Derivation of Maxwell's Equations Using Field-Impulses

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***Abstract*—Recently, new fundamental equations for electromagnetics have been presented using field-impulses as physical field-integrators. The field-impulses are physically real, causal and gauge-independent for aptly describing electromagnetics. This paper presents the derivation of Maxwell's (Faraday and Ampere) equations using field-impulses. The derivation assumes that the equations are not available in complete explicit form beforehand, and only makes references to some earlier key findings before Maxwell. It also exploits judiciously the definition and relation of field-impulses, the physical reality of their time/spatial derivatives and their mathematical (e.g. solenoidal) properties. Being the necessary and sufficient fundamental physical quantities, the field-impulses find usefulness in field derivation as well as many aspects of classical and quantum electromagnetics.**

I. Introduction

Electromagnetics has long been described using electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ governed by Maxwell's equations [1], [2]. Many have also made use of electromagnetic potentials $\boldsymbol{A}$ and ϕ to describe the fields, where $\boldsymbol{A}$ is usually called the magnetic vector potential and ϕ the electric scalar potential. Often the potentials have been regarded merely as some mathematical entities that are of no physical reality since they are not unique, gauge-dependent and not directly measurable like fields. On the other hand, the Aharonov-Bohm effect has brought to light the significance of potentials in quantum theory [3], [4]. For a long time, there has been no consensus whether fields or potentials are more fundamental physical quantities. Recently, the author has presented new fundamental equations for electromagnetics using field-impulses that are the integrators (time-integrals) of physical fields [5]. The field-impulses constitute the long sought-after fundamental physical quantities that are necessary and sufficient to aptly describe electromagnetics. They are just like fields being physically real, causal and gauge-independent, and unlike potentials that may or may not be real or causal depending on the gauge. The field-impulse equations also bear much resemblance to Maxwell's equations, preserving much symmetry, Lorentz-invariance and convenience.

At first glance, some may think that the field-impulse equations are simply the consequence of Maxwell's equations, since the former may be derived readily by carrying out the time-integrals at both sides of the latter. As a matter of fact, the field-impulse equations should be the more fundamental ones with Maxwell's equations being their derivatives. In this paper, we shall present the derivation of Maxwell's equations using field-impulses, assuming that the equations are not available in complete explicit form beforehand. In addition, just like Maxwell had to rely on some previous works of others, we shall also make references to some earlier key findings before him. Although there are many other ways for derivation of Maxwell's equations, such as using relativity and Coulomb's or Gauss's law, etc. [6], [7], they will not be discussed herein since they do not involve field-impulses explicitly.

II. Key findings before Maxwell

In this section, we review some key findings before Maxwell, noting that during his time, some of them may not (yet) be in today's complete explicit form:

- The electromotive force (EMF) induced by magnetic effect is proportional to the rate of change of magnetic field (flux), cf. Faraday's law of electromagnetic induction:

$$\text{EMF} \propto \frac{d}{dt}(\text{magnetic}). \tag{1}$$

- The induced EMF is such as to oppose the change of magnetic field (flux), cf. Lenz's law:

$$\text{EMF} \propto -\Delta(\text{magnetic}). \tag{2}$$

- (Incomplete) Ampere's law relating the magnetic field to current:

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}. \tag{3}$$

- Charge conservation and current continuity equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0. \tag{4}$$

- Gauss's law for electric field (flux density):

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho. \tag{5}$$

- The magnetic field (flux density) is solenoidal:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{6}$$

III. Derivation using field-impulses

In this section, we present the derivation of Maxwell's (Faraday and Ampere) equations using field-impulses instead of traditional fields or potentials. While referring to the findings above, the derivation exploits judiciously the definition and relation of field-impulses, the physical reality of their ownselves and time/spatial derivatives, as well as their mathematical (e.g. solenoidal) properties.

