Efficient Robust Fuzzy Model Predictive Control of Discrete Nonlinear Time-Delay Systems via Razumikhin Approach

Long Teng, Youyi Wang, Wenjian Cai, and Hua Li

Abstract—**In this work, two efficient robust fuzzy model predictive control algorithms are investigated for discrete nonlinear systems with multiple time delays and bounded disturbances. The famous T-S fuzzy systems are utilized to represent nonlinear systems. Instead of Lyapunov-Krasovskii functional, Lyapunov-Razumikhin function is adopted to deal with time delays because it involves invariant sets in the original state space of the system. A sequence of explicit control laws corresponding to a sequence of constraint sets are computed off-line that the online computational burden associated with classical model predictive control algorithms is significantly reduced. In particular, the set invariance theory behind Razumikhin approach, which is more complicated than the one for non-delayed systems, is directly observed. And it is proved that all (delayed) states can enter the terminal set in finite time. Moreover, robust positive invariance and input-to-state stability for time-delay systems concerning disturbances are realized. In addition, an online optimization algorithm is also provided based on the off-line computed ellipsoidal sets. Thus the conservatism induced by Razumikhin approach is relaxed, while the computational cost is not significantly increased.**

Index Terms—**Time delay, model predictive control (MPC), T-S fuzzy systems, optimal control, Lyapunov-Razumikhin.**

I. Introduction

Utilization of Lyapunov theory for time-delay systems can be categorized as two different approaches in general, i.e., the Lyapunov-Krasovskii functional (LKF) and the Lyapunov-Razumikhin function (LRF) [1], [2]. For discrete-time systems, the Krasovskii approach makes use of an augmentation of the state vector with all delayed states, which yields the applications of classical Lyapunov methods to an augmented system without delay. As a result, the Krasovskii approach is indirect, and the complexity of it will increase with the size of delay such that it is impractical for large delays [3]. In addition, as the Lyapunov function is required to decrease monotonically along all the trajectories, which makes it too strict that it may be non-tractable for systems with severe complexities or under constraints. As an alternative, the Razumikhin approach relies on a function that does not need to decrease all the time. Although it is conservative but it involves a Lyapunov function for the original non-augmented system. Therefore, it has a good potential to get rid of the complexity associated with the Krasovskii approach especially for systems with large delays and disturbances. It is noted that extensive works can be seen on stabilization of time-delay systems by adopting LKF, while LRF-based approaches have not been widely investigated until in recent years, see [3]–[8].

Moreover, as an optimization control approach, model predictive control (MPC) has attracted extensive research interests from both academic field and industry. Many results have been achieved in MPC of non-delayed systems, in contrast, there are only a few works concerning time-delay systems. Several early results can be seen in [9]–[11], however, these approaches are only applied to linear systems and disturbance is not considered. MPC of multivariable time-delay systems is investigated in [12]. Recently, an LRF-based synthesis approach is proposed for unconstrained linear delay difference inclusions (DDIs) [5], based on which, a receding horizon controller for systems with state constraints is also provided. In [6], MPC based on time-varying Lyapunov functions is investigated for linear DDIs that the conservatism of Razumikhin approach can be avoided. For nonlinear case, a combination of Lyapunov-Razumikhin and Lyapunov-Krasovskii conditions is adopted to determine the terminal constraint set and terminal cost function, respectively, for continuous time-delay systems [13]. In [14], based on non-quadratic LKF, MPC of nonlinear delayed systems modeled as T-S fuzzy systems is investigated. It can be seen that MPC for time-delay systems is not completely investigated that many problems still remain unsolved [13].

One of the main drawbacks of classical MPC strategies is that it requires solving online an optimization problem, which prevents their applications to some contexts due to the high computational cost. Thus several MPC schemes have been developed to reduce the online computational burden, i.e., the explicit MPC and efficient MPC algorithms, see [15] and references therein. For the efficient implementation of MPC, a sequence of explicit control laws corresponding to a sequence of constraint sets are computed off-line. As a consequence, the online computation is reduced to look up the smallest constraint set where the state belongs and apply the pre-determined control laws. It is noted that reasonable works can be seen on efficient MPC for non-delayed systems [15]–[17]. In addition, as an alternative to the explicit MPC, time-optimal control has been investigated to reduce the complexity

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while achieve similar control performance [18]. However, considering efficient solutions for time-delay systems, there are only a few works for linear time-delay systems [19], [20], where only the Krasovskii functional is investigated that these stabilization approaches are very complex.

In addition, in the aforementioned contributions on MPC of time-delay systems, either classical formulations or efficient ones, disturbance is not investigated and only a few results are applied to nonlinear systems [13]. Besides, it is worth mentioning that an off-line stabilizing synthesis approach based on LRF is investigated for linear DDIs in [3] that the ISS regarding disturbance is realized. In [21], an online robust MPC algorithm for discrete nonlinear systems with multiple time delays and persistent disturbances is investigated. Furthermore, time-varying delay is investigated in [22]. On the other hand, the T-S fuzzy method offers an alternative modeling approach for nonlinear systems to any degree of accuracy in any convex compact set [23], and then the classical control theories for linear systems can be adopted, which greatly simplifies the analysis and synthesis of nonlinear systems. It is noted that a combination of T-S fuzzy approach and MPC, i.e., fuzzy model predictive control (FMPC), has been extensively investigated for non-delayed systems [24]–[28]. However, research results are still quite limited concerning FMPC for time-delay systems, despite [14].

In contrast to our previous work in [21] which requires online optimization, two efficient robust MPC approaches for time-delay systems are provided in this paper. Compared with LKF, LRF is not a simple extension of traditional Lyapunov function to time-delay systems, thus new conditions for robust positive invariance and finite-time control performance should be established. Firstly, an off-line robust MPC method is proposed for discrete nonlinear systems where time delays and disturbances are involved simultaneously. The LRF is adopted that a sequence of robust constraint sets associated with a sequence of explicit control laws are constructed in the original state space, with which system all (delayed) states are guaranteed to enter the terminal constraint set in finite time and ISS is achieved. Furthermore, based on the obtained robust constraint sets, a simple online optimization algorithm is also introduced to reduce the conservatism caused by the off-line approach. Compared with several existing FMPC approaches, less conservative results are achieved via the proposed algorithms. Particularly, the special control theory and the positive invariance behind Razumikhin approach are revealed. The relevant constraint sets, where system current state and all (delayed) states belong, can be observed directly by the proposed method. Thus one can see the non-monotonic characteristic of LRF which makes it more flexible and easier, when compared to LKF, especially for systems with severe complexities and constraints where it may be difficult to find a monotonic Lyapunov function. It is valuable to note that MPC based on flexible set-membership constraints is investigated for non-delayed systems in [29] such that the non-monotonic convergence to the terminal set is achieved.

