Lyapunov and Matrix Norm Stability Analysis of ADI-FDTD Schemes for Doubly Lossy Media

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***Abstract*—Lyapunov and matrix norm stability analysis is applied on various alternating-direction-implicit finite-difference time-domain (ADI-FDTD) schemes for doubly lossy media. The stability analysis is performed rigorously in both time and Fourier domains. Among the schemes considered are averaging, forward-backward, backward-forward, forward-forward, exponential time differencing and backward-backward. From the analysis, it is found that all schemes except backward-backward scheme are unconditionally stable. For backward-backward scheme, the condition for stability is determined.**

***Index Terms*—Alternating-direction-implicit finite-difference time-domain (ADI-FDTD), lossy media, stability.**

I. INTRODUCTION

The alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method [1], [2] has been developed to remove the Courant-Friedrich-Lewy (CFL) constraint on the chosen time step, hence the unconditional stability feature. Such feature makes the ADI-FDTD method attractive for further extension into lossy media using different schemes [3]-[7]. Some stability analysis using the Von Neumann eigenvalue approach has been provided in [5] and [6] for several ADI-FDTD schemes for 1-D and 2-D electrically lossy media. The Von Neumann eigenvalue method, which converts the difference equation into Fourier domain for determining the magnitude of the eigenvalues is straightforward and popular for stability analysis. While this approach is still applicable for lossless ADI-FDTD method, or perhaps simpler specific case of electrically lossy in 1-D or 2-D, it is often tedious in 3-D doubly lossy media (both electric and magnetic conductivities are nonzero) due to the complexity of the updating matrix of the ADI-FDTD schemes.

Of late, the energy-based method has been widely used as an alternative method to the Von Neumann eigenvalue approach. The unconditionally stable feature of the ADI-FDTD in lossless media has been shown in [8], while the most commonly used averaging scheme of ADI-FDTD method for lossy media is presented in [9]. The matrix norm approach for stability analysis is another viable option which has been applied in lossless Crank-Nicolson, split-step and ADI-FDTD method [10], [11]. Nevertheless, it appears that the full 3-D stability analysis of various possible ADI-FDTD schemes for the most general case of doubly lossy media are still lacking.

In this paper, we shall present the Lyapunov and matrix norm stability analysis applied to various ADI-FDTD schemes for doubly lossy media. The stability analysis is performed rigorously in both time and Fourier domains. Among the schemes considered are averaging, forward-backward, backward-forward, forward-forward, exponential time differencing and backward-backward (These schemes will be elaborated further in Section IV). Averaging, exponential time differencing, forward-backward and backward-forward schemes have second order temporal accuracy. For forward-forward and backward-backward schemes, the temporal accuracy is only of first order. Generally, exponential time differencing has the highest accuracy for its closest resemblance to the solution of first-order differential equation, while averaging scheme is easier to formulate and is most commonly used. Forward-backward and backward-forward schemes, on the other hand, can be used for higher efficiency implementation due to involvement of fewer conductivity terms.

The Lyapunov method, somewhat related to energy-based method, and matrix norm method are ideal stability analysis approaches for various ADI-FDTD schemes due to some unique features in its updating matrices, which will be shown later. For Lyapunov stability analysis, it will be shown that one does not need to solve the discrete Lyapunov equation to deduce the stability of the particular scheme. Instead, one would only require to determine the positive definiteness of a certain matrix. From there, the subtle relationship between the Lyapunov and energy-based methods can also be seen. On the other hand, the matrix norm stability analysis yields the same condition to ensure stability as the Lyapunov method. Hence, the unconditionally stable feature of various ADI-FDTD schemes is proven. For scheme which is found to be not unconditionally stable, the condition for stability is determined.

The organization of this paper is as follows. Section II and III present the generalized Lyapunov and matrix norm stability analysis, respectively, both in time domain. The generalized stability analysis is then applied to aforementioned various ADI-FDTD schemes for doubly lossy media in Section IV. In Section V, it is shown that the analysis can be similarly carried out in the Fourier domain using the Von Neumann method.

II. GENERALIZED LYAPUNOV STABILITY ANALYSIS

Consider a discrete system

$$\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^{n} \tag{1}$$

where $\mathbf{u}$ is the state vector with time indices n and $n+1$, and $\mathbf{M}$ is the amplification matrix. The system is said to be stable in the sense of Lyapunov if the state vector remains bounded

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near the origin for all $n \geq 0$. Furthermore, if the state vector converges to the origin as $n \to \infty$, the system is asymtotically stable. The Lyapunov theorem states that the above discrete system is stable in the sense of Lyapunov if there exists a positive definite $\mathbf{Q}$ to the discrete Lyapunov equation

$$\mathbf{M}^T\mathbf{Q}\mathbf{M} - \mathbf{Q} = -\mathbf{P} \tag{2}$$

such that $\mathbf{P}$ is positive semidefinite. If there exists a positive definite $\mathbf{Q}$ to (2) such that $\mathbf{P}$ is positive definite, the system is asymtotically stable [12]-[14]. In FDTD stability analysis, researchers are often interested in whether the solution will always remain bounded (nonincreasing) as the time marches. Therefore, it should be mentioned that throughout the paper, if the system is found to be either stable in the sense of Lyapunov, or asymtotically stable, we shall simply refer them as "stable".

In the following, the above Lyapunov theorem will be applied in the stability analysis of ADI-FDTD schemes for doubly lossy media. The discretized Maxwell equations in space describing an electromagnetics problem within a computational domain terminated by perfect electric conductor (PEC) walls can be expressed in compact matrix form as

