

The Weights in MDS Codes

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Abstract—The weights in MDS codes of length n and dimension k over the finite field $GF(q)$ are studied. Up to some explicit exceptional cases, the MDS codes with parameters given by the MDS conjecture are shown to contain all k weights in the range $n - k + 1$ to n . The proof uses the covering radius of the dual code.

Index Terms—covering radius, MDS codes, quantum codes, weight distribution

I. INTRODUCTION

One of the main properties of a linear error-correcting code is its minimum distance since this determines the maximal number of errors that can be corrected independently of the position of the errors. More information about the error-correcting properties of a code can be derived from its weight distribution.

In some cases, one does not need to know the number of codewords of a given weight, but only the set of weights of the codewords. Assmus and Mattson [1] established a connection between codes and designs based on the non-zero weights in the codes. Hill and Lizak [9], [10] derived conditions on the non-zero weights that imply that a code can be extended. Rains [13] showed that a quantum error-correcting code (QECC) of length $n' < n$ can be derived from a QECC of length n if an auxiliary code contains a word of weight n' . Using this result, Rötteler *et al.* [7], [14] constructed quantum MDS codes for all lengths $n \leq q + 1$. The construction relies on the following statement which, *e. g.*, can be found in [12, p. 320] without an explicit proof:

An MDS code with parameters $[n, k, d]_q$ has k distinct nonzero weights, $n - k + 1, \dots, n$. (1)

In this note, we show that this statement is not true in general and investigate for which parameters of MDS codes it holds.

The material is organized as follows. Section II collects the necessary definitions and notation (for further details see, *e. g.*, [11], [12]). Section III disposes of the trivial codes. Section IV studies codes with $n \leq q$. Section V and Section VI study codes of lengths $q + 1$ and $q + 2$, respectively.

II. PRELIMINARIES

A linear code with parameters $[n, k, d]_q$ of length n , dimension k , and minimum distance d is a subspace of dimension

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k of the vector space \mathbb{F}_q^n over the finite field $\mathbb{F}_q = GF(q)$ with q elements. Any two vectors in C differ in at least d places. From the Singleton bound we have $d \leq n - k + 1$, and a code establishing this bound is called a *maximum distance separable* (MDS) code. Trivial families of MDS codes are the full vector space $[n, n, 1]_q$, repetition codes $[n, 1, n]_q$ of any length, and their duals $[n, n - 1, 2]_q$. The main conjecture on MDS codes states that if there is a non-trivial MDS code with parameters $[n, k, n - k + 1]_q$ over \mathbb{F}_q , then $n \leq q + 1$, except when q is even and $k = 3$ or $k = q - 1$ in which case $n \leq q + 2$ (see, *e. g.*, [11, Chapter 7.4, p. 265]).

The weight enumerator $W_C(X, Y)$ of a code is given by the polynomial

$$W_C(X, Y) = \sum_{w=0}^n A_w X^{n-w} Y^w,$$

where A_w is the number of codewords of Hamming weight w in the code C . The weight enumerator of an MDS code is unique. It is given by $A_0 = 1$, $A_w = 0$ for $0 < w < d$, and

$$\begin{aligned} A_w &= \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} (q^{w-d+1-j} - 1) \\ &= \binom{n}{w} (q-1) \sum_{j=0}^{w-d} (-1)^j \binom{w-1}{j} q^{w-d-j}, \end{aligned} \quad (2)$$

for $d \leq w \leq n$ (see, *e. g.*, [12, Ch. 11, §3, Theorem 6]).

While this formula allows us to compute the weight enumerator of any MDS code, it is not obvious for which weights w there is a codeword of weight w in the code C , *i. e.*, $A_w > 0$.

III. TRIVIAL MDS CODES

Statement (1) is clearly true for the trivial MDS codes with parameters $[n, 1, n]_q$ and $[n, n, 1]_q$ over any field \mathbb{F}_q . For the dual of the repetition code, we have the first counterexample.

Theorem 1: The code with parameters $[n, n-1, 2]_q$ contains words of all weights $2, \dots, n$ if and only if $q > 2$ or $n = 2$.

Proof: The dual of the binary repetition code of length n is an even weight code with parameters $[n, n-1, 2]_2$. It contains only words of even weight, so statement (1) is false for $q = 2$ and $n > 2$.