The reminder of the paper is structured as follows. Section II provides some preliminary knowledge of DDIs and time-delay T-S fuzzy systems. In Section III, the efficient formulation of LRF-based robust MPC for time-delay systems is investigated. And a numerical example is given to verify the specific control theory and positive invariance behind Razumikhin approach. In addition, an online optimization algorithm is introduced in Section IV. Simulations on a CSTR system with time delay are illustrated and comparisons of several FMPC methods are discussed. Conclusions are drawn in Section V.

Notations: $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_{[z1,z2]}$, and $\mathbb{N}$ denote the set of real numbers, the set of non-negative reals, the set of integers, the set of non-negative integers, the set of integers in the interval $[z1, z2]$, and the set of natural numbers, respectively. $diag\{...\}$ stands for a block-diagonal matrix. $\|x\|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$. $\mathbf{x}_{[c1,c2]} := \{x(l)\}_{l\in\mathbb{Z}_{[c1,c2]}}$, with $c1, c2 \in \mathbb{Z}$, denotes a sequence that is ordered monotonically with respect to the index $l \in \mathbb{Z}_{[c1,c2]}$. $x \in \Omega_{a1}\backslash\Omega_{a2}$ denotes that $x \in \Omega_{a1}$ and $x \notin \Omega_{a2}$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$, i.e., $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$. $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$. The function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{KL}$, i.e., $\beta \in \mathcal{KL}$, if for each fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{K}$ and for each fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is strictly decreasing and $\lim_{s\to\infty} \beta(r, s) = 0$. The symbol $*$ represents a symmetric structure in matrix inequalities.

II. Preliminaries

In this section, some preliminaries about DDIs and LRF-based definitions, and T-S fuzzy time-delay systems are introduced in sequence.

A. DDIs and LRF Conditions

Consider the following DDI [5] with disturbances,

$$x^+ \in F(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]}, w(k)), \qquad k \in \mathbb{Z}_+, \tag{1}$$

where x^+ represents the system state in the next time instant, $\mathbf{x}_{[k-h,k]} \in \mathbb{R}^{n(h+1)}$, $\mathbf{u}_{[k-h,k]} \in \mathbb{R}^{m(h+1)}$ and $w(k) \in \mathbb{R}^c$, represent a sequence of (delayed) states, inputs, and disturbances, respectively. $h \in \mathbb{Z}_+$ is the maximal delay. The DDI (1) is linear if the set-valued map $F : (\mathbb{R}^n)^{h+1} \times (\mathbb{R}^m)^{h+1} \times \mathbb{R}^c \rightrightarrows \mathbb{R}^n$ is of the form

$$F(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}, w) := \sum_{i=-h}^{0} (A_i x(i) + B_i u(i)) + w \tag{2}$$

where $(\{A_i, B_i\}_{i\in\mathbb{Z}_{[-h,0]}}) \in AB$, the set $AB \subset (\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m})^{h+1}$ is a non-empty and compact polytope.

To stabilize DDI (1), a control law $\pi : \mathbb{R}^{n(h+1)} \rightrightarrows \mathbb{R}^m$ is adopted. The closed-loop system with the control law yields

$$x^+ \in F_\pi(\mathbf{x}_{[k-h,k]}, w(k)), \qquad k \in \mathbb{Z}_+, \tag{3}$$

where

$$F_\pi(\mathbf{x}_{[-h,0]}, w) := \left\{F(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}, w) | u(0) \in \pi(\mathbf{x}_{[-h,0]})\right\}$$

and $\mathbf{u}_{[-h,-1]}$ is assumed to be known. The dependence of F_π on $\mathbf{u}_{[k-h,k-1]}$ is omitted in (3) because it is assumed that the input sequence $\mathbf{u}_{[k-h,k-1]}$ is dependent on the state sequence $\mathbf{x}_{[k-h,k]}$ only.

Definition 1 (LRF-type Positively Invariant Set, D-invariance [30]): Consider DDI (1), a set $\Omega \subseteq \mathbb{R}^n$ is called an RPI set for the closed-loop system corresponding to the control law π, if for all $\mathbf{x}_{[k-h,k]} \in \Omega^{h+1}$, it holds $x^+ \in \Omega$.

Definition 2 (LRF-type Robust Positively Invariant (RPI) Set, Robust D-invariance): Consider DDI (1), a set $\Omega \subseteq \mathbb{R}^n$ is called an RPI set for the closed-loop system corresponding to the control law π, if for all $\mathbf{x}_{[k-h,k]} \in \Omega^{h+1}$, and $\forall w \in \mathbb{W}$, $\mathbb{W} \subseteq \mathbb{R}^c$, it holds $x^+ \in \Omega$.

Definition 3 (Input-to-State Stability (ISS) [4]): The DDI (3) is called ISS if there exist $\beta \in \mathcal{KL}$ and $\delta \in \mathcal{K}$, such that for all $k \in \mathbb{Z}_+$ it holds that

$$\|x(k)\| \leq \beta(\left\|\mathbf{x}_{[-h,0]}\right\|, k) + \delta(\left\|\mathbf{w}_{[0,k-1]}\right\|), \tag{4}$$

where $\mathbf{w}_{[0,k-1]} \in \mathbb{W}^k$ is the disturbance sequence.

Definition 4 (LRF-type ISS Lyapunov Function [4]): For LRF, suppose that there exists a positive definite function such that satisfies the following conditions:

(i) There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying for all $x \in \mathbb{R}^n$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{5}$$

(ii) For all initial conditions, there exists a function $\delta \in \mathcal{K}$ satisfying

$$V(x^+) \leq \bar{V}(x(k)) + \delta\,\|w(k)\| \tag{6}$$

where $\bar{V}(x(k)) = \max\limits_{i\in\mathbb{Z}_{[-h,0]}} \{V(x(k+i))\}$, then $V(x)$ is called an ISS Lyapunov function for DDI (3).

It is noted that detailed descriptions of LRF-type ISS and ISS Lyapunov function for DDIs are introduced in [4].

In what follows, to simplify the whole problem, we only consider the following system with state delays and disturbances,

$$x^+ \in F(\mathbf{x}_{[k-h,k]}, u(k), w(k)), \qquad k \in \mathbb{Z}_+, \tag{7}$$

B. T-S Fuzzy Time-delay Systems

Based on the aforementioned DDIs, nonlinear time-delay system (7) can be represented as T-S fuzzy time-delay systems comprised of fuzzy blending of the piecewise linear DDIs in (2):

Plant Rule l**:**
IF: z_1 is F_1^l and ..., z_v is F_v^l
THEN:

$$x^+ = \sum_{i=-h}^{0} A_{i,l}x(k+i) + B_l u(k) + E_l w(k) \tag{8}$$

where $z := [z_1, \ldots, z_v]$ are premise variables, F_1^l, ..., F_v^l are fuzzy sets, $l \in \mathbb{Z}_{[1,L]}$, with L being the number of fuzzy rules. The T-S fuzzy system (8) can be written as,

$$x^+ = \sum_{i=-h}^{0} A_{i,\mu}x(k+i) + B_\mu u(k) + E_\mu w(k) \tag{9}$$

where $A_{i,\mu} := \sum_{l=1}^{L} \mu_l(z)A_{i,l}$, $B_\mu := \sum_{l=1}^{L} \mu_l(z)B_l$, $E_\mu := \sum_{l=1}^{L} \mu_l(z)E_l$, $\mu_l(z)$ is normalized membership function. Denote $\mu_l(z)$ as μ_l for simplicity in the following parts.

