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ON DIVERSITY OF CERTAIN t-INTERSECTING FAMILIES

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Abstract. Let $[n] = \{1, 2, \dots, n\}$ and $2^{[n]}$ be the set of all subsets of $[n]$. For a family $\mathcal{F} \subseteq 2^{[n]}$, its diversity, denoted by $\operatorname{div}(\mathcal{F})$, is defined to be

$$\operatorname{div}(\mathcal{F}) = \min_{x \in [n]} \{|\mathcal{F}(\overline{x})|\},$$

where $\mathcal{F}(\overline{x}) = \{F \in \mathcal{F} : x \notin F\}$. Basically, $\operatorname{div}(\mathcal{F})$ measures how far $\mathcal{F}$ is from a trivial intersecting family, which is called a star. In this paper, we consider a generalization of diversity for t-intersecting family.

1. Introduction

Let $[n] = \{1, \dots, n\}$, and let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$. A family $\mathcal{A}$ of subsets of $[n]$ is *t-intersecting* if $|A \cap B| \geq t$ for all $A, B \in \mathcal{A}$. One of the most beautiful results in extremal combinatorics is the Erdős-Ko-Rado theorem.

Theorem 1.1 (Erdős, Ko, and Rado [8], Frankl [9], Wilson [36]). *Suppose $\mathcal{A} \subseteq \binom{[n]}{k}$ is t-intersecting and $n > 2k - t$. Then for $n \geq (k - t + 1)(t + 1)$, we have*

$$|\mathcal{A}| \leq \binom{n-t}{k-t}.$$

Moreover, if $n > (k - t + 1)(t + 1)$, then the equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{k} : T \subseteq A\}$ for some t-set T.

In the celebrated paper [1], Ahlswede and Khachatrian extended the Erdős-Ko-Rado theorem by determining the structure of all t-intersecting set systems of maximum size for all possible n (see also [3, 14–16, 22, 26, 32, 34, 35] for some related results). There have been many recent results showing that a version of the Erdős-Ko-Rado theorem holds for combinatorial objects other than set systems. For example, an analogue of the Erdős-Ko-Rado theorem for the Hamming scheme is proved in [33]. A complete solution for the t-intersection problem in the Hamming space is given in [2]. Some recent work done on this

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problem and its variants can be found in [4, 6, 7, 12, 13, 18, 24]. The Erdős-Ko-Rado type results also appear in vector spaces [5, 17], set partitions [19, 20, 23, 28–30] and weak compositions [21, 25, 27].

In this paper, we will consider the diversity of an intersecting family. Let $2^{[n]}$ be the set of all subsets of $[n]$. For each $x \in [n]$ and $\mathcal{F} \subseteq 2^{[n]}$, let

$$\mathcal{F}(\overline{x}) = \{F \in \mathcal{F} : x \notin F\}.$$

The *diversity* of a family $\mathcal{F} \subseteq 2^{[n]}$, denoted by $\mathrm{div}(\mathcal{F})$, is defined by

$$\mathrm{div}(\mathcal{F}) = \min_{x \in [n]} \{|\mathcal{F}(\overline{x})|\}.$$

We note here that an 1-intersecting family will just be called an intersecting family.

Theorem 1.2. *Let $k > 1$. There exists a positive integer $n_0 = n_0(k)$ such that if $n \geq n_0$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then*

$$\mathrm{div}(\mathcal{F}) \leq \binom{n-3}{k-2}.$$

Lemons and Palmer [31] proved Theorem 1.2 for $n > 6k^3$. Recently, Frankl [11] improved it to $n \geq 6k^2$. Let

$$\mathcal{T} = \left\{T \in \binom{[n]}{k} : |T \cap \{1, 2, 3\}| \geq 2\right\}.$$

It is not hard to see that $\mathcal{T}$ is intersecting and $\mathrm{div}(\mathcal{T}) = \binom{n-3}{k-2}$. In fact, $|\mathcal{T}(\overline{x})| = \binom{n-3}{k-2}$ for all $x \in \{1, 2, 3\}$ and $|\mathcal{T}(\overline{x})| > \binom{n-3}{k-2}$ for all $x \in [n] \setminus \{1, 2, 3\}$. So, the bound in Theorem 1.2 is tight. In [10], Frankl proved the theorem for $n > 2k$ under the additional assumption $|\mathcal{F}| \geq |\mathcal{T}|$.

Two generalizations of Theorem 1.2 were given by Frankl [11], which are Theorems 1.3 and 1.4.

Theorem 1.3. *Let $k > t > 0$. There exists a positive integer $n_0 = n_0(k, t)$ such that if $n \geq n_0$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is t-intersecting, then*

$$\mathrm{div}(\mathcal{F}) \leq \binom{n-t-2}{k-t-1}.$$

Note that when $t = 1$, Theorem 1.3 coincides with Theorem 1.2. The family

$$\mathcal{G} = \left\{G \in \binom{[n]}{k} : |G \cap [t+2]| \geq t+1\right\},$$

shows that the bound in Theorem 1.3 is tight.

For the other generalization, we need a new definition. The *independence number* of a family $\mathcal{F} \subseteq \binom{[n]}{k}$ is the maximum integer q such that $\mathcal{F}$ contains q pairwise disjoint members. The independence number of $\mathcal{F}$ will be denoted by $\nu(\mathcal{F})$. For a subset $Q \subseteq [n]$, let

$$\mathcal{F}(\overline{Q}) = \{F \in \mathcal{F} : F \cap Q = \varnothing\}.$$

Theorem 1.4. *Let $k, q > 0$. There exists a positive integer $n_0 = n_0(k, q)$ such that if $n \geq n_0$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ with $\nu(\mathcal{F}) = q$, then there is a set $Q \in \binom{[n]}{k}$ with*

$$\left|\mathcal{F}(\overline{Q})\right| \leq \sum_{2\leq i\leq q+1} \binom{q+1}{i}\binom{n-2q-1}{k-i}.$$

Note that when $q = 1$, Theorem 1.4 coincides with Theorem 1.2. The family

$$\mathcal{H} = \left\{H \in \binom{[n]}{k} : |H \cap [2q+1]| \geq 2\right\},$$

shows that the bound in Theorem 1.4 is tight.