Now let $q > 2$. The codewords of the code C with parameters $[n, n-1, 2]_q$ are exactly those vectors for which the sum of the coefficients is zero. We prove by induction that C contains vectors of all weights $w = 2, \dots, n$. For $w = 2$, the code contains the vector $\mathbf{v}^{(2)} = (1, -1, 0, 0, \dots, 0)$. Now assume that there exists a vector $\mathbf{v}^{(w)} \in C$ with $\text{wt}(\mathbf{v}^{(w)}) = w < n$. Without loss of generality, let the first w coordinates of $\mathbf{v}^{(w)}$ be non-zero. We construct a vector $\mathbf{v}^{(w+1)} \in C$ with $\text{wt}(\mathbf{v}^{(w+1)}) = w + 1$ as follows. Replace the non-zero element $v_w^{(w)}$ by a different non-zero element of \mathbb{F}_q to obtain a vector \mathbf{x} with $\sum_i x_i = s \neq 0$. Next, replace $x_{w+1} = 0$ by $-s$. The resulting vector is the desired new codeword $\mathbf{v}^{(w+1)}$. ■

IV. MDS CODES OF LENGTH $n \leq q$

In this section we show that statement (1) is true for all MDS codes of length $n \leq q$. The main tool is a relation, due to Delsarte [5], between the external distance s' of a code and the covering radius which we define first.

Definition 2 (Covering radius): The covering radius $\rho(C)$ of a linear code with parameters $[n, k, d]_q$ is the maximal distance of any vector in \mathbb{F}_q^n to the code C , *i. e.*

$$\rho(C) = \max_{\mathbf{v} \in \mathbb{F}_q^n} \min_{\mathbf{c} \in C} d(\mathbf{v}, \mathbf{c}).$$

Definition 3 (External distance): The external distance s' of a linear code with parameters $[n, k, d]_q$ is the number of non-zero weights in the dual code $C' = C^\perp$, *i. e.*

$$s' = |\{i: i = 1, \dots, n | B_i \neq 0\}|,$$

where B_i denotes the number of codewords of weight i in the dual code C^\perp . The dual code C^\perp is given by

$$C^\perp = \{\mathbf{v} \in \mathbb{F}_q^n: \mathbf{v} \cdot \mathbf{c} = \sum_{i=1}^n v_i c_i = 0 \forall \mathbf{c} \in C\}.$$

Theorem 4 (External distance bound [5, Theorem 3.3]):

For a code with parameters $[n, k, d]_q$ with external distance s' , every vector in \mathbb{F}_q^n is at distance less than or equal to s' from at least one codeword. Hence, for the covering radius $\rho(C)$, we have $\rho(C) \leq s'$.

This implies that statement (1) holds for codes C for which the covering radius $\rho(C^\perp)$ of the dual code C^\perp is at least k . Then by Theorem 4, the number s of nonzero weights in C is at least $s \geq k$. Trivially, we have $s \leq n - d + 1$ and therefore $s \leq k$ for MDS codes.

Furthermore, we use the following ‘‘supercode lemma’’ which can, *e. g.*, be found in [3, Lemma 8.2.1].

Lemma 5: Let $C_0 \subset C$ and $D(C_0, C)$ be the maximum distance of a vector in C to C_0 . Then $\rho(C_0) \geq D(C_0, C)$. In particular, $\rho(C_0) \geq d(C)$.

With this preparation, we are ready to prove the following.

Theorem 6: Any MDS code with parameters $[n, k, d]_q$ of length $n \leq q$ has k nonzero weights.

Proof: Trivially, a code with minimum distance $n - k + 1$ can have at most k non-zero weights. The extended narrow-sense Reed-Solomon (RS) codes with parameters $[q, k, q - k + 1]_q$ form a sequence of nested MDS codes [11, Theorem 5.3.2]. Shortening these codes, we obtain sequences of nested MDS codes $C_{n,k} = [n, k, n - k + 1]_q$ with $C_{n,k} \subset C_{n,k+1}$ for any length $n \leq q$. For the dual codes we have $C_{n,k}^\perp = [n, n - k, k + 1]_q \subset C_{n,k-1}^\perp = [n, n - k + 1, k]_q$. By Lemma 5 the code $C_{n,k}^\perp$ has covering radius $\rho(C_{n,k}^\perp) \geq k$. Using Theorem 4, it follows that $s \geq k$, *i. e.*, the code C has at least k non-zero weights. Recall that the weight distribution of an MDS code with parameters $[n, k, d]_q$ is uniquely determined by its parameters. Hence the result does not only hold for the MDS codes derived from extended RS codes, but for all MDS codes. ■

Remark 7: The covering radius of extended RS codes is also given in [3, Theorem 10.5.7]. The lower bound on the

covering radius of MDS codes of length $n \leq q$ can also be found in [6].