Control law for above T-S fuzzy system is given below:

Control Rule l**:**
IF: z_1 is F_1^l and ..., z_v is F_v^l
THEN:

$$u_l(k) = K_l x(k)$$

The final control output can be obtained as follows,

$$u(k) = K_\mu x(k) \tag{10}$$

where $K_\mu = \sum_{l=1}^{L} \mu_l K_l$.

Remark 1: Based on the LRF-based stabilization method in [5] (see Section 3.1 in [5]) and the robust FMPC approach for non-delayed systems in [24], an online robust FMPC approach is established for discrete time-delay systems in [21]. In contrast to the online robust FMPC approach, in what follows, two efficient off-line FMPC approaches are introduced.

III. LRF-BASED OFF-LINE MPC APPROACH

In off-line MPC schemes, the approximation constraint set can be either ellipsoidal or polytopic [31]. In this work, the former one is utilized for the construction of robust constraint sets concerning its simple representation. However, it should be noted that the latter formulation is more flexible than the former one.

For non-delayed systems, selection of an appropriate control law only requires to look up all acquired constraint sets and find the smallest one where system current state belongs, see [16]. In contrast, in this work where Razumikhin approach is adopted for time-delay systems, the appropriate control law is determined not only by the system current state but also by all delayed states. Firstly, the smallest constraint set where all (delayed) states belong should be found out, then the corresponds control law set is applied.

Moreover, for non-delayed systems it requires only one step to steer the system state from a constraint set to a neighboring one via the corresponding off-line control law, see details of "robust one-step set" for non-delayed systems in [24]. As a consequence, the system state can enter the terminal constraint set in finite time steps. The finite-time control performance is still effective for the proposed LRF-based off-line MPC approach, however, the principle behind is more complicated and it will be investigated in this work.

Remark 2: As the Krasovskii approach for time-delay systems can be regarded as an extension of the classical Lyapunov approach for non-delayed systems, thus both of them share the same principles in Lyapunov function and set invariance theory. To better clarify the special properties of the proposed LRF-based efficient MPC algorithm, above and in what follows, the review of non-delayed systems which have been well recognized is provided for comparison purpose.

A. Terminal Constraint Set

The terminal constraint set should satisfy the following two requirements:

Firstly, the terminal constraint set should be an RPI set. Let quadratic LRF $V(x) := x^T P_t x$, denote $\Omega_t \subseteq \mathbb{R}^n$ as the following ellipsoidal set

$$\Omega_t := \{x | \bar{V}(x(k)) \leq \xi\} \tag{11}$$

where $\bar{V}(x(k))$ is defined in (6), with $i \in \mathbb{Z}_{[-h,0]}$. From the definition of the quadratic LRF $V(x)$ and $\bar{V}(x)$, It can be seen that Ω_t is a convex set.

Lemma 1: The set Ω_t is an RPI set if there exists a positive scalar $\lambda \in \mathbb{R}_{(0,1)}$ such that

$$\tfrac{1}{\xi}V(x^+) - \tfrac{1-\lambda}{\xi}\bar{V}(x(k)) - \tfrac{\lambda}{\eta^2}w^T(k)w(k) \le 0 \quad (12)$$

Proof: According to Definition 2, move the last two items in (15) to the right side. As $\bar{V}(x(k)) \le \xi$ and $w^T(k)w(k) \le \eta^2$, then $\frac{1}{\xi}V(x^+) \le (1-\lambda)+\lambda = 1$, thus $V(x^+) \le \xi$, proof is completed. It can be seen that disturbance is involved through this way, and this result is extended from the one for non-delayed systems in [24].

Secondly, there exist $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$, and a positive definite function $V(x)$ such that $\forall\, x \in \Omega_t$,

$$\alpha_3(\|x\|) \le V(x) \le \alpha_4(\|x\|) \quad (13)$$

$$V(x^+) - \bar{V}(x(k)) < -x^T(k)Qx(k) - u^T(k)Ru(k) \quad (14)$$

In the following, the conditions which satisfy the aforementioned requirements are provided.

Theorem 1: Consider T-S fuzzy system (9), if there exist a positive definite matrix X_t, general matrices Y_a (or Y_b, Y_l presented below), G_b, Z, and positive scalars τ, $\lambda \in \mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix} \Xi_0 & * & * & * & * & * \\ 0 & \Xi_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \Xi_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2}I & * \\ \Theta_0 & \Theta_1 & \cdots & \Theta_h & E_a & -X_t \end{bmatrix} \le 0 \quad (15)$$

$$\begin{bmatrix} \Gamma_0 & * & * & * & * & * & * & * \\ 0 & \Gamma_1 & * & * & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * & * & * \\ 0 & 0 & \cdots & \Gamma_h & * & * & * & * \\ 0 & 0 & \cdots & 0 & -\tau\xi I & * & * & * \\ \Theta_0 & \Theta_1 & \cdots & \Theta_h & \xi E_a & -X_t & * & * \\ QG_b & 0 & \cdots & 0 & 0 & 0 & -\xi Q & * \\ RY_b & 0 & \cdots & 0 & 0 & 0 & 0 & -\xi R \end{bmatrix} < 0 \quad (16)$$

$$\begin{bmatrix} Z & * \\ {Y_b}^T & G_b + {G_b}^T - X_t \end{bmatrix} \ge 0,\ Z_{tt} \le u^2_{t,\max},\ t \in \mathbb{Z}_{[0,m]} \quad (17)$$

where $\Xi_{-i} = \gamma_i(\lambda-1)(G_b + {G_b}^T - X_t)$, $\Gamma_{-i} = \gamma_i(X_t - G_b - {G_b}^T)$, $i \in \mathbb{Z}_{[-h,0]}$, $\Theta_0 = A_{0,a}G_b + B_aY_b$, $\Theta_{-j} = A_{j,a}G_b$, $j \in \mathbb{Z}_{[-h,-1]}$, $a, b \in \mathbb{Z}_{[1,L]}$, Z_{tt} is the t-th diagonal element of matrix Z, γ_i are pre-set valued positive numbers that $\sum_{i=-h}^{0}\gamma_i = 1$, then a terminal constraint set $\Omega_t = \left\{x|x^TP_tx \le \xi\right\}$, with $P_t = \xi {X_t}^{-1}$, is obtained for system (9), and the corresponding feedback control law is given as $\pi_t = K_\mu x$, with $K_\mu = \sum_{a=1}^{L}\mu_a K_a$ and $K_a = Y_a {G_a}^{-1}$.