In this paper, we will give another generalization of Theorem 1.2. We need a definition. The *t-diversity* of a family $\mathcal{F} \subseteq 2^{[n]}$, denoted by $\mathrm{div}_t(\mathcal{F})$, is defined by

$$\mathrm{div}_t(\mathcal{F}) = \min_{X\in\binom{[n]}{t}} \left\{\left|\mathcal{F}(\overline{X})\right|\right\}.$$

Note that when $t = 1$, the t-diversity is the same as diversity. We will prove the following main theorem.

Theorem 1.5. *Let $k \geq 2t > 0$. There exists a positive integer $n_0 = n_0(k, t)$ such that if $n \geq n_0$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is t-intersecting, then*

(1) $$\mathrm{div}_t(\mathcal{F}) \leq \binom{n-3t}{k-2t}.$$

The inequality (1) *is tight.*

The bound on n obtained here is quadratic over k, that is $n_0(k,t) = \binom{2t+1}{t}^2 k^2$. The remaining of this paper is organised as follows: In Section 2.1, we show that the upper bound given in (1) is tight. In Section 2.2, we show that the main theorem holds by proving a refinement of Theorem 1.5.

2. Proof of the main theorem

2.1. Tightness of Theorem 1.5

We first consider the tightness of Theorem 1.5. The following lemma is obvious by the definition.

Lemma 2.1. *Let $\mathcal{A} \subseteq \mathcal{B} \subseteq \binom{[n]}{k}$. Then*

$$\mathrm{div}_t(\mathcal{A}) \leq \mathrm{div}_t(\mathcal{B}).$$

Let A and B be two subsets of $[n]$. We say A *avoids* B if $A \cap B = \varnothing$. Using this, we see that

$$\mathcal{F}(\overline{Q}) = \{F \in \mathcal{F} : F \text{ avoids } Q\}.$$

Let

$$\mathcal{U}_1 = \left\{U \in \binom{[n]}{k} : |U \cap [3t]| \geq 2t\right\};$$

$$\mathcal{U}_2 = \left\{ U \in \binom{[n]}{k} : |U \cap [3t]| = 2t \right\}.$$

Note that $\mathcal{U}_2 \subseteq \mathcal{U}_1$. For each family $\mathcal{F} \subseteq \binom{[n]}{k}$ with $\mathcal{U}_2 \subseteq \mathcal{F} \subseteq \mathcal{U}_1$ and each $S \in \binom{[3t]}{2t}$, let

$$\mathcal{F}[S] = \{F \in \mathcal{F} : F \cap [3t] = S\}.$$

Then, $\mathcal{F}[S] \cap \mathcal{F}[S'] = \varnothing$ for distinct S, $S' \in \binom{[3t]}{2t}$ and

$$\mathcal{U}_2 = \bigcup_{S \in \binom{[3t]}{2t}} \mathcal{F}[S]. \tag{2}$$

Furthermore, $|\mathcal{F}[S]| = \binom{n-3t}{k-2t}$. So,

$$|\mathcal{U}_2| = \sum_{S \in \binom{[3t]}{2t}} |\mathcal{F}[S]| = \sum_{S \in \binom{[3t]}{2t}} \binom{n-3t}{k-2t} = \binom{3t}{2t}\binom{n-3t}{k-2t}.$$

For each $X \in \binom{[3t]}{t}$,

$$\mathcal{F}(\overline{X}) \subseteq \mathcal{U}_1(\overline{X}) = \mathcal{U}_2[S], \tag{3}$$

where $S = [3t] \setminus X$. The following theorem shows that the bound in Theorem 1.5 is tight.

Theorem 2.2. *Suppose $k \geq 2t > 0$, $n \geq k + t$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ with $\mathcal{F} \subseteq \mathcal{U}_1$. Then, the following statements hold.*

(a) *$\mathcal{F}$ is t-intersecting.*

(b) *If $\mathcal{U}_2 \subseteq \mathcal{F}$, then $\operatorname{div}_t(\mathcal{F}) = \binom{n-3t}{k-2t}$.*

(c) *If $\mathcal{U}_2 \nsubseteq \mathcal{F}$, then $\operatorname{div}_t(\mathcal{F}) < \binom{n-3t}{k-2t}$.*

Proof. (a) Let $A, B \in \mathcal{F}$. Since $(A \cap [3t]) \cup (B \cap [3t]) \subseteq [3t]$,

$$3t \geq |A \cap [3t]| + |B \cap [3t]| - |A \cap B \cap [3t]| \geq 4t - |A \cap B \cap [3t]|.$$

This implies that $|A \cap B| \geq t$. Hence, $\mathcal{F}$ is t-intersecting.

(b) Let $X \in \binom{[3t]}{t}$. We will show that

$$|\mathcal{F}(\overline{X})| = \binom{n-3t}{k-2t}.$$

Let $U \in \mathcal{F}(\overline{X})$. Since U avoids X and $U \in \mathcal{U}_1$,

$$|U \cap ([3t] \setminus X)| \geq 2t.$$

Therefore, $U \cap [3t] = ([3t] \setminus X) = S_0$. Hence, $\mathcal{F}(\overline{X}) \subseteq \mathcal{U}_2[S_0]$. On the other hand, each element in $\mathcal{U}_2[S_0]$ avoids X. Thus, $\mathcal{U}_2[S_0] \subseteq \mathcal{F}(\overline{X})$. This implies that $\mathcal{F}(\overline{X}) = \mathcal{U}_2[S_0]$ and

$$|\mathcal{F}(\overline{X})| = |\mathcal{U}_2[S_0]| = \binom{n-3t}{k-2t}.$$

In fact, $|\mathcal{F}(\overline{X})| = \binom{n-3t}{k-2t}$ for all $X \in \binom{[3t]}{t}$.