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{D}_\epsilon & \mathbf{O}\\ \mathbf{O} & \mathbf{D}_\mu\end{bmatrix}\begin{bmatrix}\mathbf{e}\\ \mathbf{h}\end{bmatrix} = \begin{bmatrix}\mathbf{D}_\sigma & -\mathbf{C}^T\\ \mathbf{C} & \mathbf{D}_{\sigma^*}\end{bmatrix}\begin{bmatrix}\mathbf{e}\\ \mathbf{h}\end{bmatrix} \tag{3}$$

where

$$\mathbf{e} = \begin{bmatrix}E_{x(i+\frac{1}{2},j,k)} & E_{y(i,j+\frac{1}{2},k)} & E_{z(i,j,k+\frac{1}{2})}\end{bmatrix}^T \tag{4}$$

$$\mathbf{h} = \begin{bmatrix}H_{x(i,j+\frac{1}{2},k+\frac{1}{2})} & H_{y(i+\frac{1}{2},j,k+\frac{1}{2})} & H_{z(i+\frac{1}{2},j+\frac{1}{2},k)}\end{bmatrix}^T \tag{5}$$

$$\mathbf{D}_\epsilon = \operatorname{diag}\left(\epsilon_{x(i+\frac{1}{2},j,k)}, \epsilon_{y(i,j+\frac{1}{2},k)}, \epsilon_{z(i,j,k+\frac{1}{2})}\right) \tag{6}$$

$$\mathbf{D}_\mu = \operatorname{diag}\left(\mu_{x(i,j+\frac{1}{2},k+\frac{1}{2})}, \mu_{y(i+\frac{1}{2},j,k+\frac{1}{2})}, \mu_{z(i+\frac{1}{2},j+\frac{1}{2},k)}\right) \tag{7}$$

$$\mathbf{D}_\sigma = \operatorname{diag}\left(\sigma_{x(i+\frac{1}{2},j,k)}, \sigma_{y(i,j+\frac{1}{2},k)}, \sigma_{z(i,j,k+\frac{1}{2})}\right) \tag{8}$$

$$\mathbf{D}_{\sigma^*} = \operatorname{diag}\left(\sigma^*_{x(i,j+\frac{1}{2},k+\frac{1}{2})}, \sigma^*_{y(i+\frac{1}{2},j,k+\frac{1}{2})}, \sigma^*_{z(i+\frac{1}{2},j+\frac{1}{2},k)}\right). \tag{9}$$

Note that i, j and k are grid position indices within the whole computation domain along x, y and z directions, respectively. $\mathbf{C}$ is the curl matrix comprising spatial step Δx, Δy and Δz resulting from the discetization of the curl operator through central differencing approximation. $\mathbf{O}$ is the null matrix. The dimensions of all matrices and vectors are dependent on the size of the computational domain. ϵ, μ, σ and σ^* are the medium permittivity, permeability, electric and magnetic conductivities, respectively. Henceforth, the indices i, j, k of material parameters ϵ, μ, σ and σ^* shall be dropped for convenience.

Upon discretizing in time, the ADI-FDTD update equations for doubly lossy media generally read:

$$\left(\mathbf{I} + \mathbf{D}_A - \mathbf{D}_C\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} = \left(\mathbf{I} - \mathbf{D}_B + \mathbf{D}_C\mathbf{B}\right)\mathbf{u}^n \tag{10a}$$

$$\left(\mathbf{I} + \mathbf{D}_B - \mathbf{D}_C\mathbf{B}\right)\mathbf{u}^{n+1} = \left(\mathbf{I} - \mathbf{D}_A + \mathbf{D}_C\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} \tag{10b}$$

where

$$\mathbf{u} = \begin{bmatrix}\mathbf{e} & \mathbf{h}\end{bmatrix}^T \tag{11}$$

$$\mathbf{A} = \begin{bmatrix}\mathbf{O} & -\mathbf{C}_1^T\\ \mathbf{C}_1 & \mathbf{O}\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}\mathbf{O} & -\mathbf{C}_2^T\\ \mathbf{C}_2 & \mathbf{O}\end{bmatrix}. \tag{12}$$

$\mathbf{C}_1$ and $\mathbf{C}_2$ are composed of discretization coefficients of spatial derivatives matrices $\begin{bmatrix}0 & 0 & \partial/\partial y\\ \partial/\partial z & 0 & 0\\ 0 & \partial/\partial x & 0\end{bmatrix}$ and $\begin{bmatrix}0 & -\partial/\partial z & 0\\ 0 & 0 & -\partial/\partial x\\ -\partial/\partial y & 0 & 0\end{bmatrix}$, respectively. Their dimensions are also dependent on the size of computational domain. Note that $\mathbf{C}_1$ and $\mathbf{C}_2$ are split from $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$ such that they yield computationally efficient tridiagonal matrices in both substeps of (10). $\mathbf{D}_A$, $\mathbf{D}_B$ and $\mathbf{D}_C$ are all diagonal matrices which comprise Δt, and material parameters matrices $\mathbf{D}_\epsilon$, $\mathbf{D}_\mu$, $\mathbf{D}_\sigma$ and $\mathbf{D}_{\sigma^*}$. For the moment, we shall let matrices $\mathbf{D}_A$, $\mathbf{D}_B$ and $\mathbf{D}_C$ be general but they will be further defined accordingly for different schemes in the subsequent sections.

Before the discrete Lyapunov equation is applied, the following transfomation of vector

$$\mathbf{v} = \mathbf{D}_C^{-\frac{1}{2}}\mathbf{u} \tag{13}$$

is used to yield the overall ADI-FDTD update equations of

$$\begin{aligned}\mathbf{v}^{n+1} &= \left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right)^{-1}\left(\mathbf{I} - \mathbf{D}_A + \mathbf{A}_C\right)\\ &\quad \times \left(\mathbf{I} + \mathbf{D}_A - \mathbf{A}_C\right)^{-1}\left(\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C\right)\mathbf{v}^n\\ &= \mathbf{M}\mathbf{v}^n\end{aligned} \tag{14}$$

where

$$\mathbf{A}_C = \mathbf{D}_C^{\frac{1}{2}}\mathbf{A}\mathbf{D}_C^{\frac{1}{2}} \tag{15}$$

$$\mathbf{B}_C = \mathbf{D}_C^{\frac{1}{2}}\mathbf{B}\mathbf{D}_C^{\frac{1}{2}}. \tag{16}$$

Note that as long as we can find a pair of positive definite (or positive semidefinite) $\mathbf{P}$ and positive definite $\mathbf{Q}$ satisfying (2), the sufficiency for stability is guaranteed. To that end, we choose $\mathbf{Q}$ as

$$\mathbf{Q} = \left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right)^T\left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right). \tag{17}$$

It can be shown that (Appendix I), along with given $\mathbf{M}$ in (14) and $\mathbf{Q}$ in (17), $\mathbf{P}$ can be determined from (2) as

$$\mathbf{P} = 4\left(\mathbf{Y}^T\mathbf{D}_A\mathbf{Y} + \mathbf{D}_B\right) \tag{18}$$

where $\mathbf{Y} = \left(\mathbf{I} + \mathbf{D}_A - \mathbf{A}_C\right)^{-1}\left(\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C\right)$.

