isotropic points of  $T(\Gamma)$ . Denote by  $\Omega_i$  the locally  $M_\Gamma$  subgraph of  $\Gamma$ , which corresponds to  $p_i$  ( $i = 1, 2$ ). For each  $\Omega_i$  we have exactly two vertices  $v_{ij} \in \Theta_2(u)$  such that  $\Omega_i = \Theta(u, v_{ij})$  ( $i, j = 1, 2$ ). Considering the neighbourhood of  $v_{11}$ , we see that both  $v_{21}$  and  $v_{22}$  cannot be adjacent to  $v_{11}$ . On the other hand  $k(\Xi) = k(\Upsilon)$ . Hence one of  $v_{21}$  and  $v_{22}$  must be adjacent to  $v_{11}$ . We have shown that the mapping  $v \mapsto \Theta(u, v)$ , where  $v \in V\Xi$ , is a covering from  $\Upsilon$  to  $\Xi$ . □

The latter statement implies that  $\Xi$  possesses an involutory automorphism  $g_u$  which interchanges any  $v, w \in V\Xi$  such that  $\Theta(u, v) = \Theta(u, w)$ , and fixes  $\{u\} \cup \Theta(u)$  pointwise.

Consider the subgroup  $G_u$  of  $\text{Aut}(\Theta)$  generated by  $g_x, x \in \Theta(u)$ . We have  $G_u \cong \text{Aut}(\Gamma)$ . Therefore  $\Theta$  is the collinearity graph of a  $c^k \cdot C_2$ -geometry satisfying the conditions of result 1.2. Hence our result follows from it. However, we would like to give a complete proof of theorem 1.1 here.

Our claim is that  $\{g_v | v \in V\Theta\}$  is a class of 3-transpositions in  $\text{Aut}(\Theta)$ . Indeed, clearly for any  $x \in \Theta(u)$  the involutions  $g_u$  and  $g_x$  commute. Now let  $y \in \Theta_2(u)$ . We must prove  $\tau = (g_u g_y)^3 = 1$ . Note that  $\tau$  belongs to the kernel of the action of the stabilizer of every  $v \in \Theta(u, y)$  on  $\Theta(u)$ . Therefore it fixes every vertex of  $\Theta$ . Our claim is proved. The use of the classification of 3-transposition groups given in [8] completes the proof of theorem 1.1.  $\square$

Note: Hans Cuypers (personal communication) has suggested another idea how to complete the proof, which is much more geometric. Namely, it may be easily shown that the partial linear space on  $V\Theta$ , whose lines are triples, is an irreducible Fischer space (see introduction) (or a locally polar geometry with affine planes, see Cuypers and Pasini [7]). Then the use of classification of these objects [6] (resp. [7]) completes the proof.

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