To complete the proof for part (b), we just need to show that

$$|\mathcal{F}(\overline{Y})| > \binom{n-3t}{k-2t}$$

for all $Y \in \binom{[n]}{t} \setminus \binom{[3t]}{t}$. Note that $|Y \cap [3t]| \leq t-1$. Let

$$\mathcal{S} = \binom{[3t] \setminus Y}{2t}.$$

For each $S \in \mathcal{S}$,

$$\mathcal{F}[S] = \{F \in \mathcal{F} : F \cap [3t] = S\} \subseteq \mathcal{F}(\overline{Y}).$$

Recall that $\mathcal{F}[S] \cap F[S'] = \varnothing$ for $S \neq S'$. Therefore,

$$\begin{aligned} |\mathcal{F}(\overline{Y})| &\geq \sum_{S \in \mathcal{S}} |\mathcal{F}[S]| = |\mathcal{S}| \binom{n-3t}{k-2t} = \binom{3t-|Y|}{2t}\binom{n-3t}{k-2t} \\ &\geq \binom{2t+1}{2t}\binom{n-3t}{k-2t} > \binom{n-3t}{k-2t}. \end{aligned}$$

(c) Since $\mathcal{U}_2 = \bigcup_{S \in \binom{[3t]}{2t}} \mathcal{U}_2[S]$ (see (2)), there is an $S_0 \in \binom{[3t]}{2t}$ such that $\mathcal{U}_2[S_0] \nsubseteq \mathcal{F}$. Let $X_0 = [3t] \setminus S_0$. Note that $\mathcal{F}(\overline{X}_0) \subseteq \mathcal{U}_1(\overline{X}_0) = \mathcal{U}_2[S_0]$. Since $\mathcal{U}_2[S_0] \nsubseteq \mathcal{F}$,

$$|\mathcal{F}(\overline{X}_0)| < |\mathcal{U}_2[S_0]| = \binom{n-3t}{k-2t}.$$

Hence, $\operatorname{div}_t(\mathcal{F}) < \binom{n-3t}{k-2t}$. □

2.2. Proof of Theorem 1.5

We now prove Theorem 1.5.

Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be t-intersecting. A set $T \subseteq [n]$ is said to be a *transversal* for $\mathcal{F}$ if $|T \cap F| \geq t$ for all $F \in \mathcal{F}$. Furthermore, T is said to be a *minimal transversal* for $\mathcal{F}$ if T is a transversal and if $K \subseteq T$ is a transversal, then $K = T$. So, given a transversal T, we can obtain a minimal transversal $K \subseteq T$ by removing elements from T. The minimal transversal K is said to be *originated* from T. Since $\mathcal{F}$ is t-intersecting, all $F \in \mathcal{F}$ are transversal for $\mathcal{F}$. Let $\mathcal{K}(\mathcal{F})$ be the set of all minimal transversals originated from some element in $\mathcal{F}$, i.e., $K \in \mathcal{K}(\mathcal{F})$ if and only if

(a) K is a minimal transversal for $\mathcal{F}$;
(b) K is originated from a $F \in \mathcal{F}$.

A t-intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ is said to be *saturated* if $\mathcal{F} \cup \{G\}$ is not t-intersecting for all $G \in \binom{[n]}{k} \setminus \mathcal{F}$.

Lemma 2.3. *Suppose $k \geq t > 0$ and $n \geq 2k$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is t-intersecting and saturated, then $\mathcal{K}(\mathcal{F})$ is t-intersecting.*

Proof. Suppose $\mathcal{K}(\mathcal{F})$ is not t-intersecting. Then, there are $A, B \in \mathcal{K}(\mathcal{F})$ such that $|A \cap B| \leq t - 1$. Since A originated from some element in $\mathcal{F}$, $|A| \leq k$. Similarly, $|B| \leq k$. Now, choose $2k - (|A| + |B|)$ elements from $[n] \setminus (A \cup B)$. Put $k - |A|$ of them in a set, say C and the remaining in a set, say D. Note that $C \cap D = \varnothing$ and both $(A \cup C)$ and $(B \cup D)$ are in $\binom{[n]}{k}$. Since A is a transversal, $|(A \cup C) \cap F| \geq |A \cap F| \geq t$ for all $F \in \mathcal{F}$. This means $(A \cup C) \in \mathcal{F}$, for $\mathcal{F}$ is saturated. Similarly, $(B \cup D) \in \mathcal{F}$. Thus,

$$t \leq |(A \cup C) \cap (B \cup D)| = |A \cap B|,$$

a contradiction. Hence, $\mathcal{K}(\mathcal{F})$ is t-intersecting. □

Let

$$\tau(\mathcal{F}) = \min_{K \in \mathcal{K}(\mathcal{F})} \{|K|\}$$

be called the *transversal number* of a t-intersecting family $\mathcal{F}$. Note that

$$\tau(\mathcal{F}) \geq t.$$

Lemma 2.4. *Suppose $n \geq k \geq t > 0$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is t-intersecting. If $\tau(\mathcal{F}) \leq 2t - 1$, then $\mathrm{div}_t(\mathcal{F}) = 0$.*

Proof. Suppose $\tau(\mathcal{F}) = l \leq 2t - 1$. Let $K_0 \in \mathcal{K}(\mathcal{F})$ with $|K_0| = l$. Choose a subset $X \subseteq K_0$ of size t. Let $Y = K_0 \setminus X$. Then, $K_0 = X \cup Y$ and $X \cap Y = \varnothing$. We claim that $\mathcal{F}(\overline{X}) = \varnothing$. Suppose $\mathcal{F}(\overline{X}) \neq \varnothing$. Then, there is a $U \in \mathcal{F}(\overline{X})$. Since K_0 is a transversal for $\mathcal{F}$, $|U \cap K_0| \geq t$. Now, U avoids X implies that $U \cap K_0 = U \cap Y$. Thus, $|Y| \geq t$ and $l = |K_0| = |X| + |Y| \geq t + t = 2t$, a contradiction. Hence, $\mathcal{F}(\overline{X}) = \varnothing$ and $\mathrm{div}_t(\mathcal{F}) = 0$. □

Given a t-intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ and $1 \leq i \leq k$, let

$$\mathcal{K}_i(\mathcal{F}) = \{K \in \mathcal{K}(\mathcal{F}) : |K| = i\}.$$

We consider two cases based on the definition of $\mathcal{K}_i(\mathcal{F})$ separately: the first case is when

$$\mathcal{K}_{2t}(\mathcal{F})(\overline{X}) \neq \varnothing \text{ for all } X \in \binom{[n]}{t},$$

and the second case is when

$$\mathcal{K}_{2t}(\mathcal{F})(\overline{X}) = \varnothing \text{ for some } X \in \binom{[n]}{t}.$$

We first deal with the first case.