The first term inside the bracket in (18) involves similarity transformation of $\mathbf{D}_A$ which preserves its eigenvalues (definiteness). Hence, if $\mathbf{D}_A$ and $\mathbf{D}_B$ are positive definite (positive semidefinite), $\mathbf{P}$ is positive definite (positive semidefinite). Furthermore, the specified $\mathbf{Q}$ in (17) is always positive definite for nonsingular $\left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right)$ (otherwise the whole amplification matrix $\mathbf{M}$ in (14) is undefined) [15]. Therefore, the sufficient condition for stability of ADI-FDTD schemes for doubly lossy media is such that $\mathbf{D}_A$ and $\mathbf{D}_B$ are positive definite or positive semidefinite. Note that $\mathbf{D}_C$ has no bearing over the stability of the ADI-FDTD schemes.

III. GENERALIZED MATRIX NORM STABILITY ANALYSIS

For the same discrete system in (1), $\mathbf{u}$ will always be bounded (stable) as the time marches if we can find any induced matrix norm (e.g. 1-norm, 2-norm, ∞-norm) to the amplification matrix, $\mathbf{M}$ such that $\|\mathbf{M}\| \leq 1$ [16].

For matrix norm analysis, we proceed from (14) and apply transformation vector of

$$\mathbf{w} = (\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C)\,\mathbf{v} \tag{19}$$

to yield

$$\begin{aligned}\mathbf{w}^{n+1} &= (\mathbf{I} - \mathbf{D}_A + \mathbf{A}_C)\,(\mathbf{I} + \mathbf{D}_A - \mathbf{A}_C)^{-1} \\ &\quad \times (\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C)\,(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C)^{-1}\,\mathbf{w}^n \\ &= \mathbf{M}^{'}\mathbf{w}^n \\ &= \mathbf{M}_A\mathbf{M}_B\mathbf{w}^n \end{aligned} \tag{20}$$

where

$$\mathbf{M}_A = (\mathbf{I} - \mathbf{D}_A + \mathbf{A}_C)\,(\mathbf{I} + \mathbf{D}_A - \mathbf{A}_C)^{-1} \tag{21}$$

$$\mathbf{M}_B = (\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C)\,(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C)^{-1}\,. \tag{22}$$

It should be noted that evaluating the matrix norm directly from $\mathbf{M}$ in (14) is difficult. The rationale behind transforming $\mathbf{v}$ further into $\mathbf{w}$ in (19) is to transform $\mathbf{M}$ in (14) into $\mathbf{M}^{'}$ in (20). $\mathbf{M}_A$ and $\mathbf{M}_B$ can then be extracted and their respective norm can be evaluated separately.

From the definition of induced matrix norm, the 2-norm (euclidean norm) of $\mathbf{M}_A$ can be expressed as

$$\|\mathbf{M}_A\|_2 = \sqrt{\sup_{\forall \mathbf{x}, \mathbf{x} \neq 0} \frac{[(\mathbf{I} - \mathbf{T}_A)\,\mathbf{x}]^T\,(\mathbf{I} - \mathbf{T}_A)\,\mathbf{x}}{[(\mathbf{I} + \mathbf{T}_A)\,\mathbf{x}]^T\,(\mathbf{I} + \mathbf{T}_A)\,\mathbf{x}}} \tag{23}$$

where $\mathbf{T}_A = \mathbf{D}_A - \mathbf{A}_C$.

Since $\mathbf{A}_C$ is skew-symmetric ($\mathbf{A}_C = -\mathbf{A}_C^T$), (23) can be reduced into

$$\|\mathbf{M}_A\|_2 = \sqrt{\sup_{\forall \mathbf{x}, \mathbf{x} \neq 0} \frac{\mathbf{x}^T\,[(\mathbf{I} - \mathbf{D}_A)^T(\mathbf{I} - \mathbf{D}_A) + \mathbf{R}_A]\,\mathbf{x}}{\mathbf{x}^T\,[(\mathbf{I} + \mathbf{D}_A)^T(\mathbf{I} + \mathbf{D}_A) + \mathbf{R}_A]\,\mathbf{x}}} \tag{24}$$

where $\mathbf{R}_A = \mathbf{A}_C^T\mathbf{A}_C - \mathbf{A}_C^T\mathbf{D}_A - \mathbf{D}_A\mathbf{A}_C$. We can see from (24) that if $\mathbf{D}_A$ is positive definite or positive semidefinite, $(\mathbf{I} - \mathbf{D}_A)^T(\mathbf{I} - \mathbf{D}_A) \leq (\mathbf{I} + \mathbf{D}_A)^T(\mathbf{I} + \mathbf{D}_A)$, and thus, $\|\mathbf{M}_A\|_2 \leq 1$.

Similarly, by replacing $\mathbf{A}_C$ with $\mathbf{B}_C$ and taking the skew-symmetric nature of $\mathbf{B}_C$ ($\mathbf{B}_C = -\mathbf{B}_C^T$) into consideration, we obtain

$$\|\mathbf{M}_B\|_2 = \sqrt{\sup_{\forall \mathbf{x}, \mathbf{x} \neq 0} \frac{\mathbf{x}^T\,[(\mathbf{I} - \mathbf{D}_B)^T(\mathbf{I} - \mathbf{D}_B) + \mathbf{R}_B]\,\mathbf{x}}{\mathbf{x}^T\,[(\mathbf{I} + \mathbf{D}_B)^T(\mathbf{I} + \mathbf{D}_B) + \mathbf{R}_B]\,\mathbf{x}}} \tag{25}$$

where $\mathbf{R}_B = \mathbf{B}_C^T\mathbf{B}_C - \mathbf{B}_C^T\mathbf{D}_B - \mathbf{D}_B\mathbf{B}_C$. Here, $\|\mathbf{M}_B\|_2 \leq 1$ if $\mathbf{D}_B$ is positive definite or positive semidefinite.