A. Derivation of Maxwell-Faraday equation

Following the force-impulse being the integrator of force in mechanics, the electric field-impulse $\boldsymbol{I}_E$ can be defined as the integrator of electric field $\boldsymbol{E}$ as

$$\boldsymbol{I}_E = \int \boldsymbol{E}\, dt, \qquad \boldsymbol{E} = \frac{\partial \boldsymbol{I}_E}{\partial t}. \tag{7}$$

Similar arguments apply to other field-impulses, flux-impulses and current integrator by replacing $\boldsymbol{E}$ in (7) with $\boldsymbol{H}$, $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{J}$. Note that based on such definition, $\boldsymbol{I}_E$ is physical just like its time derivative counterpart $\boldsymbol{E}$. Likewise, the spatial derivatives (curl) of $\boldsymbol{I}_E$ should also be physical, which may be denoted by $\boldsymbol{X}$ (to be determined) as

$$\nabla \times \boldsymbol{I}_E = \boldsymbol{X}. \tag{8}$$

Since the divergence of curl is zero, one may relate $\boldsymbol{X}$ (proportionally) to the magnetic flux density $\boldsymbol{B}$ that is also solenoidal upon referring to (6):

$$\boldsymbol{X} = \alpha \boldsymbol{B}. \tag{9}$$

Taking into account the opposing (negative) effect in (2), we can choose the factor $\alpha = -1$ to yield

$$\nabla \times \boldsymbol{I}_E = -\boldsymbol{B}. \tag{10}$$

Using the magnetic flux-impulse $\boldsymbol{I}_B$ that is the integrator of magnetic flux density $\boldsymbol{B}$, we can express (10) in the form involving time derivative like (1) as

$$\nabla \times \boldsymbol{I}_E = -\frac{\partial \boldsymbol{I}_B}{\partial t}. \tag{11}$$

Equation (11) corresponds to the field-impulse equation upon which one can take derivative to obtain Maxwell-Faraday equation readily:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{12}$$

B. Derivation of Maxwell-Ampere equation

Instead of the usual charge conservation and current continuity equation in (4), one can write

$$\rho = -\nabla \cdot \boldsymbol{I}_J \tag{13}$$

where $\boldsymbol{I}_J$ is the integrator of current $\boldsymbol{J}$. Substituting (13) into the Gauss's law (5), we have

$$\nabla \cdot \big(\boldsymbol{D} + \boldsymbol{I}_J\big) = 0. \tag{14}$$

Since the bracketed term in (14) is solenoidal, it may be expressed in terms of $\boldsymbol{Y}$ (to be determined) as

$$\boldsymbol{D} + \boldsymbol{I}_J = \nabla \times \boldsymbol{Y}. \tag{15}$$

In view of (3), $\boldsymbol{Y}$ can be identified as the magnetic field-impulse $\boldsymbol{I}_H$ that is the integrator of magnetic field $\boldsymbol{H}$:

$$\boldsymbol{Y} = \boldsymbol{I}_H. \tag{16}$$

The electric flux density $\boldsymbol{D}$ can also be related to the electric flux-impulse $\boldsymbol{I}_D$, which gives rise to

$$\nabla \times \boldsymbol{I}_H = \frac{\partial \boldsymbol{I}_D}{\partial t} + \boldsymbol{I}_J. \tag{17}$$

Equation (17) corresponds to the field-impulse equation upon which one can take derivative to obtain Maxwell-Ampere equation readily:

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}. \tag{18}$$

IV. Conclusion

This paper has presented the derivation of Maxwell's (Faraday and Ampere) equations using field-impulses instead of traditional fields or potentials. It has been assumed that the equations are not available in complete explicit form beforehand, and references have been made only to some earlier key findings before Maxwell. Some physical and mathematical arguments have been discussed to exploit the definition and relation of field-impulses, the physical reality of their time/spatial derivatives, as well as their mathematical (e.g. solenoidal) properties. The field-impulses find usefulness in field derivation as well as many aspects of classical and quantum electromagnetics, such as Lorentz force, momentum, interaction energy, wave equation, quantum-particle interaction, etc. They set forth a fundamental change in the physical entities for electromagnetics and resolve the longstanding issues pertaining to field-potential dissensus. The field-impulse equations would facilitate many subsequent applications through the computational methods adaptable from those developed previously for fields, e.g., [8]–[11].

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