Proof: Firstly, we show that (18) implies (15). Resorting to the dilation lemma [32],

$$-{G_b}^T {X_t}^{-1} G_b \le X_t - {G_b}^T - G_b \quad (18)$$

The following inequality is obtained from (18),

$$\begin{bmatrix} \mathrm{M}_0 & * & * & * & * & * \\ 0 & \mathrm{M}_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \mathrm{M}_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2}I & * \\ \Theta_0 & \Theta_1 & \cdots & \Theta_h & E_a & -X_t \end{bmatrix} \le 0 \quad (19)$$

where $\mathrm{M}_{-i} = \gamma_i(\lambda-1){G_b}^T {X_t}^{-1} G_b$. By multiplying $diag\left\{{G_b}^{-T}, {G_b}^{-T}, \cdots {G_b}^{-T}, I, I\right\}$ and its transpose from both sides of (22), respectively, yields that

$$\begin{bmatrix} \Delta_0 & * & * & * & * & * \\ 0 & \Delta_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \Delta_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2}I & * \\ \tilde{A}_0 & \tilde{A}_{-1} & \cdots & \tilde{A}_{-h} & E_a & -X_t \end{bmatrix} \le 0 \quad (20)$$

where $\Delta_{-i} = \gamma_i(\lambda-1){X_t}^{-1}$, $\tilde{A}_0 = A_{0,a} + B_aK_b$, $\tilde{A}_j = A_{j,a}$, $j \in \mathbb{Z}_{[-h,-1]}$. Based on which one has

$$\sum_{a=1}^{L}\sum_{b=1}^{L}\mu_a\mu_b \begin{bmatrix} \Delta_0 & * & * & * & * & * \\ 0 & \Delta_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \Delta_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2}I & * \\ \tilde{A}_0 & \tilde{A}_{-1} & \cdots & \tilde{A}_{-h} & E_a & -X_t \end{bmatrix} \le 0 \quad (21)$$

where $\sum_{a=1}^{L}\mu_a = 1$, $\sum_{b=1}^{L}\mu_b = 1$.

The above inequality can be written as

$$\begin{bmatrix} \Delta_0 & * & * & * & * & * \\ 0 & \Delta_1 & * & * & * & * \\ \vdots & \vdots & \ddots & * & * & * \\ 0 & 0 & \cdots & \Delta_h & * & * \\ 0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2}I & * \\ \tilde{A}_{0,\mu} & \tilde{A}_{-1,\mu} & \cdots & \tilde{A}_{-h,\mu} & E_\mu & -X_t \end{bmatrix} \le 0 \quad (22)$$

where $\tilde{A}_{0,\mu} = A_{0,\mu} + B_\mu K_\mu$, $\tilde{A}_{j,\mu} = A_{j,\mu}$, $j \in \mathbb{Z}_{[-h,-1]}$. Applying schur complement to (25), and multiplying $\left[x^T(k), x^T(k-1), \cdots, x^T(k-h), w^T(k)\right]$ and its transpose from both sides of the resulting matrix inequality, respectively, yields that

$$\varpi \begin{bmatrix} \Delta_0 & & & & \\ & \Delta_1 & & & \\ & & \ddots & & \\ & & & \Delta_h & \\ & & & & -\frac{\lambda}{\eta^2}I \end{bmatrix} \varpi^T + \varpi\Lambda {X_t}^{-1}\Lambda^T\varpi^T \le 0 \quad (23)$$

where $\varpi = \left[x^T(k), x^T(k-1), \cdots, x^T(k-h), w^T(k)\right]$, $\Lambda = \left[\tilde{A}_{0,\mu}, \tilde{A}_{-1,\mu}, \cdots, \tilde{A}_{-h,\mu}, E\right]^T$.

By substituting X_t^{-1} with P_t/ξ, (26) is equivalent to

$$\begin{array}{l}(\lambda-1)\sum\limits_{i=-h}^{0}\gamma_i x^T(k+i)X_t^{-1}x(k+i)-\frac{\lambda}{\eta^2}w^T(k)w(k)\\+(\sum\limits_{i=-h}^{0}x^T(k+i)(\tilde{A}_{i,\mu})^T)X_t^{-1}(\sum\limits_{i=-h}^{0}x^T(k+i)(\tilde{A}_{i,\mu})^T)^T\\=\frac{(\lambda-1)}{\xi}\sum\limits_{i=-h}^{0}\gamma_i x^T(k+i)P_t x(k+i)\\-\frac{\lambda}{\eta^2}w^T(k)w(k)+\frac{1}{\xi}x^{+T}P_t x^{+}\\=\frac{(\lambda-1)}{\xi}\sum\limits_{i=-h}^{0}\gamma_i V(x(k+i))-\frac{\lambda}{\eta^2}w^T(k)w(k)+\frac{1}{\xi}V(x^{+})\\\leq 0\end{array} \tag{24}$$

As $\sum_{i=-h}^{0}\gamma_i V(x(k+i))\leq\bar{V}(x(k))$ due to $\sum_{i=-h}^{0}\gamma_i=1$, thus (15) can be got.

It is noted that (16) can be easily got by resorting to the eigenvalues of P_t. (19) which guarantees (17) can be verified by following a similar proof for (18). The input constraint is satisfied by (20) and the proof can be seen in [33].

Theorem 2: With the computed terminal constraint set Ω_t and the corresponding state feedback control law $u(k)=K_\mu x(k)$, the closed-loop system (9) is ISS with respect to disturbance w.

Proof: It is noted that (18) guarantees the RPI property of Ω_t, and (19) guarantees the satisfaction of the optimization problem. From (18)-(20), one can see that the computation of Ω_t and the corresponding state feedback control law is independent of a specific state, which indicates that the corresponding state feedback control law is feasible for any state in Ω_t. Meanwhile, the RPI property is guaranteed that system states are trapped in the terminal set for all time. And the feedback control law guarantees the satisfaction of (17) at each time instant. It is noted that (17) leads to

$$V(x^{+})<\bar{V}(x(k))+w^T(k)w(k) \tag{25}$$

which implies (6) is guaranteed. In addition, (5) can be easily satisfied by resorting to the eigenvalues of P_t. Therefore, from Definition 4 it can be got that $V(x)$ is an ISS Lyapunov function for the closed-loop system (9). It is noted that Definition 4 leads to Definition 3, see details in [4], thus ISS is achieved.