Using norm inequality of

$$\left\|\mathbf{M}^{'}\right\|_2 \leq \|\mathbf{M}_A\|_2 \cdot \|\mathbf{M}_B\|_2\,, \tag{26}$$

we note that both $\|\mathbf{M}_A\|_2 \leq 1$ and $\|\mathbf{M}_B\|_2 \leq 1$ imply that $\left\|\mathbf{M}^{'}\right\|_2 \leq 1$, which guarantees stability. It can be seen that the condition to guarantee algorithm stability formulated from matrix norm stability analysis is exactly the same as those formulated from Lyapunov stability analysis in the previous section. Both analyses show that the positive definiteness (or positive semidefiniteness) of $\mathbf{D}_A$ and $\mathbf{D}_B$ are paramount to ensure stability of ADI-FDTD schemes for doubly lossy media.

IV. STABILITY ANALYSIS OF VARIOUS ADI-FDTD SCHEMES FOR DOUBLY LOSSY MEDIA

We now proceed to analyze the stability of various ADI-FDTD schemes for doubly lossy media using the above generalized stability analysis. The schemes considered in this section include averaging, forward-backward, backward-forward, exponential time differencing, forward-forward, backward-backward and lossless ADI-FDTD.

A. Averaging

The averaging scheme [3], [9] of ADI-FDTD method for doubly lossy media is one of the most common schemes used where the conductivity terms are averaged between two time indices in both substeps. The scheme calls for the following update procedures:

$$\begin{aligned}&\left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} \\ &= \left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{B}\right)\mathbf{u}^{n} \end{aligned} \tag{27a}$$

$$\begin{aligned}&\left(\mathbf{I} + \frac{\Delta t}{4}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{B}\right)\mathbf{u}^{n+1} \\ &= \left(\mathbf{I} - \frac{\Delta t}{4}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} \end{aligned} \tag{27b}$$

where

$$\mathbf{D}_{\epsilon\mu} = \begin{bmatrix}\mathbf{D}_\epsilon & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_\mu\end{bmatrix}, \quad \mathbf{D}_{\sigma\sigma^*} = \begin{bmatrix}\mathbf{D}_\sigma & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_{\sigma^*}\end{bmatrix}. \tag{28}$$

Note that the averaging scheme is second-order in temporal accuracy.

Comparing (27) to (10) in the generalized stability analysis, it is found that

$$\mathbf{D}_A = \mathbf{D}_B = \frac{\Delta t}{4}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*} \tag{29}$$

$$\mathbf{D}_C = \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}. \tag{30}$$

If ϵ_ξ, μ_ξ, σ_ξ, σ_ξ^* ($\xi = x, y, z$) of all grid points (inhomogeneous media) within the computational domain and Δt are positive, $\mathbf{D}_A$ and $\mathbf{D}_B$ are always positive definite. Note that σ_ξ and σ_ξ^* can also be zero at certain grid points to represent combination of lossless and lossy media, or electrically lossy ($\sigma^* = 0$) and magnetically lossy ($\sigma = 0$) media likewise. In this case, $\mathbf{D}_A$ and $\mathbf{D}_B$ are always positve semidefinite. As far as Lyapunov stability analysis is concerned, we can see from (18) that $\mathbf{P}$ is always positive definite, or positive semidefinite. On the other hand, from matrix norm stability analysis, it can be found from (24) and (25) that

$$\|M_A\|_2 \leq 1, \quad \|M_B\|_2 \leq 1 \tag{31}$$

always hold as long as $\mathbf{D}_A$ and $\mathbf{D}_B$ are positive definite or positive semidefinite.

Therefore, it can be concluded concurrently from both Lyapunov and matrix norm stability analysis that the averaging scheme is unconditionally stable as there is no restriction imposed on the chosen time step Δt.

B. Forward-Backward

In forward-backward scheme [7], the conductivity terms are applied at the forward time $(n+\frac{1}{2})$ in the first substep and backward time $(n+\frac{1}{2})$ in the second. The scheme calls for the following update procedures:

$$\begin{aligned}&\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}\\&=\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{B}\right)\mathbf{u}^{n}\end{aligned}\tag{32a}$$

$$\begin{aligned}&\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{B}\right)\mathbf{u}^{n+1}\\&=\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}+\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}.\end{aligned}\tag{32b}$$

The forward-backward scheme is second-order in temporal accuracy.

Comparing (32) to (10) in the generalized stability analysis, it is found that

$$\mathbf{D}_A=\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}\tag{33}$$

$$\mathbf{D}_B=\mathbf{O}\tag{34}$$

$$\mathbf{D}_C=\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}.\tag{35}$$

Here, similar arguments as in the averaging scheme applies. $\mathbf{D}_A$ and $\mathbf{D}_B$ are either always positive definite or positive semidefinite. As a result, from Lyapunov stability analysis, $\mathbf{P}$ is always positive definite or positive semidefinite. From matrix norm stability analysis, we also find that

$$\|M_A\|_2\le 1,\quad \|M_B\|_2=1\tag{36}$$

always hold as long as $\mathbf{D}_A$ and $\mathbf{D}_B$ are positve definite or positive semidefinite. Therefore, forward-backward scheme is also unconditionally stable.

C. Backward-Forward

The backward-forward scheme is essentially a dual of the forward-backward scheme. In this scheme, the conductivity terms are applied at the backward time (n) in the first substep and forward time $(n+1)$ in the second. The scheme calls for the following update procedures:

$$\begin{aligned}&\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}\\&=\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}+\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{B}\right)\mathbf{u}^{n}\end{aligned}\tag{37a}$$

$$\begin{aligned}&\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{B}\right)\mathbf{u}^{n+1}\\&=\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}.\end{aligned}\tag{37b}$$

The backward-forward scheme is second-order in temporal accuracy.

Comparing (37) to (10) in the generalized stability analysis, it is found that

$$\mathbf{D}_A=\mathbf{O}\tag{38}$$

$$\mathbf{D}_B=\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}\tag{39}$$

$$\mathbf{D}_C=\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}.\tag{40}$$

Again, $\mathbf{D}_A$ and $\mathbf{D}_B$ are either always positive definite or positive semidefinite. As a result, from Lyapunov stability analysis, $\mathbf{P}$ is always positive definite or positive semidefinite. From matrix norm stability analysis, we also find that

$$\|M_A\|_2=1,\quad \|M_B\|_2\le 1\tag{41}$$

always hold as long as $\mathbf{D}_A$ and $\mathbf{D}_B$ are positve definite or positive semidefinite. Therefore, backward-forward scheme is also unconditionally stable.