V. CODES OF LENGTH $q + 1$

In [6, Theorem 2] it has been shown that the covering radius of some MDS codes with parameters $[q + 1, k, d]_q$ is $\rho(C) = d - 2$. In this case, Theorem 4 only implies that there are at least $k - 1$ non-zero weights, and there is indeed a family of MDS codes of length $q + 1$ for which statement (1) is false, namely the simplex codes with parameters $[q + 1, 2, q]_q$ over \mathbb{F}_q which contain only words of weight zero or q (see, *e. g.*, [11, Theorem 2.7.5]).

Furthermore, if $\rho(C) = d - 2$, we can not have a sequence of nested MDS codes of length $q + 1$ with co-dimension one. However, there are sequences of nested MDS codes with co-dimension two. For this, we recall the explicit construction of MDS codes of length $q + 1$ as cyclic or consta-cyclic codes (see also [8], [14]).

Theorem 8: For any k , $1 \leq k \leq q + 1$, there exists an MDS code $C_{q+1,k}$ over \mathbb{F}_q with parameters $[q + 1, k, q - k + 2]_q$ that is either cyclic or consta-cyclic. The codes of even dimension and the codes of odd dimension form two sequences of nested codes, *i. e.*, $C_{q+1,k} \subset C_{q+1,k+2}$.

Proof: Let ω denote a primitive element of \mathbb{F}_{q^2} . Hence $\alpha := \omega^{q-1}$ is a primitive $(q + 1)$ -th root of unity.

First we consider the case when $q + 1 - k$ is odd. We define the following polynomial of degree $2\mu + 1$:

$$g_1(z) := \prod_{i=-\mu}^{\mu} (z - \alpha^i).$$

Its zeros α^i and α^{-i} are conjugates of each other since $\alpha^q = \alpha^{-1}$. Hence, $g_1(z)$ is a polynomial over \mathbb{F}_q . The resulting cyclic code C over \mathbb{F}_q has length $q + 1$ and dimension $q - 2\mu$. The generator polynomial $g_1(z)$ has $2\mu + 1$ consecutive zeros, so the BCH bound yields $d \geq 2\mu + 2$. Therefore C is an MDS code $[q + 1, q - 2\mu, 2\mu + 2]_q$.

If $q + 1 - k$ is even and q is even too, the polynomial

$$g_2(z) := \prod_{i=q/2-\mu}^{q/2+1+\mu} (z - \alpha^i) = \prod_{i=q/2-\mu}^{q/2} (z - \alpha^i)(z - \alpha^{-i})$$

has degree $2\mu + 2$. It is a polynomial over \mathbb{F}_q with $2\mu + 2$ consecutive zeros, so the resulting code is an MDS code with parameters $[q + 1, q - 1 - 2\mu, 2\mu + 3]_q$.

Finally, if $q + 1 - k$ is even and q is odd, consider the polynomial

$$g_3(z) := \prod_{i=1}^{\mu} (z - \omega\alpha^i)(z - \omega\alpha^{1-i})$$

of degree 2μ . The roots $\omega\alpha^i$ and $\omega\alpha^{1-i}$ are conjugates of each other as $(\omega\alpha^i)^q = \omega^{(1+(q-1)i)q} = \omega^{q+(1-q)i} = \omega^{1+(q-1)(1-i)} = \omega\alpha^{1-i}$, so $g_3(z)$ is a polynomial over \mathbb{F}_q . Furthermore, $g_3(z)$ divides $z^{q+1} - \omega^{q+1} \in \mathbb{F}_{q^2}[z]$ as $(\omega\alpha^i)^{q+1} = \omega^{q+1}$. Therefore $g_3(z)$ defines a consta-cyclic code C of length $q + 1$ and dimension $q + 1 - 2\mu$ over \mathbb{F}_q . Note that C can be considered as a shortened subcode of the cyclic code of length $q^2 - 1$ over \mathbb{F}_{q^2} generated by $g_3(z)$. For

the latter, the BCH bound yields $d \geq 2\mu + 1$, hence C is an MDS code with parameters $[q + 1, q + 1 - 2\mu, 2\mu + 1]_q$.