B. A Sequence of Robust Constraint Sets

Denote the obtained terminal set Ω_t and corresponding control law π_t, respectively, as Ω_0 and π_0. A robust constraint set Ω_1 ($\Omega_1\supseteq\Omega_0$) is employed such that system all (delayed) states $x(k+i)_{\forall i\in\mathbb{Z}_{[-h,0]}}\in\Omega_1$. According to Definition 2, a control law π_1 is given such that the system state in the next instant $x^{+}\in\Omega_0$, and it is easy to see that the positive invariance is guaranteed due to $\Omega_0\subseteq\Omega_1$. After Ω_1 and corresponding control law π_1 are obtained, in a similar way, a sequence of robust constraint sets $\{\Omega_0,\Omega_1,\cdots\Omega_e,\cdots\Omega_{\tilde{N}}\}$ and corresponding control laws $\{\pi_0,\pi_1,\cdots\pi_e,\cdots\pi_{\tilde{N}}\}$ can be got, $\Omega_e\subseteq\mathbb{R}^n$ with $e\in\mathbb{Z}_{[1,\tilde{N}]}$, $\tilde{N}$ ($\tilde{N}\geq 1$) represents the number of constraint sets.

Firstly, without loss of generality, at instant k, assume system all (delayed) states $x(k+i)_{\forall i\in\mathbb{Z}_{[-h,0]}}\in\Omega_e$, the design principle requires that $x^{+}\in\Omega_{e-1}$. It can be seen that the RPI property is guaranteed due to $\Omega_e\supseteq\Omega_{e-1}$. Therefore, based on which one has

$$\begin{array}{l}x^{+T}X_{e-1}^{-1}x^{+}-(1-\lambda)\max\limits_{i\in\mathbb{Z}_{[-h,0]}}\left\{x^T(k+i)X_e^{-1}x(k+i)\right\}\\-\dfrac{\lambda}{\eta^2}w^T(k)w(k)\leq 0\end{array} \tag{26}$$

where X_e and X_{e-1} are associated with two neighboring robust constraint sets $\Omega_e=\left\{x|x^TX_e^{-1}x\leq 1\right\}$ and $\Omega_{e-1}=\left\{x|x^TX_{e-1}^{-1}x\leq 1\right\}$, respectively. Similar to (15), (29) guarantees $x^{+T}X_{e-1}^{-1}x^{+}\leq 1$, provided that $\max\limits_{i\in\mathbb{N}_{[-h,0]}}\left\{x^T(k+i)X_e^{-1}x(k+i)\right\}\leq 1$.

Secondly, the optimization control is formulated as follows,

$$\begin{array}{l}x^{+T}P_e x^{+}-\max\limits_{i\in\mathbb{Z}_{[-h,0]}}\left\{x^T(k+i)P_e x(k+i)\right\}\\<-x^T(k)Qx(k)-u^T(k)Ru(k)\end{array} \tag{27}$$

Theorem 3: Assume constraint set Ω_{e-1} is already known, if there exist positive definite matrix X_e, general matrices Y_a (or Y_b, Y_l presented below), Z, and positive scalars ξ_e, $\lambda\in\mathbb{R}_{(0,1)}$, such that the following matrix inequalities are feasible

$$\begin{bmatrix}\Psi_0 & * & * & * & * & *\\0 & \Psi_1 & * & * & * & *\\\vdots & \vdots & \ddots & * & * & *\\0 & 0 & \cdots & \Psi_h & * & *\\0 & 0 & \cdots & 0 & -\frac{\lambda}{\eta^2}I & *\\\Pi_0 & \Pi_1 & \cdots & \Pi_h & E_a & -X_{e-1}\end{bmatrix}\leq 0 \tag{28}$$

$$\begin{bmatrix}\Phi_0 & * & * & * & * & * & *\\0 & \Phi_1 & * & * & * & * & *\\\vdots & \vdots & \ddots & * & * & * & *\\0 & 0 & \cdots & \Phi_h & * & * & *\\\Pi_0 & \Pi_1 & \cdots & \Pi_h & -X_e & * & *\\QX_e & 0 & \cdots & 0 & 0 & -\xi_e Q & *\\RY_b & 0 & \cdots & 0 & 0 & 0 & -\xi_e R\end{bmatrix}<0 \tag{29}$$

$$\begin{bmatrix}Z & *\\{Y_b}^T & X_e\end{bmatrix}\geq 0,\ Z_{tt}\leq u_{t,\max}^2,\ t\in\mathbb{Z}_{[0,m]} \tag{30}$$

where $\Psi_{-i}=\gamma_i(\lambda-1)X_e$, $\Phi_{-i}=-\gamma_i X_e$, $i\in\mathbb{Z}_{[-h,0]}$, $\Pi_0=A_{0,a}X_e+B_aY_b$, $\Pi_{-j}=A_{j,a}X_e$, $j\in\mathbb{Z}_{[-h,-1]}$, $a,b\in\mathbb{Z}_{[1,L]}$, Z_{tt} is the t-th diagonal element of matrix Z, γ_i are pre-set valued positive numbers that $\sum_{i=-h}^{0}\gamma_i=1$, then the set $\Omega_e:=\left\{x|x^TP_ex\leq\xi_e\right\}$, with $P_e=\xi_e{X_e}^{-1}$, is a robust constraint set for T-S fuzzy system (9), and the corresponding feedback control law π_e will steer system state to Ω_{e-1} at the next instant, $\pi_e=K_\mu x$, with $K_\mu=\sum_{l=1}^{L}\mu_lK_l$ and $K_l=Y_l{X_e}^{-1}$.

Proof: It is noted that (31) and (32) satisfy (29) and (30), respectively. (33) fulfills the input constraint. The proof is very similar to that of Theorem 1. Thus it is omitted here.

C. Control Algorithm

Based on the above results, the whole control algorithm is summarized to complete the off-line FMPC method.

Algorithm 1:

1. Off-line computation

Step 1. Compute a terminal constraint set Ω_0 and the corresponding control law π_0 by adopting Theorem 1;

Step 2. Based on the obtained terminal constraint set, a sequence of constraint sets $\{\Omega_0, \Omega_1, \cdots, \Omega_{\tilde{N}}\}$ together with corresponding control laws can be got by Theorem 3;

2. Online computation

Step 1. Consider an initial condition $x(i)_{\forall i \in \mathbb{Z}_{[-h,0]}}$ which satisfies $x(i)_{\forall i \in \mathbb{Z}_{[-h,0]}} \in \Omega_{\tilde{N}}$. Let $k = 0$ and go to Step 2;

Step 2. At time instant k, obtain system all (delayed) states $x(k+i)_{\forall i \in \mathbb{Z}_{[-h,0]}}$, implement the search to decide $x(k+i)_{\forall i \in \mathbb{Z}_{[-h,0]}} \in \Omega_e \backslash \Omega_{e-1}$, if $\Omega_e = \Omega_0$, then the corresponding control law for Ω_0 is applied all the time. Otherwise, the corresponding control law for Ω_e is applied, let $k = k+1$ and repeat Step 2.