D. Exponential Time Differencing

Apart from being adopted in explicit FDTD scheme [17], [18], the exponential time differencing (ETD) scheme can also be incorporated into the ADI-FDTD method, which calls for the following update procedures:

$$\left(\mathbf{I}-\mathbf{D}_e\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}=\left(e^{-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}}+\mathbf{D}_e\mathbf{B}\right)\mathbf{u}^{n}\tag{42a}$$

$$\left(\mathbf{I}-\mathbf{D}_e\mathbf{B}\right)\mathbf{u}^{n+1}=\left(e^{-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}}+\mathbf{D}_e\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}\tag{42b}$$

where $\mathbf{D}_e=\mathbf{D}_{\sigma\sigma^*}^{-1}\left(\mathbf{I}-e^{-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}}\right)$ is diagonal. The ETD scheme is also second-order in temporal accuracy.

Note that the general stability analysis procedure in the previous sections cannot be applied directly to (42). In order to apply the analysis procedure, we have to first consider the following mapping of matrices:

$$e^{-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}}=\left(\mathbf{I}+\mathbf{D}_{m1}\right)^{-1}\left(\mathbf{I}-\mathbf{D}_{m1}\right).\tag{43}$$

Using the above mapping along with some manipulation, (42) can be rewritten as

$$\begin{aligned}&\left(\mathbf{I}+\mathbf{D}_{m1}-\mathbf{D}_e\left(\mathbf{I}+\mathbf{D}_{m1}\right)\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}\\&=\left(\mathbf{I}-\mathbf{D}_{m1}+\mathbf{D}_e\left(\mathbf{I}+\mathbf{D}_{m1}\right)\mathbf{B}\right)\mathbf{u}^{n}\end{aligned}\tag{44a}$$

$$\begin{aligned}&\left(\mathbf{I}+\mathbf{D}_{m1}-\mathbf{D}_e\left(\mathbf{I}+\mathbf{D}_{m1}\right)\mathbf{B}\right)\mathbf{u}^{n+1}\\&=\left(\mathbf{I}-\mathbf{D}_{m1}+\mathbf{D}_e\left(\mathbf{I}+\mathbf{D}_{m1}\right)\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}\end{aligned}\tag{44b}$$

Comparing (44) to (10), we obtain

$$\mathbf{D}_A=\mathbf{D}_B=\mathbf{D}_{m1}\tag{45}$$

$$\mathbf{D}_C=\mathbf{D}_e\left(\mathbf{I}+\mathbf{D}_{m1}\right)\tag{46}$$

where $\mathbf{D}_{m1}$ can be solved from (43) as

$$\mathbf{D}_{m1}=\left(\mathbf{I}+e^{-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}}\right)^{-1}\left(\mathbf{I}-e^{-\frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}}\right).\tag{47}$$

We can see from (47) that $\mathbf{D}_{m1}$ and hence, $\mathbf{D}_A$ and $\mathbf{D}_B$ are always positive definite or positive semidefinite. As a result, from Lyapunov stability analysis, $\mathbf{P}$ is always positive definite or positive semidefinite. From matrix norm stability analysis, we also find that

$$\|M_A\|_2 \leq 1, \quad \|M_B\|_2 \leq 1 \tag{48}$$

always hold as long as $\mathbf{D}_A$ and $\mathbf{D}_B$ are positive definite or positive semidefinite. Therefore, the ETD scheme is unconditionally stable.

E. Forward-Forward

In forward-forward scheme [4], the conductivity terms are applied at forward time for both substeps ($n+\frac{1}{2}$ for the first and $n+1$ for second). The scheme calls for the following update procedures:

$$\begin{aligned}
&\left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} \\
&= \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{B}\right)\mathbf{u}^{n} && \text{(49a)} \\
&\left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{B}\right)\mathbf{u}^{n+1} \\
&= \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}. && \text{(49b)}
\end{aligned}$$

Note however that forward-forward scheme is only first-order in temporal accuracy.

In forward-forward scheme, we now consider the mapping of

$$\left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}\right)^{-1} = \left(\mathbf{I} + \mathbf{D}_{m2}\right)^{-1}\left(\mathbf{I} - \mathbf{D}_{m2}\right). \tag{50}$$

Using the above mapping along with some manipulation, (49) can be rewritten as

$$\begin{aligned}
&\left(\mathbf{I} + \mathbf{D}_{m2} - \mathbf{D}_f\left(\mathbf{I} + \mathbf{D}_{m2}\right)\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} \\
&= \left(\mathbf{I} - \mathbf{D}_{m2} + \mathbf{D}_f\left(\mathbf{I} + \mathbf{D}_{m2}\right)\mathbf{B}\right)\mathbf{u}^{n} && \text{(51a)} \\
&\left(\mathbf{I} + \mathbf{D}_{m2} - \mathbf{D}_f\left(\mathbf{I} + \mathbf{D}_{m2}\right)\mathbf{B}\right)\mathbf{u}^{n+1} \\
&= \left(\mathbf{I} - \mathbf{D}_{m2} + \mathbf{D}_f\left(\mathbf{I} + \mathbf{D}_{m2}\right)\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} && \text{(51b)}
\end{aligned}$$

where $\mathbf{D}_f = \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}\right)^{-1}$.

Comparing (51) to (10), it is observed that

$$\mathbf{D}_A = \mathbf{D}_B = \mathbf{D}_{m2} \tag{52}$$

$$\mathbf{D}_C = \mathbf{D}_f\left(\mathbf{I} + \mathbf{D}_{m2}\right) \tag{53}$$

where $\mathbf{D}_{m2}$ can be solved from (50) as

$$\mathbf{D}_{m2} = \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}\left(2\mathbf{I} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}\right)^{-1}. \tag{54}$$

Equation (54) indicates that $\mathbf{D}_{m2}$ and hence, $\mathbf{D}_A$ and $\mathbf{D}_B$ are always positive definite or positive semidefinite. As a result, from Lyapunov stability analysis, $\mathbf{P}$ is always positive definite or positive semidefinite. From matrix norm stability analysis, we also find that

$$\|M_A\|_2 \leq 1, \quad \|M_B\|_2 \leq 1 \tag{55}$$

always hold as long as $\mathbf{D}_A$ and $\mathbf{D}_B$ are positive definite or positive semidefinite. Therefore, the forward-forward scheme is unconditionally stable.