Lemma 2.5. *Let $k \geq 2t > 0$, $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be t-intersecting and saturated. If $\mathcal{K}_{2t}(\mathcal{F})(\overline{X}) \neq \varnothing$ for all $X \in \binom{[n]}{t}$, then*

$$\operatorname{div}_t(\mathcal{F}) = \binom{n-3t}{k-2t}.$$

Proof. Clearly, $\mathcal{K}_{2t}(\mathcal{F}) \neq \varnothing$. Relabelling if necessary, we may assume that

$$A_0 = \{1, 2, 3, \ldots, 2t\} \in \mathcal{K}_{2t}(\mathcal{F}).$$

Let

$$X_1 = \{1, 2, 3, \ldots, t\};$$
$$X_2 = \{t+1, t+2, t+3, \ldots, 2t\}.$$

Since $\mathcal{K}_{2t}(\mathcal{F})(\overline{X}_1) \neq \varnothing$, there is a $B_0 \in \mathcal{K}_{2t}(\mathcal{F})$ that avoids X_1. By Lemma 2.3, $\mathcal{K}_{2t}(\mathcal{F})$ is t-intersecting. Therefore $|A_0 \cap B_0| \geq t$. This implies that $A_0 \cap B_0 = X_2$. Relabelling if necessary, we may assume that

$$B_0 = X_2 \cup X_3,$$

where $X_3 = \{2t+1, 2t+2, 2t+3, \ldots, 3t\}$. Next, $\mathcal{K}_{2t}(\mathcal{F})(\overline{X}_2) \neq \varnothing$ implies that there is a $C_0 \in \mathcal{K}_{2t}(\mathcal{F})$ that avoids X_2. Since $\mathcal{K}_{2t}(\mathcal{F})$ is t-intersecting, we must have $|C_0 \cap A_0| \geq t$ and $|C_0 \cap B_0| \geq t$. Thus,

$$C_0 = X_1 \cup X_3.$$

Let $U \in \mathcal{F}(\overline{X}_3)$. Since B_0 is a transversal for $\mathcal{F}$, $|U \cap B_0| \geq t$. So, $X_2 \subseteq U$ for U avoids X_3. Similarly, C_0 is a transversal for $\mathcal{F}$ implies that $X_1 \subseteq U$. Thus, $A_0 \subseteq U$, and

$$\mathcal{F}(\overline{X}_3) \subseteq \mathcal{Y} = \left\{ Y \in \binom{[n]}{k} : A_0 \subseteq Y \text{ and } Y \text{ avoids } X_3 \right\}.$$

Since $\mathcal{F}$ is saturated and A_0 is a transversal for $\mathcal{F}$, we have $\mathcal{F}(\overline{X}_3) = \mathcal{Y}$. So $|\mathcal{F}(\overline{X}_3)| = |\mathcal{Y}| = \binom{n-3t}{k-2t}$.

It is left to show that $|\mathcal{F}(\overline{X})| \geq \binom{n-3t}{k-2t}$ for all $X \in \binom{[n]}{t}$. Since $\mathcal{K}_{2t}(\mathcal{F})(\overline{X}) \neq \varnothing$, there is a $D \in \mathcal{K}_{2t}(\mathcal{F})$ that avoids X. Let

$$\mathcal{Z} = \left\{ Z \in \binom{[n]}{k} : D \subseteq Y \text{ and } D \text{ avoids } X \right\}.$$

Since $\mathcal{F}$ is saturated and D is a transversal for $\mathcal{F}$, $\mathcal{Z} \subseteq \mathcal{F}(\overline{X})$. Hence,

$$|\mathcal{F}(\overline{X})| \geq |\mathcal{Z}| = \binom{n-3t}{k-2t}.$$

This completes the proof of the lemma. □

Next we consider the second case. We will use the following computation in the proof of Lemma 2.7

Claim 2.6. *Let $k \geq 2t > 0$ and $n \geq \binom{2t+1}{t}^2 k^2$. If $2t+1 \leq u_0 \leq i \leq k$, then*

$$\frac{\binom{n-i-t}{k-i}}{\binom{n-3t}{k-2t}} < \frac{1}{k^{i-2t}\binom{u_0}{t}^2}.$$

Proof. Note that

$$\begin{aligned} k^{i-2t}\left(\frac{\binom{n-i-t}{k-i}}{\binom{n-3t}{k-2t}}\right) &= k^{i-2t}\left(\frac{(k-2t)(k-2t-1)\cdots(k-i+1)}{(n-3t)(n-3t-1)\cdots(n-i-t+1)}\right) \\ &= \prod_{1\leq j\leq i-2t} \frac{k(k-i+j)}{n-i-t+j}. \end{aligned}$$

Next,

$$\begin{aligned} &\binom{2t+1}{t}^2 (k(k-i+j)) + (i-j+t) \\ &\leq \binom{2t+1}{t}^2 k^2 - \left(\binom{2t+1}{t}^2 k - 1\right)(i-j) + t \\ &\leq \binom{2t+1}{t}^2 k^2 - \left(\binom{2t+1}{t}^2 k - 1\right)(2t) + t \\ &\leq \binom{2t+1}{t}^2 k^2 - 2t + t \\ &< \binom{2t+1}{t}^2 k^2 \leq n. \end{aligned}$$

Therefore, $\frac{k(k-i+j)}{n-i-t+j} < \frac{1}{\binom{2t+1}{t}^2}$ and

$$\begin{aligned} k^{i-2t}\left(\frac{\binom{n-i-t}{k-i}}{\binom{n-3t}{k-2t}}\right) &< \prod_{1\leq j\leq i-2t} \frac{1}{\binom{2t+1}{t}^2} \\ &= \frac{1}{\binom{2t+1}{t}^{2(i-2t)}} \leq \frac{1}{\binom{2t+1}{t}^{2(u_0-2t)}}. \end{aligned}$$

Now, we shall show that

$$\binom{2t+1}{t}^{2(u_0-2t)} \geq \binom{u_0}{t}^2. \tag{4}$$

Clearly, it is true for $u_0 = 2t+1$. Assume that it is true for some $u_0 \geq 2t+1$. Note that

$$\begin{aligned} \binom{u_0+1}{t}^2 &= \left(\binom{u_0}{t} + \binom{u_0}{t-1}\right)^2 \\ &= \binom{u_0}{t}^2 + 2\binom{u_0}{t}\binom{u_0}{t-1} + \binom{u_0}{t-1}^2 \end{aligned}$$

$$\leq 4\binom{u_0}{t}^2 \leq 4\binom{2t+1}{t}^{2(u_0-2t)} \leq \binom{2t+1}{t}^{2(u_0+1-2t)}.$$