Remark 3: In Step 2 of off-line computation of robust constraint set, we use $-\log\ \det(X_e)$ as the objective function and minimize it to get the constraint set with maximal size [34].

Theorem 4: If a sequence of robust constraint sets $\{\Omega_0, \Omega_1, \cdots, \Omega_{\tilde{N}}\}$ can be computed by Theorem 1 and Theorem 3, with the initial conditions satisfying $x(i)_{\forall i \in \mathbb{Z}_{[-h,0]}} \in \Omega_{\tilde{N}}$, then it requires no more than $\tilde{N} \cdot (h+1)$ steps to steer system all (delayed) states to the terminal constraint set.

Proof: Without loss of generality, in the current time instant k, suppose the system all (delayed) states $x(k+i)_{\forall i \in \mathbb{Z}_{[-h,0]}} \in \Omega_e$. Firstly, the control law π_e is applied to drive $x(k+1)$ to Ω_{e-1}. As a result, at the next time instant $k+1$, it can be seen that $x(k+1) \in \Omega_{e-1}$, while $x(k+1+i)_{\forall i \in \mathbb{Z}_{[-h,-1]}} \in \Omega_e$. Therefore, the control law π_e, rather than π_{e-1}, will still be adopted at the next instant $k+1$ to drive $x(k+2)$ to Ω_{e-1}. By repeating the process, it can be concluded that it takes $h+1$ steps to drive system all (delayed) states from Ω_e to Ω_{e-1}, which is different from the context for non-delayed systems where only one step is required. Therefore, in the case of $\tilde{N}$ robust constraint sets, it takes no more than $\tilde{N} \cdot (h+1)$ steps to drive system all (delayed) states to the terminal constraint set. Proof is thus completed.

Remark 4: It is noted that ellipsoidal MPC algorithms have been widely investigated for non-delayed systems [15]–[17], [24], [35]. For time-delay systems, a few results can be seen for linear parameter-varying (LPV) systems [19], [20], where LKF is utilized that the control synthesis and computation are usually complex, and disturbance is not involved in these works.

Remark 5: Utilization of LKF conditions for the construction of ellipsoidal sets for time-delay systems can be regarded as an extension of the classical Lyapunov theory for non-delayed systems [19], although the LKF-based solutions are more complicated as LKF contains several segments. In contrast, the proposed off-line MPC algorithm based on LRF is very different from that for both non-delayed systems and time-delay systems based on LKF. As mentioned in Theorem 4, in the case of h delayed states, it takes $h+1$ steps to steer system all (delayed) states to the neighboring subset, thus it is more complicated than the concept of "robust one-step set" for non-delayed systems in [24]. We will show the special RPI property and finite-time control theory based on LRF through the following example.

D. Example 1

In this subsection, simulation on a numerical example is presented to verify the effectiveness of the proposed method. The simulation is implemented with MATLAB and all matrix inequalities are solved by YALMIP toolbox [36] with the solver PENBMI [37].

Consider the following T-S fuzzy time-delay systems with $h = 2$,

Rule 1: If $x_1(k)$ is F_1, then

$$x^+ = \sum_{i=-2}^{0} A_{i,1}x(k+i) + B_1u(k) + E_1w(k)$$

Rule 2: If $x_1(k)$ is F_2, then

$$x^+ = \sum_{i=-2}^{0} A_{i,2}x(k+i) + B_2u(k) + E_2w(k)$$

where

$$A_{0,1} = \begin{bmatrix} 1.5 & 2.5 \\ 0 & 1.5 \end{bmatrix}, A_{0,2} = \begin{bmatrix} 0.8 & 1.5 \\ 0 & 0.8 \end{bmatrix},$$
$$A_{-1,1} = A_{-2,1} = 0.05A_{0,1},$$
$$A_{-1,2} = A_{-2,2} = 0.05A_{0,2},$$
$$B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, E_1 = E_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

The membership functions are given as follows:

$$\begin{cases} F_1 = 0.5 + \sin(x_1)\cos(x_1), \\ F_2 = 1 - F_1. \end{cases}$$

Q and R are selected as $diag\{1, 1\}$ and 0.1, respectively. The input constraint and bounded disturbance are given as: $\|u\| \leq 2$, $\|w\| \leq 0.1$. As the constraint $\sum_{i=-h}^{0} \gamma_i = 1$ should be satisfied, let $\gamma_0 = 0.7$, $\gamma_{-1} = 0.15$, $\gamma_{-2} = 0.15$.

The simulation results are shown in Fig. 1. The number of robust constraint sets including the terminal constraint set is given as 25. It can be seen that, despite different initial conditions, the system all (delayed) states enter the terminal constraint set in the end.

Fig. 2 provides a clearer view of the relevant constraint sets where all (delayed) states and the current state belong, respectively, at each time instant. It can be seen that:

(1) In the existence of delayed states, at each instant, the constraint set for current state is always a (non-strict) subset of the constraint set for all (delayed) states.

(2) Consider the constraint set for all (delayed) states at each instant, except the terminal set, the corresponding control effort will steer system current state to a strict subset at the next instant.

(3) It takes no more than 3 (i.e., $h+1$, here $h = 2$) steps to drive system all (delayed) states from a constraint set to a strict subset. Thus the finite-time control performance in Theorem 4 is verified.

(4) The right histogram also shows that system current state at instant 2 enters a superset at the next instant. Thus

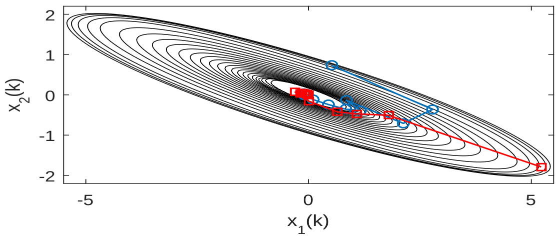

Fig. 1. Trajectories in state space for two different initial conditions: $x(i)_{\forall i \in \mathbb{Z}_{[-2,0]}} = [0.52; 0.73]$, $x(i)_{\forall i \in \mathbb{Z}_{[-2,0]}} = [5.22; -1.79]$, respectively.

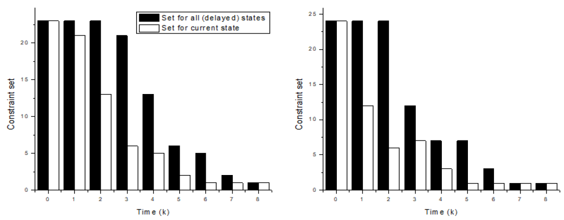

Fig. 2. Constraint sets for all (delayed) states and the current state at each time instant for two different initial conditions. The left and the right histograms correspond to $x(i)_{\forall i \in \mathbb{Z}_{[-2,0]}} = [0.52; 0.73]$, $x(i)_{\forall i \in \mathbb{Z}_{[-2,0]}} = [5.22; -1.79]$, respectively. Set 1 represents the smallest constraint set (terminal constraint set) in Fig. 1, while set 25 represents the largest constraint set.

it verifies that the Lyapunov function can be non-monotone in the Razumikhin approach.