F. Backward-Backward

In backward-backward scheme, the conductivity terms are applied at backward time for both substeps (n for the first and $n+\frac{1}{2}$ for second). The scheme calls for the following update procedures:

$$\begin{aligned}
&\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} \\
&= \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{B}\right)\mathbf{u}^{n} && \text{(56a)} \\
&\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{B}\right)\mathbf{u}^{n+1} \\
&= \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*} + \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}. && \text{(56b)}
\end{aligned}$$

In backward-backward scheme, we consider the following mapping of

$$\mathbf{I} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}\mathbf{D}_{\sigma\sigma^*} = \left(\mathbf{I} + \mathbf{D}_{m3}\right)^{-1}\left(\mathbf{I} - \mathbf{D}_{m3}\right). \tag{57}$$

Note that the mapping applied to backward-backward scheme, c.f. (57), as well as the mappings applied to ETD and forward-forward schemes in the previous subsections, c.f. (43) and (50) are used to transform their update equations into the form of (10) so that the general stability analysis procedure in Sections II and III can be carried out easily.

Using the above mapping along with some manipulation, (56) can be written as

$$\begin{aligned}
&\left(\mathbf{I} + \mathbf{D}_{m3} - \mathbf{D}_g\left(\mathbf{I} + \mathbf{D}_{m3}\right)\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} \\
&= \left(\mathbf{I} - \mathbf{D}_{m3} + \mathbf{D}_g\left(\mathbf{I} + \mathbf{D}_{m3}\right)\mathbf{B}\right)\mathbf{u}^{n} && \text{(58a)} \\
&\left(\mathbf{I} + \mathbf{D}_{m3} - \mathbf{D}_g\left(\mathbf{I} + \mathbf{D}_{m3}\right)\mathbf{B}\right)\mathbf{u}^{n+1} \\
&= \left(\mathbf{I} - \mathbf{D}_{m3} + \mathbf{D}_g\left(\mathbf{I} + \mathbf{D}_{m3}\right)\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} && \text{(58b)}
\end{aligned}$$

where $\mathbf{D}_g = \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}$.

Comparing (58) to (10), it is observed that

$$\mathbf{D}_A = \mathbf{D}_B = \mathbf{D}_{m3} \tag{59}$$

$$\mathbf{D}_C = \mathbf{D}_g\left(\mathbf{I} + \mathbf{D}_{m3}\right) \tag{60}$$

where $\mathbf{D}_{m3}$ can be solved from (57) as

$$\mathbf{D}_{m3} = \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}\left(2\mathbf{I} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}\right)^{-1}. \tag{61}$$

However, in this case, we can see from (61) that $\mathbf{D}_{m3}$, and hence $\mathbf{D}_A$ and $\mathbf{D}_B$ are not always positive definite or positive semidefinite even when ϵ_ξ, μ_ξ, σ_ξ, σ_ξ^* of all grid points and Δt are positive. Therefore, backward-backward scheme is not unconditionally stable from Lyapunov stability analysis as $\mathbf{P}$ is not always positive definite or positive semidefinite. In order to ensure positive definiteness or positive semidefiniteness of $\mathbf{D}_{m3}$, the following condition must hold:

$$\left(2\mathbf{I} - \frac{\Delta t}{2}\mathbf{D}_{\epsilon\mu}^{-1}\mathbf{D}_{\sigma\sigma^*}\right)^{-1} > 0. \tag{62}$$

From (62), we can see that the choice of Δt is somehow bounded by ϵ_ξ, μ_ξ, σ_ξ and σ_ξ^* within the computational domain. Solving (62) for Δt, the condition to ensure stability for backward-backward scheme is thus given by

$$\Delta t < 4 \left(\min_{\xi=x,y,z,\forall i,\forall j,\forall k} \left\{ \frac{\epsilon_\xi}{\sigma\xi}, \frac{\mu_\xi}{\sigma^*_\xi} \right\} \right) \tag{63}$$

where ϵ_ξ, μ_ξ, σ_ξ, σ^*_ξ and Δt are all positive. Equation (63) implies that the time step should sufficiently resolve the minimum relaxation time of the medium, where the electric and magnetic relaxation time are commonly defined as ϵ/σ and μ/σ^*, respectively. It should be pointed out that unlike the explicit FDTD method, the stability criterion of backward-backward scheme of ADI-FDTD is only dependent on the relaxation time of the medium and *independent* of the mesh size.

From matrix norm stability analysis, we now find that

$$\|M_A\|_2 \leq 1, \quad \|M_B\|_2 \leq 1 \tag{64}$$

only hold if the condition in (63) is satisfied.

G. Lossless

For completeness, the stability analysis for the lossless ADI-FDTD method is performed in the realm of Lyapunov and matrix norm stability. The lossless ADI-FDTD method calls for the following update procedures:

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{D}^{-1}_{\epsilon\mu}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{D}^{-1}_{\epsilon\mu}\mathbf{B}\right)\mathbf{u}^{n} \tag{65a}$$

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{D}^{-1}_{\epsilon\mu}\mathbf{B}\right)\mathbf{u}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{D}^{-1}_{\epsilon\mu}\mathbf{A}\right)\mathbf{u}^{n+\frac{1}{2}}. \tag{65b}$$

It can be seen that

$$\mathbf{D}_A = \mathbf{D}_B = \mathbf{O} \tag{66}$$

$$\mathbf{D}_C = \frac{\Delta t}{2}\mathbf{D}^{-1}_{\epsilon\mu}. \tag{67}$$

Due to skew-symmetric nature of $\mathbf{A}_C$, we find that $(\mathbf{I} + \mathbf{A}_C)(\mathbf{I} - \mathbf{A}_C)^{-1}$ is in fact Cayley transform matrix which is orthogonal. Thus, matrix $\mathbf{P}$ can be determined directly from the Lyapunov equation. The Lyapunov equation for lossless ADI-FDTD read:

$$\begin{aligned} \mathbf{M}^T\mathbf{Q}\mathbf{M} - \mathbf{Q} &= (\mathbf{I} + \mathbf{B}_C)^T(\mathbf{I} + \mathbf{B}_C) - (\mathbf{I} - \mathbf{B}_C)^T(\mathbf{I} - \mathbf{B}_C) \\ &= \mathbf{O} \end{aligned} \tag{68}$$

where we have also taken into account the skew-symmetric nature of $\mathbf{B}_C$. In lossless case, $\mathbf{P}$ is a null matrix which is positive semidefinite. As from matrix norm stability analysis, since both $\mathbf{D}_A$ and $\mathbf{D}_B$ are null matrices, we find that

$$\|M_A\|_2 = 1, \quad \|M_B\|_2 = 1 \tag{69}$$

is always true. Therefore, the ADI-FDTD scheme in lossless media is unconditionally stable.