Here, we use the fact that $\binom{u_0}{t-1} \leq \binom{u_0}{t}$ (for $u_0 \geq 2t+1$). Hence, (4) holds and

$$k^{i-2t}\left(\frac{\binom{n-i-t}{k-i}}{\binom{n-3t}{k-2t}}\right) \leq \frac{1}{\binom{u_0}{t}^2}.$$

This completes the proof of the claim. □

Lemma 2.7. *Let $k \geq 2t > 0$, $n \geq \binom{2t+1}{t}^2 k^2$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be t-intersecting and saturated. If $\mathcal{K}_{2t}(\mathcal{F})(\overline{X}_0) = \varnothing$ for some $X_0 \in \binom{[n]}{t}$, then*

$$\operatorname{div}_t(\mathcal{F}) < \binom{n-3t}{k-2t}.$$

Proof. By Lemma 2.4, we may assume that $\tau(\mathcal{F}) \geq 2t$. This means $\mathcal{K}_i(\mathcal{F}) = \varnothing$ for $1 \leq i \leq 2t-1$. If there is an $X \in \binom{[n]}{t}$ with $\mathcal{F}(\overline{X}) = \varnothing$, then $\operatorname{div}_t(\mathcal{F}) = 0$. So, we may assume that $\mathcal{F}(\overline{X}) \neq \varnothing$ for all $X \in \binom{[n]}{t}$. Let $\mathcal{Y} = \binom{[n]}{k}$. For each $A \subseteq [n]$ and $X \in \binom{[n]}{t}$, let

$$\mathcal{Y}(A) = \{U \in \mathcal{Y} : A \subseteq U\};$$
$$\mathcal{Y}(A)(\overline{X}) = \{U \in \mathcal{Y}(A) : U \text{ avoids X}\}.$$

For each $F \in \mathcal{F}$, there is a $K \in \mathcal{K}(\mathcal{F})$ such that K is originated from F. So, $F \in \mathcal{Y}(K)$. Since $\mathcal{F}$ is saturated, $\mathcal{Y}(K) \subseteq \mathcal{F}$. Therefore,

$$\mathcal{F} = \bigcup_{2t \leq i \leq k} \bigcup_{K \in \mathcal{K}_i(\mathcal{F})} \mathcal{Y}(K),$$

and thus,

$$\mathcal{F}(\overline{X}) = \bigcup_{2t \leq i \leq k} \bigcup_{K \in \mathcal{K}_i(\mathcal{F})(\overline{X})} \mathcal{Y}(K)(\overline{X}) \tag{5}$$

for all $X \in \binom{[n]}{t}$. Note that if $K \in \mathcal{K}_i(\mathcal{F})(\overline{X})$ and $X \in \binom{[n]}{t}$, then

$$|\mathcal{Y}(K)(\overline{X})| = \binom{n-|K|-|X|}{k-i} = \binom{n-i-t}{k-i}. \tag{6}$$

For each $X \in \binom{[n]}{t}$ with $\mathcal{K}_{2t}(\mathcal{F})(\overline{X}) = \varnothing$, let u_X be the smallest positive integer such that

$$\mathcal{K}_i(\mathcal{F})(\overline{X}) = \varnothing \text{ for all } 2t \leq i \leq u_X - 1 \text{ and } \mathcal{K}_{u_X}(\mathcal{F})(\overline{X}) \neq \varnothing.$$

Since $\mathcal{F}(\overline{X}) \neq \varnothing$, by (5), $\mathcal{K}_i(\mathcal{F})(\overline{X}) \neq \varnothing$ for some $i \geq 2t+1$. So, $2t+1 \leq u_X \leq k$. Let

$$u_0 = \max_{X \in \binom{[n]}{t}} \{u_X\}.$$

Then, $2t+1 \le u_0 \le k$ and there is an $X_0 \in \binom{[n]}{t}$ such that

$$\mathcal{K}_i(\mathcal{F})(\overline{X}_0) = \varnothing \quad \text{for all } 2t \le i \le u_0 - 1 \text{ and } \quad \mathcal{K}_{u_0}(\mathcal{F})(\overline{X}_0) \ne \varnothing.$$

By (5) and (6),

$$|\mathcal{F}(\overline{X}_0)| \le \sum_{u_0 \le i \le k} \quad \sum_{K \in \mathcal{K}_i(\mathcal{F})(\overline{X}_0)} |\mathcal{Y}(K)(\overline{X}_0)| = \sum_{u_0 \le i \le k} w_i \binom{n-i-t}{k-i},$$

where $w_i = |\mathcal{K}_i(\mathcal{F})(\overline{X}_0)|$. By using the 'branching algorithm' of Frankl [11, Proposition 6.1], we will show that

$$\sum_{u_0 \le i \le k} k^{k-i} w_i \le \binom{u_0}{t}^2 k^{k-2t}. \tag{7}$$

Before we proceed to prove (7), let us use it to show that

$$\operatorname{div}_t(\mathcal{F}) < \binom{n-3t}{k-2t}.$$

By Lemma 2.6,

$$\frac{|\mathcal{F}(\overline{X}_0)|}{\binom{n-3t}{k-2t}} \le \sum_{u_0 \le i \le k} w_i \frac{\binom{n-i-t}{k-i}}{\binom{n-3t}{k-2t}} < \sum_{u_0 \le i \le k} w_i \frac{1}{k^{i-2t}\binom{u_0}{t}^2} \le 1,$$

where the last inequality follows from (7). Hence, $|\mathcal{F}(\overline{X}_0)| < \binom{n-3t}{k-2t}$ and $\operatorname{div}_t(\mathcal{F}) < \binom{n-3t}{k-2t}$.

Now, it is left to prove (7) holds. A sequence of length j over $[n]$ will be denoted by

$$(a_1, a_2, \ldots, a_j),$$

where $a_i \in [n]$. For each j $(2t \le j \le k)$, we will construct a family $\mathcal{L}_j$ of sequences of length j over $[n]$ such that

(a) $|\mathcal{L}_j| \le \binom{u_0}{t}^2 k^{j-2t}$,

(b) for each $(a_1, a_2, \ldots, a_j, a_{j+1}) \in \mathcal{L}_{j+1}$, we have $(a_1, a_2, \ldots, a_j) \in \mathcal{L}_j$.