The statement $C_{q+1,k} \subset C_{q+1,k+2}$ follows from the particular form of the polynomials $g_i(z)$. ■

Remark 9: Theorem 8 is a slightly modified version of Theorem 9 in [12, Ch. 11, §5]. There only cyclic codes are considered; the construction fails when both q and k are odd (see also the preface to the third printing of [12]).

Theorem 10: An MDS code with parameters $[q + 1, k, d]_q$ has k nonzero weights, except when $k = 2$.

Proof: The statement is clearly true for $k = 1$, and it does not hold for $k = 2$ since the code with parameters $[q + 1, 2, q]_q$ is a simplex code. For $k > 2$, let $C^{(p)} = [q, k, d - 1]_q$ and $C^{(s)} = [q, k - 1, d]_q$ be the MDS codes obtained by puncturing and shortening of C , respectively. Without loss of generality, we may assume that we have deleted the last position. By Theorem 6, $C^{(s)}$ contains codewords of all $k - 1$ weights $q - k + 2, \dots, q$. Appending zero to the codewords in $C^{(s)}$, we obtain a subcode of C with the same weight distribution as that of $C^{(s)}$. Thus it remains to show that C contains a word of weight $q + 1$. If the dimension of C is odd, by Theorem 8 the code C contains a subcode $[q + 1, 1, q + 1]_q$ and hence a word of weight $q + 1$. If the dimension of C is even, we know that statement (1) is false for dimension $k = 2$. If the dimension is at least four, the code C contains a subcode $[q + 1, 4, q - 2]_q$. For this code of dimension four, using (2) we compute

$$A_{q+1} = (q - 1) \sum_{j=0}^3 (-1)^j \binom{q}{j} q^{3-j} = \frac{1}{3} q (q - 1) (q^2 - 1).$$

This shows that the code with parameters $[q + 1, 4, q - 3]_q$ contains words of weight $q + 1$. ■

VI. CODES OF LENGTH $q + 2$

MDS codes of length $q + 2$ are known for $q = 2^m$, and $k = 3$ or $k = 2^m - 1$ (see, e. g., [12, Theorem 10, Ch. 11, §5]). For $m > 1$, a generator matrix or parity check matrix is given by

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 & 0 \\ 1 & \alpha & \alpha^2 & \alpha^4 & \dots & \alpha^{q-2} & 0 & 1 & 0 \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & \dots & \alpha^{2(q-2)} & 0 & 0 & 1 \end{pmatrix}, \quad (3)$$

where α is a primitive element of the field $GF(2^m)$.

First consider the code with parameters $[2^m + 2, 3, 2^m]_{2^m}$. Assume that there is a codeword $v \in C$ of weight $n - 1 = 2^m + 1$. Shortening C at the position where v is zero, we get a code $C^{(s)} = [2^m + 1, 2, 2^m]_{2^m}$ which contains a codeword v' of weight $2^m + 1$. But we already know that the code $C^{(s)}$ is a q -ary simplex code which does not contain a codeword of weight $2^m + 1$. Hence the code with parameters $[2^m + 2, 3, 2^m]_{2^m}$ does not contain a word of weight $2^m + 1$. Using (2), the non-zero coefficients of the weight enumerator are computed as $A_{2^m} = (2^{2^m} - 1)(2^{m-1} + 1)$, $A_{2^m+2} = 2^{m-1}(2^m - 1)^2$, and $A_0 = 1$.

Next, consider the dual code with parameters $[2^m + 2, 2^m - 1, 4]_{2^m}$ with the parity check matrix H given in (3).

Theorem 11: The MDS code with parameters $[2^m + 2, 2^m - 1, 4]_{2^m}$ contains words of all weights $w = 4, \dots, 2^m + 2$ if and only if $m \neq 2$.