Therefore, from above example, one can observe the special control theory and positive invariance behind Razumikhin approach that all these characteristics are different from those for non-delayed systems. And therefore, as discussed in Remarks 2 & 5, they are also different from the contexts of the Krasovskii approach for time-delay systems.

However, in the aforementioned LRF-based ellipsoidal offline FMPC approach, the obtained result is conservative due to several reasons: One reason is the intrinsic conservatism induced by the Razumikhin approach; Another one is that, the control law is off-line computed thus the system information (the delayed states $\mathbf{x}_{[k-h,k]}$) is not completely used for controller design at each instant. In the next section, a combination of the aforementioned off-line computed robust constraint sets and an online optimization algorithm is investigated to reduce the conservatism.

IV. Online optimization approach

A. Online Optimization Algorithm

Consider the following disturbance-free prediction model,

$$\begin{aligned} & x(k+s+1|k) \\ & = \sum_{i=-h}^{0} A_{i,\mu} x(k+i+s|k) + B_\mu u(k+s|k) \end{aligned} \tag{31}$$

A finite horizon cost function is optimized at each time instant, that is

$$J_N(k) = \sum_{s=0}^{N-1} \ell(k+s|k) + V_T(x(k+N|k)) \tag{32}$$

Denote

$$\begin{aligned} J_1(k) := & \, x^T(k) Q x(k) + u^T(k) R u(k) \\ & + x^T(k+1|k) P_{e-1} x(k+1|k) \end{aligned} \tag{33}$$

and minimization of $J_1(k)$ is carried out online. Without loss of generality, assume all (delayed) states $x(k+i)_{\forall i \in \mathbb{Z}_{[-h,0]}} \in \Omega_e$ at time k. The optimization problem can be formulated as

$$\min_{u(k)} \ \psi, \tag{34}$$

subject to

$$J_1(k) \le \psi, \tag{35}$$

$$x(k+1|k) \in \Omega_{e-1}, \tag{36}$$

$$u(k) \in \mathbb{U}. \tag{37}$$

where $\Omega_{e-1} := \left\{ x | x^T P_{e-1} x \le \xi_{e-1} \right\}$. In the following, we will show that the constraints of the optimization problem (37) can be represented by some linear matrix inequalities (LMIs).

Substituting $x^+ = \sum_{i=-h}^{0} A_{i,\mu} x(k+i) + B_\mu u(k)$ into (38), then it is not difficult to get that (38) is equivalent to the following inequality,

$$\begin{bmatrix} -\psi + x^T(k) Q x(k) & * & * \\ \sum_{i=-h}^{0} A_{i,\mu} x(k+i) + B_\mu u(k) & -P_{e-1}^{-1} & * \\ u(k) & 0 & -R^{-1} \end{bmatrix} \le 0 \tag{38}$$

Considering (39), which is equivalent to

$$x^{+T} P_{e-1} x^+ \le \xi_{e-1} \tag{39}$$

Taking into account the disturbance, then (42) holds if

$$x^{+T} P_{e-1} x^+ - \xi_{e-1} - \lambda (w^T(k) w(k) - \eta^2) \le 0 \tag{40}$$

As $x^+ = \sum_{i=-h}^{0} A_{i,\mu} x(k+i) + B_\mu u(k) + E_\mu w(k)$, the above inequality holds if

$$\begin{bmatrix} -\xi_{e-1} + \lambda \eta^2 & * & * \\ 0 & -\lambda & * \\ \sum_{i=-h}^{0} A_{i,\mu} x(k+i) + B_\mu u(k) & E_\mu & -P_{e-1}^{-1} \end{bmatrix} \le 0 \tag{41}$$

The input constraint is satisfied by

$$\begin{bmatrix} Z & u(k) \\ u^T(k) & I \end{bmatrix} \ge 0 \tag{42}$$

In summary, the constrained online optimization problem (37) is given as

$$\min_{u(k), \psi, \lambda, Z} \quad \psi, \tag{43}$$

subject to (41), (44), and (45), with the delayed states $\mathbf{x}_{[k-h,k]}$ known.

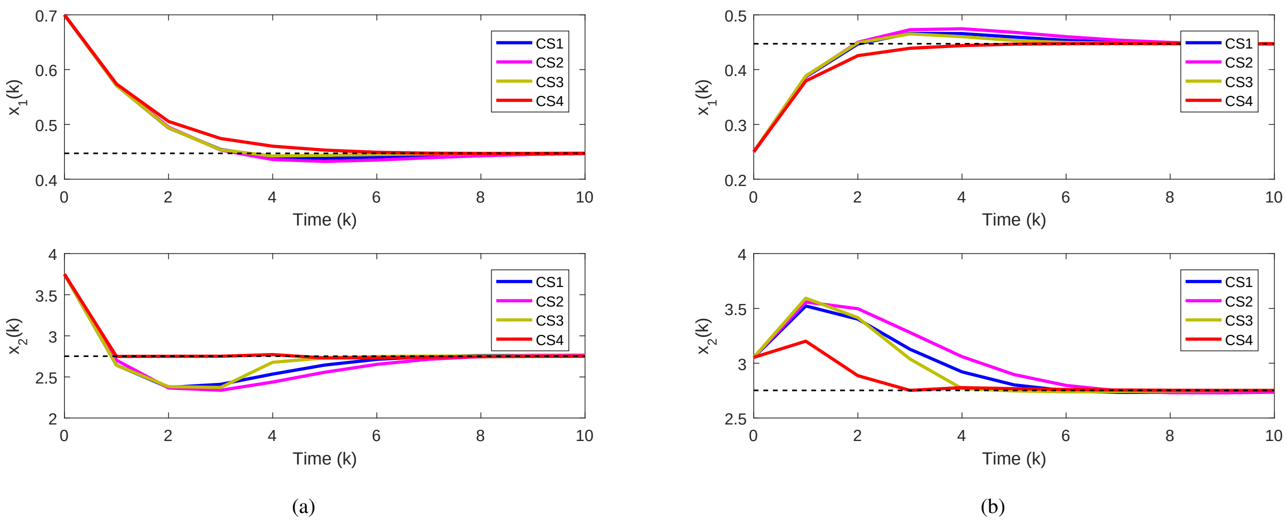

Fig. 3. State responses for two different initial conditions: (a) Initial condition (1), (b) Initial condition (2).

Therefore, the whole algorithm can be summarized as follows:

Algorithm 2:

1. Off-line computation

The off-line computation is focused on generating a sequence of robust constraint sets, the steps are the same as the ones in Algorithm 1.