Let $K_0 \in \mathcal{K}_{u_0}(\mathcal{F})(\overline{X}_0)$. Note that such an element exists for $\mathcal{K}_{u_0}(\mathcal{F})(\overline{X}_0) \ne \varnothing$. For each $Y \subseteq K_0$ with $|Y| = t$, we have $\mathcal{F}(\overline{Y}) \ne \varnothing$. So, by (5), $\mathcal{K}_i(\overline{Y}) \ne \varnothing$ for some $2t \le i \le k$. If $\mathcal{K}_{2t}(\overline{Y}) \ne \varnothing$, then choose a $K_Y \in \mathcal{K}_{2t}(\overline{Y})$. If $\mathcal{K}_{2t}(\overline{Y}) = \varnothing$, then choose a $K_Y \in \mathcal{K}_i(\overline{Y})$ where i is the smallest positive integer such that $\mathcal{K}_i(\overline{Y}) \ne \varnothing$ and $\mathcal{K}_j(\overline{Y}) = \varnothing$ for $2t \le j \le i-1$. By the choice of u_0, $i \le u_0$. Thus, $|K_Y| \le u_0$. Let $\mathcal{L}_{2t}$ be the family of all sequences of length $2t$ over $[n]$ of the following form

$$L = (a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t),$$

where $a_1 < a_2 < \cdots < a_t$, $b_1 < b_2 < \cdots < b_t$, $\{a_1, a_2, \ldots, a_t\} = Y \subseteq K_0$ and $\{b_1, b_2, \ldots, b_t\} = Z \subseteq K_Y$. Therefore, $|\mathcal{L}_{2t}| \le \binom{u_0}{t}^2$.

Choose an arbitrary set $V_0 \in \binom{[n]}{k}$. We are ready to construct the family $\mathcal{L}_{2t+1}$.

Algorithm:

(i) Partition $\mathcal{L}_{2t}$ into two parts, say $\mathcal{L}_{2t}(1)$ and $\mathcal{L}_{2t}(2) = \mathcal{L}_{2t} \setminus \mathcal{L}_{2t}(1)$ where $(c_1, c_2, \ldots, c_{2t}) \in \mathcal{L}_{2t}(1)$ if and only if

$$|\{c_1, c_2, \ldots, c_{2t}\} \cap F| \geq t \qquad \text{for all } F \in \mathcal{F}.$$

(ii) For each sequence $\mathbf{c} = (c_1, c_2, \ldots, c_{2t}) \in \mathcal{L}_{2t}(1)$ and all $v \in V_0$, put $(\mathbf{c}, v) = (c_1, c_2, \ldots, c_{2t}, v) \in \mathcal{L}_{2t+1}$.

(iii) For each sequence $\mathbf{c} = (c_1, c_2, \ldots, c_{2t}) \in \mathcal{L}_{2t}(2)$, we have

$$|\{c_1, c_2, \ldots, c_{2t}\} \cap F_{\mathbf{c}}| \leq t - 1 \qquad \text{for some } F_{\mathbf{c}} \in \mathcal{F}.$$

So, for each $v \in F_{\mathbf{c}} \setminus \{c_1, c_2, \ldots, c_{2t}\}$, put $(\mathbf{c}, v) \in \mathcal{L}_{2t+1}$.

Since $|V_0| = |F_{\mathbf{c}}| = k$, $|\mathcal{L}_{2t+1}| \leq |\mathcal{L}_{2t}|k \leq \binom{u_0}{t}^2 k$.

Suppose we have constructed the families $\mathcal{L}_j$ for $2t \leq j < k$ such that (a) and (b) hold. Now, we shall construct the family $\mathcal{L}_{j+1}$.

Algorithm:

(i) Partition $\mathcal{L}_j$ into two parts, say $\mathcal{L}_j(1)$ and $\mathcal{L}_j(2) = \mathcal{L}_j \setminus \mathcal{L}_j(1)$ where $(c_1, c_2, \ldots, c_j) \in \mathcal{L}_j(1)$ if and only if

$$|\{c_1, c_2, \ldots, c_j\} \cap F| \geq t \qquad \text{for all } F \in \mathcal{F}.$$

(ii) For each sequence $\mathbf{c} = (c_1, c_2, \ldots, c_j) \in \mathcal{L}_j(1)$ and all $v \in V_0$, put $(\mathbf{c}, v) = (c_1, c_2, \ldots, c_{2t}, v) \in \mathcal{L}_{j+1}$.

(iii) For each sequence $\mathbf{c} = (c_1, c_2, \ldots, c_j) \in \mathcal{L}_j(2)$, we have

$$|\{c_1, c_2, \ldots, c_j\} \cap F_{\mathbf{c}}| \leq t - 1 \qquad \text{for some } F_{\mathbf{c}} \in \mathcal{F}.$$

So, for each $v \in F_{\mathbf{c}} \setminus \{c_1, c_2, \ldots, c_j\}$, put $(\mathbf{c}, v) \in \mathcal{L}_{j+1}$.

Since $|V_0| = |F_{\mathbf{c}}| = k$, $|\mathcal{L}_{j+1}| \leq |\mathcal{L}_j|k \leq \binom{u_0}{t}^2 k^{j+1-2t}$. Note that $|\mathcal{L}_k| \leq \binom{u_0}{t}^2 k^{k-2t}$.

Let $K \in \mathcal{K}_j(\mathcal{F})(\overline{X}_0)$ $(u_0 \leq j \leq k)$. We claim that there is a sequence $\mathbf{d} = (d_1, d_2, \ldots, d_j) \in \mathcal{L}_j$ such that

$$\{d_1, d_2, \ldots, d_j\} = K.$$

By Lemma 2.3, $\mathcal{K}(\mathcal{F})$ is t-intersecting. Therefore, $|K \cap K_0| \geq t$, and there is a $Y = \{a_1, a_2, \ldots, a_t\} \subseteq K \cap K_0$. We may assume that $a_1 < a_2 < \cdots < a_t$. Recall that K_Y avoids Y. So, $|K \cap K_Y| \geq t$ implies that there is a $\{b_1, b_2, \ldots, b_t\} \subseteq K \cap K_Y$. We may assume that $b_1 < b_2 < \cdots < b_t$. Thus, $(a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t) \in \mathcal{L}_{2t}$ and

$$\{a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t\} \subseteq K.$$

So, there is a $\mathbf{d}_0 = (d_1, d_2, \ldots, d_{j_0}) \in \mathcal{L}_{j_0}$ such that