Proof: For $m = 1$, the code is the binary repetition code $[4, 1, 4]_2$ for which the statement holds. For $m = 2$, we obtain the hexacode, a famous two-weight code (see [4, Chapter 3, (2.5.2)]). So let $m > 2$. Similar to the proof of Theorem 10, considering the shortened code $C^{(s)} = [2^m + 1, 2^m - 2, 4]_{2^m}$ we find that the code C contains words of all weights $w = 4, \dots, 2^m + 1$. It remains to show that C contains a word v of weight $2^m + 2$. Consider the following vector

$$v = (\alpha, \underbrace{1, \dots, 1}_{q-2}, \alpha, \alpha + 1, \alpha + 1),$$

where $q = 2^m$. In order to show that v is in the kernel of H and hence $v \in C$, we note that $\sum_{i=0}^{q-2} (\alpha^j)^i = 0$ for $0 < j < q - 1$. ■

VII. CONCLUSIONS

In summary, we have the following:

- The trivial MDS codes with parameters $[n, n, 1]_q$, $[n, 1, n]_q$, and $[n, n - 1, 2]_q$ have k non-zero weights with the exception of the dual of the binary repetition code of length $n > 2$ which contains only words of even weights.
- The MDS codes with parameters $[n, k, d]_q$ of length $n \leq q + 1$ have k non-zero weights, with the exception of the q -ary simplex code with parameters $[q + 1, 2, q]_q$ which contains only words of weight zero or q .
- For $m \neq 2$, the codes with parameters $[2^m + 2, 2^m - 1, 4]_{2^m}$ have $k = 2^m - 1$ non-zero weights. These codes are quasi perfect with covering radius 2.
- The code with parameters $[2^m + 2, 3, 2^m]_{2^m}$ has only non-zero codewords of weight 2^m and $2^m + 2$, with $A_{2^m} = (2^{2^m} - 1)(2^{m-1} + 1)$ and $A_{2^m+2} = 2^{m-1}(2^m - 1)^2$. These two-weight codes are known as family TF1 in [2].

Our result covers the parameters of all non-trivial MDS codes given by the MDS conjecture. In general, if a non-trivial MDS code with parameters $[n, k, d]_q$ exists, then $2 \leq k \leq \min\{n - 2, q - 1\}$ and $n \leq q + k - 1 \leq 2q - 2$ (see [11, Corollary 7.4.4]).

Finally, we note that our result confirms the construction of quantum MDS codes given in [7], [14] as statement (1) holds for all MDS codes used therein to derive shortened quantum codes.

APPENDIX: AN ALTERNATIVE APPROACH

After the preliminary draft of this paper was made public, we have received useful comments and suggestions, some of which are instructive to better understand the weights in MDS codes.

Ludo Tolhuizen showed in 1988 in an unpublished research report that $A_w \geq f(q, n, d, w)$ where

$$f(q, n, d, w) = \binom{n}{w} (q - d)(q - 1)^{w-d} > 0$$

for $d < q$.

A referee observed that most results shown in this paper can also be derived through a careful analysis of Equation (2). Let us rewrite the said equation as

$$A_w = \binom{n}{w} (q-1) \sum_{j=0}^{w-d} (-1)^j a_j, \text{ with } a_j = \binom{w-1}{j} q^{w-d-j} \quad (4)$$

for $d \leq w \leq n$ and investigate the cases for which $A_w = 0$.

First we show that for the $[n, n-1, 2]_2$ -code C with $n > 2$, $A_w = 0$ if and only if w is odd. Note that if l is an odd positive integer and

$$S := \sum_{j=0}^l (-1)^j \binom{l+1}{j} 2^{l-j},$$

we have

$$2S + 1 = \sum_{j=0}^{l+1} (-1)^j \binom{l+1}{j} 2^{l+1-j} = (2-1)^{l+1} = 1,$$

which implies that $S = 0$. If l is an even positive integer, it follows that

$$S := \sum_{j=0}^l (-1)^j \binom{l+1}{j} 2^{l-j} = l + 1.$$

Thus, we have an alternative proof for the assertion regarding the dual of the binary repetition code in Theorem 1.

For the nontrivial cases, observe that

$$R_j := \frac{a_j}{a_{j+1}} = q \frac{j+1}{w-1-j}.$$

Clearly, for $w \leq q$ we have $R_j > 1$, which implies that $A_w > 0$ for $n \leq q$.

For $n = q + 1$, $R_j = 1$ if and only if $j = 0$ and $w = q + 1$ since $q(j+1) > w-1-j$ for $j \geq 1$. Hence, $d = q$ and $k = 2$. Therefore, an MDS code with parameters $[q+1, 2, q]_q$ has $A_{q+1} = 0$.

Similarly for $n = q + 2$ with $q = 2^m$. $R_j = 1$ if and only if $j = 0$ and $w = q + 1$, forcing $d = q$ and $k = 3$. There are no codewords of weight $2^m + 1$ in any MDS code with parameters $[2^m + 2, 3, 2^m]_{2^m}$.

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