2. Online computation

Step 1. Similar to Algorithm 1, we consider an initial condition which satisfies $x(i)_{\forall i\in\mathbb{Z}_{[-h,0]}} \in \Omega_{\tilde{N}}$. Let $k=0$ and go to Step 2;

Step 2. At time instant k, obtain the system all (delayed) states $x(k+i)_{\forall i\in\mathbb{Z}_{[-h,0]}}$, execute the search to decide $x(k+i)_{\forall i\in\mathbb{Z}_{[-h,0]}} \in \Omega_e\backslash\Omega_{e-1}$, if $\Omega_e = \Omega_0$, then the corresponding control law for Ω_0 is applied all the time. Otherwise, go to Step 3.

Step 3. Solve the optimization problem (46), obtain the control input and then feed the plant with it, let $k=k+1$ and return to Step 2.

Remark 6: Although only a finite horizon cost function is minimized at each time instant, the finite-time control performance derived in Theorem 4 is still satisfied because an additional constraint (39) is also added to the online optimization problem. Thus the system state is guaranteed to enter the neighboring subset at the next instant.

B. Example 2

Consider the CSTR system investigated in [21]. There are three nominal states when $u=0$: $x_{s1}=[0.1440;0.8862]$, $x_{s2}=[0.4472;2.7520]$, x_{s3}=[0.7646;4.7052]. The input constraint and bounded disturbance are set as: $\|u\| \leq 10$, $\|w\| \leq 1$. In this work, the steady state is chosen as x_{s2}. The following FMPC strategies are adopted in the simulation:

CS1: The online robust FMPC algorithm for time-delay systems in [21]. The prediction horizon is chosen as 1, and the weighting matrices Q and R for the cost function are selected as $diag\{1e-6, 1e-8\}$ and 0.001, respectively. Since the constraint $\sum_{i=-h}^{0}\gamma_i = 1$ should be satisfied, let $\gamma_0 = 0.8$, $\gamma_{-1} = 0.2$.

CS2: The robust FMPC for the augmented system without delay. Inspired by [9], the time-delay system is modeled as an augmented system without delay, and then the online robust FMPC approach in [24] is applied to the augmented system. In which case, Q and R for the cost function of the augmented system are selected as $diag\{1e-6, 1e-8, 1e-6, 1e-8\}$ and 0.001, respectively.

CS3: The proposed off-line FMPC algorithm (Algorithm 1). The simulation parameters are the same as the ones in CS1.

CS4: The proposed off-line FMPC algorithm with online optimization (Algorithm 2). The prediction horizon is chosen as 1, and the simulation parameters are the same as those in CS1.

Two different initial conditions are considered: (1) $x(i)_{\forall i\in\mathbb{Z}_{[-1,0]}} = [0.7; 3.75]$, (2) $x(i)_{\forall i\in\mathbb{Z}_{[-1,0]}} = [0.25; 3.0533]$. The simulation results are shown in Figs. 3-5. Table 1 shows the computational cost of different FMPC approaches. Comparisons of control performance by different approaches are shown in Fig. 3. It can be seen that, among all approaches, the best control performance is achieved by adopting CS4, while CS2 which makes use of the augmented system has longest settling time. In addition, from Table 1 one can see that, compared with CS3 and CS4, the computational cost of CS1 and CS2 are much heavier, especially for CS2 where augmented system dynamics are involved. Compared with CS3, an additional online optimization algorithm is involved in CS4, however, the online optimization is not very complex. Moreover, with the online optimization added to the off-line MPC approach in CS4, the online optimization is reduced to solving LMI problem, while both CS1 and CS2 have to solve bilinear matrix inequality (BMI) problems. The upper bounds of the cost function estimated by different approaches are shown in Fig. 4. It is shown that the upper bound estimated by CS1 and CS2 is lower than that estimated by CS4, which indicates that CS4 is less optimal than CS1 and CS2 from

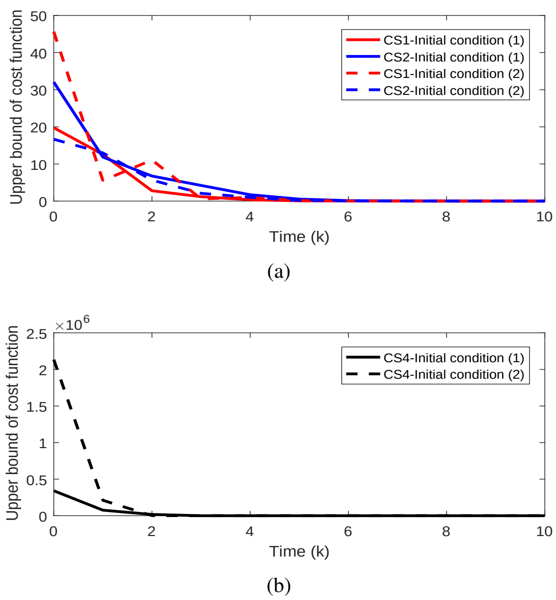

Fig. 4. The upper bound of cost function: (a) The upper bound of cost function by CS1 and CS2, (b) The upper bound of cost function by CS4.

the view of cost function. Therefore, CS4 can be regarded as a trade-off between optimality and computational burden. Fig. 5 shows the trajectories in the state space by adopting the proposed two algorithms, where the number of robust constraints sets are set as 25. From Fig. 5 one can see that system all (delayed) states are guaranteed to enter the terminal set.

TABLE I
COMPUTATIONAL COMPLEXITY OF DIFFERENT MPC STRATEGIES

Methods	Types	Dimension of the matrix inequality
CS1	online (BMI)	495×495
CS2	online (BMI)	666×666
CS3	off-line (BMI)	–
CS4	off-line (BMI) + online (LMI)	13×13 (for online part)

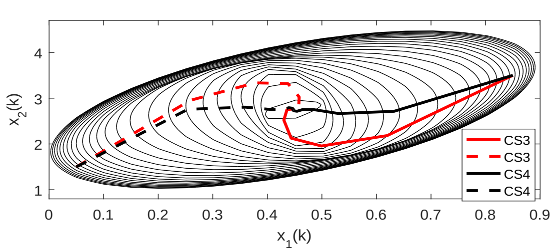

Fig. 5. Trajectories in state space for initial conditions (1) and (2) by adopting the proposed two approaches.

V. CONCLUSION

FMPC of discrete nonlinear systems with multiple time delays and persistent disturbances is investigated in this work. Based on the Lyapunov-Razumikhin approach, an efficient FMPC approach is developed that a sequence of explicit control laws corresponding to a sequence of robust constraint sets are computed off-line. The system all (delayed) states are guaranteed to the terminal constraint set in finite time, given the initial states in a specific region. In addition, an online optimization algorithm is investigated to reduce the conservatism of the off-line MPC approach. Simulation results show that the proposed two methods are effective and have advantages in control performance and computational cost over several MPC strategies. Particularly, the special control theory and the positive invariance behind Razumikhin approach are revealed by the proposed method.

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