$$\{d_1, d_2, \ldots, d_{j_0}\} \subseteq K.$$

We may assume that j_0 is the largest with such property. If $\{d_1, d_2, \ldots, d_{j_0}\} = K$, we are done. Suppose $\{d_1, d_2, \ldots, d_{j_0}\} \subsetneq K$. Since K is a minimal transversal, $\{d_1, d_2, \ldots, d_{j_0}\}$ is not a transversal. So, $|\{d_1, d_2, \ldots, d_{j_0}\} \cap F| \leq t-1$ for some $F \in \mathcal{F}$. This means $\mathbf{d}_0 \in \mathcal{L}_{j_0}(2)$ and

$$|\{d_1, d_2, \ldots, d_{j_0}\} \cap F_{\mathbf{d}_0}| \leq t-1.$$

Now, $|K \cap F_{\mathbf{d}_0}| \geq t$ implies that there is a $d_{j_0+1} \in (K \cap F_{\mathbf{d}_0}) \setminus \{d_1, d_2, \ldots, d_{j_0}\}$. By construction ('branching algorithm'), $(\mathbf{d}_0, d_{j_0+1}) \in \mathcal{L}_{j_0+1}$ and

$$\{d_1, d_2, \ldots, d_{j_0}, d_{j_0+1}\} \subseteq K.$$

This contradicts the choice of j_0. Hence, for each $K \in \mathcal{K}_j(\mathcal{F})(\overline{X}_0)$, there is a sequence $\mathbf{d}_K = (d_1, d_2, \ldots, d_j) \in \mathcal{L}_j$ such that

$$\{d_1, d_2, \ldots, d_j\} = K.$$

The sequence $\mathbf{d}_K$ is said to be *associated* to K.

A sequence $\mathbf{c} = (c_1, c_2, \ldots, c_k) \in \mathcal{L}_k$ is said to be an *extension* of a sequence $\mathbf{d} = (d_1, d_2, \ldots, d_j) \in \mathcal{L}_j$ if $c_i = d_i$ for all $1 \leq i \leq j$, i.e., $\mathbf{c} = (\mathbf{d}, c_{j+1}, c_{j+2}, \ldots, c_k)$. For each $K \in \mathcal{K}_j(\mathcal{F})(\overline{X}_0)$, let

$$\mathcal{Z}_K = \{\mathbf{c} \in \mathcal{L}_k : \mathbf{c} \text{ is an extension of } \mathbf{d}_K\}.$$

By construction, $|\mathcal{Z}_K| = k^{k-j}$. Furthermore,

$$\bigcup_{u_0 \leq j \leq k} \bigcup_{K \in \mathcal{K}_j(\mathcal{F})(\overline{X}_0)} \mathcal{Z}_K \subseteq \mathcal{L}_k.$$

Now, we shall show that $\mathcal{Z}_{K_1} \cap \mathcal{Z}_{K_2} = \varnothing$ for all $K_1 \in \mathcal{K}_{j_1}(\mathcal{F})(\overline{X}_0)$ and $K_2 \in \mathcal{K}_{j_2}(\mathcal{F})(\overline{X}_0)$ where $j_1 \neq j_2$. Let $\mathbf{c} \in \mathcal{Z}_{K_1} \cap \mathcal{Z}_{K_2}$. Then,

$$\mathbf{c} = (\mathbf{d}_{K_1}, c_{j_1+1}, c_{j_1+2}, \ldots, c_k);$$
$$\mathbf{c} = (\mathbf{d}_{K_2}, c_{j_2+1}, c_{j_2+2}, \ldots, c_k).$$

Without loss of generality, we may assume that $j_2 \geq j_1$. So, $\mathbf{d}_{K_2}$ is an extension of $\mathbf{d}_{K_1}$, and $K_1 \subseteq K_2$. Since K_1 and K_2 are minimal transversal, we must have $j_2 = j_1$ and $K_1 = K_2$. Thus, $\mathcal{Z}_{K_1} \cap \mathcal{Z}_{K_2} = \varnothing$.

So,

$$\begin{aligned}\binom{u_0}{t}^2 k^{k-2t} \geq |\mathcal{L}_k| \geq \left| \bigcup_{u_0 \leq j \leq k} \bigcup_{K \in \mathcal{K}_j(\mathcal{F})(\overline{X}_0)} \mathcal{Z}_K \right| &\geq \sum_{u_0 \leq j \leq k} \sum_{K \in \mathcal{K}_j(\mathcal{F})(\overline{X}_0)} |\mathcal{Z}_K| \\ &= \sum_{u_0 \leq j \leq k} \sum_{K \in \mathcal{K}_j(\mathcal{F})(\overline{X}_0)} k^{k-j} = \sum_{u_0 \leq j \leq k} k^{k-j} w_j.\end{aligned}$$

This proves (7). □

Finally, Theorem 1.5 follows from Lemmas 2.1, 2.4, 2.5 and 2.7. In fact by Lemmas 2.1, 2.4, 2.5 and 2.7, we have the following refinement of Theorem 1.5.

Theorem 2.8. *Let $k \geq 2t > 0$ and $n \geq \binom{2t+1}{t}^2 k^2$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is t-intersecting, then*

$$\operatorname{div}_t(\mathcal{F}) \leq \binom{n-3t}{k-2t}.$$

Moreover, if $\mathcal{F}$ is saturated, then equality holds if and only if $\mathcal{K}_{2t}(\mathcal{F})(\overline{X}) \neq \varnothing$ for all $X \in \binom{[n]}{t}$.

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References

[1] R. Ahlswede and L. H. Khachatrian, *The complete intersection theorem for systems of finite sets*, European J. Combin. **18** (1997), no. 2, 125–136. https://doi.org/10.1006/eujc.1995.0092

[2] ———, *The diametric theorem in Hamming spaces—optimal anticodes*, Adv. in Appl. Math. **20** (1998), no. 4, 429–449. https://doi.org/10.1006/aama.1998.0588

[3] C. Bey, *On cross-intersecting families of sets*, Graphs Combin. **21** (2005), no. 2, 161–168. https://doi.org/10.1007/s00373-004-0598-4

[4] P. J. Cameron and C. Y. Ku, *Intersecting families of permutations*, European J. Combin. **24** (2003), no. 7, 881–890. https://doi.org/10.1016/S0195-6698(03)00078-7

[5] A. Chowdhury and B. Patkós, *Shadows and intersections in vector spaces*, J. Combin. Theory Ser. A **117** (2010), no. 8, 1095–1106. https://doi.org/10.1016/j.jcta.2009.10.010

[6] D. Ellis, *Stability for t-intersecting families of permutations*, J. Combin. Theory Ser. A **118** (2011), no. 1, 208–227. https://doi.org/10.1016/j.jcta.2010.04.005

[7] D. Ellis, E. Friedgut, and H. Pilpel, *Intersecting families of permutations*, J. Amer. Math. Soc. **24** (2011), no. 3, 649–682. https://doi.org/10.1090/S0894-0347-2011-00690-5

[8] P. Erdős, C. Ko, and R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320. https://doi.org/10.1093/qmath/12.1.313

[9] P. Frankl, *The Erdős-Ko-Rado theorem is true for $n = ckt$*, in Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I, 365–375, Colloq. Math. Soc. János Bolyai, **18**, North-Holland, Amsterdam, 1978.