V. VON NEUMANN METHOD

Instead of analyzing in time domain, the Von Neumann method is often adopted for homogeneous media where all field components are transformed into Fourier domain. Doing so, (10) in Fourier domain read:

$$\left(\tilde{\mathbf{I}} + \tilde{\mathbf{D}}_A - \tilde{\mathbf{D}}_C\tilde{\mathbf{A}}\right)\mathbf{u}^{n+\frac{1}{2}} = \left(\tilde{\mathbf{I}} - \tilde{\mathbf{D}}_B + \tilde{\mathbf{D}}_C\tilde{\mathbf{B}}\right)\mathbf{u}^{n} \tag{70a}$$

$$\left(\tilde{\mathbf{I}} + \tilde{\mathbf{D}}_B - \tilde{\mathbf{D}}_C\tilde{\mathbf{B}}\right)\mathbf{u}^{n+1} = \left(\tilde{\mathbf{I}} - \tilde{\mathbf{D}}_A + \tilde{\mathbf{D}}_C\tilde{\mathbf{A}}\right)\mathbf{u}^{n+\frac{1}{2}}. \tag{70b}$$

where

$$\tilde{\mathbf{u}} = \begin{bmatrix}\tilde{E}_x & \tilde{E}_y & \tilde{E}_z & \tilde{H}_x & \tilde{H}_y & \tilde{H}_z\end{bmatrix}^T \tag{71}$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -jK_y \\ 0 & 0 & 0 & -jK_z & 0 & 0 \\ 0 & 0 & 0 & 0 & -jK_x & 0 \\ 0 & -jK_z & 0 & 0 & 0 & 0 \\ 0 & 0 & -jK_x & 0 & 0 & 0 \\ -jK_y & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{72}$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & jK_z & 0 \\ 0 & 0 & 0 & 0 & 0 & jK_x \\ 0 & 0 & 0 & jK_y & 0 & 0 \\ 0 & 0 & jK_y & 0 & 0 & 0 \\ jK_z & 0 & 0 & 0 & 0 & 0 \\ 0 & jK_x & 0 & 0 & 0 & 0 \end{bmatrix} \tag{73}$$

$$K_\xi = 2\sin\left(k_\xi\Delta_\xi/2\right)/\Delta_\xi. \tag{74}$$

Note that we have added tilde sign to all related matrices and vectors to indicate Fourier domain entries. Now, $\tilde{\mathbf{D}}_A$, $\tilde{\mathbf{D}}_B$ and $\tilde{\mathbf{D}}_C$ are all 6×6 diagonal matrices which comprise material parameter matrices $\tilde{\mathbf{D}}_{\epsilon\mu} = \operatorname{diag}\left(\epsilon_x, \epsilon_y, \epsilon_z, \mu_x, \mu_y, \mu_z\right)$ and $\tilde{\mathbf{D}}_{\sigma\sigma^*} = \operatorname{diag}\left(\sigma_x, \sigma_y, \sigma_z, \sigma^*_x, \sigma^*_y, \sigma^*_z\right)$.

The Lyapunov and matrix norm stability analysis detailed above is still applicable. This can be done by substituting all related matrices and vectors with their associated complex counterpart, and replacing all matrix and vector transpose operator with hermitian (complex). It should be pointed out that the complex matrices $\tilde{\mathbf{A}}_c = \tilde{\mathbf{D}}_C^{\frac{1}{2}}\tilde{\mathbf{A}}\tilde{\mathbf{D}}_C^{\frac{1}{2}}$ and $\tilde{\mathbf{B}}_c = \tilde{\mathbf{D}}_C^{\frac{1}{2}}\tilde{\mathbf{B}}\tilde{\mathbf{D}}_C^{\frac{1}{2}}$ are now skew-hermitian ($\tilde{\mathbf{A}}_c = -\tilde{\mathbf{A}}_c^H$, $\tilde{\mathbf{B}}_c = -\tilde{\mathbf{B}}_c^H$). Using the same procedures in Sections II and III, it can be shown in Fourier domain that averaging, forward-backward, backward-forward, ETD, forward-forward and lossless schemes are all unconditionally stable. For backward-backward scheme, we arrive at the following mapped $\tilde{\mathbf{D}}_{m3}$ in Fourier domain from (61) as

$$\tilde{\mathbf{D}}_{m3} = \frac{\Delta t}{2}\tilde{\mathbf{D}}^{-1}_{\epsilon\mu}\tilde{\mathbf{D}}_{\sigma\sigma^*}\left(2\tilde{\mathbf{I}} - \frac{\Delta t}{2}\tilde{\mathbf{D}}^{-1}_{\epsilon\mu}\tilde{\mathbf{D}}_{\sigma\sigma^*}\right)^{-1}. \tag{75}$$

Similar to the analysis in time domain, the positive definiteness of $\tilde{\mathbf{D}}_{m3}$ is required to ensure stability. In order for $\tilde{\mathbf{D}}_{m3}$ to be positive definite, the following condition must hold:

$$\left(2\tilde{\mathbf{I}} - \frac{\Delta t}{2}\tilde{\mathbf{D}}^{-1}_{\epsilon\mu}\tilde{\mathbf{D}}_{\sigma\sigma^*}\right)^{-1} > 0 \tag{76}$$

Solving (76) for Δt, the condition to ensure stability for backward-backward scheme in Fourier domain is given by

$$\Delta t < 4 \left(\min_{\xi = x,y,z} \left\{ \frac{\epsilon_\xi}{\sigma_\xi}, \frac{\mu_\xi}{\sigma_\xi^*} \right\} \right) \tag{77}$$

where ϵ_ξ, σ_ξ, μ_ξ, σ_ξ^* and Δt are all positive. For inhomogeneous media, ϵ_ξ, σ_ξ, μ_ξ, σ_ξ^* are typically chosen such that $\frac{\epsilon_\xi}{\sigma_\xi}$ and $\frac{\mu_\xi}{\sigma_\xi^*}$ are the minimum among different media considered within the whole computational domain. Thus, the condition (77) in Fourier domain is similar to (63) in time domain.