[10] ———, *Erdős-Ko-Rado theorem with conditions on the maximal degree*, J. Combin. Theory Ser. A **46** (1987), no. 2, 252–263. https://doi.org/10.1016/0097-3165(87)90005-7

[11] ———, *Antichains of fixed diameter*, Mosc. J. Comb. Number Theory **7** (2017), no. 3, 3–33.

[12] P. Frankl and A. Kupavskii, *Erdős-Ko-Rado theorem for $\{0, \pm 1\}$-vectors*, J. Combin. Theory Ser. A **155** (2018), 157–179. https://doi.org/10.1016/j.jcta.2017.11.003

[13] P. Frankl, S. J. Lee, M. Siggers, and N. Tokushige, *An Erdős-Ko-Rado theorem for cross t-intersecting families*, J. Combin. Theory Ser. A **128** (2014), 207–249. https://doi.org/10.1016/j.jcta.2014.08.006

[14] P. Frankl, M. Shinohara, and N. Tokushige, *Multiply union families in $\mathbb{N}^n$*, European J. Combin. **58** (2016), 66–74. https://doi.org/10.1016/j.ejc.2016.05.007

[15] P. Frankl and N. Tokushige, *On r-cross intersecting families of sets*, Combin. Probab. Comput. **20** (2011), no. 5, 749–752. https://doi.org/10.1017/S0963548311000289

[16] ______, *Intersection problems in the q-ary cube*, J. Combin. Theory Ser. A **141** (2016), 90–126. https://doi.org/10.1016/j.jcta.2016.02.006

[17] P. Frankl and R. M. Wilson, *The Erdős-Ko-Rado theorem for vector spaces*, J. Combin. Theory Ser. A **43** (1986), no. 2, 228–236. https://doi.org/10.1016/0097-3165(86)90063-4

[18] C. Godsil and K. Meagher, *A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations*, European J. Combin. **30** (2009), no. 2, 404–414. https://doi.org/10.1016/j.ejc.2008.05.006

[19] C. Y. Ku and D. Renshaw, *Erdős-Ko-Rado theorems for permutations and set partitions*, J. Combin. Theory Ser. A **115** (2008), no. 6, 1008–1020. https://doi.org/10.1016/j.jcta.2007.12.004

[20] C. Y. Ku and K. B. Wong, *On cross-intersecting families of set partitions*, Electron. J. Combin. **19** (2012), no. 4, #49.

[21] ______, *An analogue of the Erdős-Ko-Rado theorem for weak compositions*, Discrete Math. **313** (2013), no. 21, 2463–2468. https://doi.org/10.1016/j.disc.2013.07.012

[22] ______, *On r-cross intersecting families of sets*, Far East J. Math. Sci. **75** (2013), 295–300.

[23] ______, *An analogue of the Hilton-Milner theorem for set partitions*, J. Combin. Theory Ser. A **120** (2013), no. 7, 1508–1520. https://doi.org/10.1016/j.jcta.2013.05.001

[24] ______, *An Erdős-Ko-Rado theorem for permutations with fixed number of cycles*, Electron. J. Combin. **21** (2014), no. 3, #P3.16.

[25] ______, *On r-cross t-intersecting families for weak compositions*, Discrete Math. **338** (2015), no. 7, 1090–1095. https://doi.org/10.1016/j.disc.2015.01.015

[26] ______, *A Deza-Frankl type theorem for set partitions*, Electron. J. Combin. **22** (2015), no. 1, #P1.84.

[27] ______, *An analogue of the Hilton-Milner theorem for weak compositions*, Bull. Korean Math. Soc. **52** (2015), no. 3, 1007–1025. https://doi.org/10.4134/BKMS.2015.52.3.1007

[28] ______, *A non-trivial intersection theorem for permutations with fixed number of cycles*, Discrete Math. **339** (2016), no. 2, 646–657. https://doi.org/10.1016/j.disc.2015.09.032

[29] ______, *An Erdős-Ko-Rado theorem for minimal covers*, Bull. Korean Math. Soc. **54** (2017), no. 3, 875–894. https://doi.org/10.4134/BKMS.b160334

[30] ______, *Erdős-Ko-Rado theorems for set partitions with certain block size*, J. Combin. Theory Ser. A **172** (2020), 105180, 27 pp. https://doi.org/10.1016/j.jcta.2019.105180

[31] N. Lemons and C. Palmer, *The unbalance of set systems*, Graphs Combin. **24** (2008), no. 4, 361–365. https://doi.org/10.1007/s00373-008-0793-9

[32] M. Matsumoto and N. Tokushige, *The exact bound in the Erdős-Ko-Rado theorem for cross-intersecting families*, J. Combin. Theory Ser. A **52** (1989), no. 1, 90–97. https://doi.org/10.1016/0097-3165(89)90065-4

[33] A. Moon, *An analogue of the Erdős-Ko-Rado theorem for the Hamming schemes $H(n, q)$*, J. Combin. Theory Ser. A **32** (1982), no. 3, 386–390. https://doi.org/10.1016/0097-3165(82)90054-1

[34] L. Pyber, *A new generalization of the Erdős-Ko-Rado theorem*, J. Combin. Theory Ser. A **43** (1986), no. 1, 85–90. https://doi.org/10.1016/0097-3165(86)90025-7

[35] N. Tokushige, *A product version of the Erdős-Ko-Rado theorem*, J. Combin. Theory Ser. A **118** (2011), no. 5, 1575–1587. https://doi.org/10.1016/j.jcta.2011.01.010

[36] R. M. Wilson, *The exact bound in the Erdős-Ko-Rado theorem*, Combinatorica **4** (1984), no. 2-3, 247–257. https://doi.org/10.1007/BF02579226

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