VI. CONCLUSION

Lyapunov and matrix norm stability analysis has been applied on various alternating-direction-implicit finite-difference time-domain (ADI-FDTD) schemes for doubly lossy media. Among the schemes considered are averaging, forward-backward, backward-forward, forward-forward, exponential time differencing and backward-backward. From the analysis, it has been found that all schemes except backward-backward scheme are unconditionally stable. For backward-backward scheme, the condition for stability has been determined.

APPENDIX I.
DERIVATION OF MATRIX P FROM DISCRETE LYAPUNOV EQUATION

An arbitrary vector $\mathbf{v}$ and its transpose $\mathbf{v}^T$ are multiplied from the left and right at both sides of (2) to yield an energy-like equation

$$(\mathbf{v}^n)^T \mathbf{M}^T \mathbf{Q} \mathbf{M} \mathbf{v}^n - (\mathbf{v}^n)^T \mathbf{Q} \mathbf{v}^n = -(\mathbf{v}^n)^T \mathbf{P} \mathbf{v}^n. \tag{78}$$

Noting that $\mathbf{v}^{n+1} = \mathbf{M}\mathbf{v}^n$, (78) is rewritten as

$$(\mathbf{v}^{n+1})^T \mathbf{Q} \mathbf{v}^{n+1} - (\mathbf{v}^n)^T \mathbf{Q} \mathbf{v}^n = -(\mathbf{v}^n)^T \mathbf{P} \mathbf{v}^n. \tag{79}$$

Substituting Q in (17) into (79) and upon some manipulation, one obtains

$$\begin{aligned}
&(\mathbf{v}^{n+1})^T \left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right)^T \left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right) \mathbf{v}^{n+1} \\
&- (\mathbf{v}^n)^T \left(\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C\right)^T \left(\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C\right) \mathbf{v}^n \\
&- [(\mathbf{v}^n)^T \left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right)^T \left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right) \mathbf{v}^n \\
&- (\mathbf{v}^n)^T \left(\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C\right)^T \left(\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C\right) \mathbf{v}^n] \\
&= -(\mathbf{v}^n)^T \mathbf{P} \mathbf{v}^n.
\end{aligned} \tag{80}$$

From ADI-FDTD update procedures, we know that

$$\left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right) \mathbf{v}^{n+1} = \left(\mathbf{I} - \mathbf{D}_A + \mathbf{A}_C\right) \mathbf{v}^{n+\frac{1}{2}} \tag{81}$$

$$\left(\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C\right) \mathbf{v}^{n} = \left(\mathbf{I} + \mathbf{D}_A - \mathbf{A}_C\right) \mathbf{v}^{n+\frac{1}{2}}. \tag{82}$$

Substituting these into the first two terms of above equation, one gets

$$\begin{aligned}
&(\mathbf{v}^{n+\frac{1}{2}})^T \left(\mathbf{I} - \mathbf{D}_A + \mathbf{A}_C\right)^T \left(\mathbf{I} - \mathbf{D}_A + \mathbf{A}_C\right) \mathbf{v}^{n+\frac{1}{2}} \\
&- (\mathbf{v}^{n+\frac{1}{2}})^T \left(\mathbf{I} + \mathbf{D}_A - \mathbf{A}_C\right)^T \left(\mathbf{I} + \mathbf{D}_A - \mathbf{A}_C\right) \mathbf{v}^{n+\frac{1}{2}} \\
&- [(\mathbf{v}^n)^T \left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right)^T \left(\mathbf{I} + \mathbf{D}_B - \mathbf{B}_C\right) \mathbf{v}^n \\
&- (\mathbf{v}^n)^T \left(\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C\right)^T \left(\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C\right) \mathbf{v}^n] \\
&= -(\mathbf{v}^n)^T \mathbf{P} \mathbf{v}^n.
\end{aligned} \tag{83}$$

Noting that $\mathbf{A}_C$ and $\mathbf{B}_C$ are skew-symmetric ($\mathbf{A}_C = -\mathbf{A}_C^T$ and $\mathbf{B}_C = -\mathbf{B}_C^T$), (83) is reduced into

$$(\mathbf{v}^{n+\frac{1}{2}})^T 4(-\mathbf{D}_A) \mathbf{v}^{n+\frac{1}{2}} - (\mathbf{v}^n)^T 4\mathbf{D}_B \mathbf{v}^n = -(\mathbf{v}^n)^T \mathbf{P} \mathbf{v}^n. \tag{84}$$

Taking into consideration the relationship between $\mathbf{v}^{n+\frac{1}{2}}$ and $\mathbf{v}^n$ in (82), one finally arrives at

$$(\mathbf{v}^n)^T \mathbf{Y}^T 4(-\mathbf{D}_A) \mathbf{Y} \mathbf{v}^n - (\mathbf{v}^n)^T 4\mathbf{D}_B \mathbf{v}^n = -(\mathbf{v}^n)^T \mathbf{P} \mathbf{v}^n \tag{85}$$

where $\mathbf{Y} = \left(\mathbf{I} + \mathbf{D}_A - \mathbf{A}_C\right)^{-1} \left(\mathbf{I} - \mathbf{D}_B + \mathbf{B}_C\right)$.

Thus, $\mathbf{P}$ is finally derived as

$$\mathbf{P} = 4\left(\mathbf{Y}^T \mathbf{D}_A \mathbf{Y} + \mathbf{D}_B\right). \tag{86}$$

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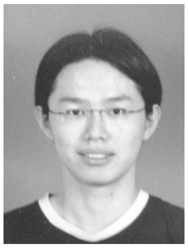

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